UNIVERSITY OF ILLINOIS URBANA-CHAMPAIGN

Illinois Center for Advanced Studies of the Universe

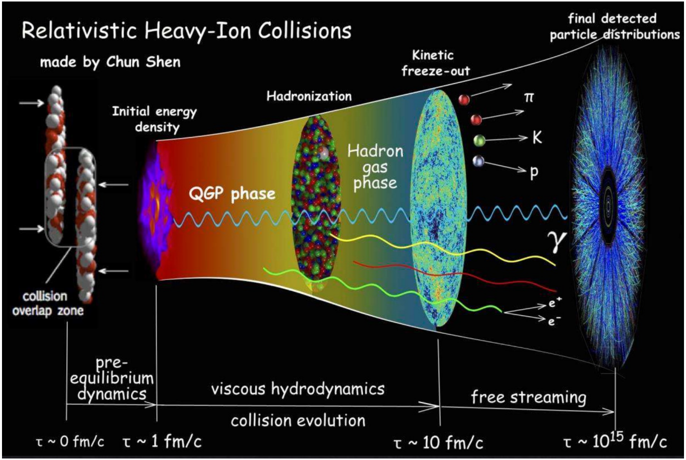
Spin Hydrodynamics: Status and Future Perspective

Enrico Speranza

Workshop on Chirality, Vorticity and Magnetic Field in Heavy Ion Collisions

UCLA - December 2, 2022

Relativistic heavy-ion collisions

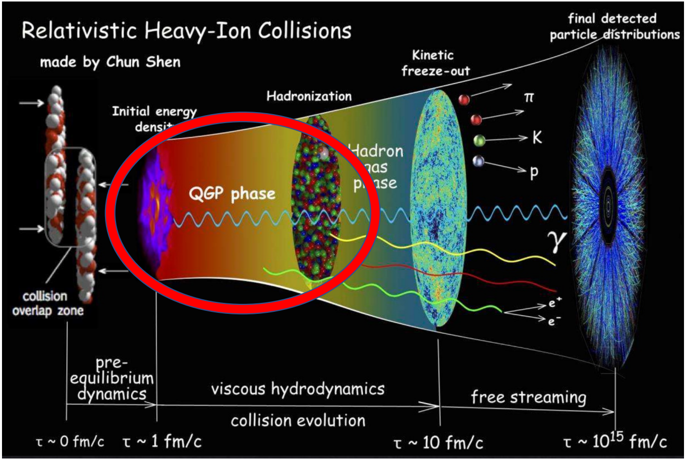


Picture by Chun Shen

- Relativistic hydrodynamics is a powerful effective theory

  Heinz, Snellings, Ann. Rev. Nucl. Part. Sci. 63, 123 (2013)

Relativistic heavy-ion collisions



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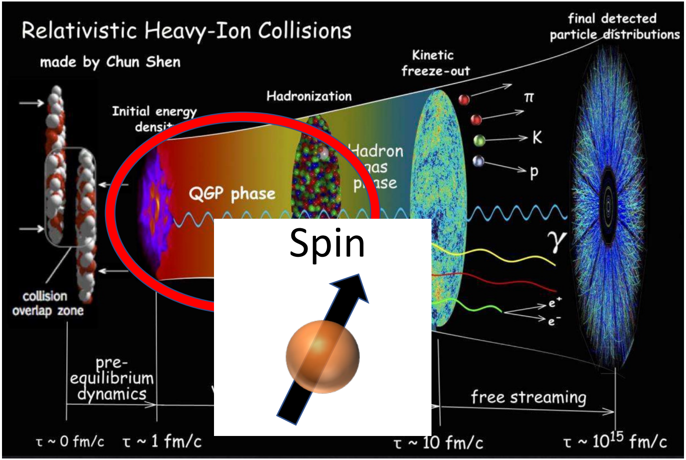


Picture by Chun Shen

- Relativistic hydrodynamics is a powerful effective theory

Heinz, Snellings, Ann. Rev. Nucl. Part. Sci. 63, 123 (2013)

Assumptions of hydrodynamics

- **Conservation laws** in long-wavelength, low-energy limit
- **Separation of length scales**: Microscopic λ_{mfp}, Macroscopic L_{hydro}

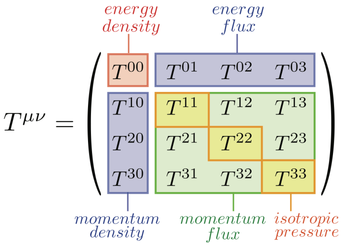
Knudsen number: $\text{Kn} \equiv \dfrac{\lambda_{\text{mfp}}}{L_{\text{hydro}}} \ll 1$ $\Longrightarrow$ **FLUID**

- Expansion in Kn $\Longrightarrow$ Gradient expansion

Relativistic hydrodynamics

- Energy-momentum tensor $T^{\mu\nu} = \langle \hat{T}^{\mu\nu} \rangle$, Current $J^{\mu} = \langle \hat{J}^{\mu} \rangle$
- Conservation of energy, momentum and charge

$$\boxed{\begin{aligned} \partial_\mu T^{\mu\nu} &= 0 \\ \partial_\mu J^{\mu} &= 0 \end{aligned}}$$



picture from Rezzolla, Zanotti, Relativistic Hydrodynamics

$$T^{\mu\nu} = \underbrace{\varepsilon u^\mu u^\nu + P\Delta^{\mu\nu}}_{\text{ideal part}} + \underbrace{\Pi^{\mu\nu}}_{\text{dissipation}}$$

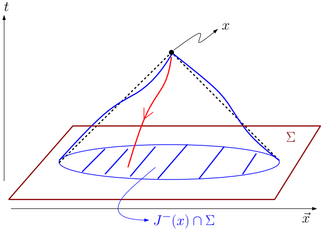
$$J^{\mu} = \underbrace{n u^\mu}_{\text{ideal part}} + \underbrace{\mathcal{J}^{\mu}}_{\text{dissipation}}$$

- Variables ideal fluid: ε, u^μ $(u^\mu u_\mu = -1)$, n

Relativistic dissipative hydrodynamics

Any physical theory of hydrodynamics must be

- **Stable** – Systems slightly away from equilibrium will return to it
- **Causal** – Information cannot propagate at speeds greater than the speed of light



Bemfica, Disconzi, Noronha, 2018

- Causality is necessary for stability
  Bemfica, Disconzi, Noronha, PRX (2022); Gavassino, Class. Quant. Grav. (2021)
- Landau-Lifshitz theory ($\Pi = -\zeta \partial_\mu u^\mu$) is acausal and unstable!
  $\Longrightarrow$ Not suitable for numerical calculations!

How does one formulate **causal** and **stable** relativistic hydrodynamics?

Israel-Stewart theory

Israel, Stewart, Ann. Phys. 118 341?372 (1979)

- Energy-momentum tensor out of equilibrium

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu + (P + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}$$

$\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$

- Π, $\pi^{\mu\nu}$ are promoted to be dynamical variables
  $\Longrightarrow$ Relaxation-type equations

$$\begin{aligned} \tau_\Pi \dot{\Pi} + \Pi &= -\zeta \partial_\mu u^\mu + \dots \\ \tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} &= -2\eta\sigma^{\mu\nu} + \dots \end{aligned}$$

- Causal and stable
  Hiscock, Lindblom, Annals of Physics (1983); Bemfica, Disconzi, Hoang, Noronha, Radosz, PRL (2021)

- Different formulations: BRSSS, DNMR, ...
  Baier, Romatschke, Son, Starinets, Stephanov JHEP (2008); Denicol, Niemi, Molnar, Rischke, PRD (2012)

In heavy-ion simulations one solves Israel-Stewart hydrodynamics

New causal and stable theories

- Most general decomposition out of equilibrium.

$$T^{\mu\nu} = (\varepsilon + \mathcal{A})u^\mu u^\nu + (P + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu} + \mathcal{Q}^\mu u^\nu + \mathcal{Q}^\nu u^\mu$$

- **Generalized Israel-Stewart theory**: $\mathcal{A}$, Π, $\mathcal{Q}^\mu$, $\pi^{\mu\nu}$ are promoted to be dynamical variables, e.g.,
Noronha, Spalinski, ES, PRL **128**, 252302 (2022)

$$\tau_{\mathcal{A}} u_\alpha \partial^\alpha \mathcal{A} + \mathcal{A} = \varphi T u_\alpha \partial^\alpha \left(\frac{1}{T}\right)$$

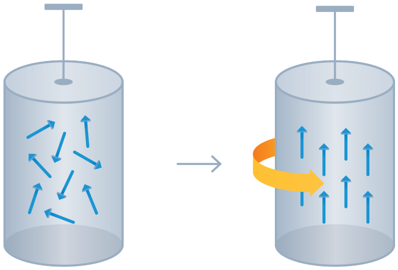
- **First-order (BDNK) theory**: write $\mathcal{A}$, Π, $\mathcal{Q}^\mu$, $\pi^{\mu\nu}$ with the most general expression using first-order derivatives of u^μ and T, e.g. Bemfica, Disconzi, Noronha, PRD **98**, 104064 (2018); PRD **100**, 104020 (2019); PRX **12**, 021044 (2022); Kovtun JHEP **10**, 034 (2019); Hoult, Kovtun, JHEP **06**, 067 (2020)

$$\mathcal{A} = a_1 u_\alpha \partial^\alpha \left(\frac{1}{T}\right) + a_2 \partial_\alpha u^\alpha$$

- Gradient expansion of Generalized Israel-Stewart originates BDNK

Rotation and polarization in heavy-ion collisions

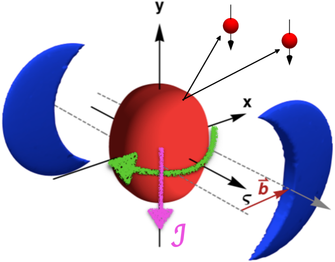
- **Barnett effect:** Ferromagnet gets magnetized when it rotates



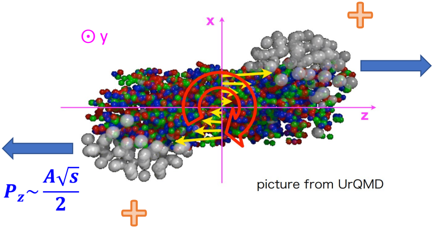
Effective interaction $\sim$ $-$Spin $\cdot$ Rotation

$\sim$ Quantum $\cdot$ Classical

Noncentral heavy-ion collisions



picture from Florkowski, Ryblewski, Kumar, Prog. Part. Nucl. Phys. **108**, 103709 (2019)



picture from UrQMD

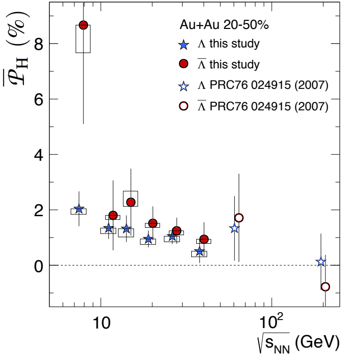
Large global angular momentum

$$J \sim \frac{A\sqrt{s}}{2}\, b \sim 10^5 \hbar$$

$\Rightarrow$ Vorticity of hot and dense matter $\Rightarrow$ particle polarization along vorticity

Experimental observation - Global Λ polarization

- Polarization along global angular momentum



L. Adamczyk et al. (STAR), Nature 548 62-65 (2017)

- Weak decay: $\Lambda \to p + \pi^-$ angular distr.: $dN/d\cos\theta = \frac{1}{2}(1 + \alpha|\vec{\mathcal{P}}_H|\cos\theta)$
- Quark-gluon plasma is the "most vortical fluid ever observed"

$$\omega = (P_\Lambda + P_{\bar{\Lambda}})k_B T/\hbar \sim 10^{21}\,\mathrm{s}^{-1}$$

Polarization observable in heavy-ion collisions

- **Assumptions**:
  1. Local thermodynamic equilibrium of spin degrees of freedom
  2. Polarization determined at freeze-out - No spin dynamics

Becattini, Chandra, Del Zanna, Grossi, Annals. Phys. 338, 32 (2013)

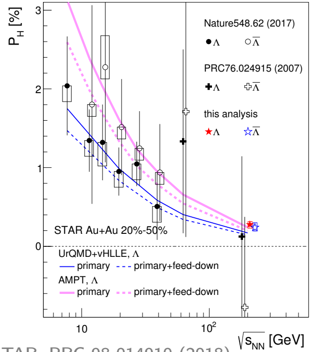
- Relativistic generalization of spin vector at **local equilibrium**

$$\langle \hat{\Pi}_\mu(p) \rangle = -\frac{\hbar^2}{8m}\epsilon_{\mu\nu\alpha\beta}p^\nu \frac{\int d\Sigma_\lambda p^\lambda f(1-f)\varpi^{\alpha\beta}}{\int d\Sigma_\lambda p^\lambda f}$$

$\varpi^{\alpha\beta} = -\frac{1}{2}(\partial^\alpha \beta^\beta - \partial^\beta \beta^\alpha)$ - Thermal vorticity
$\beta^\mu = u^\mu / T$, u^μ - Fluid velocity, T - Temperature
p^μ - particle momentum
f - Fermi-Dirac distribution function
Σ_λ - Space-time hypersurface (Freeze-out)

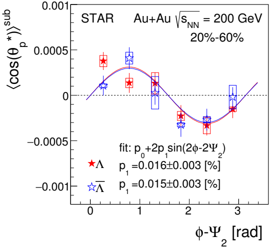
Experiments vs theory: Λ polarization (i)

Global - along J



STAR, PRC 98 014910 (2018)

Longitudinal - along beam axis

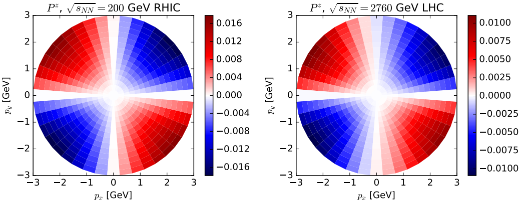


J. Adam *et al.* [STAR Collaboration], PRL 123, 132301 (2019)

$$\Pi^{\mu}(x,p) \propto (1-f)\epsilon^{\mu\nu\rho\tau} p_{\nu} \varpi_{\rho\tau}$$

$$\varpi_{\rho\tau} = -\frac{1}{2}(\partial_{\rho}\beta_{\tau} - \partial_{\tau}\beta_{\rho})$$

Becattini et al An. Phys. (2013)



F. Becattini, I Karpenko, PRL 120, 012302

- "Sign problem" between theory and experiments for longitudinal polarization!

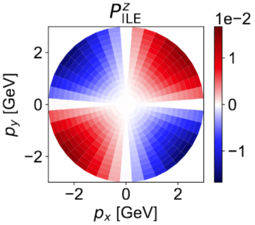
Experiments Vs theory: Λ polarization (ii)

Becattini, Buzzegoli, Inghirami, Karpenko, Palermo, PRL 127 (2021) 272302
Liu, Yin, JHEP 07 (2021) 188

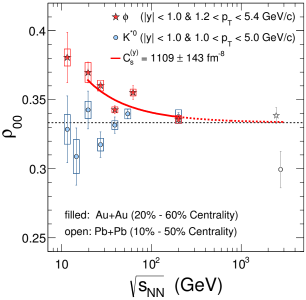
- Modified local-equilibrium formula

$$\langle \hat{\Pi}_\mu(p) \rangle = -\frac{\hbar^2}{8m}\epsilon^{\mu\nu\sigma\tau} p_\nu \frac{\int d\Sigma \cdot p f(1-f)\varpi_{\sigma\tau}}{\int d\Sigma \cdot p\, f} - \frac{1}{4m}\epsilon^{\mu\rho\sigma\tau}\frac{p_\tau p^\lambda}{\varepsilon}\frac{\int_\Sigma d\Sigma \cdot p\, f(1-f)\hat{t}_\rho(\partial_\sigma\beta_\lambda + \partial_\lambda\beta_\sigma)}{\int_\Sigma d\Sigma \cdot p\, f}$$

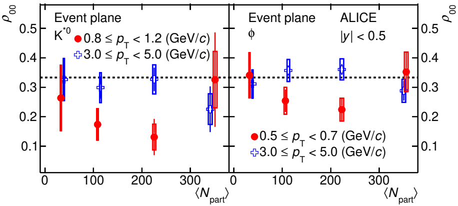
- Describes experimental data if temperature gradients are neglected on freeze-out Σ



ϕ and $K^{\star 0}$ spin alignment



STAR, 2204.02302 (2022)



ALICE, PRL 125, 012301 (2020)

- Theoretical prediction: $\rho_{00} \sim \frac{1}{3} - \frac{1}{9}\left(\frac{\hbar\,\omega}{k_B\,T}\right)^2$

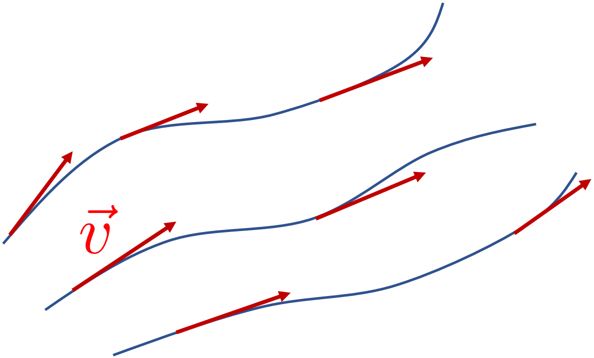
Becattini et al, (2008) Liang, Wang (2018): Yang et al (2018); Xia et al (2021), and many others

From Λ-polarization $\frac{\hbar\,\omega}{k_B\,T} \sim 10^{-2} \Rightarrow \rho_{00} \sim \left(\frac{1}{3} - 10^{-4}\right) \neq$ data

Better theoretical modeling is needed!

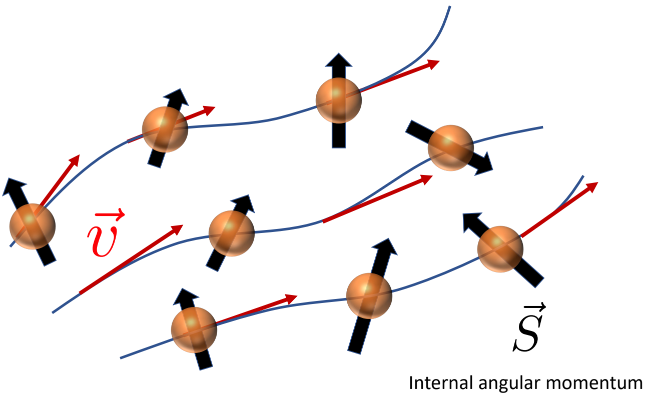
What happens if spin is a dynamical variable in hydrodynamics?

Hydrodynamics with internal angular momentum



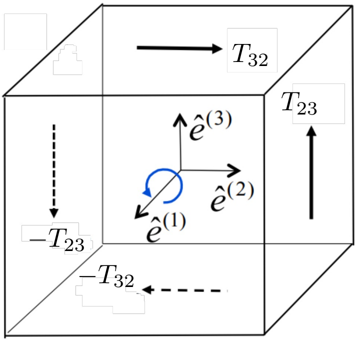
Hydrodynamic fields: Mass density $\rho(t, \vec{x})$, Fluid velocity $\vec{v}(t, \vec{x})$

Hydrodynamics with internal angular momentum



Hydrodynamic variables: Mass density $\rho(t, \vec{x})$, Fluid velocity $\vec{v}(t, \vec{x})$, Internal angular momentum $\vec{S}(t, \vec{x})$ can be classical or **quantum**

Symmetric Vs asymmetric stress tensor



- Antysimmetric stress tensor $T_{ij} \neq T_{ji} \implies$ Nonzero net torque!
- Symmetric stress tensor $\implies$ Angular momentum conservation is redundant

Hydrodynamics with internal angular momentum

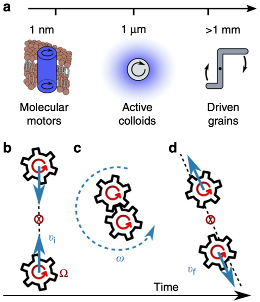
- Conservation of total angular momentum
  Lukaszewicz, Micropolar Fluids, Theory and Applications (Birkhäuser Boston, 1999)

$$\rho(\partial_t + v_j\partial_j)S_i = \partial_j C_{ji} + \tilde{T}_i$$

S_i - Internal angular momentum (not necessarily of quantum nature)
C_{ij} - Couple stress tensor, $\tilde{T}_i = \epsilon_{ijk} T_{jk}$

- Change of internal angular momentum due to C^{ji} and $\epsilon^{ijk} T^{jk}$

- Many applications: Micropolar fluids, Spintronics, chiral active fluids, ...
  R. Takahashi et al, Nature Physics 12, 52 (2016)
  D. Banerjee et al, Nature Commun 8, 1 (2017)



Relativistic spin hydrodynamics

Florkowski, Friman, Jaiswal, Speranza, PRC 97, 041901 (2018)
ES, Weickgenannt, EPJA 57, 155 (2021) (Review)

- Hydrodynamic densities from quantum field theory

$$T^{\mu\nu} = \langle \hat{T}^{\mu\nu} \rangle \qquad S^{\lambda,\mu\nu} = \langle \hat{S}^{\lambda,\mu\nu} \rangle$$

$\hat{S}^{\lambda,\mu\nu}$ - Spin tensor

- **10 hydro eqs.**: 4 Energy-momentum + 6 Total angular momentum cons.

$$\partial_\mu T^{\mu\nu} = 0 \qquad \hbar\, \partial_\lambda S^{\lambda,\mu\nu} = T^{\nu\mu} - T^{\mu\nu}$$

- **10 unknowns**: Lagrange multipliers $\beta^\mu = u^\mu / T$ and $\Omega^{\mu\nu}$
- Choice for a decomposition of orbital and spin angular momentum

How can one derive the equations of motion?

Pseudo-gauge transformations

ES, Weickgenannt, EPJA, 57, 155 (2021) (Review)

Densities are not uniquely defined $\Longrightarrow$ Relocalization

F. W. Hehl, Rep. Mat. Phys. 9, 55 (1976)

$$\hat{T}'^{\mu\nu}(x) = \hat{T}^{\mu\nu}_C(x) + \frac{1}{2}\partial_\lambda \left[\hat{\Phi}^{\lambda,\mu\nu}(x) + \hat{\Phi}^{\mu,\nu\lambda}(x) + \hat{\Phi}^{\nu,\mu\lambda}(x)\right]$$

$$\hat{S}'^{\lambda,\mu\nu} = \hat{S}^{\lambda,\mu\nu}_C(x) - \hat{\Phi}^{\lambda,\mu\nu}(x) + \partial_\rho \hat{Z}^{\mu\nu,\lambda\rho}(x)$$

$\hat{\Phi}^{\lambda,\mu\nu} = -\hat{\Phi}^{\lambda,\nu\mu}$, $\hat{Z}^{\mu\nu,\lambda\rho} = -\hat{Z}^{\nu\mu,\lambda\rho} = -\hat{Z}^{\mu\nu,\rho\lambda}$

- Global charges are left invariant
- Conservation laws $\quad \partial_\mu \hat{T}'^{\mu\nu} = 0$, $\partial_\lambda \hat{S}'^{\lambda,\mu\nu} = \hat{T}'^{\nu\mu} - \hat{T}'^{\mu\nu}$

Quantum kinetic theory with nonlocal collisions

Weickgenannt, ES, Sheng, Wang, Rischke, PRL **127**, 052301 (2021); PRD **104**, 016022 (2021)

$$p \cdot \partial \mathfrak{f}(x, p, \mathfrak{s}) = m \, \mathfrak{C}[\mathfrak{f}]$$

$$\mathfrak{C}[\mathfrak{f}] = \mathfrak{C}_{\text{local}}[\mathfrak{f}] + \hbar \, \mathfrak{C}_{\text{nonlocal}}[\mathfrak{f}]$$

- Long calculation $\Longrightarrow$ Intuitive result in low-density approximation:

$$\mathfrak{C}[\mathfrak{f}] = \int d\Gamma_1 d\Gamma_2 d\Gamma' \, \mathcal{W} \left[\mathfrak{f}(x+\Delta_1, p_1, \mathfrak{s}_1)\mathfrak{f}(x+\Delta_2, p_2, \mathfrak{s}_2) - \mathfrak{f}(x+\Delta, p, \mathfrak{s})\mathfrak{f}(x+\Delta', p', \mathfrak{s}')\right]$$
$$+ \int d\Gamma_2 \, dS_1(p) \, \mathfrak{W} \, \mathfrak{f}(x+\Delta_1, p, \mathfrak{s}_1)\mathfrak{f}(x+\Delta_2, p_2, \mathfrak{s}_2)$$

$d\Gamma \equiv d^4p \, dS(p)$

- Structure: Momentum and spin exchange + Spin exchange only
- Nonlocal Collisions $\Longrightarrow$ Displacement $\Delta \sim \mathcal{O}(\hbar) \sim \mathcal{O}(\partial)$
- $\mathcal{W}$, $\mathfrak{W}$ vacuum transition probabilities, depend on phase-space spins

Equilibrium distribution function

Weickgenannt, ES, Sheng, Wang, Rischke, PRL **127**, 052301 (2021); PRD **104**, 016022 (2021)

- **Equilibrium condition:** $\mathfrak{C}[\mathfrak{f}] = 0$
- Ansatz for distribution function
  F. Becattini, V. Chandra, L. Del Zanna, and E. Grossi, AP. 338, 32 (2013)
  W. Florkowski, R. Ryblewski, and A. Kumar, Prog. Part. Nucl. Phys. 108, 103709 (2019)

$$\mathfrak{f}_{eq}(x, p, \mathfrak{s}) \propto \exp\Big[\underbrace{-\beta(x)\cdot p}_{\text{Energy-momentum}} \underbrace{+\frac{\hbar}{4}\Omega_{\mu\nu}(x)\Sigma_{\mathfrak{s}}^{\mu\nu}}_{\text{Total angular momentum}} \Big]\delta(p^2 - M^2)$$

  - M - mass (possibly modified by interactions)
  - $\beta^\mu = u^\mu / T$, Spin potential $\Omega^{\mu\nu}$
  - Spin-dipole-moment tensor $\Sigma_{\mathfrak{s}}^{\mu\nu} \equiv -\frac{1}{m}\epsilon^{\mu\nu\alpha\beta} p_\alpha \mathfrak{s}_\beta$

- Insert into $\mathfrak{C}[\mathfrak{f}]$ and expand up to $\mathcal{O}(\hbar)$

Condition for $\mathfrak{C}[\mathfrak{f}] = 0 \Longrightarrow$ **Global equilibrium**

$$\partial_\mu \beta_\nu + \partial_\nu \beta_\mu = 0 \qquad \Omega_{\mu\nu} = -\frac{1}{2}(\partial_\mu \beta_\nu - \partial_\nu \beta_\mu) = \text{const.}$$

System gets polarized through rotations!

Equilibrium distribution function

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System gets polarized through rotations!

Spin tensor: Canonical Vs HW pseudogauge

ES, Weickgenannt, Eur. Phys. J. A **57**, no.5, 155 (2021)
Weickgenannt, Wagner ES, Phys. Rev. D **105**, no.11, 116026 (2022)

- **Canonical** pseudogauge

$$\partial_\lambda S^{\lambda,\mu\nu}_{C,eq} = \frac{1}{(2\pi\hbar)^3}\hbar\sigma \int dP p^{[\mu}\varpi^{\nu]\lambda} p^\rho \varpi_{\lambda\rho} e^{-\beta\cdot p} + \mathcal{O}(\hbar^2)$$

Problem: non conserved even in global equilibrium

- **HW** case (Hilgevoord, Wouthuysen, NP 40 (1963))

$$\partial_\mu T^{\mu\nu}_{HW} = \int d\Gamma\, p^\nu\, \mathfrak{C}[f] = 0$$

$$\hbar\, \partial_\lambda S^{\lambda,\mu\nu}_{HW} = \int d\Gamma\, \hbar\sigma \Sigma^{\mu\nu}_{\mathfrak{s}}\, \mathfrak{C}[f] = T^{[\nu\mu]}_{HW}$$

Conserved in global equilibrium and in absence of nonlocal collisions

Scales

$$\Delta \ll \lambda_{\mathrm{mfp}} \ll L_{\mathrm{hydro}}$$

- Conventional Knudsen number

$$\mathrm{Kn} = \frac{\lambda_{\mathrm{mfp}}}{L_{\mathrm{hydro}}}$$

- "Quantum" Knudsen number

$$\kappa = \frac{\Delta}{L_{\mathrm{hydro}}}$$

- For transient theory - Inverse Reynolds number

$$\mathrm{Re}^{-1} = \frac{\delta f}{f} \left(\text{for example} \sim \frac{|\Pi|}{P} \right)$$

Method of moments

- Definition of local equilibrium

$$\mathfrak{C}[\mathfrak{f}_{\text{eq}}] = \mathcal{O}(\Delta/L_{\text{hydro}})$$

- Expansion distribution function

$$f(x,p,\mathfrak{s}) \equiv f_{\text{eq}}(x,p,\mathfrak{s}) + \delta f_{p\mathfrak{s}}(x,p,\mathfrak{s})$$

- Definition of moments

$$\rho_n^{\mu_1\cdots\mu_l} \equiv \langle E_p^n\, p^{\langle\mu_1}\cdots p^{\mu_l\rangle}\rangle_\delta$$
$$\tau_n^{\mu,\mu_1\cdots\mu_l} \equiv \langle E_p^n\, \mathfrak{s}^\mu p^{\langle\mu_1}\cdots p^{\mu_l\rangle}\rangle_\delta$$

- Expand $\delta f_{p\mathfrak{s}}$ in moments

Spin tensor in HW pseudogauge

$$S^{\lambda,\mu\nu} = u^{\lambda}\tilde{\mathfrak{N}}^{\mu\nu} + \Delta^{\lambda}_{\alpha}\tilde{\mathfrak{P}}^{\alpha\mu\nu} + u_{(\alpha}\tilde{\mathfrak{H}}^{\lambda)\mu\nu\alpha} + \tilde{\mathfrak{Q}}^{\lambda\mu\nu} - \frac{\hbar}{4m^2}\partial^{[\nu}T^{\mu]\lambda}$$

$\tilde{\mathfrak{N}}^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta}\mathfrak{N}_{\alpha\beta}$, $\tilde{\mathfrak{P}}^{\alpha\mu\nu} = \epsilon^{\alpha\mu\nu\beta}\mathfrak{P}_{\beta}$, $\tilde{\mathfrak{H}}^{\mu\nu\lambda\alpha} = \epsilon^{\nu\lambda\alpha\beta}\mathfrak{H}^{\mu}{}_{\beta}$, $\tilde{\mathfrak{Q}}^{\lambda\mu\nu} = \epsilon^{\mu\nu\alpha\beta}\mathfrak{Q}^{\mu}{}_{\alpha\beta}$

$$\mathfrak{N}^{\mu\nu} \equiv -\frac{1}{2m}u^{\mu}\langle E_p^2\,\mathfrak{s}^{\nu}\rangle_{\mathrm{eq}} - \frac{1}{2m}u^{\mu}\tau_2^{\nu} \quad \text{Spin-energy tensor}$$

$$\mathfrak{P}^{\mu} \equiv -\frac{1}{6m}\langle\Delta^{\rho\sigma}p_{\rho}p_{\sigma}\,\mathfrak{s}^{\mu}\rangle_{\mathrm{eq}} - \frac{1}{6m}\left(m^2\tau_0^{\mu} - \tau_2^{\mu}\right) \quad \text{Spin-pressure vector}$$

$$\mathfrak{H}^{\lambda\mu} \equiv -\frac{1}{2m}\langle E_p p^{\langle\lambda\rangle}\mathfrak{s}^{\mu}\rangle_{\mathrm{eq}} - \frac{1}{2m}\tau_1^{\mu,\lambda} \quad \text{Spin-diffusion tensor}$$

$$\mathfrak{Q}^{\lambda\mu\nu} \equiv -\frac{1}{2m}\tau_0^{\lambda\mu\nu} \quad \text{Spin-stress tensor}$$

Matching conditions

$$T^{\mu\nu} u_\nu = T^{\mu\nu}_{\text{eq}} u_\nu$$
$$J^{\lambda,\mu\nu} u_\lambda = J^{\lambda,\mu\nu}_{\text{eq}} u_\lambda$$

- Determine β^μ, $\Omega^{\mu\nu}$ via conservation laws

$$\partial_\mu T^{\mu\nu} = 0 \qquad \partial_\lambda S^{\lambda,\mu\nu} = T^{\nu\mu} - T^{\mu\nu}$$

- Transient (Israel-Stewart) theory
  $\Longrightarrow$ Relaxation-type equations for dissipative currents Π, $\pi^{\mu\nu}$ and spin moments τ_0^α, τ_2^α, $\tau_1^{\beta,\mu}$ and $\tau_0^{\beta,\mu\nu}$

Equations of motion for spin moments

$$\dot{\tau}_r^{\langle\mu\rangle} - \mathfrak{C}_{r-1}^{\langle\mu\rangle} = \dots$$

with $\mathfrak{C}_{r-1}^{\mu,\langle\mu_1\cdots\mu_n\rangle} \equiv \int d\Gamma\, E_p^{r-1} p^{\langle\mu_1}\cdots p^{\mu_n\rangle}\mathfrak{s}^\mu \bar{\mathfrak{C}}[f]$

$$\begin{aligned}
\mathfrak{C}_{r-1}^{\mu} &= \mathfrak{C}_{r-1,\text{local}}^{\mu} + \mathfrak{C}_{r-1,\text{nonlocal}}^{\mu} \\
\mathfrak{C}_{r-1,\text{local}}^{\mu} &= -\sum_n B_{r\,n}^{(\ell)} \tau_n^\mu \\
\mathfrak{C}_{r-1,\text{nonlocal}}^{\mu} &= \int d\Gamma_1 d\Gamma_2 d\Gamma d\Gamma' \mathcal{W} E_{\mathbf{p}}^{r-1} f_{0\mathbf{p}} f_{0\mathbf{p}'} \mathfrak{s}^\mu \\
&\quad \times \left[-\frac{\hbar}{4m}(\varpi_{\alpha\beta} - \Omega_{\alpha\beta})\epsilon^{\alpha\beta\lambda\sigma} p_\lambda \mathfrak{s}_\sigma + \partial_{(\alpha}\beta_{\beta)}\Delta^\alpha p^\beta\right]
\end{aligned}$$

- We want to isolate $\tau_n^\mu \implies$ Invert $B_{r\,n}^{(\ell)} \implies$ Relaxation times
- $\mathfrak{C}_{r-1,\text{nonlocal}}^{\mu}$ gives rise to Navier-Stokes terms

Truncation: (14+24)-moment approximation

Infinite set of moment equations needs to be truncated

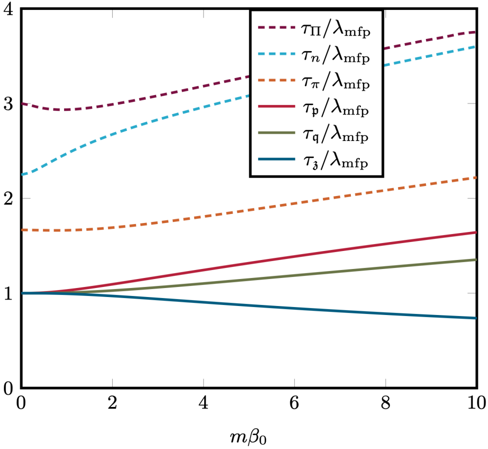
- Lowest order truncation:
  14 standard hydro moments (including charge current)
  + 24 moments appearing in the spin tensor

Independent spin moments: $\mathfrak{p}^{\mu} \equiv \tau_0^{\mu}$, $\mathfrak{z}^{\mu\nu} = \tau_1^{\langle\mu\rangle,\nu} + \tau_1^{\langle\nu\rangle,\mu}$, $\mathfrak{q}^{\lambda\mu\nu} \equiv \tau_0^{\lambda,\mu\nu}$,

$$\tau_{\mathfrak{p}} \dot{\mathfrak{p}}^{\langle\nu\rangle} + \mathfrak{p}^{\langle\mu\rangle} = \mathfrak{e} \left(\tilde{\Omega}^{\mu\nu} - \tilde{\varpi}^{\mu\nu} \right) u_{\nu} + \dots$$
$$\tau_{\mathfrak{z}} \dot{\mathfrak{z}}^{\langle\lambda\rangle\langle\rho\rangle} + \mathfrak{z}^{\mu\nu} = \dots$$
$$\tau_{\mathfrak{q}} \dot{\mathfrak{q}}^{\langle\rho\rangle\alpha\beta} + \mathfrak{q}^{\langle\mu\rangle\nu\lambda} = -\mathfrak{d} \beta_0 \sigma_{\rho}{}^{\langle\nu} \epsilon^{\lambda\rangle\mu\alpha\rho} u_{\alpha} + \dots$$

- Navies-Stokes terms come from nonlocal collisions $\sim \kappa \equiv \Delta / L_{\text{hydro}}$
- **Note**: Different pseudogauge $\Longrightarrow$ Different truncation
  $\Longrightarrow$ Different evolution

Spin relaxation times



- Spin relaxation times of the same order of conventional Π, $\pi^{\mu\nu}$
- Spin relaxes to Navier-Stokes as fast as Π, $\pi^{\mu\nu}$

Can we simulate relativistic spin hydrodynamics and give predictions?

Not yet.
We don't know if it's causal and stable

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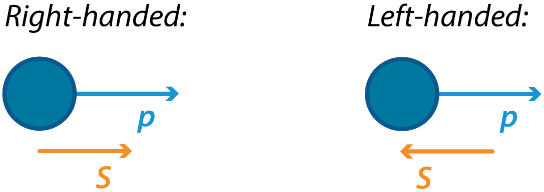
Causality and chiral hydrodynamics



Chiral hydrodynamics from kinetic theory

e.g., Chen, Son, Stephanov, PRL **115**, no.2, 021601 (2015); Yang, PRD **98**, no.7, 076019 (2018)

- Consider ensemble of particles (and antiparticles) with chirality $\pm$
- Distribution function

$$f_{\mathrm{eq},\pm}(x,p) = [\exp(g_\pm) + 1]^{-1}$$

$$g_\pm(x,p) = -\beta \cdot p - \frac{\mu_\pm}{T} - \underbrace{\frac{1}{2} S^{\mu\nu} \varpi_{\mu\nu}}_{\text{Spin-vorticity coupling}}$$

  $S^{\mu\nu}$ - Rank-2 spin tensor,
  Thermal vorticity - $\varpi^{\mu\nu} = -\frac{1}{2}(\partial^\mu \beta^\nu - \partial^\nu \beta^\mu)$
- Hydrodynamic densities from distribution function: energy-momentum tensor $T^{\mu\nu}$, vector and axial vector currents J_V^μ, J_A^μ

$$\partial_\mu T^{\mu\nu} = 0 \qquad \partial_\mu J_V^\mu = 0 \qquad \partial_\mu J_A^\mu = 0$$

This is great. But can one solve it?

Ideal chiral hydrodynamics from kinetic theory

Consider the full nonlinear system: $\partial_\mu T^{\mu\nu} = 0,\ \partial_\mu J_V^\mu = 0,\ \partial_\mu J_A^\mu = 0$

ES, Bemfica, Disconzi, Noronha, 2104.02110 (2021)

$$
\begin{aligned}
T^{\mu\nu} &= \varepsilon u^\mu u^\nu + P\Delta^{\mu\nu} + \xi_T(\omega^\mu u^\nu + \omega^\nu u^\mu) \\
J_V^\mu &= n_V u^\mu + \xi_V\,\omega^\mu \\
J_A^\mu &= n_A u^\mu + \xi_A\,\omega^\mu
\end{aligned}
$$

e.g., Chen, Son, Stephanov, PRL **115**, no.2, 021601 (2015); Yang, PRD **98**, no.7, 076019 (2018)

Vorticity - $\omega^\mu = (1/2)\epsilon^{\mu\nu\alpha\beta}u_\nu\partial_\alpha u_\beta$

- One can prove:

Characteristic determinant $= 0$

System is ill posed:
No unique solution exists for **arbitrary** initial data!

- **Conventional ideal case**: $\xi_T, \xi_V, \xi_A = 0 \implies$ Well-posed, causal and stable

Ideal chiral hydrodynamics in Landau frame

ES, Bemfica, Disconzi, Noronha, 2104.02110 (2021)

- Shift of velocity

$$u^\mu = u_L^\mu - \frac{\xi_T \omega_L^\mu}{\varepsilon + P}$$

- Constitutive relations in Landau frame

$$\begin{aligned} T^{\mu\nu} &= \varepsilon u_L^\mu u_L^\nu + P\Delta^{\mu\nu} \\ J_V^\mu &= n_V u_L^\mu + \xi_V \omega_L^\mu \\ J_A^\mu &= n_A u_L^\mu + \xi_A \omega_L^\mu \end{aligned}$$

The theory is well-posed, causal and stable!

- Definition of hydrodynamic variables (hydrodynamic frames) matter even in the ideal case

First-order viscous chiral hydrodynamics

Abboud, Noronha, ES (to appear)

- Constitutive relations in a general hydrodynamic frame

$$T^{\mu\nu} = (\varepsilon + \mathcal{A})\left(u^\mu u^\nu + \frac{\Delta^{\mu\nu}}{3}\right) + \mathcal{Q}^\mu u^\nu + \mathcal{Q}^\nu u^\mu + \pi^{\mu\nu}$$
$$J_V^\mu = (n_V + \mathcal{N}_V)u^\mu + \mathcal{J}_V^\mu$$
$$J_A^\mu = (n_A + \mathcal{N}_A)u^\mu + \mathcal{J}_A^\mu$$

- Consider the theory

$$\mathcal{A} = a_1 \mathcal{D}\varepsilon, \quad \mathcal{Q}^\mu = b_1 \Delta^{\mu\lambda}\mathcal{D}_\lambda \varepsilon, \quad \pi^{\mu\nu} = -2\eta\sigma^{\mu\nu}$$
$$\mathcal{N}_V = c_{V1}\mathcal{D}\varepsilon + c_{V2}\mathcal{D}n_V + c_{V3}\mathcal{D}n_A$$
$$\mathcal{J}_V^\mu = e_{V1}\Delta^{\mu\lambda}\mathcal{D}_\lambda\varepsilon + e_{V2}\Delta^{\mu\lambda}\mathcal{D}_\lambda n_V + e_{V3}\Delta^{\mu\lambda}\mathcal{D}_\lambda n_A + \xi_V \omega^\mu$$
$$\mathcal{N}_A = c_{A1}\mathcal{D}\varepsilon + c_{A2}\mathcal{D}n_V + c_{A3}\mathcal{D}n_A$$
$$\mathcal{J}_A^\mu = e_{A1}\Delta^{\mu\lambda}\mathcal{D}_\lambda\varepsilon + e_{A2}\Delta^{\mu\lambda}\mathcal{D}_\lambda n_V + e_{A3}\Delta^{\mu\lambda}\mathcal{D}_\lambda n_A + \xi_A \omega^\mu$$

$\mathcal{D}^\mu$ - Weyl derivative

One can prove: System is causal and stable!

How about massive spin-1 particles?

Spin-density matrix

- **Pure state**: $|\psi\rangle = \sum_\lambda c_\lambda |\lambda\rangle$
  Expectation value of an operator $\langle O\rangle = \langle\psi|O|\psi\rangle$
- **Mixed state**: incoherent mixture of $|\psi_i\rangle$ with statistical weight a_i

$$\rho = \sum_i a_i |\psi_i\rangle\langle\psi_i| = \sum_{\lambda,\lambda'} \rho_{\lambda\lambda'} |\lambda\rangle\langle\lambda'|$$

$\rho_{\lambda\lambda'} = \sum_i a_i\, c_\lambda^{(i)} c_{\lambda'}^{(i)*}$. Expectation value: $\langle O\rangle = \mathrm{Tr}(\rho\, O)$

Spin-1/2 particle: $(2 \times 2$ hermitian matrix):

$$\rho = \frac{1}{2}(1 + \vec{P}\cdot\vec{\sigma})$$

- Spin polarization vector: $\vec{P} = \langle\vec{\sigma}\rangle = \mathrm{Tr}(\rho\,\vec{\sigma})$

$$\begin{aligned} |\vec{P}| &= 1 && \text{Pure state} \\ 0 < |\vec{P}| &< 1 && \text{Mixed state} \\ |\vec{P}| &= 0 && \text{Completely unpolarized mixed state} \end{aligned}$$

Spin-density matrix for spin-1 particles

- Three polarization states (in rest frame)

$$\text{Transverse to } \vec{q}: \quad \epsilon(\pm 1) = \mp \frac{1}{\sqrt{2}}(0,\, 1,\, \pm i,\, 0)$$

$$\text{Longitudinal to } \vec{q}: \quad \epsilon(0) = (0,\, 0,\, 0,\, 1)$$

- **Spin-density matrix**: hermitian 3×3 matrix

$$\rho = \frac{1}{3}\left[1 + \frac{3}{2}\vec{P}\cdot\vec{S} + \sqrt{\frac{3}{2}}\sum_{i,j} T_{ij}(S_i S_j + S_j S_i)\right]$$

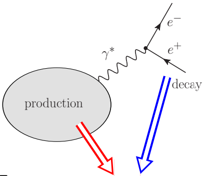
$\vec{S}$ are the spin-1 operators

- $\mathrm{Tr}\rho = 1$ (8 parameters)
- Vector polarization: $\vec{P} = \langle \vec{S} \rangle$ (3 parameters)
- Tensor polarization: $T_{ij} = \frac{1}{2}\sqrt{\frac{3}{2}}\left(\langle S_i S_j + S_j S_i\rangle - \frac{4}{3}\delta_{ij}\right)$, $\quad \sum_i T_{ii} = 0$
  (5 parameters)

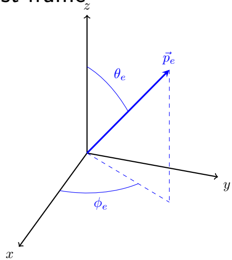
One can have tensor polarization and vanishing vector polarization

Lepton angular distribution

spin-1 $\rightarrow$ spin-$\frac{1}{2}$ + spin-$\frac{1}{2}$



Photon rest frame



$$\frac{d\sigma}{d\Omega_e} \propto \mathrm{Tr}(\,\rho\, O^{\mathrm{dec}}\,)$$

$$= \mathcal{N}\Big(1 + \lambda_\theta \cos^2\theta_e + \lambda_\phi \sin^2\theta_e \cos 2\phi_e + \lambda_{\theta\phi} \sin 2\theta_e \cos\phi_e$$

$$+ \lambda_\phi^\perp \sin^2\theta_e \sin 2\phi_e + \lambda_{\theta\phi}^\perp \sin 2\theta_e \sin\phi_e + \text{parity violating terms}\Big)$$

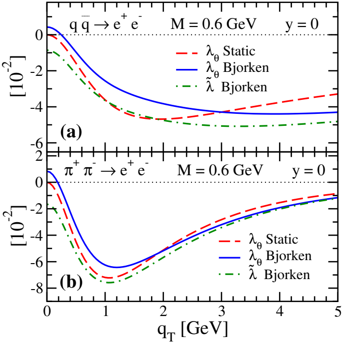
$$\lambda_\theta = \frac{\rho_T - \rho_L}{\rho_T + \rho_L}$$

- Transverse: $\rho_T = \rho_{-1-1} + \rho_{+1+1}$
  Longitudinal: $\rho_L = 2\rho_{00}$
- Completely **transverse** polarized: $\lambda_\theta = +1$
  Completely **longitudinal** polarized: $\lambda_\theta = -1$
- Photon polarization reflected in angular distribution

Virtual photon polarization without vorticity

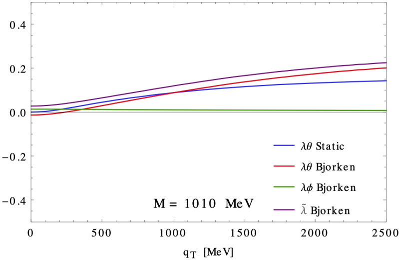
- Medium in global equilibrium without rotation emits **tensor polarized** virtual photons $\gamma^\star \to e^+ e^-$

$$\frac{d\Gamma}{d^4 q d\Omega_e} = \mathcal{N}\left(1 + \lambda_\theta \cos^2\theta_e + \lambda_\phi \sin^2\theta_e \cos 2\phi_e + \lambda_{\theta\phi} \sin 2\theta_e \cos\phi_e\right)$$



ES, Jaiswal, Friman, PLB **782**, 395
see also, Baym, Hatsuda, Strickland, PRC **95**, 044907

More realistic models



Friman, Galatyuk, Rapp, ES, van Hees, Wambach (in preparation)

- ϕ and $K^{\star 0}$ can be polarized even in absence of vorticity and dissipation
ES, Jaiswal, Friman, PLB **782**, 395; Li, Liu, 2206.11890

Tensor polarization from shear stress

Wagner, Weickgenannt, ES, 2207.14060; Li, Liu, 2206.11890

$$\rho_{00}(k) = \frac{1}{3} - \sqrt{\frac{2}{3}} \epsilon_\mu^{(0)} \epsilon_\nu^{(0)} \Theta^{\mu\nu}(k) \, .$$

Tensor polarization

$$\Theta^{\mu\nu}(k) = \frac{1}{2}\sqrt{\frac{3}{2}} \frac{1}{N(k)} \int d\Sigma_\lambda k^\lambda \int \d{S}(k) K^{\mu\nu}_{\alpha\beta} \mathfrak{s}^\alpha \mathfrak{s}^\beta f(x,k,\mathfrak{s})$$

$$\rho_{00}(k) = \frac{1}{3} - \frac{4}{15} \frac{\int d\Sigma_\alpha k^\alpha \xi \, \beta_0 f_{0\mathbf{k}} \epsilon_\mu^{(0)} \epsilon_\nu^{(0)} \left[\mathcal{H}^{(2,0)}_{\mathbf{k}1} \pi^{\mu\nu} - \mathcal{H}^{(2,1)}_{\mathbf{k}0} \frac{1}{3} \left(m^2 \mathcal{F}^{(2,0)}_{11} - 1 \right) u^{(\mu} \pi^{\nu)\lambda} k_\lambda \right]}{\int d\Sigma_\alpha k^\alpha f_{0\mathbf{k}} \left(1 - 3\mathcal{H}^{(0,0)}_{\mathbf{k}0} \Pi / m^2 + \mathcal{H}^{(0,2)}_{\mathbf{k}0} \pi^{\mu\nu} k_{\langle\mu} k_{\nu\rangle} \right)}$$

- Independent of vorticity $\Rightarrow$ One could look at different quantization axis than the global angular momentum
- Other developments regarding spin alignment, see Qun Wang's talk

Summary

- Relaxation times for spin degrees of freedom of the same order as relaxation times of conventional dissipative quantities
  $\Longrightarrow$ The use of spin hydro is justified
- Vorticity brings conceptual issues in the initial-value problem of chiral hydrodynamics
- Spin alignment might not have anything to do with vorticity

Challenges

- What is the decomposition of orbital and spin angular momentum of a relativistic fluid?
- Causal and stable relativistic spin hydrodynamics?
- Application of spin hydro to polarization observables?

EXTRA SLIDES

Dissipative corrections to the spin vector

- Pauli-Lubanski vector

$$\Pi^\mu(p) = \frac{1}{2\mathcal{N}} \int d\Sigma_\lambda p^\lambda \mathcal{A}^\mu(x,p) \ ,$$

- Axial-vector current

$$\mathcal{A}^\mu(x,p) = \int dS(p)\, \mathfrak{s}^\mu f(x,p,\mathfrak{s})$$

- Using our Navier-Stokes values

$$\Pi^\mu_{\rm NS} \sim \int_{\Sigma_{\rm FO}} d\Sigma \cdot p f_0 \Bigg\{ \epsilon^{\mu\nu\rho\sigma} p_\nu \Omega_{\rho\sigma} + \left(\delta^\mu_\nu - \frac{u^\mu p_{\langle\nu\rangle}}{E_{\mathbf{p}}} \right) \times \left[\chi_1 \epsilon^{\nu\rho\alpha\beta} \left(\tilde{\Omega}^{\alpha\beta} - \tilde{\varpi}^{\alpha\beta} \right) u_\rho - \chi_2 \beta_0 \sigma_\alpha{}^{\langle\rho} \epsilon^{\sigma\rangle\nu\alpha\beta} u_\beta p_{\langle\rho} p_{\sigma\rangle} \right] \Bigg\}$$

Virtual photon emission from a thermal medium

$$q\bar{q} \to \gamma^* \to e^+e^-$$
$$\pi^+\pi^- \to \gamma^* \to e^+e^-$$

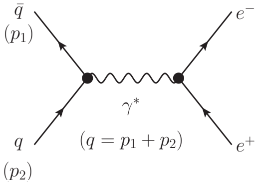
- Thermal average of initial particles momenta p through Fermi or Bose distribution
$$f(p) = \frac{1}{e^{(u\cdot p)/T} \pm 1}$$
- Fluid rest frame $u^\mu = (1, 0, 0, 0)$ $\Rightarrow$ Distribution is spherical symmetric

Photon momentum $\vec{q}$ breaks spherical symmetry, but not azimuthal symmetry

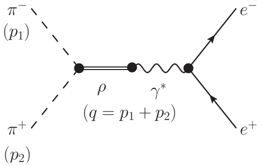
- Photons are only tensor polarized
- $|\vec{q}| \to 0$ $\Rightarrow$ No anisotropy $\Rightarrow$ No photon polarization

Boltzmann limit

$q + \bar{q} \to \gamma^* \to e^+ e^-$



$\pi^+ \pi^- \to \gamma^* \to e^+ e^-$



$$\int \frac{d^3 p_1}{E_1} \frac{d^3 p_2}{E_2} \frac{1}{e^{(u \cdot p_1)/T} \pm 1} \frac{1}{e^{(u \cdot p_2)/T} \pm 1} \sim e^{-(u \cdot q)/T} \int \frac{d^3 p_1}{E_1} \frac{d^3 p_2}{E_2}$$

- No photon polarization independently of photon momentum

Photon polarization is due to quantum statistics!