Exact Polarization in Relativistic Fluids at Global Equilibrium

Based on JHEP 10 (2021) 077
in collaboration with
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Chirality, Vorticity and Magnetic Field in Heavy Ion Collisions
UCLA 12/2022

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Main results

Exact spin density matrix for spin-S particles at general global equilibrium:

$$\Theta(p) = \frac{\sum_{n=1}^{\infty} (-1)^{2S(n+1)} e^{-nb\cdot p} D^{(S)}(\Lambda^n)}{\sum_{n=1}^{\infty} (-1)^{2S(n+1)} e^{-nb\cdot p} \mathrm{tr}\left(D^{(S)}(\Lambda^n)\right)}$$

Exact spin vector for Dirac field at global equilibrium

$$\theta^\mu = -\frac{1}{2m}\epsilon^{\mu\nu\rho\sigma}\varpi_{\nu\rho}p_\sigma \qquad S^\mu(p) = \frac{1}{2}\frac{\theta^\mu}{\sqrt{-\theta^2}}\frac{\sinh\left(\frac{\sqrt{-\theta^2}}{2}\right)}{\cosh\left(\frac{\sqrt{-\theta^2}}{2}\right) + e^{-b\cdot p+\zeta}}$$

Including **all quantum corrections** in vorticity

Global equilibrium

Density operator at global equilibrium:

$$\widehat{\rho} = \frac{1}{Z} \exp\left[-b_\mu \widehat{P}^\mu + \frac{1}{2}\varpi_{\mu\nu}\widehat{J}^{\mu\nu}\right] \qquad \langle\widehat{O}\rangle = \mathrm{Tr}\left(\widehat{\rho}\widehat{O}\right)$$

The vector b is constant and the thermal vorticity ϖ is a constant antisymmetric tensor. The four-temperature β vector is a Killing vector:

$$\beta^\mu(x) = b^\mu + \varpi^{\mu\nu}x_\nu \equiv \frac{u^\mu}{T}$$

At global equilibrium:

$$\frac{A^\mu}{T} = \varpi^{\mu\nu}u_\nu$$

Acceleration

$$\frac{\omega^\mu}{T} = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\varpi_{\nu\rho}u_\sigma$$

Angular velocity

The generators of the Poincaré group appear in the density operator.

Analytic continuation of the thermal vorticity: $\varpi \mapsto -i\phi$

$$\widehat{\rho} = \frac{1}{Z} \exp\left[-b_\mu \widehat{P}^\mu - \frac{i}{2}\phi_{\mu\nu}\widehat{J}^{\mu\nu}\right]$$

P $\mapsto$ translations
J$\mapsto$ Lorentz transformations

Factorization of the density operator:

$$\widehat{\rho} = \frac{1}{Z} \exp\left[-\widetilde{b}_\mu(\phi)\widehat{P}^\mu\right] \exp\left[-i\frac{\phi_{\mu\nu}}{2}\widehat{J}^{\mu\nu}\right] \equiv \frac{1}{Z}\exp\left[-\widetilde{b}_\mu(\phi)\widehat{P}^\mu\right]\widehat{\Lambda}$$

$$\widetilde{b}^\mu(\phi) = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \underbrace{(\phi^\mu{}_{\alpha_1}\phi^{\alpha_1}{}_{\alpha_2}\dots\phi^{\alpha_{k-1}}{}_{\alpha_k})}_{k \text{ times}} b^{\alpha_k} \qquad \widehat{\Lambda} \equiv \mathrm{e}^{-i\frac{\phi_{\mu\nu}}{2}\widehat{J}^{\mu\nu}}$$

We can use **group theory** to calculate **thermal expectation values.**

Any thermal expectation value in a free quantum field theory is obtained from:

$$\langle \widehat{a}^\dagger_s(p)\widehat{a}_t(p')\rangle = \frac{1}{Z}\mathrm{Tr}\left(\exp[-\tilde{b}_\mu(\phi)\widehat{P}^\mu]\widehat{\Lambda}\ \widehat{a}^\dagger_s(p)\widehat{a}_t(p')\right)$$

$$[\widehat{a}^\dagger_s(p),\widehat{a}_t(p')]_\pm = 2\varepsilon\delta^3(\boldsymbol{p}-\boldsymbol{p'})\delta_{st}$$

Using Poincaré transformation rules and (anti)commutation relations (particle with spin $\mathbb{S}$):

$$\begin{aligned}\langle \widehat{a}^\dagger_s(p)\widehat{a}_t(p')\rangle =&(-1)^{2S}\sum_r D^S(W(\Lambda,p))_{rs}\mathrm{e}^{-\widetilde{b}\cdot\Lambda p}\langle \widehat{a}^\dagger_r(\Lambda p)\widehat{a}_t(p')\rangle + \\ &+ 2\varepsilon\,\mathrm{e}^{-\widetilde{b}\cdot\Lambda p}D^S(W(\Lambda,p))_{ts}\delta^3(\Lambda\boldsymbol{p}-\boldsymbol{p}')\end{aligned}$$

$D(W)=[\Lambda p]^{-1}\Lambda[p]$ is the “Wigner rotation” in the $\mathbb{S}$-spin representation.

We find a solution by iteration:

I
$$\langle \widehat{a}_s^\dagger(p)\widehat{a}_t(p')\rangle \sim 2\varepsilon\, \mathrm{e}^{-\widetilde{b}\cdot\Lambda p} D^S(W(\Lambda,p))_{ts}\delta^3(\Lambda\boldsymbol{p}-\boldsymbol{p}')$$

II
$$\langle \widehat{a}_s^\dagger(p)\widehat{a}_t(p')\rangle \sim 2\varepsilon\,(-1)^{2S} D^S(W(\Lambda^2,p))_{ts}\mathrm{e}^{-\widetilde{b}\cdot\left(\Lambda p+\Lambda^2 p\right)}\delta^3(\Lambda^2\boldsymbol{p}-\boldsymbol{p}')+ + 2\varepsilon\,\mathrm{e}^{-\widetilde{b}\cdot\Lambda p} D^S(W(\Lambda,p))_{ts}\delta^3(\Lambda\boldsymbol{p}-\boldsymbol{p}')$$

∞
$$\langle \widehat{a}_s^\dagger(p)\widehat{a}_t(p')\rangle = 2\varepsilon' \sum_{n=1}^{\infty}(-1)^{2S(n+1)}\delta^3(\Lambda^n\boldsymbol{p}-\boldsymbol{p}')D^S(W(\Lambda^n,p))_{ts}\mathrm{e}^{-\widetilde{b}\cdot\sum_{k=1}^{n}\Lambda^k p}$$

For vanishing vorticity (i.e. Λ=I) we recover Bose and Fermi statistics:

$$\langle \widehat{a}_s^\dagger(p)\widehat{a}_t(p')\rangle = 2\varepsilon' \sum_{n=1}^{\infty}(-1)^{2S(n+1)}\delta^3(\boldsymbol{p}-\boldsymbol{p}')\delta_{ts}\,\mathrm{e}^{-nb\cdot p} = \frac{2\varepsilon\,\delta^3(\boldsymbol{p}-\boldsymbol{p}')\delta_{ts}}{\mathrm{e}^{b\cdot p}+(-1)^{2S+1}}$$

Wigner function

The Wigner function for free fermions:

$$W(x,k) = -\frac{1}{(2\pi)^4}\int \mathrm{d}^4y\, \mathrm{e}^{-ik\cdot y}\langle :\Psi(x-y/2)\overline{\Psi}(x+y/2):\rangle$$

Expectation values are expressed as integrals of W:

$$j_A^\mu(x) = \int \mathrm{d}^4k\, \mathrm{tr}\left(\gamma^\mu\gamma_5 W(x,k)\right)$$

Wigner equation, a constraint:

$$\left(m - \not{k} - \frac{i\hbar\not{\partial}}{2}\right)W(x,k) = 0$$

Holds regardless of the density operator

Exact Wigner function for free fermions at global equilibrium:

$$W(x,k) = \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3p}{2\varepsilon}\sum_{n=1}^{\infty}(-1)^{n+1}\mathrm{e}^{-n\tilde{\beta}(in\phi)\cdot p}\times$$
$$\left[\mathrm{e}^{-in\frac{\phi:\Sigma}{2}}(m+\not{p})\delta^4\left(k-(\Lambda^n p+p)/2\right)+(m-\not{p})\mathrm{e}^{in\frac{\phi:\Sigma}{2}}\delta^4\left(k+(\Lambda^n p+p)/2\right)\right]$$

Where $\Lambda = \mathrm{e}^{-i\frac{\phi}{2}:J}$ is in the four-vector representation.
Solves the Wigner equation! Full summation of the "$\hbar$ expansion".

Differs from previous **ansatz:**
[F.Becattini, V.Chandra, L.Del Zanna, E.Grossi, Annals Phys. 338 (2013) 32-49]

$$W_A(x,k) = \int \frac{\mathrm{d}^3p}{2\varepsilon}\delta^4(k-p)(\not{p}+m)\left(\mathrm{e}^{\beta\cdot p}e^{-\frac{\varpi:\Sigma}{2}}+\mathbb{I}\right)^{-1}(\not{p}+m)$$
$$+\delta^4(k+p)(m-\not{p})\left(e^{\beta\cdot p}e^{\frac{\varpi:\Sigma}{2}}+\mathbb{I}\right)(m-\not{p})$$

We can use the exact solution to compute **exact expectation values.**

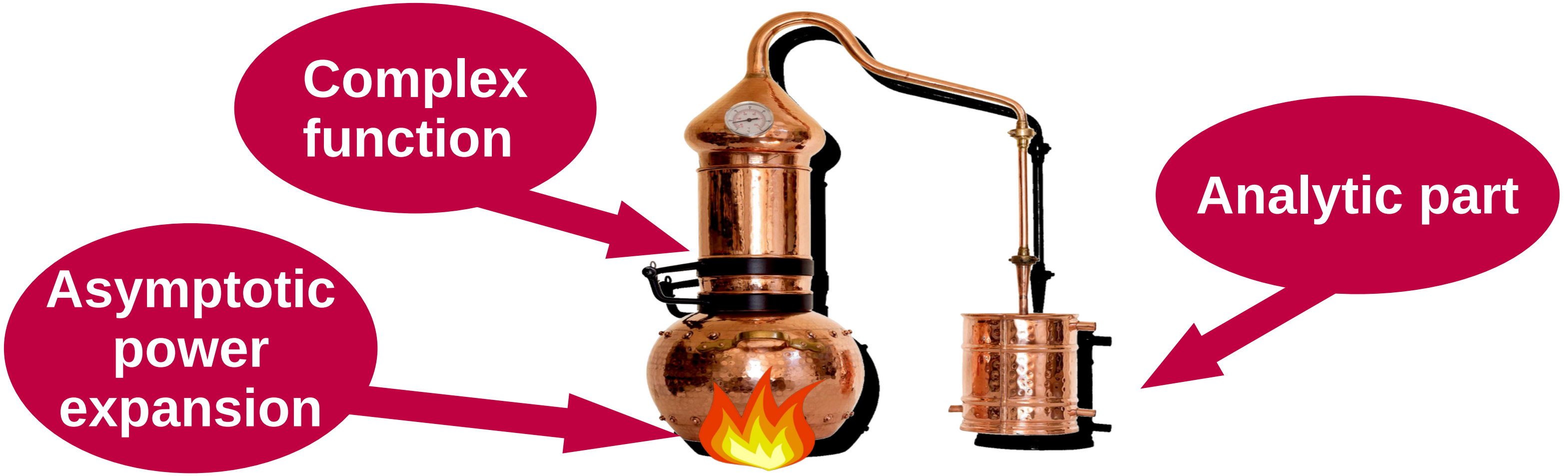
Energy density for massless fermions, equilibrium with acceleration ($\phi = ia/T$)

$$\rho = \frac{3T^4}{8\pi^2} \sum_{n=1}^{\infty} (-1)^{n+1} \phi^4 \frac{\sinh n\phi}{\sinh^5 (n\phi/2)}$$

The series is finite as long as ϕ is real. For real thermal vorticity it diverges!

The series includes terms which are non analytic at $\phi = 0$.

Analytic distillation:



The series boils down to polynomials: $\alpha^\mu = \frac{A^\mu}{T}$ $w^\mu = \frac{\omega^\mu}{T}$

$$\rho = \frac{7\pi^2}{60\beta^4} - \frac{\alpha^2}{24\beta^4} - \frac{17\alpha^4}{960\pi^2\beta^4}$$

Expectation values vanish at the Unruh temperature $T_U = \sqrt{-A \cdot A}/2\pi$
[G.Prokhorov, O. Teryaev, V. Zakharov, JHEP03(2020)137]

Axial current under rotation: [V. Ambrus, E. Winstanley, 1908.10244]

$$j_A^\mu = T^2 \left(\frac{1}{6} - \frac{w^2}{24\pi^2} - \frac{\alpha^2}{8\pi^2} \right) \frac{w^\mu}{\sqrt{\beta^2}}$$

First exact results at equilibrium with **both rotation and acceleration.**
[V. Ambrus, E. Winstanley Symmetry 2021, 13(11)]

$$\rho = T^4 \left(\frac{7\pi^2}{60} - \frac{\alpha^2}{24} - \frac{w^2}{8} - \frac{17\alpha^4}{960\pi^2} + \frac{w^4}{64\pi^2} + \frac{23\alpha^2 w^2}{1440\pi^2} + \frac{11(\alpha \cdot w)^2}{720\pi^2} \right)$$

New contribution!

Spin vector

Spin vector of massive Dirac fermions:

$$S^\mu(p) = \frac{1}{2}\frac{\int \mathrm{d}\Sigma \cdot p \; \mathrm{tr}\left(\gamma^\mu\gamma_5 W_+(x,p)\right)}{\int \mathrm{d}\Sigma \cdot p \; \mathrm{tr}\left(W_+(x,p)\right)}$$

Exact spin vector at global equilibrium:

$$S^\mu(p) = \frac{1}{2m}\frac{\sum_{n=1}^{\infty}(-1)^{n+1}\mathrm{e}^{-n\tilde{b}(in\phi)\cdot p}\mathrm{tr}\left(\gamma^\mu\gamma_5\mathrm{e}^{-in\frac{\phi:\Sigma}{2}}\not{p}\right)\delta^3(\Lambda^n p - p)}{\sum_{n=1}^{\infty}(-1)^{n+1}\mathrm{e}^{-n\tilde{b}(in\phi)\cdot p}\mathrm{tr}\left(\mathrm{e}^{-in\frac{\phi:\Sigma}{2}}\right)\delta^3(\Lambda^n p - p)}$$

How to handle a ratio of series of δ-functions? Where does it come from?

In quantum field theory the spin density matrix is defined:

$$\Theta(p)_{rs} = \frac{\langle \widehat{a}_s^\dagger(p)\widehat{a}_r(p)\rangle}{\sum_t \langle \widehat{a}_t^\dagger(p)\widehat{a}_t(p)\rangle}$$

From the **analytic continuation** of the density operator:

$$\langle \widehat{a}_s^\dagger(p)\widehat{a}_t(p)\rangle = 2\varepsilon \sum_{n=1}^{\infty}(-1)^{2S(n+1)}\delta^3(\Lambda^n \boldsymbol{p} - \boldsymbol{p})D^S(W(\Lambda^n, p))_{ts}\mathrm{e}^{-\widetilde{b}\cdot\sum_{k=1}^{n}\Lambda^k p}$$

The spin density matrix is singular unless $\Lambda \mathrm{p}=\mathrm{p}$. The analytic continuation of the density operator is in the little group of p.

Transformations such that Λp=p can be described using a space-like vector. For real vorticity it reads:

$$\theta^\mu = -\frac{1}{2m}\epsilon^{\mu\nu\rho\sigma}\varpi_{\nu\rho}p_\sigma$$

The series can be summed and the exact result is

$$S^\mu_E(p) = \frac{1}{2}\frac{\theta^\mu}{\sqrt{-\theta^2}}\frac{\sinh\left(\frac{\sqrt{-\theta^2}}{2}\right)}{\cosh\left(\frac{\sqrt{-\theta^2}}{2}\right)+e^{-b\cdot p+\zeta}}$$

Corrections to **all orders in vorticity**

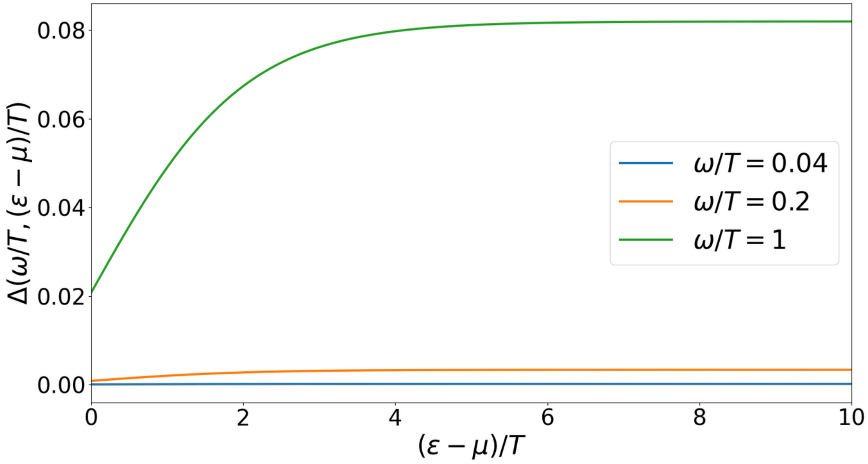
For small vorticity (linear approximation) the previous literature is reproduced

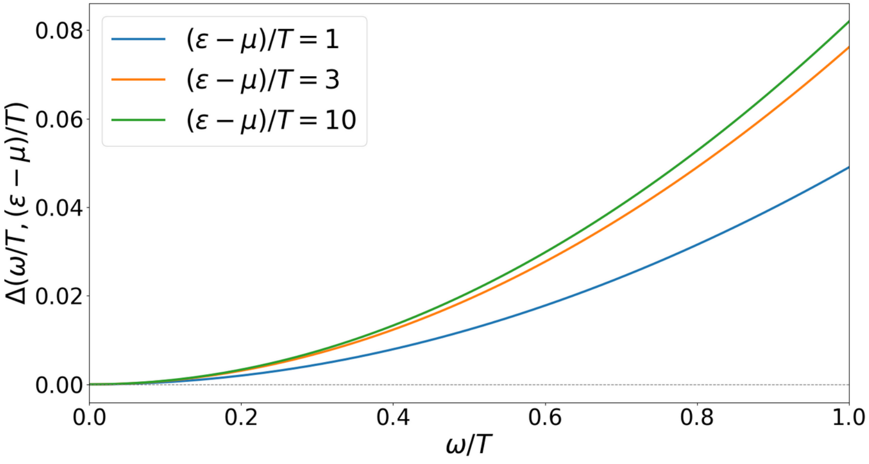
$$S^\mu_L(p) = \frac{\theta^\mu}{4}\frac{1}{1+e^{-b\cdot p+\zeta}} = -\frac{1}{8m}\epsilon^{\mu\nu\rho\sigma}\varpi_{\nu\rho}p_\sigma(1-n_F)$$

Exact polarization in heavy-ion collisions

In relativistic heavy ion collisions $\omega \sim 10^{22}$ s^{-1} and $\omega/T \sim 0.04$.

$$\Delta = \left| \frac{S_E - S_L}{S_E} \right|$$





The difference is very small in most physical cases.

Similar arguments can be repeated for a generic spin-S field

$$\Theta(p) = \frac{\sum_{n=1}^{\infty}(-1)^{2S(n+1)}e^{-nb\cdot p}D^{(S)}(\Lambda^n)}{\sum_{n=1}^{\infty}(-1)^{2S(n+1)}e^{-nb\cdot p}\mathrm{tr}\left(D^{(S)}(\Lambda^n)\right)}$$

From here we can compute the spin-vector and alignment.

Spin-vector for any spin:

$$S^\mu(p) = \frac{\theta^\mu}{2\sqrt{-\theta^2}}\frac{\sum_{n=1}^{\infty}(-1)^{2S(n+1)}e^{-nb\cdot p+\zeta}\cosh^2\left(\frac{n\sqrt{-\theta^2}}{2}\right)\left[S\sinh\left(n(1+S)\sqrt{-\theta^2}\right)-(1+S)\sinh\left(nS\sqrt{-\theta^2}\right)\right]}{\sum_{n=1}^{\infty}(-1)^{2S(n+1)}e^{-nb\cdot p+\zeta}\mathrm{csch}\left(\frac{n\sqrt{-\theta^2}}{2}\right)\sinh\left(n\left(S+\frac{1}{2}\right)\sqrt{-\theta^2}\right)}$$

Conclusions & Outlook

Exact Wigner function at general global equilibrium with thermal vorticity.

Exact spin polarization vector and **spin density matrix**

Higher order corrections in vorticity cannot solve the polarization sign puzzle

Outlook:

- Polarization in the case $\varpi_{\mu\nu} p^\nu \neq 0$?

  Conjecture: the spin vector depends only on θ even in that case. Perturbative calculations to higher orders in vorticity could be used as a check.

Thanks for the attention!

Polarization-team:
Firenze marathon 2022