The hadronic vacuum polarization contribution to the muon anomalous magnetic moment

Christoph Lehner (BNL)

RBC and UKQCD Collaborations

March 9, 2016 – LGT2016 at BNL
The anomalous magnetic moment

Potential of particle in magnetic field

\[ V(x) = -\vec{\mu} \cdot \vec{B}(x) \] (1)

with

\[ \vec{\mu} = g \left( \frac{e}{2m} \right) \vec{S}, \] (2)

where \( \vec{S} \) is the spin of the particle.

Relativistic description with classical photon (Dirac) yields

\[ g = 2 \] (3)

but taking into account QFT yields non-zero anomalous magnetic moment

\[ a = (g - 2)/2. \] (4)
The anomalous magnetic moment

These anomalous moments are measured very precisely. For the electron (Hanneke, Fogwell, Gabrielse 2008)

\[ a_e = 0.00115965218073(28) \]  \hspace{1cm} (5)

yielding the currently most precise determination of the fine structure constant

\[ \alpha = 1/137.035999157(33) \]  \hspace{1cm} (6)

via a 5-loop QED computation (Aoyama, Hayakawa, Kinoshita, Nio 2015).
The anomalous magnetic moment

\( a \neq 0 \) requires QFT: \( a \) can be expressed in terms of scattering of particle off a classical photon background

\[
(-ie) \left[ \gamma_\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q^\nu}{2m} F_2(q^2) \right]
\]

(7)

with \( F_2(0) = a \).
The muon anomalous magnetic moment promises to be useful to discover new physics beyond the standard model (SM) of particle physics.

In general, new physics contributions to $a_\ell$ are given by $a_\ell - a_\ell^{SM} \propto (m_\ell^2/\Lambda_{NP}^2)$ for lepton $\ell = e, \mu, \tau$ and new physics scale $\Lambda_{NP}$.

With $\ell = \tau$ being experimentally inaccessible, $\ell = \mu$ promises good sensitivity to new physics.

Example contributions: one-loop MSSM neutralino/sgluon and chargino/sneutrino contributions to $a_\mu$.
The muon anomalous magnetic moment

Currently a tension of more than $3\sigma$ exists:

\[
\begin{align*}
\text{Total SM prediction} & : 11 659 181.5 (4.9) \\
\text{BNL E821 result} & : 11 659 209.1 (6.3)
\end{align*}
\]

\[
a^\text{EXP}_\mu - a^\text{SM}_\mu = (27.6 \pm 8.0) \times 10^{-10} \quad (8)
\]
And a new experiment (Fermilab E989) promises a $4 \times$ reduction in experimental uncertainty:
And a new experiment (Fermilab E989) promises a $4 \times$ reduction in experimental uncertainty:

$$\ddot{\omega}_a = -\frac{q}{m} \left[ a_\mu \vec{B} - \left( a_\mu - \frac{1}{\gamma^2 - 1} \right) \frac{\vec{\beta} \times \vec{E}}{c} \right]$$
Hadronic contributions to $a_\mu$

<table>
<thead>
<tr>
<th>Contribution</th>
<th>Value $\times 10^{10}$</th>
<th>Uncertainty $\times 10^{10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>QED (5 loops)</td>
<td>11 658 471.895</td>
<td>0.008</td>
</tr>
<tr>
<td>EW</td>
<td>15.4</td>
<td>0.1</td>
</tr>
<tr>
<td><strong>HVP LO</strong></td>
<td>692.3</td>
<td><strong>4.2</strong></td>
</tr>
<tr>
<td>HVP NLO</td>
<td>-9.84</td>
<td>0.06</td>
</tr>
<tr>
<td>HVP NNLO</td>
<td>1.24</td>
<td>0.01</td>
</tr>
<tr>
<td><strong>Hadronic light-by-light</strong></td>
<td>10.5</td>
<td><strong>2.6</strong></td>
</tr>
<tr>
<td>Total SM prediction</td>
<td>11 659 181.5</td>
<td>4.9</td>
</tr>
<tr>
<td>BNL E821 result</td>
<td>11 659 209.1</td>
<td>6.3</td>
</tr>
<tr>
<td>Fermilab E989 target</td>
<td></td>
<td>$\approx 1.6$</td>
</tr>
</tbody>
</table>

A reduction of uncertainty for HVP and HLbL is needed. For HLbL only model estimations exist. ⇒ First-principles non-perturbative determination desired.
The hadronic vacuum polarization from the lattice
On-going efforts by ETMC, HPQCD+MILC, RBC+UKQCD, ... 

HPQCD2016(CON) neglects the systematic error estimates for the HVP disconnected and QED/isospin-breaking corrections.
The Hadronic Vacuum Polarization

- Quark-connected piece with $>90\%$ of the contribution with by far dominant part from up and down quark loops

- Quark-disconnected piece with $\approx 1.5\%$ of the contribution ($1/5$ suppression already through charge factors); arXiv:1512.09054

- QED and isospin-breaking corrections, estimated at the few-per-cent level
HVP quark-connected contribution

Biggest challenge is to control statistics and potentially large finite-volume errors (Estimated at $O(10\%)$ Aubin et al. 2015)

Finite-volume errors are exponentially suppressed in the simulation volume but seem to be sizeable in QCD boxes with $m_\pi L = 4$

Statistics: for strange and charm solved issue, for up and down quarks existing methodology (such as HPQCD moments approach) less effective
HVP quark-connected contribution

Starting from

\[ \sum_x e^{iqx} \langle J_\mu(x) J_\nu(0) \rangle = (\delta_{\mu\nu} q^2 - q_\mu q_\nu) \Pi(q^2) \] (9)

with vector current \( J_\mu(x) = i \sum_f Q_f \bar{\Psi}_f(x) \gamma_\mu \Psi_f(x) \) and using the subtraction prescription of Bernecker-Meyer 2011

\[ \Pi(q^2) - \Pi(q^2 = 0) = \sum_t \left( \frac{\cos(qt) - 1}{q^2} + \frac{1}{2} t^2 \right) C(t) \] (10)

with \( C(t) = \frac{1}{3} \sum_{\vec{x}} \sum_{j=0,1,2} \langle J_j(\vec{x}, t) J_j(0) \rangle \) we may write

\[ a_{\mu}^{\text{HVP}} = \sum_{t=0}^\infty w_t C(t) , \] (11)

where \( w_t \) captures the QED part of the diagram.
Integrand $w_T C(T)$ for the light-quark connected contribution:

$m_\pi = 140$ MeV, $a = 0.11$ fm (RBC/UKQCD $48^3$ ensemble)

Statistical noise from long-distance region (HPQCD2016 only used lattice data up to 0.5fm–1.5fm)
A closer look at the NLO FV ChPT prediction (1-loop sQED):

We show the partial sum $\sum_{t=0}^{T} w_t C(t)$ for different geometries and volumes:

![Graph showing the partial sum for different geometries and volumes.](image-url)
Compare difference of integrand of $48 \times 48 \times 96 \times 48$ (spatial) and $48 \times 48 \times 48 \times 96$ (temporal) geometries with NLO FV ChPT: 

$m_\pi = 140$ MeV, $a = 0.11$ fm (RBC/UKQCD $48^3$ ensemble)
Similar agreement from Aubin et al. 2015 (arXiv:1512.07555v2)

MILC lattice data with \( m_\pi L = 4.2, m_\pi \approx 220 \text{ MeV} \); Plot difference of \( \Pi(q^2) \) from different irreps of 90-degree rotation symmetry of spatial components versus NLO FV ChPT prediction (red dots)

While the absolute value of \( a_\mu \) is poorly described by the two-pion contribution, the volume dependence may be described sufficiently well to use ChPT to control FV errors at the 1\% level; this needs further scrutiny
Regarding statistics:

It is potentially helpful to define stochastic estimator for strict upper and lower bounds of $a_\mu$, which has reduced statistical fluctuations C.L. et al. 2016

up and down loop shown here: data shown here is from early stages of computation with 5% statistical error, currently at around 2% statistical error. Within the next year our current setup can produce a continuum limit with 1% statistical error.
HVP quark-disconnected contribution

First results at physical pion mass with a statistical signal
RBC/UKQCD arXiv:1512.09054

Statistics is clearly the bottleneck

New stochastic estimator allowed us to get result

\[ a_\mu^{\text{HVP (LO) DISC}} = -9.6(3.3)_{\text{stat}}(2.3)_{\text{sys}} \times 10^{-10} \]  \hspace{1cm} (12)

from 20 configurations at physical pion mass and 45 propagators/configuration.

\[
C(t) = \frac{1}{3V} \sum_{j=0,1,2} \sum_{t'} \langle V_j(t + t')V_j(t') \rangle_{SU(3)}
\]  

(13)

where \( V \) stands for the four-dimensional lattice volume, \( V_\mu = (1/3)(V^{u/d}_\mu - V^s_\mu) \), and

\[
V^{f}_\mu(t) = \sum_{\vec{x}} \text{Im Tr}[D_{\vec{x},t;\vec{x},t}^{-1}(m_f)\gamma_\mu].
\]  

(14)

We separate 2000 low modes (up to around \( m_s \)) from light quark propagator as \( D^{-1} = \sum_n v^n(w^n)^\dagger + D^{-1}_{\text{high}} \) and estimate the high mode stochastically and the low modes as a full volume average Foley 2005.

We use a sparse grid for the high modes similar to Li 2010 which has support only for points \( x_\mu \) with \( (x_\mu - x^{(0)}_\mu) \mod N = 0 \); here we additionally use a random grid offset \( x^{(0)}_\mu \) per sample allowing us to stochastically project to momenta.
Combination of both ideas is crucial for noise reduction at physical pion mass!

Fluctuation of $\mathcal{V}_\mu$ ($\sigma$):

Since $C(t)$ is the autocorrelator of $\mathcal{V}_\mu$, we can create a stochastic estimator whose noise is potentially reduced linearly in the number of random samples, hence the normalization in the lower panel.
Result for partial sum $L_T = \sum_{t=0}^{T} w_t C(t)$:

For $t \geq 15$ $C(t)$ is consistent with zero but the stochastic noise is $t$-independent and $w_t \propto t^4$ such that it is difficult to identify a plateau region based only on this plot.
Resulting correlators and fit of $C(t) + C_s(t)$ to $c_\rho e^{-E_\rho t} + c_\phi e^{-E_\phi t}$ in the region $t \in [t_{\text{min}}, \ldots, 17]$ with fixed energies $E_\rho = 770$ MeV and $E_\phi = 1020$. $C_s(t)$ is the strange connected correlator.

We fit to $C(t) + C_s(t)$ instead of $C(t)$ since the former has a spectral representation.

We could use this model alone for the long-distance tail to help identify a plateau but it would miss the two-pion tail
We therefore additionally calculate the two-pion tail for the disconnected diagram in ChPT:

![Graph showing the sum of $L_T$ and $F_T$ with a plateau from which we read our final result for $\alpha_{\text{HVP (LO) DISC}}^\mu$. The lower panel compares the partial sums $L_T$ for all values of $T$ with our final result for $\alpha_{\text{HVP (LO) DISC}}^\mu$ with its statistical error band.](image)

We report our final result $\alpha_{\text{HVP (LO) DISC}}^\mu = 9.6(3.3)(2.3) \times 10^{-10}$, where the first error is statistical and the second systematic.

Before concluding, we note that our result appears to be dominated by very low energy scales. This is not surprising since the signal is expressed explicitly as difference of light-quark and strange-quark Dirac propagators. We therefore expect energy scales significantly above the strange mass to be suppressed. We already observed this above in the dominance of low modes of the Dirac operator for our signal. Furthermore, our result is statistically consistent with the one-loop ChPT two-pion contribution of Fig. 6.
We then pick a point in the potential plateau region such as $T = 20$ and use a combined estimate of the resonance model and the two-pion tail to estimate $\sum_{t=\infty}^{T} w_t C(t)$ as a systematic uncertainty.

Combined with an estimate of discretization errors, we find

$$a_{\mu}^{\text{HVP (LO) DISC}} = -9.6(3.3)_{\text{stat}}(2.3)_{\text{sys}} \times 10^{-10}. \quad (15)$$
Largely unexplored but finite-volume errors are likely substantial

New methods with potential to control large finite-volume errors in lattice QCD+QED simulations may prove useful (C* boundary conditions Lucini et al. 2015, massive QED Endres et al. 2015, QED∞ C.L. et al. Lattice 2015)

We are actively working on this measurement using technology similar to our on-going hadronic light-by-light calculation (next talk by Luchang)
First-principles determination of the HVP contribution comparable with Fermilab E989 uncertainty (0.3% uncertainty on HVP) is very challenging.

Substantial progress in the last year both for the HVP light connected and disconnected contributions.

Active effort on necessary sub-leading contributions such as QED/isospin-breaking corrections.
First-principles predictions for the HVP on time-scale of and with errors comparable to Fermilab E989 appear possible!
Thank you
Experimental setup: muon storage ring with tuned momentum of muons to cancel leading coupling to electric field

\[
\tilde{\omega}_a = -\frac{q}{m} \left[ a_\mu \vec{B} - \left( a_\mu - \frac{1}{\gamma^2 - 1} \right) \frac{\vec{\beta} \times \vec{E}}{c} \right]
\]

Because of parity violation in weak decay of muon, a correlation between muon spin and decay electron direction exists, which can be used to measure the anomalous precession frequency \( \omega_a \):

![Graph showing time distribution of electron counts for muon decays]