Reformulation of anomaly inflow on the lattice and construction of lattice chiral gauge theories

Juan W. Pedersen (The Univ. of Tokyo, Komaba)

Based on [arXiv:22mm.*****] with Y.Kikukawa

Nov. 6th 2022, Riken-BNL Research Center Seminar

Outline

Goal **Construction of lattice chiral gauge theories (LCGTs)**

- Analyze the integrability condition of the chiral determinant of overlap fermions
- Derive necessary and sufficient conditions to construct LCGTs (without gauge anomalies)

Discussion **Reformulate the modern theory of anomalies on the lattice**

- Dai-Freed theorem $\longleftrightarrow$ 5-dim lattice DW fermions
- APS index theorem $\longleftrightarrow$ 6-dim lattice DW fermions

  $\Longrightarrow$ Bordism invariance of the lattice η invariant
- Triviality of the lattice η invariant $\Longrightarrow$ Integrability conditions

Table of contents

Anomaly in continuum theory

Anomaly inflow

Anomaly inflow = Anomaly cancellation from the bulk

(e.g.) 4-dim chiral fermion + 5-dim Chern-Simons [Callan,Harvey(1985)]

Consider chiral fermion on 4-dim manifold X:

$$Z_\psi[A] = \int [D\psi][D\bar{\psi}] \exp\left[-\int d^4x \bar{\psi} i \sigma^\mu \left(\partial_\mu + iA_\mu\right)\psi \right]$$

and gauge transformation:

$$A_\mu \to A_\mu^g = gA_\mu g^{-1} + g\partial_\mu\left(g^{-1}\right)$$

$\Longrightarrow$ Partition function transforms as

$$Z_\psi\left[A^g\right] = e^{i\varphi(g)} Z_\psi[A]$$

$\Longrightarrow$Phase ambiguity depending gauge fields

... (perturbative) gauge anomaly

Anomaly inflow

Chern-Simons action on 5-dim manifold Y:

$$S_{CS} = k \int_Y A \wedge F \wedge F = k \int_Y \epsilon^{\mu\nu\rho\sigma\tau} A_\mu F_{\nu\rho} F_{\sigma\tau}$$

Under (5-dim) gauge transformation $A_\mu \to A_\mu + \partial_\mu \chi$:

$$\delta \left[k \int_Y \epsilon^{\mu\nu\rho\sigma\tau} A_\mu F_{\nu\rho} F_{\sigma\tau} \right] = k \int_{\partial Y} \chi \wedge F \wedge F$$

$\Longrightarrow$ Gauge dependent at the boundary

Exactly match with the anomaly of 4-dim chiral fermion

$\Longrightarrow$ 4-dim chiral fermion $+$ 5-dim CS theory $=$ Anomaly free

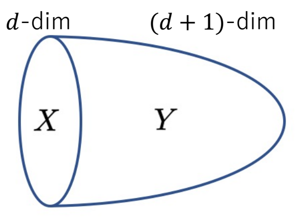
(No phase ambiguity of partition function)

η invariant $\Longrightarrow$ perturbative + non-perturbative anomaly inflow

(Atiyah-Patodi-Singer) η invariant

Dirac operator on bulk Y : $\mathcal{D}_Y$
and eigenvalues $\{\lambda_k\}$
Suppose APS boundary condition

$$\eta(Y) = \frac{1}{2}\left(\sum_k \mathrm{sign}\,(\lambda_k)\right)_{\mathrm{reg}}$$

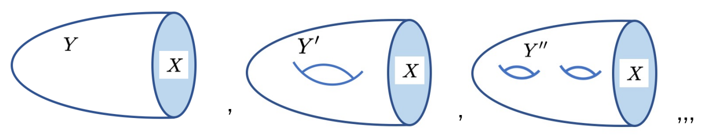


Dai-Freed theorem states... [Witten(2016)][Yonekura(2016)]
"Anomaly of d-dim theory + (d+1)-dim $\exp(-2\pi i\eta(Y))$ $\Longrightarrow$anomaly cancellation"

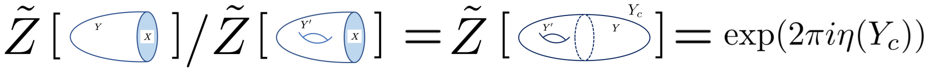
APS b.c.

Anomaly cancelation condition

- d-dim Anomaly $+$ (d+1)-dim $\exp(-2\pi i\eta(Y)) \Longrightarrow$ anomaly cancellation
- The combination can be adopted as the partition function of d-dim (anomalous) theory **if it is trivial in the bulk**



partition function:



Anomaly cancellation condition:

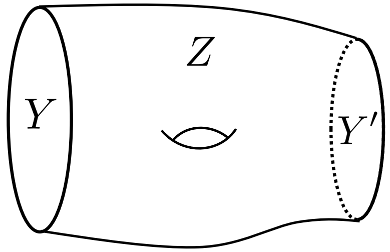
“ $\exp\left(2\pi i\eta\left(Y_c\right)\right) = 1$ for any (d+1)-dim closed mfd Y_c whose cross-section is X”

Bordism

Bordism invariance $\Longrightarrow$calculate $\exp\left(2\pi i\eta\left(Y_c\right)\right)$ for any closed mfd efficiently

Bordism invariance

- (d+1)-dim manifolds Y and Y' are **bordant (bordism equivalent)** if there exist a (d+2)-dim manifold Z such that $\partial Z = Y_1 \sqcup Y_2$
- An amount $\alpha(Y)$ is **bordism invariant** if $\alpha(Y_1) = \alpha(Y_2)$ for bordant Y_1 and Y_2



Bordism invariance of η invariant

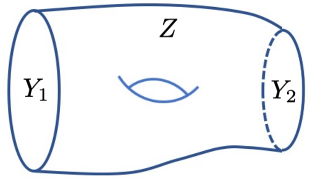
APS index theorem

$$\mathrm{Index}\,(\mathcal{D}_Z) = \int_Z I_{d+2}(F,R) - \eta(Y)$$

Z : (d+2)-dim manifold such that $\partial Z = Y$

$I_{d+2}(R,F) = \hat{A}(R)\,\mathrm{ch}(F)\Big|_{d+2}$ (R : Riemann curvature, F : Field strength)

Under the perturbative anomaly cancellation condition $\sum \mathrm{Tr}\left(T^a\left\{T^b,T^c\right\}\right) = 0$ ($\Longleftrightarrow \quad \int_Z I_{d+2}(F,R) = 0$),



$$\exp\left(2\pi i\eta\left(Y_1\right)\right)\exp\left(-2\pi i\eta\left(Y_2\right)\right) = \exp(2\pi i\mathcal{I}) = 1$$

$\Longrightarrow \exp\left(2\pi i\eta\left(Y_c\right)\right)$ is bordism invariant

Lattice fermions

Chiral symmetry on the lattice

- Nielsen-Ninomiya theorem
  Chiral relation $\{\gamma_5, D\} = 0 \quad \Longrightarrow$ Doubling problem
- Ginsparg-Wilson(GW relation)
  ... “Redefinition” of chiral symmetry on the lattice

$$\{\gamma_5, D\} = 2aD\gamma_5 D$$

- Exact symmetry on the lattice
  Infinitesimal transformation:

$$\delta\psi = \gamma_5(1 - aD)\psi \quad , \quad \delta\overline{\psi} = \overline{\psi}\gamma_5 \quad \Rightarrow \quad \delta S = 0$$

Overlap Dirac operator

Overlap operator

$$D_{\mathrm{ov}} = \frac{1}{2a}\left(1 + X\frac{1}{\sqrt{X_{\mathrm{w}}^{\dagger}X_{\mathrm{w}}}}\right)$$

$$\left(X_{\mathrm{w}} = aD_{\mathrm{w}} - m_0 \quad , \quad D_{\mathrm{w}} = -\gamma_{\mu}\frac{1}{2}\left(\nabla_{\mu} - \nabla_{\mu}^{\dagger}\right) + \frac{a}{2}\nabla_{\mu}\nabla_{\mu}^{\dagger} \right)$$

- Gauge-invariant solution of GW relation
- Exponentially local:

$$\|D_{\mathrm{ov}}(x,y)\| \leq C|x-y|^{\sigma}\mathrm{e}^{-|x-y|/\lambda}$$

  under the **admissibility condition**
- Low energy effective action of Domain-wall fermion

Admissibility condition

Admissibility condition

Restrict the configuration space of link fields to a subspace:

$$\mathfrak{U} \;=\; \Big\{ \{U(x,\mu)\} \;|\; \|1 - P_{\mu\nu}(x)\| < \epsilon \;\; {}^{\forall}(x,\mu,\nu) \Big\}$$

$$\left(\epsilon \;<\; \frac{2}{5d(d-1)} \quad , \quad d = \text{dimension} \right)$$

- Locality of overlap fermion (DW fermion)
- Topological structure of link fields

Weyl fermion on the lattice

- Ginsparg-Wilson relation $\{\gamma_5, D\} = 2aD\gamma_5 D$

$$\Rightarrow \hat{\gamma}_5 = \gamma_5(1-2aD)\ ,\ \hat{P}_\pm = \frac{1}{2}(1 \pm \hat{\gamma}_5)$$

- Lattice Weyl fermion

$$\psi_-(x) = \hat{P}_-\psi_-(x) \quad , \quad \overline{\psi}_-(x) = \overline{\psi}_-(x) P_+$$

- Action

$$S = a^4 \sum \overline{\psi}_-(x) D \psi_-(x)$$

- Mode expansion

$$\psi_-(x) = \sum v_i(x) c_i \quad , \quad \hat{P}_- v_i(x) = v_i(x)$$
$$\overline{\psi}_-(x) = \sum \overline{c}_k \overline{v}_k(x) \quad , \quad \overline{v}_k(x) P_+ = \overline{v}_k(x)$$

Path integral

- Path integral measure

$$\mathcal{D}\left[\psi_-\right]\mathcal{D}\left[\overline{\psi}_-\right] = \prod dc_j \prod d\bar{c}_k$$

- Partition function

$$\begin{aligned} Z &= \int \mathcal{D}\left[\psi_-\right]\mathcal{D}\left[\bar{\psi}_-\right] \mathrm{e}^{-a^4 \sum_x \bar{\psi}_-(x) D\psi_-(x)} \\ &= \int \prod_i dc_i \prod_i d\bar{c}_j \mathrm{e}^{-\sum_{ij} \bar{c}_j M_{ji} c_i} = \det M_{ji} \end{aligned}$$

$$\left(M_{ji} = a^4 \sum_x \bar{v}_j D v_i(x) \right)$$

... Chiral determinant of overlap Dirac operator

Gauge anomaly of overlap Weyl fermion

- $$Z = \det\left(\bar{v} D v\right) = \det M_{ji}$$
- Unitary transformation of basis

$$\begin{aligned} \tilde{v}_i(x) &= v_l(x)\left(\mathcal{Q}^{-1}\right)_{li}, \quad \tilde{c}_j = \sum_l \mathcal{Q}_{jl} c_l \\ \mathcal{D}\left[\psi_-\right] &\to \mathcal{D}\left[\psi_-\right]\det\mathcal{Q} \quad , \quad \det M_{ji} \to \det M_{ji} \det\mathcal{Q} \end{aligned}$$

$\Longrightarrow$ Phase ambiguity **depending gauge fields**

... Gauge anomaly on the lattice

Construction of LCGTs

- (Kaplan's formulation)
  5-dim Wilson fermion + "kink" mass term

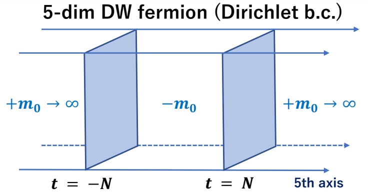
$$S = \sum_{x,t} \bar{\psi}(x,t) X^{(5)}_{DW} \psi(x,t)$$

$$X^{(5)}_{\mathrm{DW}} = D^{(5)}_{\mathrm{w}} + m\theta(t) \quad , \quad (t = \mathrm{5th\ axis})$$

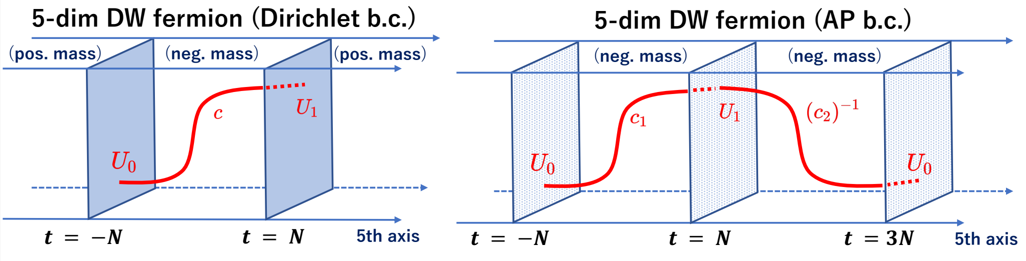
- (Shamir's formulation)
  Domain-wall fermion in a finite space

$$S = \sum_{x,t} \bar{\psi}(x,t) X^{(5)}_{\mathrm{w}} \psi(x,t)$$

$$X^{(5)}_{\mathrm{w}} = D^{(5)}_{\mathrm{w}} - m_0 \quad , \quad m_0 \in (0,2)$$



Domain-wall Fermion (Chiral set up)



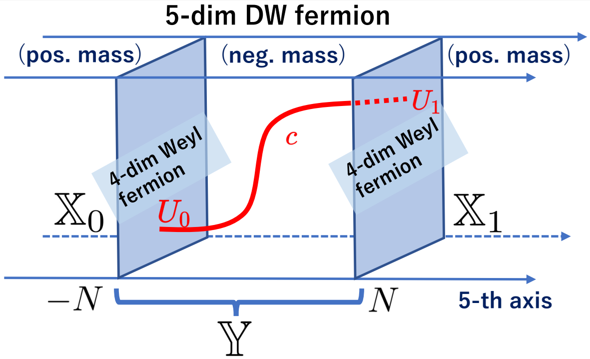
- Dirichlet b.c. $\psi_-(x,t)|_{t=-N,N+1} = 0 \quad , \quad \bar{\psi}_-(x,t)\big|_{t=-N,N+1} = 0$
- AP b.c. $\psi_-(x,t)|_{t=-N} = \psi_-(x,t)|_{t=3N} \quad , \quad \bar{\psi}_-(x,t)\big|_{t=-N} = \bar{\psi}_-(x,t)\big|_{t=3N}$
- Set up for chiral fermion ($U_0 \sim$ reference gauge field)

$$U(x,t,\mu)|_{t=-N+1} = U_0(x,\mu) \quad , \quad U(x,t,\mu)|_{t=N} = U_1(x,\mu)$$

5-dim DW fermions and anomaly inflow (Dai-Freed theorem)

Partition function of 5-dim DW fermions with Dirichlet b.c.

$+$ Chiral determinant of 4-dim Weyl fermions on the boundaries



Use “master equation” for calculating the DW partition function and get:

5-dim DW fermions $+$ **4-dim Weyl fermions**

$$\lim_{N\to\infty} \frac{\det X_{\mathrm{w}}^{(5)}\Big|_{\mathrm{Dir}}^{c}}{\left|\det X_{\mathrm{w}}^{(5)}\Big|_{\mathrm{AP}}^{c\cdot c^{-1}}\right|^{1/2}} = \det\left(\overline{v} D_{\mathrm{ov}} v^{1}\right) \det\left(\overline{v} D_{\mathrm{ov}} v^{0}\right)^{*} \frac{\det\left(v^{1\dagger}\prod_{t\in\tilde{c}} T_t v^{0}\right)}{\left|\det\left(v^{1\dagger}\prod_{t\in\tilde{c}} T_t v^{0}\right)\right|}$$

$$=: \exp\left(\Gamma\left(\mathbb{X}_1 \cup \overline{\mathbb{X}}_0\right)\right) \exp\left(i2\pi\eta_{\mathrm{DF}}\left(\mathbb{Y}|_{\mathrm{Dir}}\right)\right)$$

▸ master equation

5-dim DW fermions and anomaly inflow (Dai-Freed theorem)

5-dim DW fermions $+$ **4-dim Weyl fermions**

$$\lim_{N\to\infty} \frac{\det X_{\mathrm{w}}^{(5)}\Big|_{\mathrm{Dir}}^{c}}{\left|\det X_{\mathrm{w}}^{(5)}\Big|_{\mathrm{AP}}^{c\cdot c^{-1}}\right|^{1/2}} = \exp\left(\Gamma\left(\mathbb{X}_1 \cup \overline{\mathbb{X}}_0\right)\right) \exp\left(i2\pi\eta_{\mathrm{DF}}\left(\mathbb{Y}|_{\mathrm{Dir}}\right)\right)$$

Boundary part : $\exp\left(\Gamma\left(\mathbb{X}_1 \cup \overline{\mathbb{X}}_0\right)\right)$ ...effective action of overlap Weyl fermion
Bulk part : $\exp\left(i2\pi\eta_{\mathrm{DF}}\left(\mathbb{Y}|_{\mathrm{Dir}}^{c}\right)\right)$ ...(c.f. "APS-η invariant")

They have same $\{v_i\}$ dependence and cancel them out

$\Longrightarrow$ Anomaly inflow based on Dai-Freed theorem

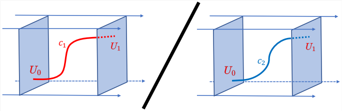
Anomaly $\leftrightarrow$ bulk dependency $\exp\left(i2\pi\eta_{\mathrm{DF}}\left(\mathbb{Y}|_{\mathrm{Dir}}\right)\right)$

The lattice η invariant and integrability conditions

- Define the lattice η invariant with the phase of 5-dim overlap fermions [Aoyama,YK(1999)] :

$$e^{i2\pi\eta\left(\mathbb{Y}|_{\mathrm{Dir/AP}}\right)} := \lim_{N\to\infty}\left[\frac{\det D^{(5)}_{\mathrm{ov}}\Big|_{\mathrm{Dir/AP}}}{\left|\det D^{(5)}_{\mathrm{ov}}\Big|_{\mathrm{Dir/AP}}\right|}\right]^2 = \lim_{N\to\infty}\frac{\det X^{(5)}_{\mathrm{w}}\Big|_{\mathrm{Dir/AP}}}{\left|\det X^{(5)}_{\mathrm{w}}\Big|_{\mathrm{Dir/AP}}\right|}$$

- Bulk dependence



$$= \frac{e^{i2\pi\eta_{\mathrm{DF}}\left(\mathbb{Y}|^{c_1}_{\mathrm{Dir}}\right)}}{e^{i2\pi\eta_{\mathrm{DF}}\left(\mathbb{Y}|^{c_2}_{\mathrm{Dir}}\right)}} = e^{i2\pi\eta\left(\mathbb{Y}|^{c_1 c_2^{-1}}_{\mathrm{AP}}\right)}$$

$\Longrightarrow$ Bulk independency $\quad e^{i2\pi\eta_{\mathrm{DF}}\left(\mathbb{Y}^{c_1}|_{\mathrm{Dir}}\right)} = e^{i2\pi\eta_{\mathrm{DF}}\left(\mathbb{Y}^{c_2}|_{\mathrm{Dir}}\right)}$

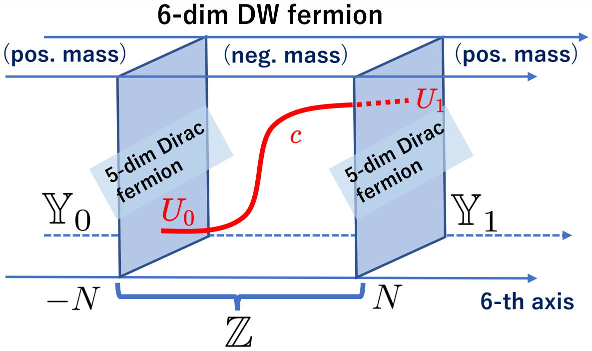
"$e^{i2\pi\eta\left(\mathbb{Y}|_{\mathrm{AP}}\right)} = 1$ for arbitrary gauge configurations "

(Integrability condition)

6-dim DW fermions and APS index theorem

Partition function of 6-dim DW fermions with Dir. b.c.

$+$ Determinant of 5-dim overlap fermions on the boundaries



6-dim DW fermions and APS index theorem

6-dim DW fermions + 5-dim overlap fermions

$$\lim_{N\to\infty} \frac{\det X_{\mathrm{w}}^{(6)}\Big|_{\mathrm{Dir}}^{c}}{\left|\det X_{\mathrm{w}}^{(6)}\Big|_{\mathrm{AP}}^{c^{-1}}\right|^{1/2}} = \det\left(\bar{v} D_{\mathrm{ov}} v^1\right) \det\left(\bar{v} D_{\mathrm{ov}} v^0\right)^* \frac{\det\left(v^{1\dagger} \prod_{t\in\tilde{c}} T_t v^0\right)}{\left|\det\left(v^{1\dagger} \prod_{t\in\tilde{c}} T_t v^0\right)\right|}$$

In 6-dim case, we can chose basis $\{v_i\}$ as follows:

$$\begin{aligned} v_i(x) &= \frac{1}{\sqrt{2}} \begin{pmatrix} X^{(5)} \frac{1}{\sqrt{X^{(5)\dagger} X^{(5)}}} \\ 1 \end{pmatrix} \phi_i(x) \\ \left(X^{(5)\dagger} X^{(5)}\right) \phi_i(x) &= \left(\lambda_i\right)^2 \phi_i(x) \\ \sum \phi_i(x)\phi_i(y)^\dagger &= I_{4\times 4}\delta_{x,y} \end{aligned}$$

▸ master equation

6-dim DW fermions and APS index theorem

6-dim DW fermions + 5-dim overlap fermions

$$\lim_{N\to\infty} \frac{\det X_{\mathrm{w}}^{(6)}\Big|^{c}_{\mathrm{Dir}}}{\left|\det X_{\mathrm{w}}^{(6)}\Big|^{cc^{-1}}_{\mathrm{AP}}\right|^{1/2}} = \det D_{\mathrm{ov}}^{(5)}\Big|_{Y_1} \left\{\det D_{\mathrm{ov}}^{(5)}\Big|_{Y_0}\right\}^{*} \times \frac{\det \mathcal{T}|^{c}_{\mathrm{APS}}}{|\det \mathcal{T}|^{c}_{\mathrm{APS}}|}$$

$$\mathcal{T}|^{c}_{\mathrm{APS}} = \frac{1}{2}\left(\frac{1}{\sqrt{X^\dagger X}}X^\dagger \quad 1\right)\Big|_{\mathbb{Y}_1} \prod_{t\in\tilde{c}} T_t^{(5)} \begin{pmatrix} X\frac{1}{\sqrt{X^\dagger X}} \\ 1 \end{pmatrix}\Bigg|_{\mathbb{Y}_0}$$

6-dim DW fermions and APS index theorem

6-dim DW fermions $+$ **5-dim overlap fermions**

$$\lim_{N\to\infty}\frac{\det X_{\rm w}^{(6)}\Big|_{\rm Dir}^{c}}{\left|\det X_{\rm w}^{(6)}\Big|_{\rm AP}^{cc^{-1}}\right|^{1/2}} = \det D_{\rm ov}^{(5)}\Big|_{\mathrm{Y}_1}\left\{\det D_{\rm ov}^{(5)}\Big|_{\mathrm{Y}_0}\right\}^{*}\times\frac{\det \mathcal{T}|_{\rm APS}^{c}}{|\det \mathcal{T}|_{\rm APS}^{c}|}$$

Focus on the phase:

$$\text{L.H.S.} = (-1)^{I(\mathbb{Z}|_{\rm Dir})} \quad , \quad I\left(\mathbb{Z}|_{\rm Dir}\right) = -\frac{1}{2}\,\mathrm{Tr}\left\{H_{\rm w}^{(6)}/\sqrt{H_{\rm w}^{(6)2}}\Big|_{\rm Dir}\right\}$$

$$\text{Boundary} = \exp\left(i\pi\eta\left(\mathbb{Y}|_{\rm AP}\right)\right)\exp\left(-i\pi\eta\left(\mathbb{Y}|_{\rm AP}\right)\right)$$ [Fukaya et al.(2020)]

$$\text{Bulk} = \exp\left(i\pi P\left(\mathbb{Z}|_{\rm APS}^{c}\right)\right) := \frac{\det \mathcal{T}|_{\rm APS}^{c}}{|\det \mathcal{T}|_{\rm APS}^{c}|}$$

$\Longrightarrow$ APS index theorem $I\left(\mathbb{Z}|_{\rm Dir}\right) = P\left(\mathbb{Z}|_{\rm APS}^{c}\right) + \eta\left(\mathbb{Y}_1|_{\rm AP}\right) - \eta\left(\mathbb{Y}_0|_{\rm AP}\right)$

Lattice topological field

- Definition of P:

$$\mathrm{e}^{i\pi P\left(\mathbb{Z}|^c_{\mathrm{APS}}\right)} \equiv \frac{\det \mathcal{T}|^c_{\mathrm{APS}}}{|\det \mathcal{T}|^c_{\mathrm{APS}}|}$$

- It can be expressed with a field $q^{(6)}(z)$:

$$P\left(\mathbb{Z}|^c_{\mathrm{APS}}\right) = \lim_{N\to\infty} \sum_{y,s\in c} q^{(6)}(z)$$

$$q^{(6)}(z) := -\frac{1}{2}\,\mathrm{tr}\left\{ \left.\frac{H_{\mathrm{w}}}{\sqrt{H_{\mathrm{w}}^2}}\right|_{\mathrm{AP}} \right\}(z,z) \quad , \quad H_{\mathrm{w}} = \gamma_7 X_{\mathrm{w}}^{(6)}$$

- $q^{(6)}(z)$ is topological:

$$\sum_{y,s\in c} \delta_\eta q^{(6)}(z) = 0$$

Cohomological problem

- $$P\left(\mathbb{Z}|^{c_1}_{\mathrm{APS}}\right) = \lim_{N\to\infty} \sum_{y,s\in c_1} q^{(6)}(z)$$
- **(The cohomological problem)** Assumption: $q^{(6)}$ can be expressed with the lattice Chern character and gauge invariant currents:

$$q^{(6)}(z) = \hat{c}_3(z) + \partial^*_\mu k_\mu(z)$$

- Under the perturbative anomaly cancellation condition $\hat{c}_3(z) = 0$ (i.e. $\Sigma_R \operatorname{Tr}_R\left[T^a\left\{T^b, T^c\right\}\right] = 0$):

$$P\left(\mathbb{Z}|^{c}_{\mathrm{APS}}\right) = \sum_y k_s(z)|_{\mathbb{Y}_1} - \sum_y k_s(z)|_{\mathbb{Y}_0}$$

Bordism invariance of the lattice η invariant

- Redefine the lattice η invariant:

$$\check{\eta}\left(\mathbb{Y}_{1,0}|_{\mathrm{AP}}\right) = \eta\left(\mathbb{Y}_{1,0}|_{\mathrm{AP}}\right) + \sum_{y} k_s(y,s)|_{\mathbb{Y}_{1,0}}$$

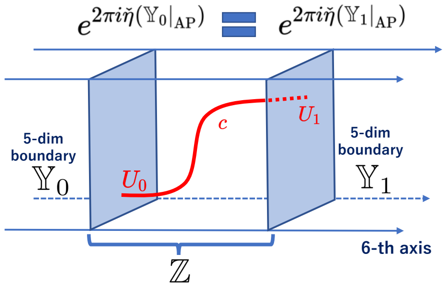
$$\Longrightarrow \quad e^{2\pi i\check{\eta}\left(\mathbb{Y}_1|_{\mathrm{AP}}\right)} e^{-2\pi i\check{\eta}\left(\mathbb{Y}_0|_{\mathrm{AP}}\right)} = (-1)^{2I\left(\mathbb{Z}|^c_{\mathsf{Dir}}\right)} = 1$$

from APS index theorem:

$$\begin{aligned}
I\left(\mathbb{Z}|_{\mathrm{Dir}}\right) &= P\left(\mathbb{Z}|^c_{\mathrm{APS}}\right) + \eta\left(\mathbb{Y}_1|_{\mathrm{AP}}\right) - \eta\left(\mathbb{Y}_0|_{\mathrm{AP}}\right) \\
I\left(\mathbb{Z}|_{\mathrm{Dir}}\right) &= \check{\eta}\left(\mathbb{Y}_1|_{\mathrm{AP}}\right) - \check{\eta}\left(\mathbb{Y}_0|_{\mathrm{AP}}\right) \\
\exp(\pi i I\left(\mathbb{Z}|_{\mathrm{Dir}}\right)) &= \exp(\pi i\check{\eta}\left(\mathbb{Y}_1|_{\mathrm{AP}}\right)) \exp(-\pi i\check{\eta}\left(\mathbb{Y}_0|_{\mathrm{AP}}\right)) \\
1 = \exp(2\pi i I\left(\mathbb{Z}|_{\mathrm{Dir}}\right)) &= \exp(2\pi i\check{\eta}\left(\mathbb{Y}_1|_{\mathrm{AP}}\right)) \exp(-2\pi i\check{\eta}\left(\mathbb{Y}_0|_{\mathrm{AP}}\right))
\end{aligned}$$

Bordism invariance of the lattice η invariant

$$e^{2\pi i\check{\eta}\left(\mathbb{Y}_1|_{\mathrm{AP}}\right)}e^{-2\pi i\check{\eta}\left(\mathbb{Y}_0|_{\mathrm{AP}}\right)}=1$$



$\Longrightarrow$ $e^{i2\pi\check{\eta}\left(\mathbb{Y}|_{\mathrm{AP}}\right)}$ is "bordism" invariant

Able to evaluate $e^{i2\pi\check{\eta}\left(\mathbb{Y}|_{\mathrm{AP}}\right)}$ in arbitrary gauge configurations

Looking back on our discussion

1. **Problem** Gauge anomaly of 4-dim lattice chiral fermion
   Solution Anomaly inflow with 5-dim object $\exp\left(i2\pi\eta_{\mathrm{DF}}\left(\mathbb{Y}|_{\mathrm{Dir}}\right)\right)$

2. **Problem** 5-dim dependency
   Solution $\exp(i2\pi\eta_{\mathrm{DF}}\left(\mathbb{Y}|_{\mathrm{Dir}}^{c_1}\right))/\exp(i2\pi\eta_{\mathrm{DF}}\left(\mathbb{Y}|_{\mathrm{Dir}}^{c_2}\right)) = \exp(i2\pi\eta\left(\mathbb{Y}|_{\mathrm{AP}}\right))$
   $\Longrightarrow$ If $\exp(i2\pi\eta\left(\mathbb{Y}|_{\mathrm{AP}}\right)) = 1$ for any gauge configurations, it doesn't depend on the bulk! (Integrability condition)

3. **Problem** Calculating $\exp(i2\pi\eta\left(\mathbb{Y}|_{\mathrm{AP}}\right))$ for any gauge configurations
   Solution Redefining $\check{\eta}\cdots$"bordism" invariant (←Cohomological problem)
   $\Longrightarrow$ Only need to calculate it on representatives of "bordism" equivalent class

Summary: Construction of LCGTs

- We derived two conditions to construct LCGTs:

  Condition-1 Triviality of $q^{(6)}(z)$ under the perturbative anomaly cancellation condition

  Condition-2 $e^{i2\pi\check{\eta}\left(\mathbb{Y}|_{\mathrm{AP}}\right)} = 1$ for representative gauge configurations of "bordism" equivalent class following the admissibility condition on the 5-dim lattice space

- It is known that Condition-1 holds in $U(1)$ and $SU(2) \times U(1)$ [Lüscher(1999)][YK,Nakayama(2001)]
  $\longrightarrow$We further confirmed that Condition-2 also holds in those cases

- We gave a proof of Condition-1 in generic non-Abelian gauge theories [JWP,YK(in prep)]

Construction of SU(2)$\times$U(1) chiral gauge theory

$SU(2) \times U(1)$ chiral gauge theory

- $SU(2) \times U(1)$ gauge theory satisfies the Condition-1
  $\Longrightarrow$ Let's check the Condition-2
- Consider a 5-dim lattice space $\mathbb{L}^5(\mathbb{L}_{\rm P}^4 \times \mathbb{L}_{\rm AP})$
- Classification of $SU(2) \times U(1)$ Gauge DOF
  [Narayanan,Neuberger(1995)][Lüscher(1999)]
  1. U(1) magnetic flux $m_{\mu\nu}$ … Defined on $\mathbb{L}_{\rm P}^2$ among $\mathbb{L}_{\rm P}^4$
  2. U(1) Wilson line W_μ … Defined on $\mathbb{L}_{\rm P}^4$ and wraps along $\mathbb{L}_{\rm AP}$ direction
  3. SU(2) Instanton ϕ … Defined on either $\mathbb{L}_{\rm P}^4$ or $\mathbb{L}_{\rm P}^3 \times \mathbb{L}_{\rm AP}$

Result: $SU(2) \times U(1)$ chiral gauge theory

U(1) mag. flux	U(1) winding	SU(2) instanton	determinant
0	0	$\mathbb{L}_{\mathrm{P}}^4$	$1 = 1^4$
0	0	$\mathbb{L}_{\mathrm{P}}^3 \times \mathbb{L}_{\mathrm{AP}}^1$	$1 = (-1)^4$
1	0	0	1
1	1	0	$\fallingdotseq 1$
1	0	$\mathbb{L}_{\mathrm{P}}^4$	1
1	0	$\mathbb{L}_{\mathrm{P}}^3 \times \mathbb{L}_{\mathrm{AP}}^1$	1
1	1	$\mathbb{L}_{\mathrm{P}}^4$	$\fallingdotseq 1$
1	1	$\mathbb{L}_{\mathrm{P}}^3 \times \mathbb{L}_{\mathrm{AP}}^1$	$\fallingdotseq 1$

comments

- Calculated original $\eta\left(\mathbb{Y}_{1,0}|_{\mathrm{AP}}\right)$, instead of $\check{\eta}\left(\mathbb{Y}_{1,0}|_{\mathrm{AP}}\right)$
- "$\fallingdotseq 1$" means $1 \pm O(10^{-2})$
- Calculated with only two configurations for the magnetic flux
- SM charge $\Rightarrow$ perturbative anomaly cancellation

$\Longrightarrow$ Condition-2 confirmed!

Thank you!

Proof of Condition-1

Setup: 6-dim, $\hat{c}_3 = 0$, $\underline{q(x) = \partial^*_\mu k_\mu(x)}$

- Taking complete axial gauge, we define link variables with a product of some independent plaquette variables (from a reference point x_0)

$$\hat{P}\left(x, \mu, \nu; x^{(0)}\right) \sim \prod_{\tau} \prod_{y=x_0}^{x} \hat{U}(y, \tau)$$

- Introduce a parameter s:

$$\hat{P}_s\left(x, \mu, \nu; x^{(0)}\right) \sim \prod_{\tau} \prod_{y=x_0}^{x} \hat{U}^s(y, \tau) \quad , \quad q_s(x) = q(x)|_{\hat{U} \to \hat{U}_s}$$

Proof of Condition-1

- Taylor expansion of $q_s(x)$:

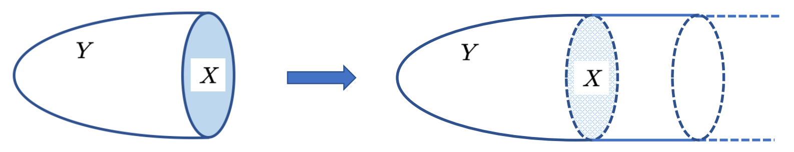
$$q_s(x) = \sum_{n=5}^{\infty} \frac{s^n}{n!} q_0^{(n)}(x)$$

- We can rewrite $q_0^{(j)}(x)$ with gauge-invariant currents form lower degree

$\Longrightarrow$ We can show the triviality of the original field $q(x)$ with those currents

Details will be discussed in our paper in prep.

APS boundary condition



- Deform a (d+1)-dim mfd : Y s.t. $\partial Y = X \quad \longrightarrow \quad Y'$ s.t. $\partial Y' = \emptyset$
- APS boundary condition of Y

  $:=$ "Square integrability of ψ on Y' "

  $=$ Following condition on Y

$$\psi_+|_X \in \mathcal{H}_-, \quad \psi_-|_X \in \mathcal{H}_+$$

$\psi_\pm$: Fields with chirality ± 1

$\mathcal{H}_\pm$: Subspace of fields with positive (negative) eigenvalues

back

5-dim DW fermions and Dai-Freed theorem

Formula for calculation of 5-dim Domain-wall partition function [YK(2002)]

$$
\det X_{\rm w}^{(5)}\Big|_{\rm Dir}^{c_1} = \det\left(P_- + P_+ \prod_{t\in\tilde{c}_1} T_t\right) \quad , \quad \det X_{\rm w}^{(5)}\Big|_{\rm AP}^{c_1\tilde{c}_2^{-1}} = \det\left(1 + \prod_{t\in c_1\tilde{c}_2^{-1}} T_t\right)
$$

$$
T_t = \frac{1 - a_5 H_t/2}{1 + a_5 H_t/2}
$$

$$
H = -\gamma_5 X_{\rm w}^{(4)} \frac{1}{1 + a_5 X_{\rm w}^{(4)}/2}
$$

▸ back

6-dim DW fermions and APS index theorem

Formula for calculation of 5-dim Domain-wall partition function [YK(2002)]

$$\det X_{\rm w}^{(6)}\Big|_{\rm Dir}^{c_1} = \det\left(P_- + P_+ \prod_{t\in\tilde{c}_1} T_t\right) \quad , \quad \det X_{\rm w}^{(6)}\Big|_{\rm AP}^{c_1\tilde{c}_2^{-1}} = \det\left(1 + \prod_{t\in c_1\tilde{c}_2^{-1}} T_t\right)$$

$$T_t = \frac{1 - a_6 H_t/2}{1 + a_6 H_t/2}$$

$$H = -\overline{\gamma}\tilde{X}_{\rm w}^{(5)} \frac{1}{1 + a_d \tilde{X}_{\rm w}^{(5)}/2} = -\begin{pmatrix} 0 & X^{(5)} \\ X^{(5)\dagger} & 0 \end{pmatrix}$$

$$X^{(5)} = X_{\rm w}^{(5)} \frac{1}{1 + a_d X_{\rm w}^{(5)}/2}$$

▸ back