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QCD+QED with C* boundary conditions

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based on:

B. Lucini, AP, A. Ramos, N. Tantalo,
Charged hadrons in local finite-volume QED+QCD with C boundary conditions*,
arXiv:1509.01636.

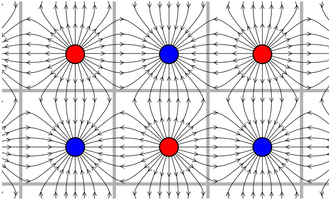
QCD+QED in finite volume

- On a torus with periodic boundary conditions, the Gauss law forbids a nonzero charge.

$$\partial_k E_k(x) = \rho(x) \quad \Rightarrow \quad Q = \int d^3x \; \rho(t, \mathbf{x}) = \int d^3x \; \partial_k E_k(t, \mathbf{x}) = 0$$

- C^* boundary conditions in some spatial direction (Wiese, Kronfeld, Polley).

$$A_\mu(x + L\mathbf{k}) = -A_\mu(x) \qquad \psi(x + L\mathbf{k}) = C^{-1}\bar{\psi}^T(x) \qquad \bar{\psi}(x + L\mathbf{k}) = -\psi^T(x)C$$



Electric flux can escape the torus and flow into the mirror charge

$$Q(t) = \int d^3x \; \rho(t, \mathbf{x}) = \int d^3x \; \partial_k E_k(t, \mathbf{x}) \neq 0$$

QCD+QED with C^* boundary conditions

- Local QFT at fixed L (time evolution of fields in x is determined only by the value of fields and their derivatives x). Locality guarantees
  - Renormalizability by power counting
  - Volume-independence of renormalization constants
  - Operator product expansion
  - Effective-theory description of long-distance physics
  - Symanzik improvement program

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- C* boundary conditions partially break:
  - Charge conservation
  - Flavour symmetry

  However:
  - A descrete symmetry group survive, and this is enough to contruct most of the interesting one-particle states.
  - Charge and flavour symmetry breaking is exponentially suppressed in the volume (even with dynamical photons).
  - Operator mixing works as if flavour symmetry were preserved (no additional mixing).

Flavour symmetries in QED+QCD$_\mathrm{C}$

$$\psi(x + L\mathbf{k}) = C^{-1}\bar{\psi}^T(x) \qquad \bar{\psi}(x + L\mathbf{k}) = -\psi^T(x)C$$

- Flavour symmetry is partially broken:

  $$\psi_f \to e^{i\alpha}\psi_f \qquad \bar{\psi}_f \to e^{-i\alpha}\bar{\psi}_f$$

  leaves the b.c.s invariant iff $e^{i\alpha} = \pm 1$. Flavour number F_f is not conserved but $(-1)^{F_f}$ is.

- Baryon number is a linear combination of flavour charges

  $$B = \frac{1}{3}\sum_f F_f$$

  Baryon number B is not conserved but $(-1)^{3B}$ is. At large enough volume only states with integer B exist (confinement). Since B is odd iff $3B$ is odd, then $(-1)^B$ is conserved.

- Electric charge is a linear combination of flavour charges

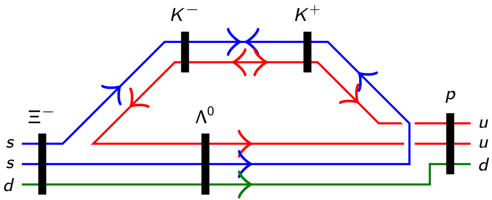
  $$Q = \sum_f \frac{\hat{q}_f}{3} F_f$$

  Electric charge Q is not conserved but $(-1)^{3Q}$ is. At large enough volume $(-1)^Q$ is conserved.

Flavour symmetries in QED+QCD$_C$

Particle	F_u	F_d	F_s	F_c	B	$(-1)^{F_u}$	$(-1)^{F_d}$	$(-1)^{F_s}$	$(-1)^{F_c}$	$(-1)^B$
π^+	1	-1	0	0	0	$-$	$-$	$+$	$+$	$+$
$\pi^+\pi^+ \to \varnothing$	2	-2	0	0	0	$+$	$+$	$+$	$+$	$+$
K^+	1	0	-1	0	0	$-$	$+$	$-$	$+$	$+$
K^0	0	1	-1	0	0	$+$	$-$	$-$	$+$	$+$
D^+	0	-1	0	1	0	$+$	$-$	$+$	$-$	$+$
D_s^+	0	0	-1	1	0	$+$	$+$	$-$	$-$	$+$
D^0	-1	0	0	1	0	$-$	$+$	$+$	$-$	$+$
p	2	1	0	0	1	$+$	$-$	$+$	$+$	$-$
n	1	2	0	0	1	$-$	$+$	$+$	$+$	$-$
Λ^0	1	1	1	0	1	$-$	$-$	$-$	$+$	$-$
Σ^+	2	0	1	0	1	$+$	$+$	$-$	$+$	$-$
$\Sigma^- \to \Sigma^+$	0	2	1	0	1	$+$	$+$	$-$	$+$	$-$
$\Xi^0 \to n$	1	0	2	0	1	$-$	$+$	$+$	$+$	$-$
$\Xi^- \to p$	0	1	2	0	1	$+$	$-$	$+$	$+$	$-$
$\Omega^- \to \Sigma^+$	0	0	3	0	1	$+$	$+$	$-$	$+$	$-$

Flavour violation in QED+QCD$_C$



- The s quarks travels around the torus and generates a $\Delta s = -2$ mixing.
- Because of confinement the s quark must be accompanied by another quark.
- This process requires a K meson traveling around the torus. Naively we expect an exponential suppression $\exp(-M_K L)$...

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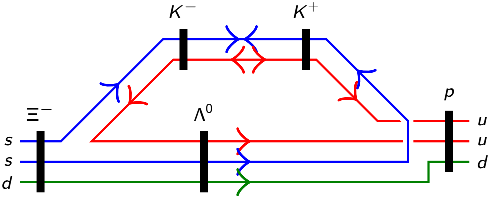

$$C(t;L) = \sum_{\mathbf{x}} \langle \Xi_+(t,\mathbf{x})^\dagger \Xi_+(0) \rangle = \sum_n A_n(L) e^{-tM_n(L)}$$

$$E_n < M_{\Xi^-} \quad : \quad |A_n(L)| < e^{-2\mu L}$$

$$\mu = \left[M_{K^\pm}^2 - \left(\frac{M_{\Xi^-}^2 - M_{\Lambda^0}^2 + M_{K^\pm}^2}{2M_{\Xi^-}} \right)^2 \right]^{1/2}$$

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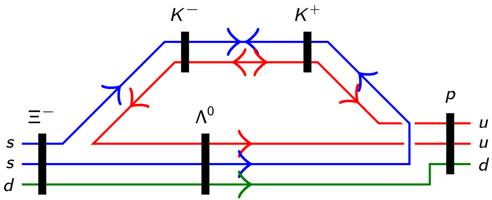

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$$E_n < M_{\Xi^-} \quad : \quad |A_n(L)| < e^{-2\mu L} \simeq 10^{-10} \quad \text{for } M_\pi L = 4$$

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Translations and charge-conjugation

C* b.c.'s in operatorial form:

$$\exp(iP_k L) = \mathcal{C}$$

- P_k momentum operator
- $\exp(iP_k L)$ unitary operator that implements a translation by L along the direction k
- $\mathcal{C}$ unitary operator that implements charge conjugation

C-even states	$\mathcal{C} = +1$	$P_k = \dfrac{2\pi n_k}{L}$
C-odd states	$\mathcal{C} = -1$	$P_k = \dfrac{\pi(2n_k + 1)}{L}$

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Let us consider a state with

$$P_k|\psi\rangle = 0 \qquad \mathcal{C}|\psi\rangle = 1 \qquad (-1)^Q|\psi\rangle = -1$$

Since this is in an eigenstate of $\mathcal{C}$ we get

$$\langle\psi|Q|\psi\rangle = 0$$

However this comes from the mixing beween states with $Q = \pm 1, \pm 3, \ldots$ but not from the mixing with states with $Q = 0$!

Finite-volume corrections to hadron masses

$$\frac{\Delta m(L)}{m} = \frac{e^2}{4\pi}\left\{\frac{q^2\xi(1)}{2mL} + \frac{q^2\xi(2)}{\pi(mL)^2} - \frac{1}{4\pi mL^4}\sum_{\ell=1}^{\infty}\frac{(-1)^{\ell}(2\ell)!}{\ell!\,L^{2(\ell-1)}}\mathcal{T}_{\ell}\,\xi(2+2\ell)\right\} + \ \dots$$

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- The non-universal corrections are related to the forward Compton amplitude for the scattering of a soft photon on the hadron at rest

$$\mathcal{T}_{\ell} = \left. \frac{d^{\ell}}{d(\mathbf{k}^2)^{\ell}}\, T_{\mu}^{\mu}(|\mathbf{k}|, \mathbf{k}) \right|_{\mathbf{k}=0}$$

Some implementation details

- Both the U(1) and SU(3) gauge fields have to satisfy C* b.c.'s

$$U_\mu(x + L\mathbf{k}) = U_\mu^*(x)$$

- Instead of the fermionic determinant, one needs to simulate

$$\mathrm{Pf}\, D_{2V} = \mathrm{Det}\, [D_{2V}^\dagger D_{2V}]^{1/4}\ \mathrm{sgn}\ \mathrm{Pf}\, D_{2V}$$

  where the Dirac operator acts on pseudofermions that are defined on the doubled lattice. RHMC is needed (this would be true also without C* b.c.'s)

- At large volume

$$|\mathrm{Pf}\, D_{2V}| = \mathrm{Det}\, [D_{2V}^\dagger D_{2V}]^{1/4} \simeq \left(\mathrm{Det}\, [D_V^\dagger D_V]^{1/4}\right)^2 \ .$$

  This is the Clark and Kennedy's n-th root acceleration.

- The Pfaffian has a mild sign problem with Wilson fermions (it is positive in the continuum limit).

- (Dirac) interpolating operators of charged states without gauge fixing. In the continuum:

$$\pi(t) = \int d^3x\ e^{-\imath \int d^3y\ \Phi(\mathbf{y}-\mathbf{x})\partial_k A_k(t,\mathbf{y})} \{\bar{u}\gamma_5 d + \bar{d}\gamma_5 u\}(t,\mathbf{x})$$

  where $\Phi(\mathbf{x})$ is the electric potential of a unit charge in a box with C* b.c.'s

$$\partial_k \partial_k \Phi(\mathbf{x}) = \delta^3(\mathbf{x})$$
$$\Phi(\mathbf{x} + L\mathbf{k}) = -\Phi(\mathbf{x})$$

Summary

- QCD+QED with C* boundary conditions is a local QFT in finite volume, and provides a framework to describe a certain class of electrically-charged states in a rigorous and gauge-invariant way.
- C* boundary conditions partially break flavour (and charge) conservation. F_f is not conserved but $(-1)^{F_f}$ is.
- Several interesting states are not affected by the finite-volume mixing (p, n, $\pi^\pm$, $K^\pm$, K_0, Λ^0, $D^\pm$, D^0, $D_s^\pm$, $B^\pm$, B^0, Σ^+)
- Some states are affected by the finite-volume mixing, e.g. Ξ^- or Ω^-, but the mixing with lighter states is exponentially suppressed with the volume (with a generally large exponent).
- Non-unversal finite-volume corrections to the masses of stable charged hadrons are $1/L^4$ rather than $1/L^3$ (thanks to locality).
- Operator mixing is not affected by breaking of flavour symmetry (thanks to locality).
- QCD+QED with C* boundary conditions can be formulated on the lattice with a compact U(1). Charged states can be described in a gauge-invariant way.