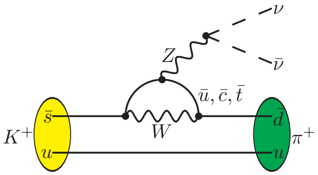
Exploratory lattice QCD study of the rare kaon decay $K^+ \to \pi^+ \nu\bar{\nu}$



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Workshop@BNL, 03/10/2016

Collaborators

- on behalf of RBC-UKQCD collaboration
- people involved in this project

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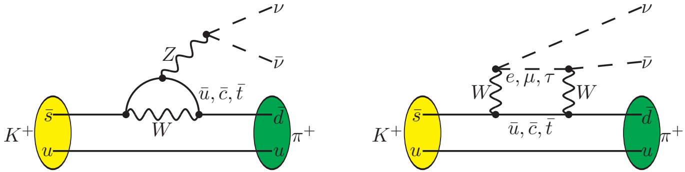
Introduction

$K^+ \to \pi^+\nu\bar{\nu}$: Experiment vs Standard model

- As FCNC process, $K \to \pi\nu\bar{\nu}$ decay through second-order weak interaction



Z-exchange

W-*W*

SM effects highly suppressed in the second order → ideal probes for NP

- Past experimental measurement is 2 times larger than SM prediction

$$\mathrm{Br}(K^+ \to \pi^+\nu\bar{\nu})_{\mathrm{exp}} = 1.73^{+1.15}_{-1.05} \times 10^{-10}$$ arXiv:0808.2459

$$\mathrm{Br}(K^+ \to \pi^+\nu\bar{\nu})_{\mathrm{SM}} = 9.11 \pm 0.72 \times 10^{-11}$$ arXiv:1503.02693

still consistent with $> 60\%$ exp. error

New experiments

- New generation of experiment: NA62 at CERN aims at
  - observation of $O(100)$ events in 2-3 years
  - 10%-precision measurement of $\mathrm{Br}(K^+ \to \pi^+ \nu \bar{\nu})$

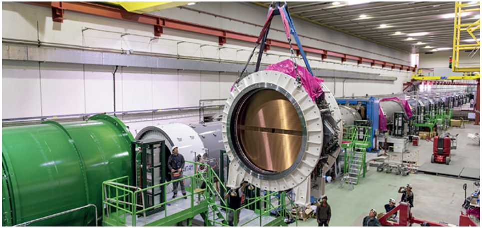
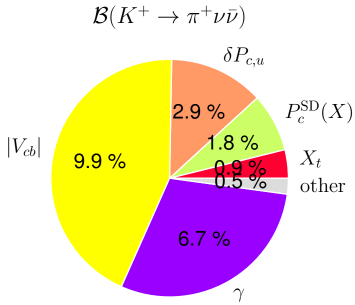


Figure: 09/2014, the final straw-tracker module is lowered into position in NA62

- $K_L \to \pi^0 \nu \bar{\nu}$
  - even more challenging since all the particles involved are neutral
  - only upper bound was set by KEK E391a in 2010
  - new KOTO experiment at J-PARC designed to observe K_L decays

SM error budget for $K^+ \to \pi^+ \nu\bar{\nu}$

Can we do better? $\Rightarrow$ continue tradition to lead exp, further reduce SM err



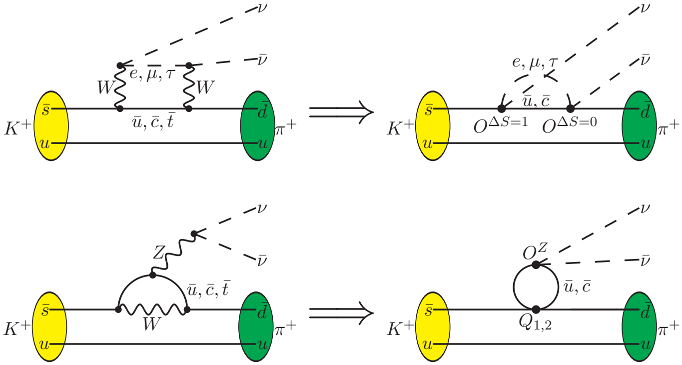
- Error budget [Buras et.al, arXiv:1503.02693]
  - $|V_{cb}|$, γ: CKM inputs for $|V_{td}|$, $|V_{ts}|$
  - $\delta P_{c,u}$: LD contribution
- Fixing CKM inputs, LD contribution is dominant theoretical uncertainty

- phenomenological ansatz invovling χPT and OPE [hep-ph/0503107] yields $\delta P_{c,u} = 0.04 \pm 0.02$ $\Rightarrow$ branching ratio enhanced by 6%
  - 50% error in $\delta P_{c,u}$ is a guess rather than a controlled error
  - $\delta P_{c,u}$ may be much larger or even smaller

Can lattice QCD do a better job?

Lattice method

Integrate out the heavy W, Z bosons and work in 4-flavor theory



W-W diagram

$$O_{q\ell}^{\Delta S=1} = (\bar{s}q)_{V-A}(\bar{\nu}\ell)_{V-A}, \quad O_{q\ell}^{\Delta S=0} = (\bar{\ell}\nu)_{V-A}(\bar{q}d)_{V-A}$$

Z-exchange diagram

$$O_q^W = C_1(\mu)Q_{1,q}(\mu) + C_2(\mu)Q_{2,q}(\mu), \quad O_\ell^Z = J_\mu^Z\, \bar{\nu}_\ell\gamma^\mu(1-\gamma_5)\nu_\ell$$

SD divergence (I)

Physical transition amplitude $K^+ \to \pi^+\nu\bar\nu$ is given by

$$\mathcal{A} = \int d^4x \, \langle \pi^+\nu\bar\nu | T[C_1^{\overline{\rm MS}} O_1^{\overline{\rm MS}}(x)\, C_2^{\overline{\rm MS}} O_2^{\overline{\rm MS}}(0)] | K^+\rangle + \langle \pi^+\nu\bar\nu | C_3^{\overline{\rm MS}} O_3^{\overline{\rm MS}}(0) | K^+\rangle$$

SD physics $\Rightarrow$ Wilson coeff $C_1^{\overline{\rm MS}}(\mu)$, $C_2^{\overline{\rm MS}}(\mu)$, $C_3^{\overline{\rm MS}}(\mu)$, known from PT

$O_3 = (\bar s d)_{V-A}(\bar\nu\nu)_{V-A}$

Lattice setup

$$O_i^{\overline{\rm MS}}(\mu) = C_i^{{\rm lat}\to\overline{\rm MS}}(\mu, a)\, O_i^{\rm lat}(a)$$

$C_i^{{\rm lat}\to\overline{\rm MS}}(\mu, a)$ has removed the lattice cutoff dependence in $O_i^{\rm lat}(a)$

However, a new lattice cutoff dependence appears in the bilocal operator

$$\int d^4x \, T[O_1^{\overline{\rm MS}}(x)\, O_2^{\overline{\rm MS}}(0)] = \int d^4x \, T[C_1^{{\rm lat}\to\overline{\rm MS}} O_1^{\rm lat}(x)\, C_2^{{\rm lat}\to\overline{\rm MS}} O_2^{\rm lat}(0)] + \bar C(a)\, O_3^{\rm lat}$$

$\bar C(a)$ term is introduced to remove the new lattice cutoff dependence

Question: How to determine $\bar C(a)$?

Non-perturbative renormalization using RI/SMOM scheme

- Lattice to $\overline{\mathrm{MS}}$ conversion for a local operator

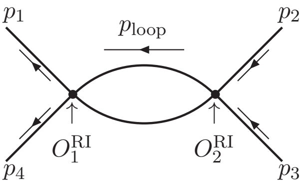
$$O_i^{\mathrm{RI}}(\mu_0) = C_i^{\mathrm{lat}\to\mathrm{RI}}(\mu_0, a)\, O_i^{\mathrm{lat}}(a) \quad \Leftarrow \text{NPR}$$
$$O_i^{\overline{\mathrm{MS}}}(\mu) = C_i^{\mathrm{RI}\to\overline{\mathrm{MS}}}(\mu, \mu_0)\, O_i^{\mathrm{RI}}(\mu_0) \quad \Leftarrow \text{PT}$$

- Lattice to $\overline{\mathrm{MS}}$ conversion for a bilocal operator
  - Step 1: Define a bilocal operator in RI/SMOM scheme

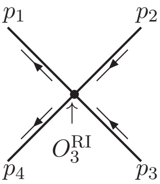
$$[O_1 O_2]^{\mathrm{RI}}(\mu_0) \equiv \int d^4x\, T\left[[O_1^{\mathrm{RI}}(\mu_0)](x)\,[O_2^{\mathrm{RI}}(\mu_0)](0)\right] - X(\mu_0, a)\, O_3^{\mathrm{RI}}(\mu_0)$$

  - Step 2: Impose the renormalization condition for amputated Green's function at external momentum $p_i^2 = \mu_0^2$

$$\langle [O_1 O_2]^{\mathrm{RI}} \rangle \Big|_{p_i^2=\mu_0^2} =$$



$$- X(\mu_0, a) \times$$



$$= 0$$

Non-perturbative renormalization using RI/SMOM scheme

- Lattice to $\overline{\mathrm{MS}}$ conversion for a bilocal operator
  - Step 3: Write $\overline{\mathrm{MS}}$ bilocal operator in terms of RI/SMOM operators

$$\int d^4x\, T\Big[[O_1^{\overline{\mathrm{MS}}}(\mu)](x)\,[O_2^{\overline{\mathrm{MS}}}(\mu)](0)\Big] = C_1^{\mathrm{RI}\to\overline{\mathrm{MS}}}(\mu,\mu_0)\,C_2^{\mathrm{RI}\to\overline{\mathrm{MS}}}(\mu,\mu_0)[O_1 O_2]^{\mathrm{RI}}(\mu_0) + Y(\mu,\mu_0)\,O_3^{\mathrm{RI}}(\mu_0)$$

  - Step 4: Evaluate amputated Green's function at $p_i^2 = \mu_0^2$

$$\langle \int d^4x\, T\Big[[O_1^{\overline{\mathrm{MS}}}(\mu)](x)\,[O_2^{\overline{\mathrm{MS}}}(\mu)](0)\Big]\rangle\Big|_{p_i^2=\mu_0^2} = \left[\frac{Z_q^{\mathrm{RI}}(\mu_0)}{Z_q^{\overline{\mathrm{MS}}}(\mu)}\right]^{\frac{n}{2}} Y(\mu,\mu_0)\langle O_3^{\mathrm{RI}}\rangle\Big|_{p_i^2=\mu_0^2}$$

- Short summary

  $\overline{\mathrm{MS}}$ bilocal operator $\Rightarrow$ RI bilocal + RI local operators

  RI bilocal operator $\Rightarrow$ bare lattice bilocal + lattice local operators

Momentum setup (I)

- $K^+ \to \pi^+\nu\bar{\nu}$: Set kaon momentum $\vec{p}_K = \vec{0}$
- To satisfy 1) on-shell condition for $K, \pi, \nu, \bar{\nu}$, 2) 4-momentum conservation

$$\vec{p}_\pi, \vec{p}_\nu, \vec{p}_{\bar{\nu}} \neq \frac{2\pi}{L}\vec{n}$$

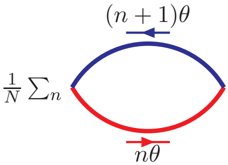
- N copy of periodic gauge fields to enlarge the lattice volume
  - quark fields live on the larger volume, obeying $q(x+NL) = q(x)$
  - gauge fields live on the smaller volume, obeying $U_\mu(x+L) = U_\mu(x)$
- The larger volume allows us to access smaller momentum $\theta = \frac{2\pi}{NL}$.
- A typical 2pt function with momentum θ

$$\frac{1}{NL}\sum_{\vec{k}=\frac{2\pi}{NL}n} \frac{1}{i(\not{k}+\not{\theta})+m}\frac{1}{i\not{k}+m}$$

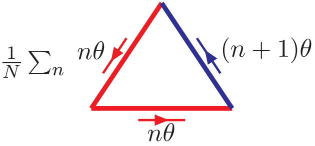
$$= \frac{1}{N}\left(\frac{1}{L}\sum_{\vec{k}=\frac{2\pi}{L}n} \frac{1}{i(\not{k}+\not{\theta})+m}\frac{1}{i\not{k}+m} + \frac{1}{L}\sum_{\vec{k}=\frac{2\pi}{L}n} \frac{1}{i(\not{k}+2\not{\theta})+m}\frac{1}{i(\not{k}+\not{\theta})+m} + \cdots\right)$$

Momentum setup (II)

- A typical 2pt function with momentum θ

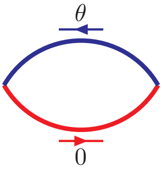


2pt function

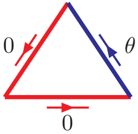


3pt function

- To enlarge from L to NL $\Rightarrow$ propagator with TBCs θ, 2θ, $\cdots$, $(N-1)\theta$
- If we only want to access small momentum θ, then one TBC is sufficient



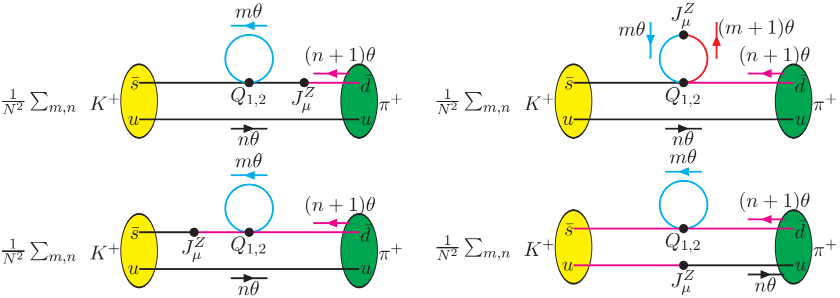
2pt function



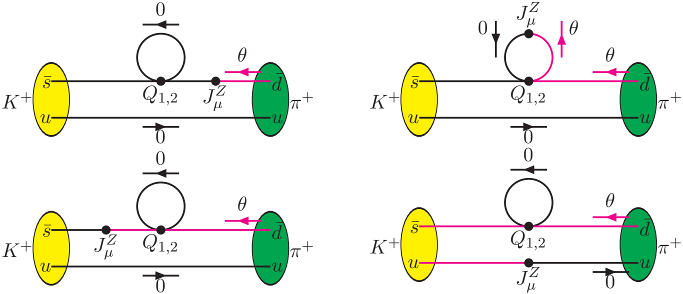
3pt function

- According to χPT [Sachrajda & Villadoro, 04], missing terms contribute only exponentially-suppressed volume correction

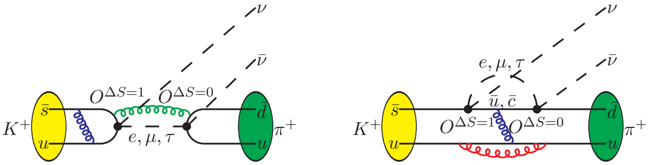
N copy of gauge fields, sum over m,n



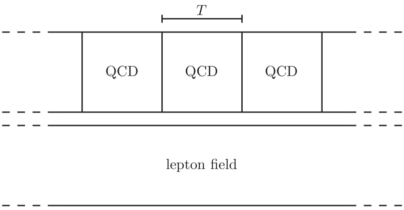
Twisted boundary condition, $m = 0$, $n = 0$



Lepton propagator



- Use overlap fermion for lepton propagator
  - Time extent for lepton is infinite



$$[D_\ell^{-1}]^{(T=\infty)}(t,\vec{x}) = \int \frac{dp_4}{2\pi}\frac{1}{L^3}\sum_{\vec{p}} D_\ell^{-1}(p_4,\vec{p})$$

Exponentially growing contamination (I)

Given a non-local matrix element in Minkowski space

$$\begin{aligned}\mathcal{T}^M &= i\int dt\,\langle f|T[O^{\Delta S=1}(t)O^{\Delta S=0}(0)]|K\rangle \\ &= \sum_{n_s}\frac{\langle f|O^{\Delta S=1}|n_s\rangle\langle n_s|O^{\Delta S=0}|K\rangle}{E_{n_s}-E_f+i\epsilon}+\sum_n\frac{\langle f|O^{\Delta S=0}|n\rangle\langle n|O^{\Delta S=1}|K\rangle}{E_n-E_K-i\epsilon}\end{aligned}$$

In Euclidean space

$$\begin{aligned}\mathcal{T}^E &= \sum_{t=-T_a}^{T_b}\langle f|T[O^{\Delta S=1}(t)O^{\Delta S=0}(0)]|K\rangle \\ &= \sum_{n_s}\frac{\langle f|O^{\Delta S=1}|n_s\rangle\langle n_s|O^{\Delta S=0}|K\rangle}{E_{n_s}-E_f}\left(1-e^{(E_f-E_{n_s})T_b}\right) \\ &\quad-\sum_n\frac{\langle f|O^{\Delta S=0}|n\rangle\langle n|O^{\Delta S=1}|K\rangle}{E_n-E_K}\left(1-e^{(E_K-E_n)T_a}\right)\end{aligned}$$

If $E_n < E_K$, remove exp growing contamination, $\mathcal{T}^E \Rightarrow \mathcal{T}^M$

Exponentially growing contamination (II)

Perform integration at both t_1 and t_2 [N. Christ et.al., arXiv:1212.5931]

$$\sum_{t_1=t_a}^{t_b}\sum_{t_2=t_a}^{t_b}\langle f|T[O^{\Delta S=1}(t_2)O^{\Delta S=0}(t_1)]|K\rangle\, e^{m_K t_1}e^{-m_f t_1}$$
$$= \sum_{n_s}\frac{\langle f|O^{\Delta S=1}|n_s\rangle\langle n_s|O^{\Delta S=0}|K\rangle}{E_{n_s}-E_f}\left(T_{\mathrm{box}}-\frac{1-e^{(E_f-E_{n_s})T_{\mathrm{box}}}}{E_{n_s}-E_f}\right)$$
$$+\sum_{n}\frac{\langle f|O^{\Delta S=0}|n\rangle\langle n|O^{\Delta S=1}|K\rangle}{E_{n}-E_K}\left(T_{\mathrm{box}}-\frac{1-e^{(E_K-E_{n})T_{\mathrm{box}}}}{E_{n}-E_K}\right)$$

Here $T_{\mathrm{box}} = t_b - t_a + 1$ is defined as size of the integral window

Evaluate the matrix element $\langle f|O^{\Delta S=0}|n\rangle\langle n|O^{\Delta S=1}|K\rangle$ for $E_n < E_K$

Remove the exponential growing contamination, and fit with $a + bT_{box}$, the slope b is what we want

Power-law finite-volume effects

- Above two-particle threshold, $\sum_n$ and $\sum_{n_s}$ shall be replaced by $\displaystyle{\not\!\!\int}_n$ and $\displaystyle{\not\!\!\int}_{n_s}$
- For infinite volume, integral is well defined using principal value

$$\mathcal{I}^\infty = \mathcal{P} {\not\!\!\int}_n \frac{\langle f|O^{\Delta S=0}|n\rangle^{\infty\,\infty}\langle n|O^{\Delta S=1}|K\rangle}{E_K - E_n}$$

- For finite volume, energy states are always discrete, we still have

$$\mathcal{I}^L = \sum_n \frac{\langle f|O^{\Delta S=0}|n\rangle^{LL}\langle n|O^{\Delta S=1}|K\rangle}{E_K - E_n}$$

- Finite-volume correction $\mathcal{I}^\infty = \mathcal{I}^L - \delta\mathcal{I}$

$$\delta\mathcal{I} = \cot(\phi(E)+\delta(E))(\phi'(E)+\delta'(E))\langle f|O^{\Delta S=0}|\pi\pi,E\rangle^{LL}\langle\pi\pi,E|O^{\Delta S=1}|K\rangle\Big|_{E=m_K}$$

  - $\phi(E)$: a known function depending on L; $\delta(E)$: $\pi\pi$ scattering phase
  - $\cot(\phi(E)+\delta(E))$ is singular at $E=E_n$; it cancels the singularity of $\mathcal{I}^L$
- For a complete derivation of $\delta\mathcal{I}$, see

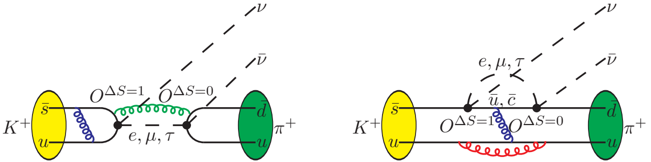
  [N. Christ, XF, G. Martinelli, C. Sachrajda, arXiv:1504.01170]

Lattice results

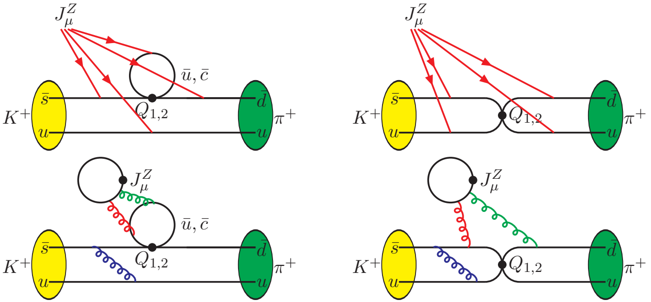
Summary of diagrams

All diagrams are calculated

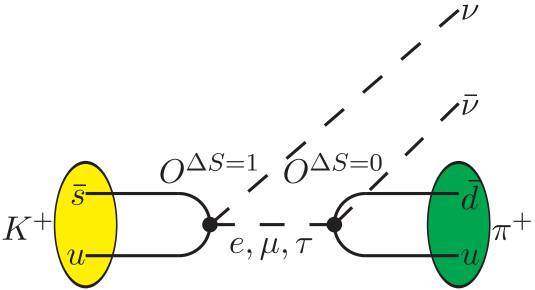
- W-W diagram:



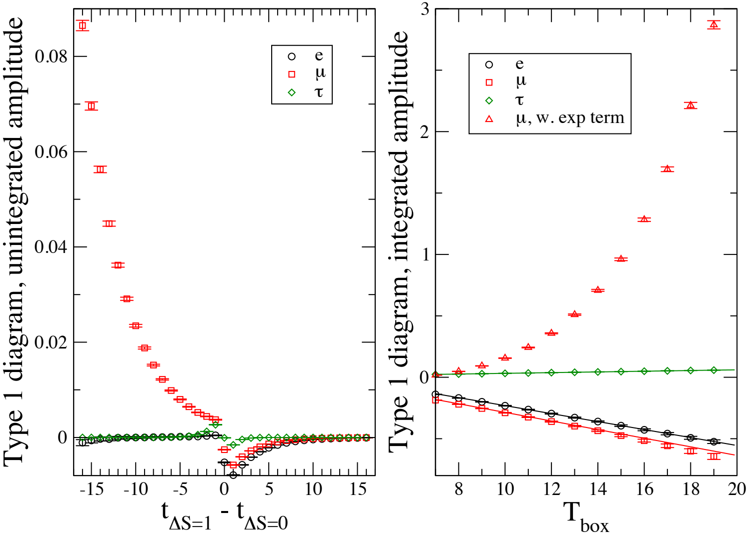
- Z-exchange diagram:



W-W: Type 1 diagram (I)

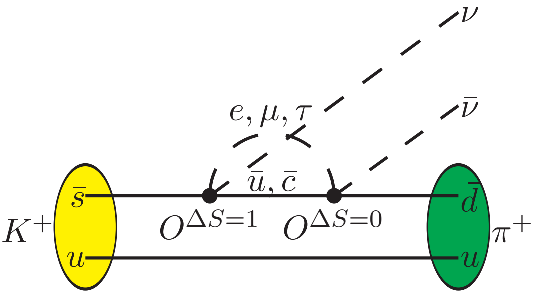


W-W: Type 1 diagram (II)

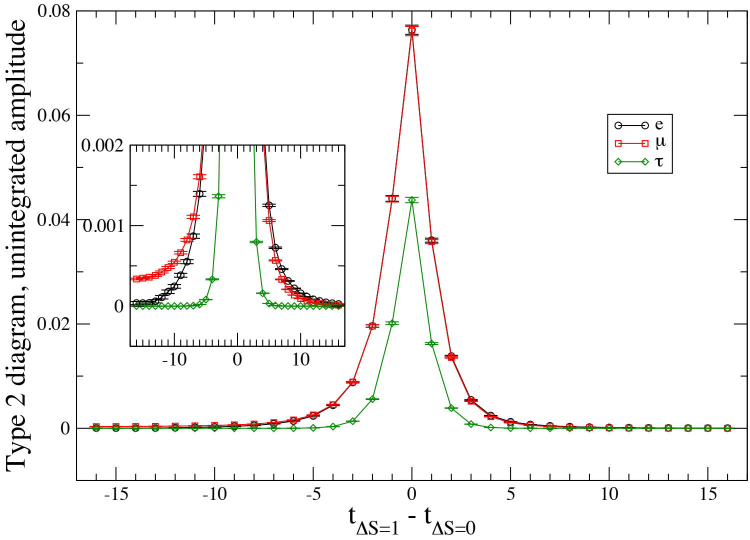


Right figure: the slope of the curve gives transition amplitude

W-W: Type 2 diagram (I)

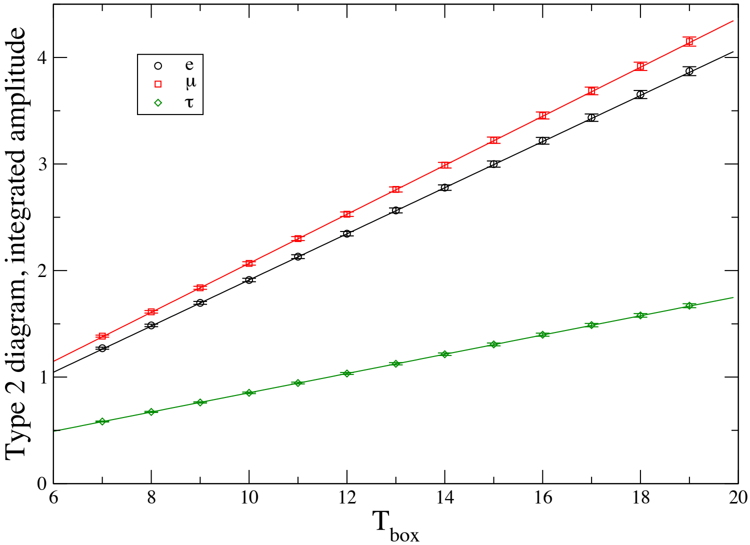


W-W: Type 2 diagram (II)



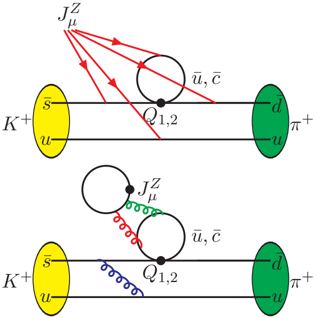
Unintegrated transition amplitude for the Type 2 diagram

W-W: Type 2 diagram (III)

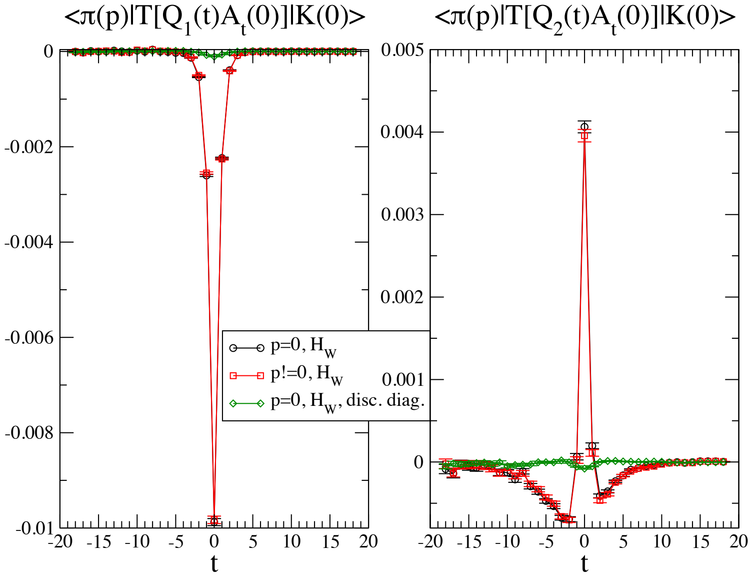


Double integrated transition amplitude for the Type 2 diagram

Z-exchange diagram (I)



Z-exchange diagram (II)



Unintegrated matrix element for Z-exchange diagram

Preliminary results

W-W diagram

F_{WW}	Type 1	model	Type 2
e	$-1.685(47) \times 10^{-2}$	$-1.740(6) \times 10^{-2}$	$7.78(17) \times 10^{-2}$
μ	$-1.818(40) \times 10^{-2}$	$-1.822(6) \times 10^{-2}$	$8.49(17) \times 10^{-2}$
τ	$1.491(36) \times 10^{-3}$	$1.471(5) \times 10^{-3}$	$2.374(74) \times 10^{-2}$

Z-exchange diagram

F_Z	conn. diag.	disc. diag.
$C_1 \cdot Q_1$	$2.143(46) \times 10^{-2}$	$0.62(7) \times 10^{-3}$
$C_2 \cdot Q_2$	$1.670(69) \times 10^{-2}$	$0.92(19) \times 10^{-3}$
$C_1 \cdot Q_1 + C_2 \cdot Q_2$	$3.834(82) \times 10^{-2}$	$1.54(24) \times 10^{-3}$

Short-distance divergences have been removed by setting $\mu < 2.15$ GeV

Statistical error controlled to be ~3% except for disc. diagrams

How far we are from destination?

Preliminary lattice result based on 800 configurations

$$P_c^{\mathrm{lat}}(\mu < 2.15\ \mathrm{GeV}) = \frac{\pi^2}{2\lambda^4 M_W^2}(\langle F_{WW}\rangle_\ell + F_Z) = 0.080(3)$$

phenomenological ansatz:

$$\delta P_{c,u} = 0.040(20)$$

Premature to make a direction comparison:

- Calculation presented here:
  $16^3 \times 32$, $m_\pi = 420$ MeV, $m_c = 860$ MeV, $a^{-1} = 1.73$ GeV
- USQCD project running on:
  $32^3 \times 64$, $m_\pi = 170$ MeV, $m_c = 750$ MeV, $a^{-1} = 1.37$ GeV
- Plan to do:
  $64^3 \times 128$, $m_\pi = 140$ MeV, $m_c = 1.3$ GeV, $a^{-1} = 2.38$ GeV

Rare kaon decay: accessible to LQCD ⇒ control systematic effects