Lattice Radial Quantization, Cubature and Hyperbolic lattices.

Herbert Neuberger

Department of Physics
Rutgers University
Piscataway, NJ08854

March 10, 2016, BNL

Curtain-Raiser

- In the mid 80's advances in Conformal Quantum Field Theory in 1d (space) and a new mechanism for anomaly cancellation converged into the String Renaissance.
- A euphoric blend of Mathematics and tantalizing hints of Quantum (Super)Gravity was marketed to everybody with unprecedented success for abstract Science. Most of the talented students and younger researchers in Theoretical Physics joined the gold rush. Some struck it rich.
- This concentration of talent produced the AdS/CFT correspondence, which, in turn, brought about a Conformal Quantum Field Theory Renaissance for dimensions $d \geq 2$. Together with advances in Hamiltonian RG for Many Body Theory, this may rejuvenate the successful overlap of some interests of Particle Theorists and Condensed Matter Theorists of the past, in the context of Critical Phenomena.

Introduction

- In 2008, Pavlos Vranas invited me to attend a workshop at Livermore on BSM Physics, although he knew that I felt that composite models in general were unlikely to become valid BSM Physics because of the high precision LEP results – and had not worked on so-called “lattice BSM”.
- The talks related to the conformal window in YM convinced me that a blind application of standard LFT techniques to a model connecting a free marginally unstable UVFP to a truly interacting IRFP would not work. One would not be able to control by numerical MC this two scale problem in the foreseeable future.
- I commented at some point that using radial quantization – inside the conformal window – would at least turn the determination of non-canonical IR scaling dimensions into a mass computation, something we know how to do well.

CFT basics

- A scale invariant Euclidean Field Theory in d-dimensional Euclidean space provides a unitary representation of its group of similarities which consists of translations P_μ, rotations $M_{\mu\nu}$ and scaling $\mathcal{D}$. Near the identity there are $d + \frac{d(d-1)}{2} + 1 = \frac{d(d+1)}{2} + 1$ parameters.
- Often, the behaviour at infinity of space is such that $\mathbf{I}$, the inversion $\vec{x} \to \frac{\vec{x}}{x^2}$ is also represented. Near the identity this adds d more generators, given by $\mathbf{I}P_\mu\mathbf{I}$, raising the number of parameters to $\frac{(d+1)(d+2)}{2}$. (Realizing $\mathbf{I}$ is not necessary.)
- The global conformal group is $(S)O(d+1,1)$. Geometrically, the group is the group of Möbius transformations of d-dimensional one-point compactified flat Euclidean space. It is generated by reflections in planes and spheres. (The component connected to the identity suffices.)

Radial Quantization basics

- The group of Euclidean distance preserving maps provides a single tempting slicing of spacetime for quantization. One chooses a “time” evolution direction. The full symmetry no longer is explicit. But, it is there, if things go well.
- The Möbius group provides more tempting ways of slicing and associated quantizations. In radial quantization one picks a point as the “origin” and ends up with an $S^{d-1} \times R$ cylinder where translations in the R direction are generated by $\mathcal{D}$, with $\mathcal{D}$ bounded from below.
- Minus the logarithm of the transfer matrix is the lattice version of $\mathcal{D}$; its spectrum is discrete, given by the set of scaling dimensions of the theory.

RUTGERS

One real free massless scalar field example

- Carry out the following steps: (1) Write down the action (2) Change from cartesian coordinates to radial ones (3) Replace r by $r_0 e^{-t}$ and change variables in the path integral to $|x|^{\frac{d-2}{2}}\phi(x) = \Phi(t, \hat{\omega})$ where $(r_0 e^{-t}, \hat{\omega})$ is the point x in spherical coordinates. (4) Ignore the Jacobian on account that it is field independent.
- Set $r_0 = 1$. The integrand of the action against dt has no explicit t-dependence. Get the "Hamiltonian" from the action and diagonalize the quadratic form in the spherical harmonics basis. Introduce creation/annihilation operators by the Euclidean version of Feynman's prescription.
- This is all worked out very clearly in Fubini, Hanson and Jackiw, PRD 7, 1732 (1972) who introduced radial quantization. For $d = 3$, for example, you find that $\mathcal{D} = \sum_{l=0,1,2\ldots,m=-|l|,-|l|+1,\ldots|l|} (l + \frac{1}{2}) a^{\dagger}_{lm} a_{lm}$ where we chose to give zero value to the lowest eigenvalue of $\mathcal{D}$.

Example contd: Towers in spectrum and interactions

- The $l(l+1)$ got changed into a $(l+1/2)^2 = l(l+1)+1/4$ by the field rescaling. This produces the right set of dimensions for the free field. The 1/4-constant is not a mass term in flat space. “Tuning to criticality” amounts to setting the mass term to a value that gives equally-spaced distinct eigenvalue towers to $\mathcal{D}$. This follows from conformal symmetry alone. “Away from criticality” means violating translational invariance. $\exists$ dilation preserving regularizations.
- For general d one gets $(l+\frac{d-2}{2})^2$. There is no shift for $d=2$ because the field is dimensionless. For $d=1$, $l=0$.
- To maintain t-independence only a potential proportional to $(\phi^2)^{\frac{d}{d-2}}$ is allowed in the original action. With it, FHJ set up Feynman perturbation theory.
- Things look fine for $d \geq 3$. The cases $d=1,2$ are special.

RUTGERS

A lattice example: d=2 Ising.

- One way to interpret the infinite power at $d = 2$ is that it forces $\phi^2 = 1$, so that $(\phi^2)^{\pm\infty} = 1$, giving the Ising model if we assume we are on a lattice. $\ddot\smile$
- Radial quantization means working on $S \times R$. It is straightforward to put this on a lattice.
- $\mathcal{D}$ becomes then the logarithm of the transfer matrix with periodic boundary conditions in the space direction.
- The transfer matrix can be exactly diagonalized. (See the Stat. Mech. book by K. Huang for example.)
- To make the model have an IRFP we choose to set the couplings to the symmetric critical value.
- We put N spins round the ring and calculate the lowest eigenvalue of $\mathcal{D}$ in the odd (under spin flip) sector.

RUTGERS

d=2 Ising continued

- The answer we need to get is that this eigenvalue is equal to 1/4, the η-exponent in this case.
- The finite N formula is $\frac{N}{\pi}[E_1(N) - E_0(N)]$ where $E_{0,1}(N)$ are the logarithms of the highest eigenvalues of the transfer matrix in the even and odd sectors respectively.
- $E_1(N) - E_0(N) = \frac{1}{2}\left[\gamma(0) + \gamma(\frac{2\pi}{N}) + ... + \gamma(2\pi - \frac{2\pi}{N})\right]$ $-\frac{1}{2}\left[\gamma(\frac{\pi}{N}) + \gamma(\frac{3\pi}{N} + ... + \gamma(2\pi - \frac{\pi}{N})\right]$ $\cosh\gamma(q) = 2 - \cos q$.
- For $N = 1$ we get 0.28055, for $N = 10$ we get 0.25106 and for $N = 20$ we get 0.25026. So it works fine in this case.

d=3 Ising

- Now we need to latticize $S^2 \times R$ and put an Ising model on that lattice.
- A first attempt was made with Rich Brower and George Fleming; I spoke about this four years ago at a BNL workshop when we had only preliminary data.
- The project produced a reasonable value for the lowest dimension in the odd sector, but the approach to the limit remained somewhat erratic.
- Also, the calculation had to be pushed to large lattices, while the initial hope was that this be a more efficient way to get critical exponents than the ones based on the usual way of quantizing the system.

Lessons from first simulation for $d = 3$ Ising

- We did not discretize S^2 but rather a piecewise flat approximation to it, a 12 faced Icosahedron which had conical singularities. These are points where the angle covered going round them is less then 2π. There is no way to avoid such singularities if one insists on preserving the topology while using only flat two dimensional pieces. These singularities would impact the approach to the limit, and, in addition, this limit won't have spherical symmetry.
- We used a cluster algorithm which limited our flexibility in choosing the action: we could not experiment with nnn couplings for example. We did not try continuous fields.
- We did not work out fully the group theoretical details.
- Having made sure that lattice radial quantization is feasible we proceeded to refine the method. There are many choices to make and it is not clear a priori which will work out best. I decided that by pursuing different lines, more ground gets covered and overall progress gets sped up.

Cubature: one way to discretize the sphere

- I place on S^2 points in complete orbits under the 120 element Icosahedral group I_h.
- Each new orbit comes with a common new weight w_i which is adjusted to enhance the symmetry under cubature.
- The criterion is that $\int_{S^2} f(\hat{\omega}) d\hat{\omega} = \sum_{i=1}^{N} w_i f(\hat{\omega}_i)$ for f's which are linear combinations of functions of $\hat{\omega}, \hat{\omega} \in S^2$ of angular momentum $L \leq L_{\max}$.
- One can add one 20, one 30 and one 60 points orbit and any number of 120 points orbits.
- One does not have to include 20,30,60 point orbits.
- The nr. of vertices needed for a given $L_{\max}$ can be found using the decompositions of $SO(3)$ irreps into I_h irreps.
- The weights w_i are determined by linear equations.

The Action.

- Start from a continuum action with one UV cutoff $1/\sqrt{s}$, $A = \int d\hat{\omega} d\hat{\omega}' \Phi(\hat{\omega}) K(\hat{\omega}, \hat{\omega}') \Phi(\hat{\omega}') + \int d\hat{\omega} V(\Phi(\hat{\omega}))$. ($t$-direction is treated in a standard way). $K(\hat{\omega}, \hat{\omega}')$ is the matrix element of the heat kernel on S^2, $K = (1 - e^{s\partial^2_{\hat{\omega}}})/s$.
- For a set of N points, set $K_{i,j} = K(\hat{\omega}_i, \hat{\omega}_j)$. The discrete A is $A = \sum_{i,j} \Phi_i K^\circ_{i,j} \Phi_j + \sum_i w_i V(\Phi_i)$. It has two UV cutoffs.

$$1 \leq i, j \leq N, \quad K^\circ_{i,j} = \begin{cases} w_i K_{i,j} w_j & \text{if } i \neq j \\ -\sum_{k \neq i} w_i K_{i,k} w_k & \text{if } i = j. \end{cases}$$

- For a Φ given by a sum of $l \leq \lambda_N$ spherical waves, A will be given by its discrete counterpart for $l \leq \lambda_N$ to a good approximation for $s = const/N$ and V polynomial.
- Preserving a discrete translational invariance in the “time” direction gives a transfer matrix as usual.

RUTGERS

Example: 3D Ising on Icosahedron

- I constructed the transfer matrix for $N = 12$ (arranged as the vertices of an Icosahedron) with Ising spin variables replacing the fields Φ_i, producing a 2^{12} dimensional Hilbert space. The transfer matrix was exactly diagonalized numerically.
- Its kernel K° was chosen to respect I_h and reproduce the continuum eigenvalues $l(l+1)$ exactly to $l = 3$. (A “mass” term is irrelevant).
- The coupling of the time hopping part was adjusted to ensure approximate equality of the lowest gaps between tower base and first excited state for the lowest towers in the odd and even sectors.
- The raw results for the excitation energies, E, are

Sect.	Sym	l	E
Even	ϵ	0	1.31428
Even	ϵ	1	2.16494
Even	$T_{\mu\nu}$	2	2.39325
Even	ϵ	2	2.78684
Even	skip 12 states		
Even	ϵ'	0	3.12558

Sect	Sym	l	E
Odd	σ	0	0.42744
Odd	σ	1	1.26897
Odd	σ	2	1.90395
Odd	σ	3 (F_{1u})	2.38953
Odd	$\partial^2\sigma$	0	2.47213

RUTGERS

Dimensions

Determining the scale from the spacings of the lowest tower members in the even and odd sector I get $\rho = 1.182(6)$ and this produces the following dimensions after multiplication by ρ.

Sym	L	dim	exact
σ	0	0.505	0.518
$\partial^2\sigma$	0	2.996	2.518

Sym	L	dim	exact
ϵ	0	1.553	1.413
$T_{\mu\nu}$	2	2.922	3
ϵ'	0	3.694	3.8

The "dim" column is the numerical result. The "exact" column is either literal or to the accuracy shown. The "exact" entry for the descendant $\partial^2\sigma$ is inferred. There is no way to credibly estimate the errors of the numerical results. The error on ρ merely is an indication for the accuracy on the scale of the lowest gaps in the towers. For a lattice of 12 points one hardly could expect better numbers. There is no reason to take these numbers seriously, except feel encouraged.

In the published paper, PRD 90, 114501 (2014), there were a few typographical errors and one state was likely misidentified. This has been corrected above.

RUTGERS NEW BRUNSWICK

Hyperbolic lattices

- On the cylinder the spectrum of the dilatation operator is directly accessible. It is a Schrödinger picture. The OPE coefficients of the scaling fields are less accessible.
- For a Heisenberg picture we ought to work on S^d. Any metric that is conformal to the flat metric would do. We need a good discretization for that metric.
- Standard LFT takes a discrete (Coxeter) subgroup of the geometric group and uses its Cayley graph as the lattice.
- Coxeter subgroups of the symmetry group of one sheet of a two-sheeted Hyperboloid in $d + 2$ dimensions provide new opportunities. The only difference to the familiar case is the bilinear form defining elementary reflections.
- The lattices of interest are non amenable. The targeted system then resides on their "boundaries". As it had to be, an ambiguity is present leaving the conformal factor undetermined. Cannot go into details at this time !

Curtain falls

- Draw your own conclusions !

- I thank the organizers for inviting me to tell you my story.