Baryon interactions from HAL QCD and Luscher's finite volume methods

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for HAL QCD Collaboration

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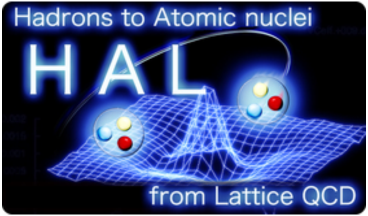
Ref. TI for HAL QCD Coll., PoS(Lattice2015), arXiv:1511.05246 [hep-lat].



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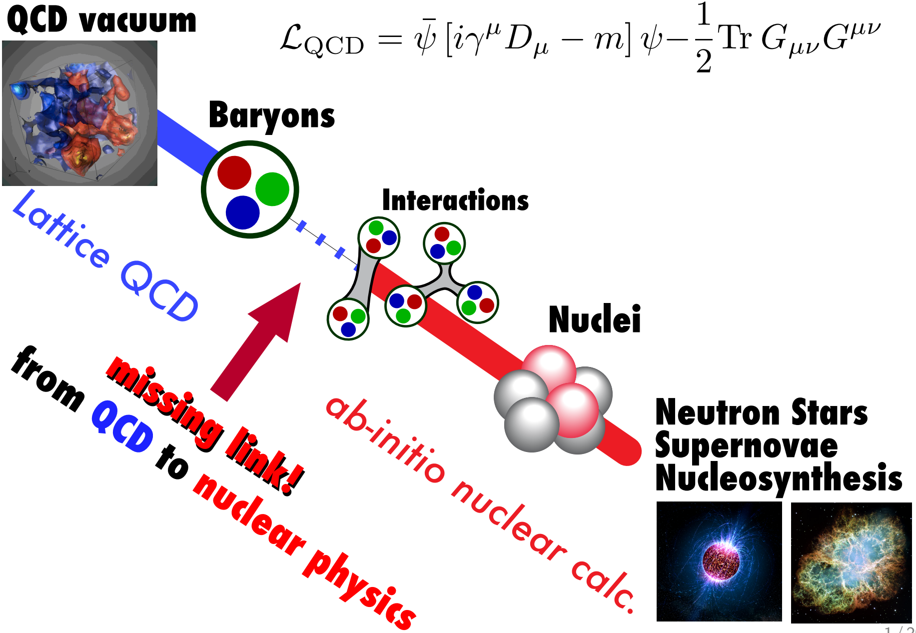
▶ inconsistency between two lattice QCD methods

	Lüscher		HAL QCD	phys. point
NN($^1S_0$)	**bound**	$\Leftrightarrow$	**unbound**	**unbound**
NN($^3S_1$)	**bound**	$\Leftrightarrow$	**unbound**	**bound**

▶ Today, we will clarify this problem,
and how to study reliable hadron interactions from lattice QCD.

Dynamics of QCD: From Quarks to Universe

$$\mathcal{L}_{\mathrm{QCD}} = \bar{\psi}\left[i\gamma^{\mu}D_{\mu} - m\right]\psi - \frac{1}{2}\mathrm{Tr}\,G_{\mu\nu}G^{\mu\nu}$$



Hadron Interaction from Lattice QCD

we know 2 methods

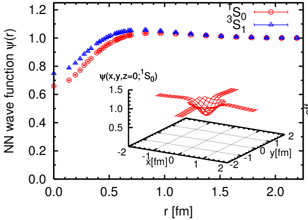
1. **Lüscher's finite volume method** — Lüscher '86, '91

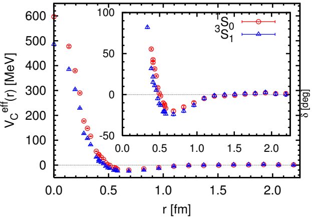
   1. energy shift of two-particle system in **"box"** ▶ 2. phase shift

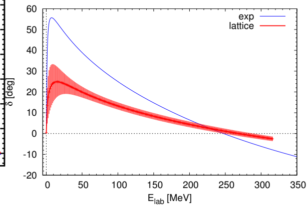
$$\Delta E_L = 2\sqrt{k^2+m^2} - 2m \implies k\cot\delta(k) = \frac{1}{\pi L}\sum_{\boldsymbol{n}\in\mathbb{Z}^3}\frac{1}{|\boldsymbol{n}|^2-(kL/2\pi)^2}$$

2. **HAL QCD method** — Ishii-Aoki-Hatsuda '07

   1. NBS wave function ▶ 2. potential ▶ 3. phase shift



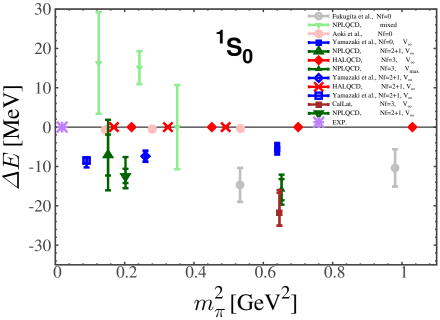


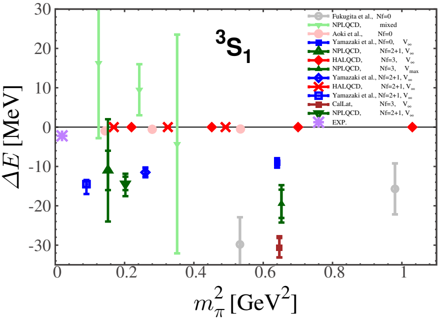


NN Interactions from Lattice QCD

	Lüscher		HAL QCD	phys. point
$^1S_0$	**bound**	$\Leftrightarrow$	**unbound**	**unbound**
$^3S_1$	**bound**	$\Leftrightarrow$	**unbound**	**bound**

- pion mass is unphysical, but there are **systematic deviations**





Lüshcer vs HAL QCD

unfortunately, systematic discrepancies are reported between

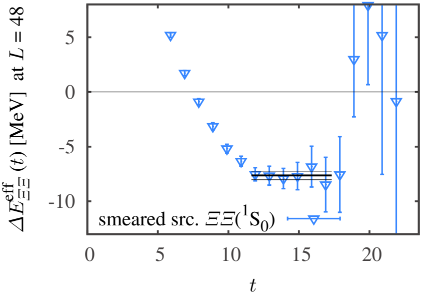
Lüscher's finite volume method and **HAL QCD method**

but those works used **different** quark mass, source (smeared and wall), ...

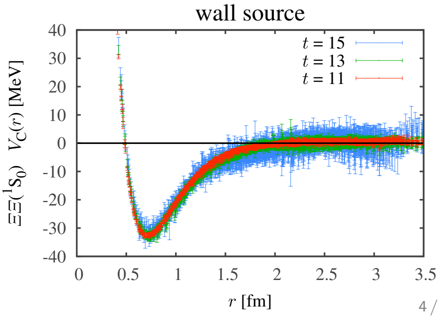
□ **check baryon interactions** from **both methods** with **the same setup**

■ which is correct ? what is the origin of discrepancies ?

■ measure energy shift



■ analyze potential



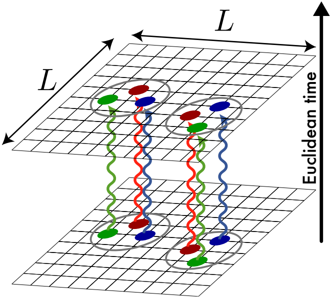
Lüscher's Finite Volume Method

- "**energy shift**" **in finite box** L^3

$$\Delta E_L = E_{BB} - 2m_B = 2\sqrt{k^2 + m_B^2} - 2m_B$$

$\Rightarrow$ **phase shift** $\delta(k)$

$$k \cot \delta(k) = \frac{1}{\pi L} \sum_{\boldsymbol{n} \in \mathbb{Z}^3} \frac{1}{|\boldsymbol{n}|^2 - (kL/2\pi)^2}$$



- **measurement**: plateau in **effective mass**

$$\Delta E_{\text{eff}}(t) = \log \frac{R(t)}{R(t+1)} \to \Delta E_L$$

$$R(t) = \frac{G_{BB}(t)}{\{G_B(t)\}^2} \to \exp\left[-\left(E_{BB} - 2m_B\right) t\right]$$

with $G_{BB}(t)(G_B(t))$: BB(B) correlators

- NN($^1S_0$) energy shift

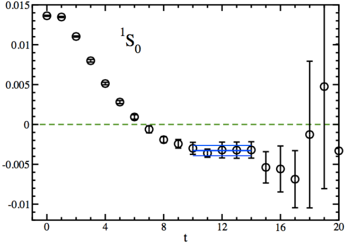
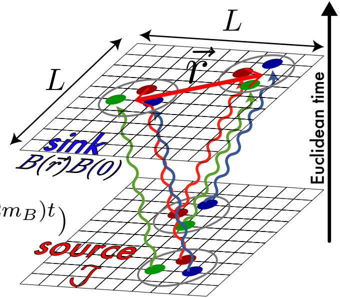


Fig. Yamazaki et al. '12

- effective mass plot

$\Longrightarrow$ **standard method** in "single particle state"

Time-dependent HAL QCD Method



■ Nambu-Bethe-Salpeter correlation function

$$R(\vec{r},t) \equiv \frac{\langle 0|T\{B(\vec{x}+\vec{r},t)B(\vec{x},t)\}\bar{\mathcal{J}}(0)|0\rangle}{\{G_B(t)\}^2}$$
$$= \sum_n A_n \psi_n(\vec{r}) e^{-(E_n-2m_B)t} + \mathcal{O}(e^{-(E_{\rm th}-2m_B)t})$$

▶ in Lüscher's method: $\sum_{\vec{r}} R(\vec{r},t) = R(t) \Longrightarrow \exp\left[-(E_{BB}-2m_B)t\right]$

□ each $\psi_n(\vec{r})e^{-E_n t} \equiv \langle 0|T\{B(\vec{x}+\vec{r},t)B(\vec{x},t)\}|2B,n\rangle$ satisfies

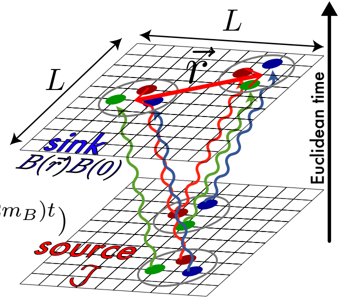
$$\left[\frac{k_n^2}{m_B} - H_0\right]\psi_n(\vec{r}) = \int d\vec{r'}\, U(\vec{r},\vec{r'})\psi_n(\vec{r'})$$

with the same non-local interaction kernel $U(\vec{r},\vec{r'})$

■ R-corr. satisfies t-dep. Schrödinger-like eq. with **elastic** saturation

$$\left[\frac{1}{4m_B}\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t} - H_0\right] R(\vec{r},t) = \int d\vec{r'}\, U(\vec{r},\vec{r'}) R(\vec{r'},t)$$

Time-dependent HAL QCD Method



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▶ **"potential"** using velocity expansion $U(r,r') \simeq V(r)\delta(r-r')$

$$V(\vec{r}) = \frac{1}{4m_B}\frac{(\partial/\partial t)^2 R(\vec{r},t)}{R(\vec{r},t)} - \frac{(\partial/\partial t)R(\vec{r},t)}{R(\vec{r},t)} - \frac{H_0 R(\vec{r},t)}{R(\vec{r},t)}$$

▶ **This method does not require the ground state saturation.**

Difficulties in Multi-Baryon System from Lattice QCD

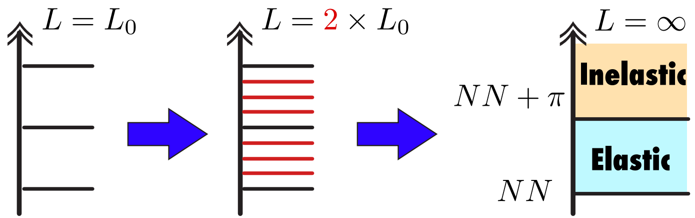
- Lüscher's method requires **ground state saturation**

$$G_{NN}(t) = c_0 \exp(-E_0^{(NN)} t) + c_1 \exp(-E_1^{(NN)} t) + \cdots \simeq c_0 \exp(-E_0^{(NN)} t)$$

  - S/N becomes worse as [mass number A] $\times$ [light quark] $\times$ [$t \to \infty$]

$$S/N \sim \exp\left[-A \times \left(m_N - \frac{3}{2} m_\pi\right) \times t\right]$$

  - smaller gaps: $\Delta E \sim \vec{p}^{\,2}/m \sim \mathcal{O}(1/L^2) \Longrightarrow$ large t



- advantage of **HAL QCD method**
  - only **elastic states saturation** — E-indep potential

$$\left[\frac{1}{4m_B}\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t} - H_0\right] R(\vec{r}, t) = \int d\vec{r'} U(\vec{r}, \vec{r'}) R(\vec{r'}, t)$$

Lattice Setup

- $2+1$ improved Wilson + Iwasaki gauge[†]

  lattice spacing $a = 0.08995(40)$ fm, $a^{-1} = 2.194(10)$ GeV

  $m_\pi = 0.51$ GeV, $m_N = 1.32$ GeV, $m_K = 0.62$ GeV, $m_\Xi = 1.46$ GeV

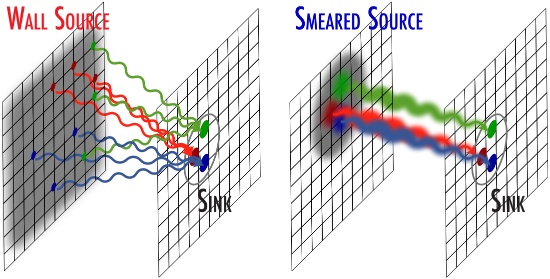
- **2-type quark sources**
  - **wall source**

    standard of HAL QCD
  - **smeared source**

    the same as Yamazaki *et al*[†].

    → **NN int.** by Lüscher

    & **Helium** binding energy



□ analyze **$\Xi\Xi(^1S_0)$-channel** — the same rep. as NN($^1S_0$) and better S/N with high stat. — ex. 48^4: (#smeared src.)/(*Yamazaki et al.*) $\simeq$ 5

volume	# conf.	# smeared src.	# wall src.
$40^3 \times 48$	200	512	48
$48^3 \times 48$	200	4 × 256	4 × 48
$64^3 \times 64$	327	256	4 × 32

† Yamazaki-Ishikawa-Kuramashi-Ukawa, arXiv:1207.4277.

$\Xi\Xi(^1S_0)$ Effective Energy Shift Plot

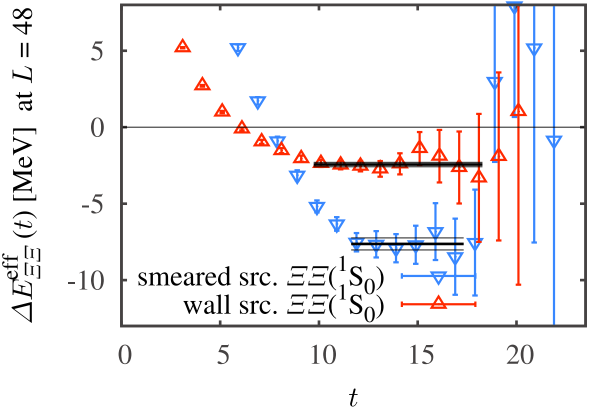
effective mass plot

$$\Delta E^{\mathrm{eff}}_{\Xi\Xi}(t) = \log R(t)/R(t+1) \to E_{\Xi\Xi} - 2m_\Xi$$

with $R(t) \equiv G_{\Xi\Xi}(t)/\{G_\Xi(t)\}^2$

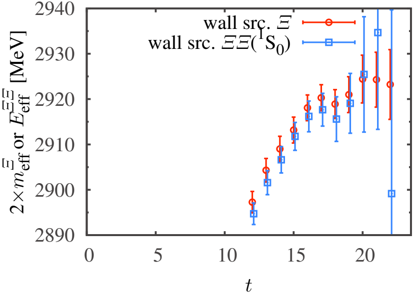
- **wall source**
  $\Delta E_L \sim -3$ MeV
- **smeared source**
  $\Delta E_L \sim -8$ MeV
- **plateau** depends on quark source

???

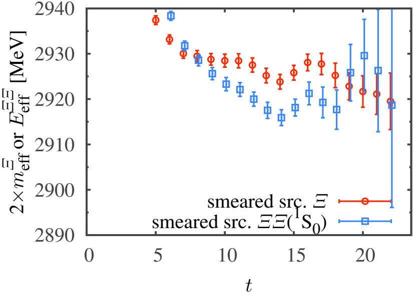


Effective Masses of ΞΞ and Ξ

wall source



smeared source



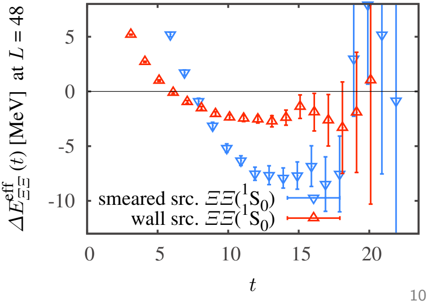
- Be careful with effective mass plot **w/o plateaux** in $E^{\rm eff}_{\Xi\Xi}(t)$ & $m^{\rm eff}_{\Xi}(t)$,

$$\Delta E_{\rm eff}(t) = E^{\Xi\Xi}_{\rm eff}(t) - 2m^{\Xi}_{\rm eff}(t)$$

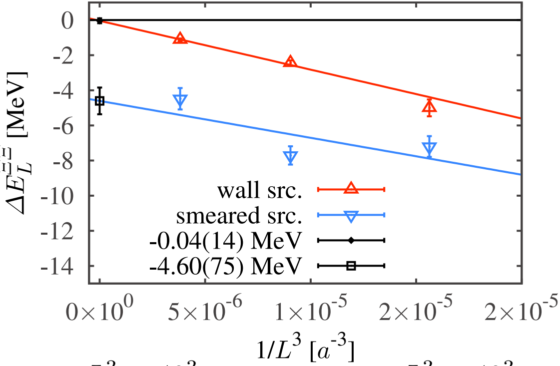
shows a **fake plateau** by cancellation

▶ we need much larger t,

and much more statistics



Do Not Trust Plateaux

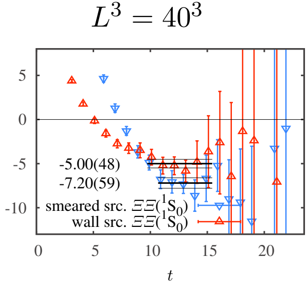


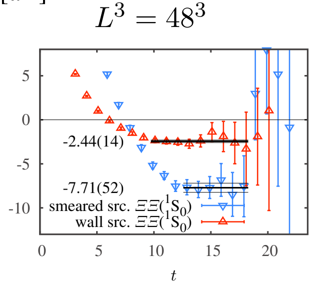
- **"direct method"**

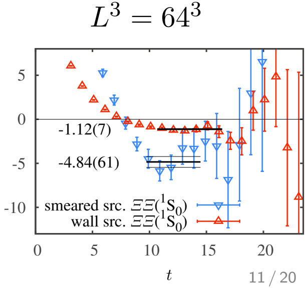
wall src. vs **smeared src.**

- different plateaux
- different conclusions
  unbound or **bound**

- **where is the true ΔE_L**

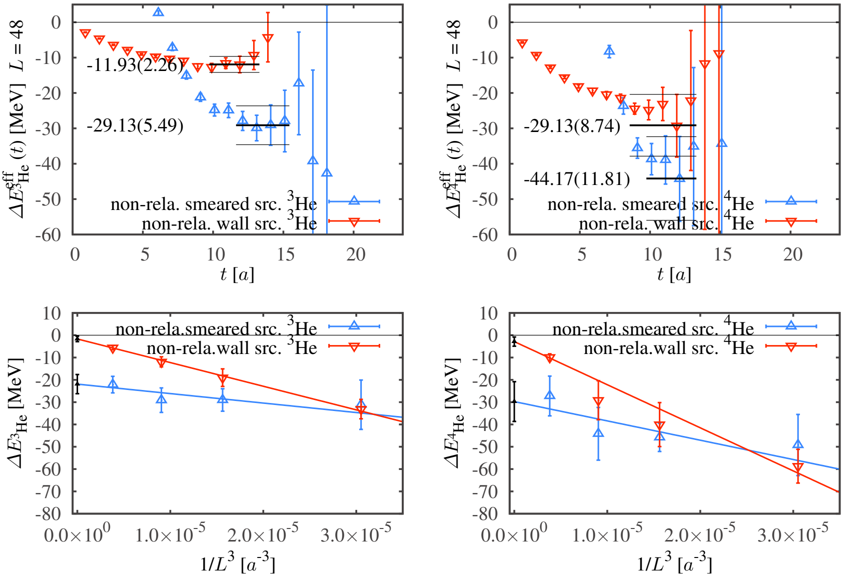




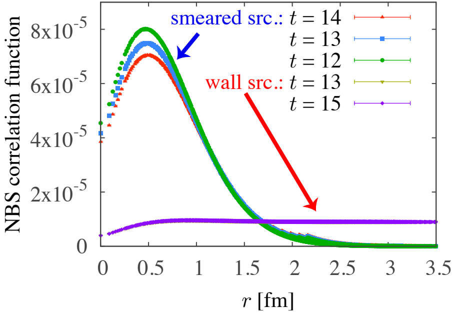


Energy Shift of Triton and Helium, and $1/L^3 \to 0$

fake plateaux problem will be much harder than two-body system



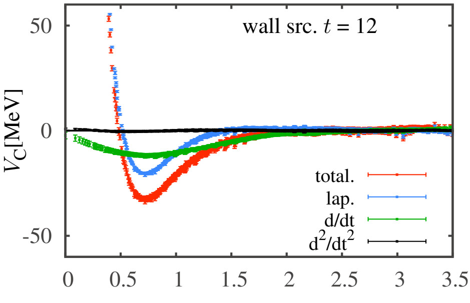
NBS Wave Function and $\Xi\Xi(^1S_0)$ Central Potential $V_c(\vec{r})$



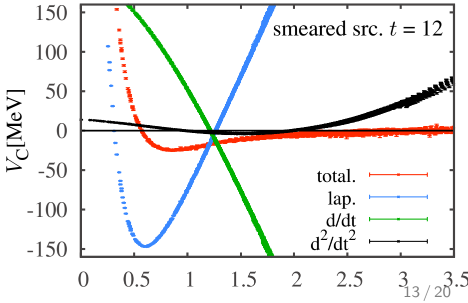
- ○ **wall src.** — weak t-dep.
- ○ **smeared. src.** — strong t-dep.
  $\Rightarrow$ $\sim \mathcal{O}(100)$ MeV cancellation
- • t-dep. HAL method works well!!

$$V_c(\vec{r}) = -\frac{H_0 R}{R} - \frac{(\partial/\partial t)R}{R} + \frac{(\partial/\partial t)^2 R}{4mR}$$

■ **wall src.**



□ **smeared src.**



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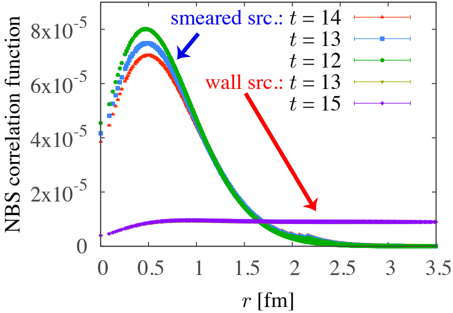

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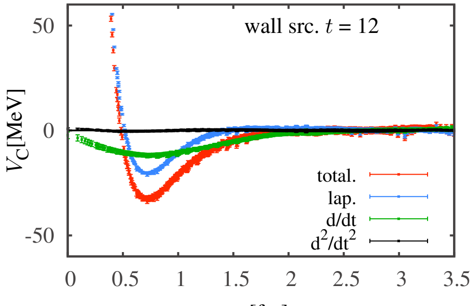
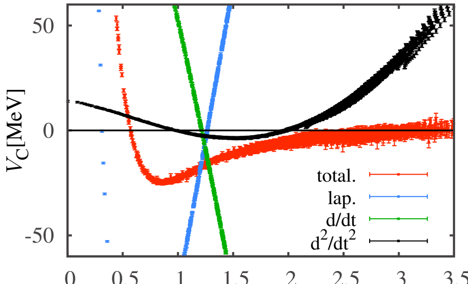
■ **wall src.**

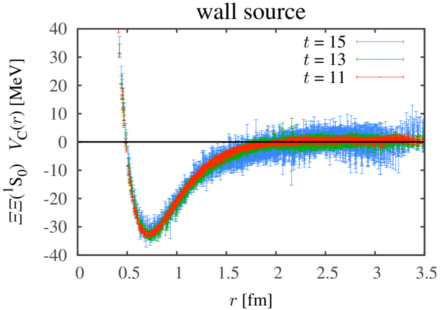


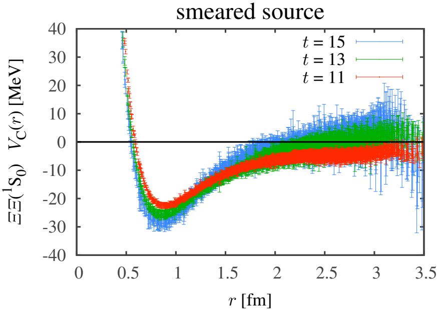
□ **smeared src.**



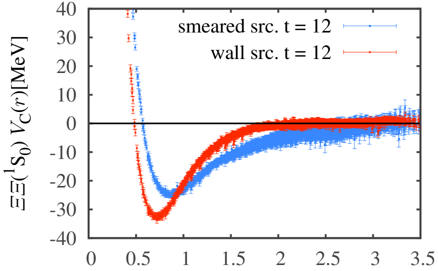
$\Xi\Xi(^1S_0)$ Potential: Wall Source vs Smeared Source

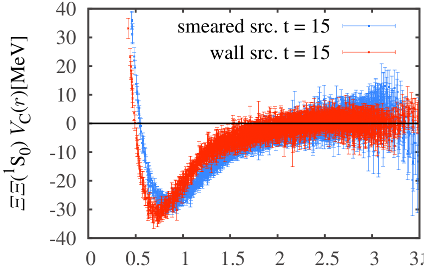
- **wall src.** t- "stable" $\Longleftrightarrow$ **smeared src.** t-depend





- **"smeared src."** converges to **"wall src."**





Residual Diff. of Pot.: Next Leading Order Correction

Derivative expansion: $U(r,r') = \{V_0(r) + V_1(r)\nabla^2\}\delta(r-r')$ (for $^1S_0$)

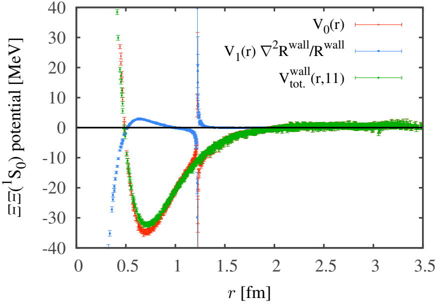
$$\left[\frac{1}{4m}\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t} - H_0\right] R(r,t) = \int d^3r' U(r,r') R(r',t)$$

$$\therefore \frac{1}{4m}\frac{(\partial^2/\partial t^2)R}{R} - \frac{(\partial/\partial t)R}{R} - \frac{H_0 R}{R} = V_0(r) + V_1(r)\frac{\nabla^2 R(r,t)}{R(r,t)} \equiv \tilde{V}_{\mathrm{total}}(r,t)$$

now, we have $R^{\mathrm{smear}}(r,t)$ and $R^{\mathrm{wall}}(r,t)$ $\Rightarrow$ solve $V_0(r)$ and $V_1(r)$

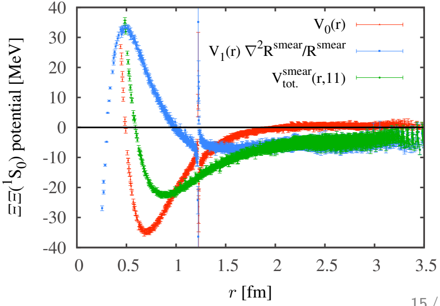
▶ **wall source**

- Good convergence of **LO pot.**



▶ **smeared source**

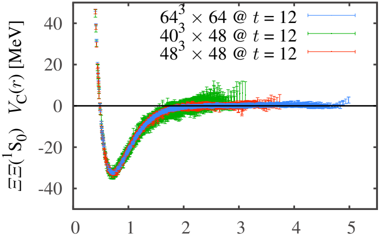
- **with NLO pot. correction**



Lüscher from HAL QCD: Energy Shift from Potential

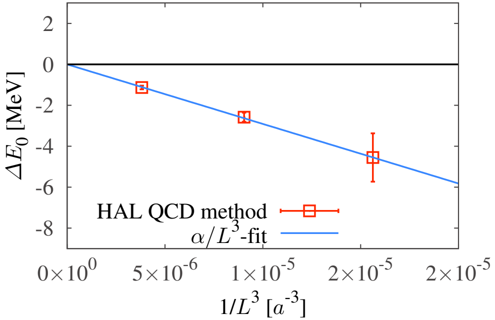
- HAL QCD gives reliable interaction **without g.s. saturation**
  quark source, time, and volume independent
- where is the true **"energy shift"** in finite volume ?

1. INPUT: potential $V(\vec{r})$



2. SOLVE: eigenvalue in finite box L^3

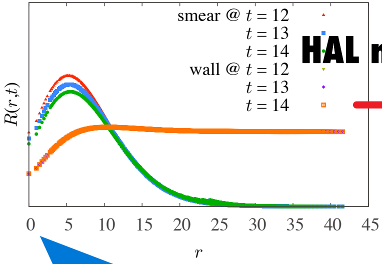
$$[H_0 + V]\,\psi = \Delta E \psi$$



- After all, $\Xi\Xi(^1\mathrm{S}_0)$ **is unbound** at $m_\pi = 510$ MeV

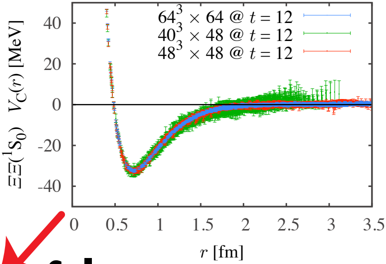
Wavefunction, Potential, Eigenvalues and Eienfunctions

NBS wave function



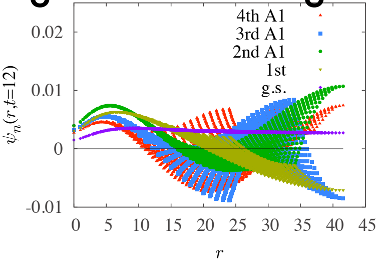
HAL method

Potential



Solve $[H_0 + V]\psi = E\psi$

Eigenvalues & Eigenfunctions



feed back

decomposition
projection

ground state

& excited states

(elastic scattering)

Excited State in NBS Wavefunction

- R-corr. decomposition by energy eigenmodes Ψ_n

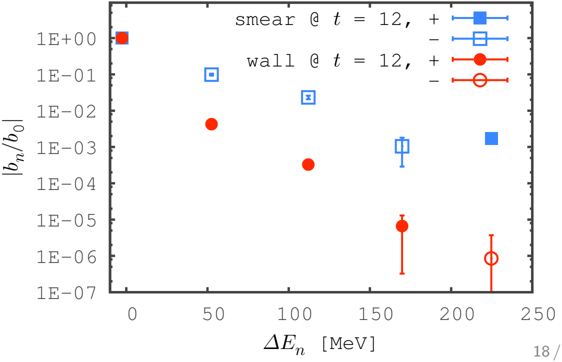
$$R(\vec{r},t) = \sum_n a_n \Psi_n(\vec{r},t) \exp\left(-\Delta E_n t\right), \qquad a_n \leftarrow \langle \Psi_n^\dagger, R\rangle \; e^{\Delta E_n t}$$

$$R(\vec{p}=0,t) = \sum_n a_n \left[\sum_{\vec{r}} \Psi_n(\vec{r})\right] \exp\left(-\Delta E_n t\right) = \sum_n b_n \exp\left(-\Delta E_n t\right)$$

☐ ex. **1st excited state**

- **wall source**
  $b_1/b_0 \ll 0.01$
- **smeared source**
  $b_1/b_0 \simeq -0.1$
- with energy gap
  $E_1 - E_0 \simeq 50$ MeV
  for $L^3 = 48^3$

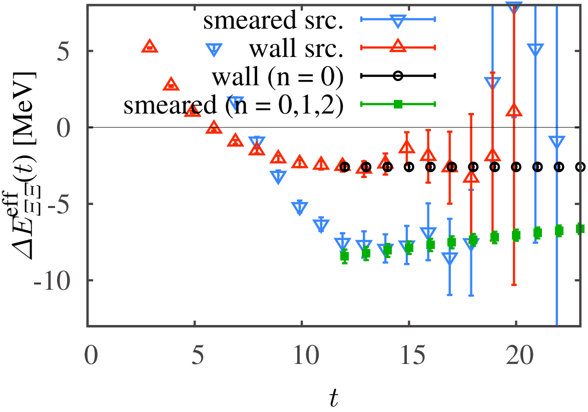
"contamination" of excited states b_n/b_0



Excited States Contribution to $\Delta E_{\mathrm{eff}}(t)$ and Fake Plateau

$$\Delta E_{\mathrm{eff}}(t) = \log \frac{\sum_n b_n \exp\left(-\Delta E_n t\right)}{\sum_n b_n \exp\left(-\Delta E_n (t+1)\right)}$$

- "direct measurement" — well reproduced by low-lying eigenmodes†
  - **"fake plateau"** of **smeared src.** around $t = 12 - 18$



† eigenvalues $\mathit{\Delta} E_n$, coefficients b_0^{wall} and b_n^{smear} for $n = 0, 1, 2$, at $t = 12$.

Excited States Contribution to $\Delta E_{\text{eff}}(t)$ and Fake Plateau

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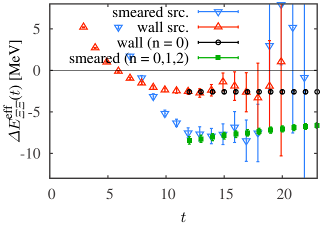
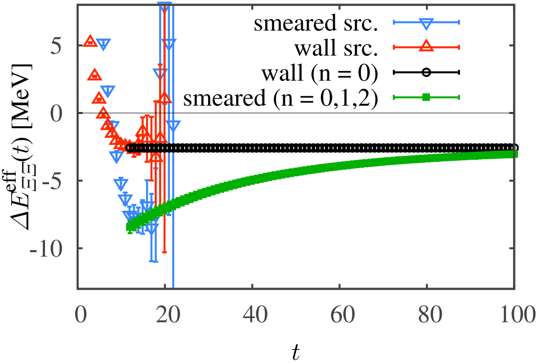
■ "direct measurement" — well reproduced by low-lying eigenmodes†

▶ **"fake plateau"** of **smeared src.** around $t = 12 - 18$

□ **g.s. saturation** of smeared source — **100 lattice units** $\sim$ **10 fm !!!**





† eigenvalues ΔE_n, coefficients b_0^{wall} and b_n^{smear} for $n = 0, 1, 2$, at $t = 12$.

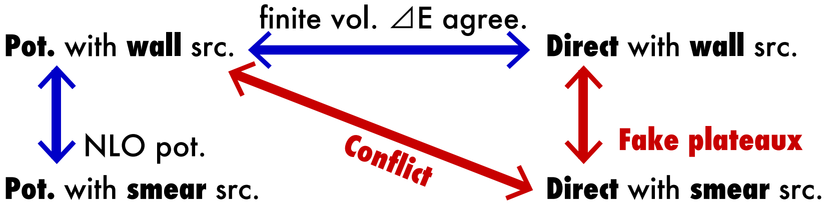
Summary: Lüscher ~~vs~~ and HAL QCD

✓ ~~"Lüscher vs HAL"~~ $\Longrightarrow$ "Lüscher(**smeared**) vs Lüshcer(**wall**)"

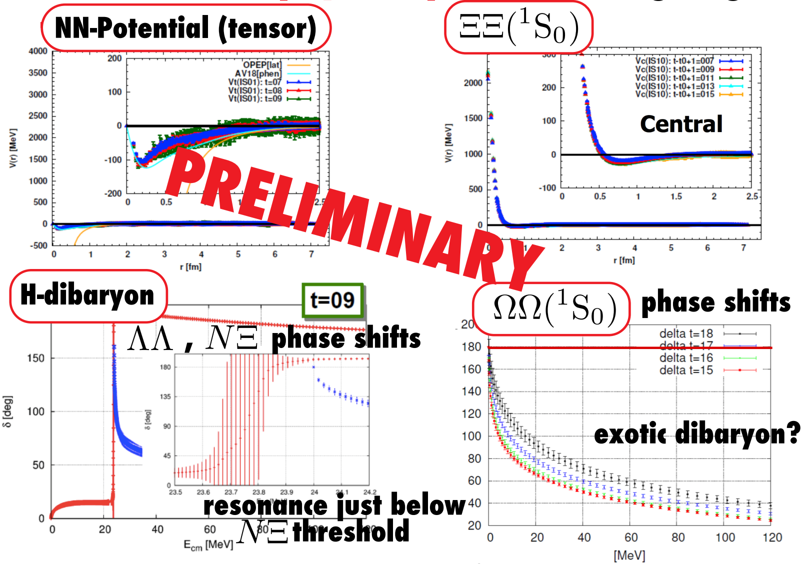
Lüshcer and **HAL QCD** methods are **consistent**

$\Longrightarrow$ **Lattice QCD can answer reliable hadron interactions.**

- "direct measurement" — **ground state saturation** is extremely hard
  $\Longrightarrow$ *do not trust plateau*, it may be a **fake signal!**
  ✓**check** other sources or variational method
- HAL QCD works well **without g.s. saturation**.
  ✓g.s. saturation at physical point is much more difficult
  $\Longrightarrow$ only HAL QCD can approach physical point
- **NBS corr.** and **"potential"** clarify excited states and **fake plateau**.



HAL QCD Coll. at physical point is now going on.

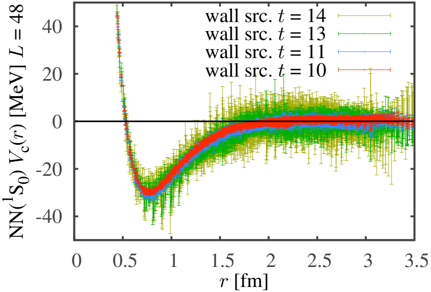


- **"direct measurement"**: $1/t \sim \Delta E \sim p^2/m_N \sim$ 20 MeV in $L = 8$ fm

$$\Rightarrow \quad S/N \sim \exp\left[-A \times \left(m_N - \frac{3}{2}m_\pi\right) \times t\right] \sim 10^{-25}$$

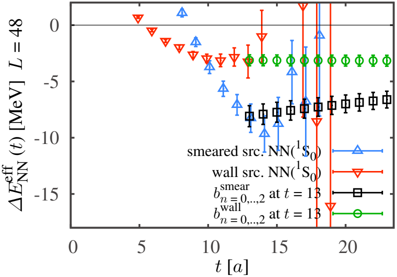
5 Appendix

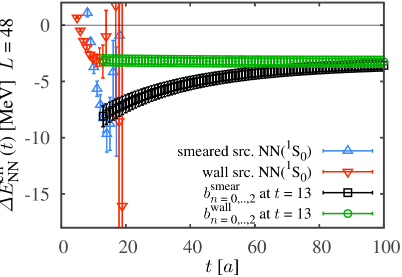
NN(^{1}S$_0$) Effective Energy Shift and Fake Plateau



the same as ΞΞ-channel analysis

1. get NN(^{1}S$_0$) potential $V_c(r)$
2. solve eigenvalues ΔE_n and eigenfunctions Ψ_n
3. decompose NBS wavefunc.

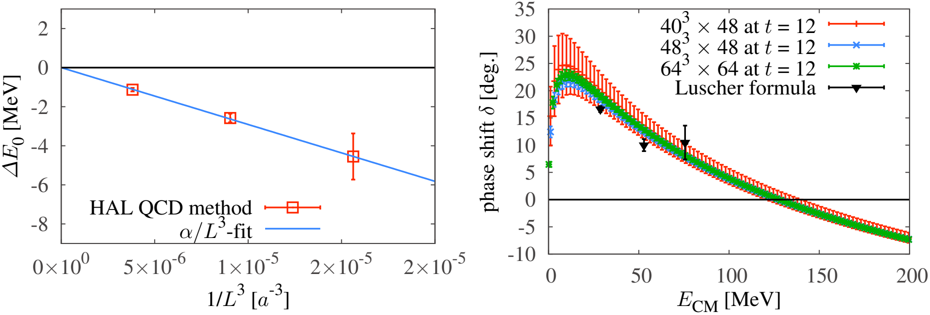




ΞΞ($^1S_0$) Phase Shift

$$k \cot \delta(k) = \frac{1}{\pi L} \sum_{n \in \mathbf{Z}^3} \frac{1}{|n|^2 - (kL/2\pi)^2}, \qquad \Delta E = 2\sqrt{m^2 + k^2} - 2m$$

volume dep. of ΔE_0 | phase shift δ

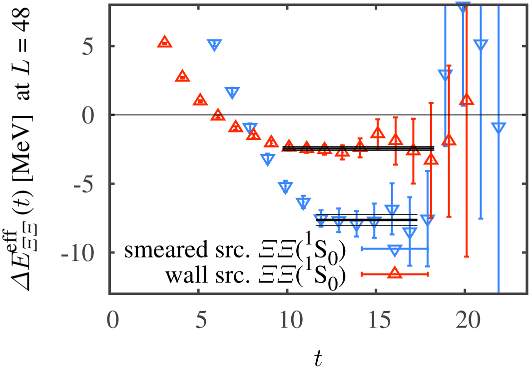


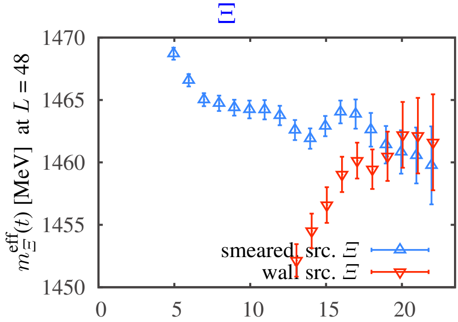
- wall src. pot. ⟹ solve eigenvalue & Lüscher formula ⟹ phase shift
- wall src. pot. ⟹ fit & solve Schrödinger eq. ⟹ phase shift
  - ○ using 2 Gaussians + (Yukawa)^2 ansatz for $V_c(\vec{r})$ fit

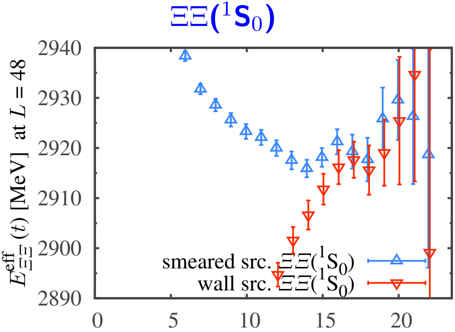
Effective Masses Plot

$$\Delta E^{\rm eff}_{\Xi\Xi}(t) = E^{\rm eff}_{\Xi\Xi}(t) - 2m^{\rm eff}_{\Xi}(t)$$

shows plateau
even without plateaux
in $m^{\rm eff}_{\Xi}(t)$ or $E^{\rm eff}_{\Xi\Xi}(t)$







Next Leading Order of Derivative Expansion

Derivative expansion: $U(r,r') = \{V_0(r) + V_1(r)\nabla^2\}\delta(r-r')$ (for $^1\mathrm{S}_0$)

$$\left[\frac{1}{4m}\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t} - H_0\right] R(r,t) = \int d^3r' U(r,r') R(r',t)$$

$$\therefore \frac{1}{4m}\frac{(\partial^2/\partial t^2)R}{R} - \frac{(\partial/\partial t)R}{R} - \frac{H_0 R}{R} = V_0(r) + V_1(r)\frac{\nabla^2 R(r,t)}{R(r,t)} \equiv \tilde{V}_{\mathrm{total}}(r,t)$$

- Now, we have R^{smear} and R^{wall}

$$\begin{cases} V_0(r) + V_1(r)\nabla^2 R^{\mathrm{smear}}/R^{\mathrm{smear}} = \tilde{V}^{\mathrm{smear}}_{\mathrm{total}}(r, t_{\mathrm{smear}}) \\ V_0(r) + V_1(r)\nabla^2 R^{\mathrm{wall}}/R^{\mathrm{wall}} = \tilde{V}^{\mathrm{wall}}_{\mathrm{total}}(r, t_{\mathrm{wall}}), \end{cases}$$

- LO $V_0(r)$ and NLO $V_1(r)$ potentials are given by

$$V_1(r) = \frac{\tilde{V}^{\mathrm{smear}}_{\mathrm{total}}(r, t_{\mathrm{smear}}) - \tilde{V}^{\mathrm{wall}}_{\mathrm{total}}(r, t_{\mathrm{wall}})}{\nabla^2 R^{\mathrm{smear}}/R^{\mathrm{smear}} - \nabla^2 R^{\mathrm{wall}}/R^{\mathrm{wall}}}$$

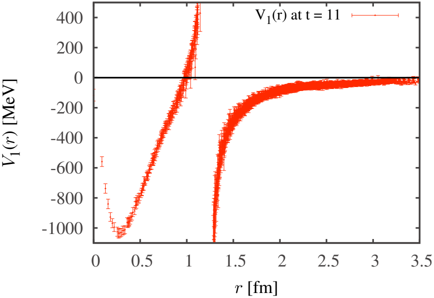
$$V_0(r) = \tilde{V}^{\mathrm{smear}}_{\mathrm{total}}(r, t_{\mathrm{smear}}) - V_1(r)\frac{\nabla^2 R^{\mathrm{smear}}}{R^{\mathrm{smear}}}.$$

- $\nabla^2 R/R - \nabla^2 R/R$ in denominator

Results: NLO Potential

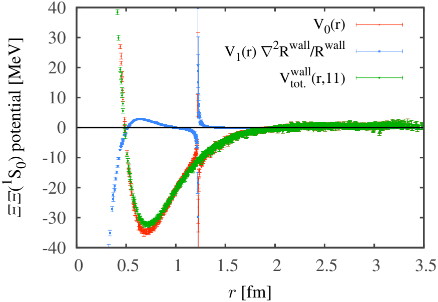
- results of HAL QCD method
  - ▶ source independent

$$V_{\text{total}}(r,t) = V_0(r) + V_1(r)\frac{\nabla^2 R(r,t)}{R(r,t)}$$



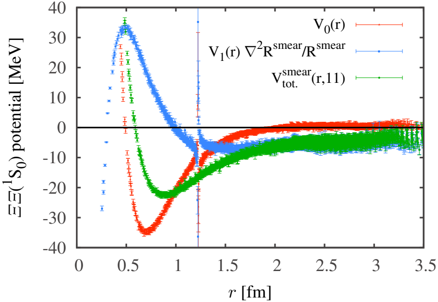
▶ **wall source**

- Good convergence of **LO pot.**



▶ **smeared source**

- **with NLO pot. correction**



Results:NLO Potential (1) $V_0(r)$ and $V_1(r)$

$$V_0(r) = \tilde{V}^{\mathrm{exp.}}_{\mathrm{total}} - V_1(r)\frac{\nabla^2 R^{\mathrm{exp.}}}{R^{\mathrm{exp.}}}, \quad V_1(r) = \frac{\tilde{V}^{\mathrm{exp.}}_{\mathrm{total}} - \tilde{V}^{\mathrm{wall}}_{\mathrm{total}}}{\nabla^2 R^{\mathrm{exp.}}/R^{\mathrm{exp.}} - \nabla^2 R^{\mathrm{wall}}/R^{\mathrm{wall}}}$$

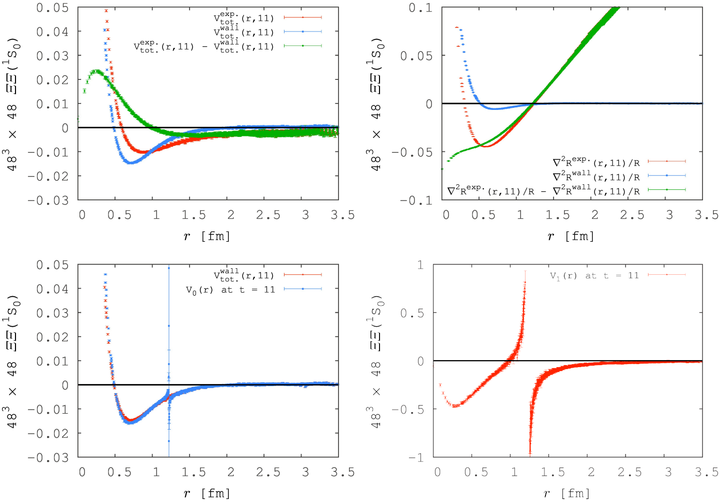


Figure: (a) diff. of $\tilde{V}_{\mathrm{total}}$ (b) diff. of $\nabla^2 R/R$ (c) $V_0(r)$ and $\tilde{V}^{\mathrm{wall}}$ (d) $V_1(r)$

Results:NLO Potential (2) V_0 and V_1 in smeared and wall

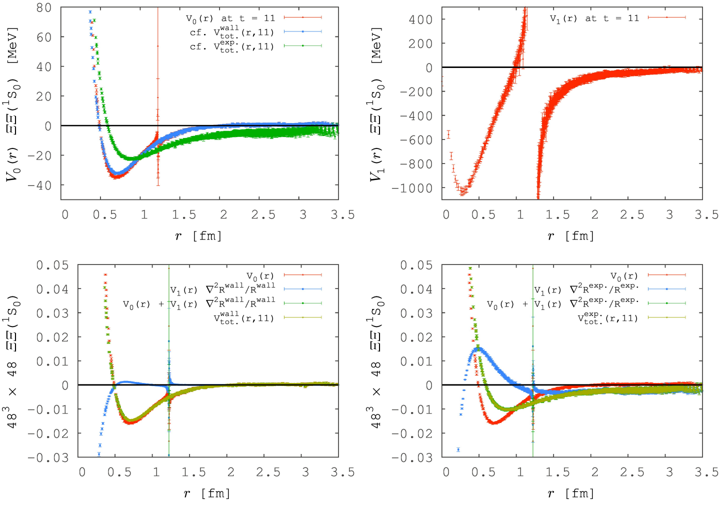


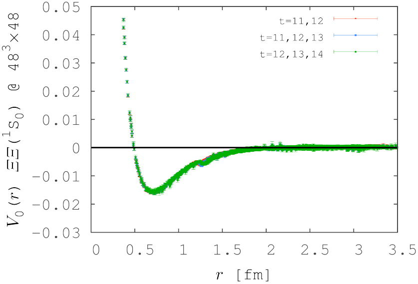
Figure: (a) $V_0(r)$ (b) $V_1(r)$ (c) wall src. summary (d) smeared src. summary

Results:NLO Potential (3) SVD

solving SVD with several t's

$$\begin{pmatrix} 1 & \nabla^2 R^{\mathrm{exp.}}(r,t_1)/R^{\mathrm{exp.}}(r,t_1) \\ 1 & \nabla^2 R^{\mathrm{wall}}(r,t_1)/R^{\mathrm{wall}}(r,t_1) \\ 1 & \nabla^2 R^{\mathrm{exp.}}(r,t_2)/R^{\mathrm{exp.}}(r,t_2) \\ 1 & \nabla^2 R^{\mathrm{wall}}(r,t_2)/R^{\mathrm{wall}}(r,t_2) \\ \vdots & \vdots \end{pmatrix} \begin{pmatrix} V_0(r) \\ V_1(r) \end{pmatrix} = \begin{pmatrix} \tilde{V}^{\mathrm{exp.}}_{\mathrm{total}}(r,t_1) \\ \tilde{V}^{\mathrm{wall}}_{\mathrm{total}}(r,t_1) \\ \tilde{V}^{\mathrm{exp.}}_{\mathrm{total}}(r,t_2) \\ \tilde{V}^{\mathrm{wall}}_{\mathrm{total}}(r,t_2) \\ \vdots \end{pmatrix}$$

$$A \begin{pmatrix} V_0(r) \\ V_1(r) \end{pmatrix} = B \longrightarrow U\Sigma V^t \begin{pmatrix} V_0(r) \\ V_1(r) \end{pmatrix} = B,$$



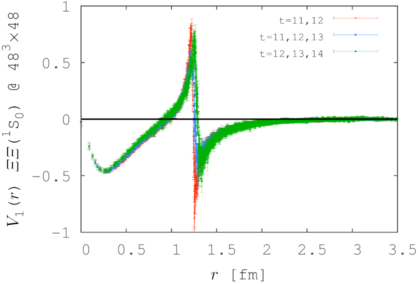


Figure: (a) $V_0(r)$ (b) $V_1(r)$ using SVD

Results:NLO Potential (4) Energy Eigenvalues

$\tilde{V}_{\mathrm{tot}}^{\mathrm{wall}}$, $V_0(r)$ and $V_0(r) + V_1(r)\nabla^2$
energy eigenvalues are consistent with each other
$\Longrightarrow$ wall source potential $\simeq$ LO

volume at $48^3 \times 48$	t	g.s. [MeV]	1st [MeV]
$\tilde{V}_{\mathrm{total}}^{\mathrm{wall}}(r)$	11	-2.30(20)	53.08(29)
	12	-2.58(22)	52.87(33)
	13	-2.70(30)	52.80(41)
$\tilde{V}_{\mathrm{total}}^{\mathrm{exp.}}(r)$	13	-5.33(59)	49.86(58)
	14	-4.76(55)	50.46(54)
$V_0(r)$	11,12,13	-2.47(23)	52.91(32)
	12,13,14	-2.62(28)	52.86(37)
$V_0(r) + V_1(r)\nabla^2$	11,12,13	-2.56(1.56)	53.07(2.73)
	12,13,14	-2.74(2.52)	51.58(3.77)

Table: The energy eigenvalues from "potential method".

Weight of Eigenstates in Wavefunction

$$R^{\text{wall/smear}}(\vec{x},t) = \sum_n a_n^{\text{wall/smear}} \Psi_n(\vec{x},t) e^{-\Delta E_n t}$$

with eigenfunctions Ψ_n and eigenvalues ΔE_n

- smeared $\Longrightarrow$ higher states are significant $|a_0^{\text{exp.}}| \sim |a_1^{\text{exp.}}| \sim |a_2^{\text{exp.}}|$
- wall $\Longrightarrow$ g.s. dominant $|a_0^{\text{wall}}| \gg |a_1^{\text{wall}}| \gg |a_2^{\text{wall}}|$

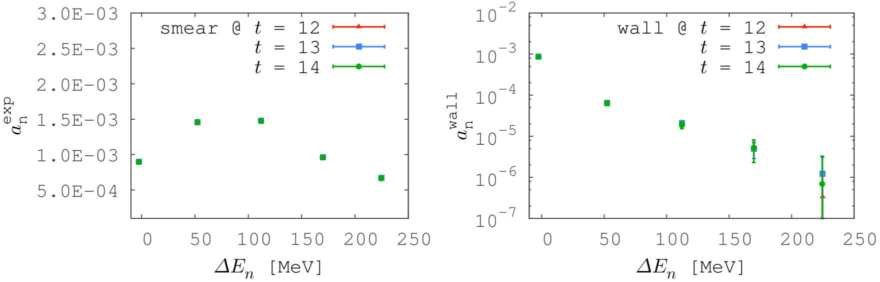


Figure: (a) smeared source. (b) wall source.

Projection of Eigenstate: Improvement of “Sink” Operator

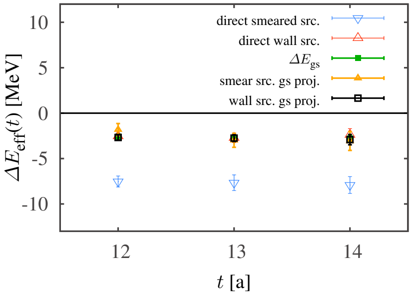
$$\tilde{R}^{(f)}(t) \equiv \sum_{\vec{r}} f(\vec{r}) \sum_{\vec{x}} \langle 0|B(\vec{r}+\vec{x},t)B(\vec{x})\bar{\mathcal{J}}(0)|0\rangle = \sum_{\vec{r}} f(\vec{r}) R(\vec{r},t)$$

and

$$\Delta\tilde{E}^{(f)}_{\mathrm{eff}}(t) = \log \frac{\tilde{R}^{(f)}(t)}{\tilde{R}^{(f)}(t+1)}$$

- ground state projection
- 1st excited state projection

$$f(\vec{r}) = \Psi^{\dagger}_{\mathrm{gs}}(\vec{r})$$



$$f(\vec{r}) = \Psi^{\dagger}_{1}(\vec{r})$$