From topological θ vacua to asymptotic freedom: Qubit models, sign problems, and anomalies

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- From qubits to QFT
- Qubit regularization of toy-models of QCD
 - UV: asymptotic freedom
 - IR: θ vacua
- Symmetries and anomalies
 - What do anomalies say about constructing new lattice/qubit regularizations?
- Summary

Quantum field theory is important...

The framework of QFT

- underpins the standard model of particle physics electroweak theory,
 OCD
- explains long distance behavior of condensed matter systems: critical phenomenon, topological phases,...

...But it is hard!

Despite the enormous importance of precise quantitative computations in QFT, our general tools are lacking

- Perturbation theory
 - Great for weakly coupled theories such as QED
 - Fails for strongly coupled theories such as QCD at low energies
- Lattice Monte Carlo
 - Great for static quantities (like hadron spectrum in QCD) when the sign problem can be solved
 - Insufficient for studying out-of-equilibrium physics, such as realtime dynamics (such as hadronization in QCD); or sectors of high particle number density (such as neutron star cores); or even for static quantities in the presence of a sign problem
- With quantum computing at scale, we might solve some of the issues which plague classical lattice calculations

"Digitization" of QFTs for quantum computers

Traditional lattice regularization for bosons = ∞-dim local Hilbert space.
 Implied by the bosonic commutation relations

$$[\phi_x, \pi_y] = i\delta_{x,y} \tag{1}$$

- But digital quantum computers need a finite dimensional local Hilbert space
- Need to truncate the Hilbert space somehow...
- Several approaches towards finding a "digitization"
 - Field-space digitization [Jordan, Lee, Preskill, 2011, ...]
 - Loop-string hadrons [Raychoudhary et al, 2020, ...]
 - Single-particle digization [Barata et al, 2020, ...]
 - Tensor networks [Meurice, 2020, ...]
 - Discrete subgroups for gauge theories [Lamm et al, ...]
 - D-theory, quantum-link models [Brower et al, 2004, ...]
 - ...

"Digitization"

• Most approaches to digitization: truncate the Hilbert space (to n qubits), then reproduce the traditional lattice Hamiltonian by taking $n \to \infty$, and then take the continuum limit like in traditional lattice models

Digitized model $\xrightarrow{n\to\infty}$ Traditional lattice model $\xrightarrow{a\to 0}$ continuum QFT (2)

• Is it necessary to do this 2-step procedure? No!

"Digitization"

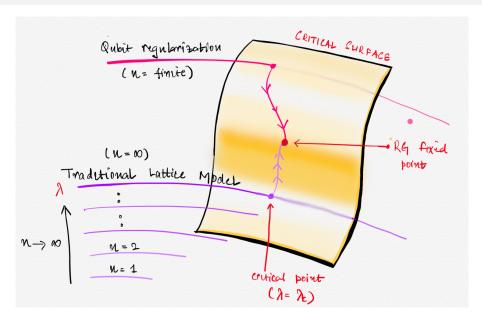
- Wilson's insight: QFT = Second-order phase transitions
- Even with finite n (#qubits per lattice site) one can obtain continuum limits of field theories

Qubit regularization of field theories

- Continuum limit: tune to a second-order critical point of a quantum lattice Hamiltonian
- This defines a procedure to obtain a continuum QFT
- Qubit regularization:

 a quantum lattice Hamiltonian acting on a finite-dimensional local
 Hilbert space (kept fixed) which reproduces a desired QFT in the vicinity of a quantum critical point.

Regularization from a Wilsonian RG perspective



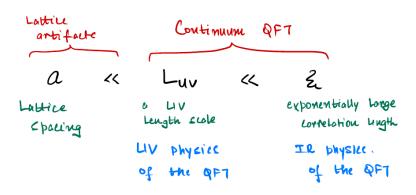
Qubit regularization of QCD

- For some QFTs, it is quite easy to construct such "qubit" models.
- For example, the Ising model famously reproduces the ϕ^4 theory in d=3,4 dimensions
- However, it is not obvious that all QFTs can be obtained in this way
- Question:

Can we recover all features of QCD from models of qubits?

- asymptotic freedom
- topological θ term

The challenge of asymptotic freedom



 To get the continuum limit, we need to recover both the IR physics and the UV physics

A toy model of QCD

- O(3) nonlinear sigma model in 1+1 dimensions
- Continuum action

$$S[\vec{n}(x)] = \frac{1}{2g^2} \int d^2x \, \partial_\mu \vec{n} \cdot \partial^\mu \vec{n}$$
 (3)

with $\vec{n} \in \mathbb{R}^3$ and $|\vec{n}| = 1$.

• toy model for QCD: asymptotic freedom, dynamical mass generation, dimensional transmutation, θ -vacua

QCD vs. (1+1)d O(3)

QCD/SU(N)-YM

- 3 + 1-dimensional
- Local gauge symmetry
- Asymptotically free
- Dimensional transmutation
- Nonperturbative mass gap
- Nontrivial topology, heta-term

$O(3) NL\sigma M$

- 1 + 1-dimensional
- Global *O*(3) symmetry
- Asymptotically free
- Dimensional transmutation
- Nonperturbative mass gap
- Nontrivial topology, θ -term

Traditional lattice regularization

- O(3) nonlinear sigma model in 1+1 dimensions
- Lattice regulated action:

$$S = \frac{1}{2g^2} \int d^2x \; \partial_\mu \vec{n} \cdot \partial^\mu \vec{n}$$
 (4)

Valve discretization

$$S = -\frac{1}{g^2} \sum_{\langle xy \rangle} \vec{n}_x \cdot \vec{n}_y \tag{5}$$

• 2d O(3) NLSM is the continuum QFT which emerges in the $g \to 0$ limit of the lattice model

Qubit Regularization of the O(3) NLSM with a θ term

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From Asymptotic Freedom to θ Vacua: Qubit Embeddings of the O(3) Nonlinear σ Model

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O(3) NLSM at arbitrary θ

- So far, we have talked about the O(3) NLSM at $\theta = 0$.
- Just like QCD, the O(3) NLSM allows for a topological θ term

$$S_{\theta}[\vec{\phi}] = \frac{1}{g^2} \int d^2x (\partial_{\mu}\vec{\phi})^2 + i\theta Q[\vec{\phi}] \tag{6}$$

where

$$Q[\vec{\phi}] = \frac{1}{8\pi} \int d^2x \, \varepsilon_{\mu\nu} \, \vec{\phi} \cdot (\partial^{\mu} \vec{\phi}) \times (\partial^{\nu} \vec{\phi}) \tag{7}$$

is the topological theta term.

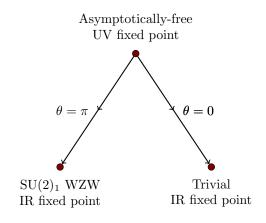
In nature,
$$\theta < 10^{-10} \implies$$
 Strong CP problem

Physics of θ

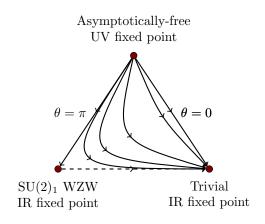
$$S_{ heta}[\vec{\phi}] = S_0 + i\theta Q[\vec{\phi}]$$

- The physics of θ is totally non-pertubative
- ullet does not show up in perturbation theory \Longrightarrow UV physics unchanged.
 - S_{θ} is an asymptotically free theory for all θ with a non-pertubatively generated energy scale.
- What about the IR physics?
 - ullet heta non-perturbatively changes IR physics
 - At $\theta=\pi$, the low-energy physics is completely different from $\theta=0$!
 - It is, in fact, massless in the IR \implies flows to the SU(2)₁ WZW CFT.
- What happens at arbitrary θ ?

RG flow



RG flow



The theory at $\theta \neq 0, \pi$

- $\theta=0,\pi$ points are very special due to their integrability [Zamolodchikov et al, 1978; 1992] and we have a very good understanding.
- At $\theta \neq 0, \pi$ the situation is not so clear.
- Many attempts to understand the behavior of the theory, but questions still remain...

The problem with θ

- It has been argued that the topological charge has ultraviolet divergences for the O(3) model and is not a physical quantity [Schwab, 1982; Luscher, 1982; Blatter et al, 1996].
- If so, it might happen that θ is an irrelevant parameter (for $\theta \neq \pi$) and simply renormalizes to zero.
- Other studies, by studying the theory about $\theta=\pi$ WZW point, have argued that that in fact there is a critical θ_c below which the theory renormalizes to zero, but is nontrivial for $\theta \geq \theta_c$ [Controzzi, Mussardo, 2003; Venuti et al 2005].
- Question: Is there a continuum QFT S_{θ} for each value of θ ?
- If these concerns can not be sorted out for the 2d O(3) NLSM, the case of QCD is even harder...

Lattice formulation

- In the conventional approach, θ introduces a severe sign problem (imaginary coefficient in Euclidean time)
- [Bögli, Niedermayer, Pepe, Wiese, 2011] studied the θ -vacua using non-standard ("topological") actions:
 - In their approach the sign problem is "mild" for smaller lattices.
 - Concluded that S_{θ} is unique for each θ .
- It would be good to have a completely sign-problem free way of studying θ vacua.

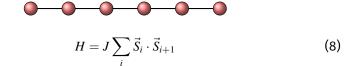
Qubit regularization

Is there a "qubit regularization" of this model?

- We need to obtain
 - UV physics (asymptotic freedom)
 - IR physics (θ -vacua)

Haldane Conjecture

- In 1981, Haldane surprised both condensed matter and high-energy communities
- Consider the antiferromagnetic spin-S Heisenberg chain



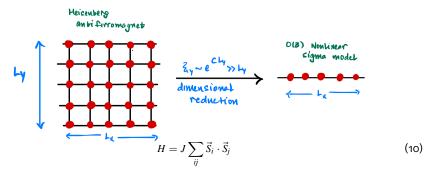
Haldane Conjecture

Spin-S chain
$$\leftrightarrow$$
 O(3) sigma model at $\theta = 2\pi S$ (9)

S=1/2 chain
$$\theta = \pi$$
 NLSM massless
S=1 chain $\theta = 0$ NLSM massive

How do we take the continuum limit (asymptotic freedom)?

UV: Asymptotic Freedom from D-theory



- For large L_Y , spontaneous symmetry breaking: $O(3) \rightarrow O(2)$
- The continuous fields $\vec{\phi}$ arise from collective Goldstone mode excitations of the spin-1/2 variables \vec{S}_i
- Study the limit: $1 \ll L_Y \lesssim L_X$
 - Asymptotic freedom of the $\mathcal{O}(3)$ model guarantees exponential convergence in \mathcal{L}_{Y}
- Dimensional reduction back to (1+1)-d theory! [Chandrasekharan, Wiese, 1997; Brower et al, 1999]

UV and IR

- D-theory provides a recipe to get the UV physics of asymptotically free theories
- But what about IR? Can we generate a θ term in the IR?

IR: θ term in spin chains

• In terms of the spin variables, it can be shown using bosonization [Affleck, 1988]

$$a^{-1}\vec{S}_n = \vec{J}_L + \vec{J}_R + i(-1)^n c(\operatorname{Tr} g)\vec{\sigma}.$$
 (11)

- Note that "charge conjugation" $g\mapsto -g$ maps to translation by one unit $S_n\mapsto S_{n+1}$
- Therefore, to generate a θ term, we must break this translation-by-one symmetry.
- For example, we can stagger the couplings on even and odd sites

$$J_{\pm} = J(1 \pm \gamma). \tag{12}$$

$$J_{\pm} \qquad J_{-} \qquad J_$$

• For this case, [Haldane, Affleck]

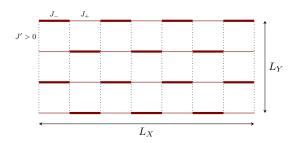
$$\theta = 2\pi S(1+\gamma). \tag{13}$$

Can be generalized to spin ladders [Sierra, 1996; Sierra et al, 1997]

Taking the continuum limit with θ term

- We can finally put the two pieces of the puzzle together
 - UV = Asymptotic freedom ⇒ Dimensional reduction
 - IR = topological θ term \implies C breaking using staggered couplings
- Therefore, we can now take the continuum limit of these models at non-trivial θ!

θ -term with D-theory



- Proposal: Continuum limit of the O(3) NLSM with θ term obtained in the $L_X \gg L_Y \gg 1$ limit
- Analysis of spin ladders ¹ suggests, for $J_{\pm}=J(1\pm\gamma)$,

$$\theta \approx 2\pi S L_Y(1+c\gamma) \implies |\theta-\pi| = c\pi \frac{\gamma L_Y}{\sqrt{L_Y}} (\text{odd } L_y)$$
 (14)

A gift: no sign problem! So we can actually numerically check this.

¹Sierra 1995; Martin-Delgado, Shankar, Sierra 1996

Probing the continuum limit for asymptotically free theories

- To probe the universal behavior of the continuum limit, we can use the **step scaling function** as a convenient tool [Luscher, Weisz, Wolff, 1991]
- Put the asymptotically free theory in a box of size L (natural length scale)
- Define a dimensionless renormalized coupling $\bar{g}^2(L)$
 - For example, we can choose $\bar{g}^2(L) = M(L)L$, where M(L) is the finite-volume mass gap
- All dimensionless observables depend only on the renormalized coupling $\bar{g}^2(L)$.

Step scaling function

• We will look at the universal function F(z) defined by

$$\frac{\xi(\beta, 2L)}{\xi(\beta, L)} \equiv F\left(\xi(\beta, L)/L\right) \tag{15}$$

where β is a bare coupling and $z=\xi(\beta,L)/L$ is the renormalized coupling

• $\xi(\beta, L)$ is a definition of finite-volume correlation length: the "second-moment" correlation length

$$\xi(L) = \frac{\sqrt{\tilde{G}(0)/\tilde{G}(2\pi/L) - 1}}{2\sin(\pi/L)}$$
 (16)

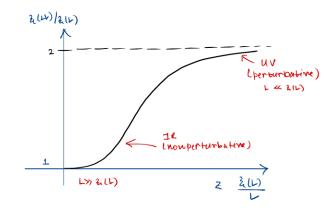
Easy to measure

Step scaling function: qualitative behavior

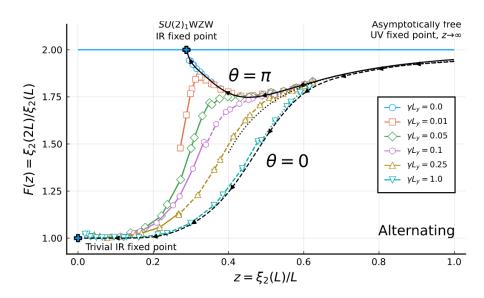
$$z = \xi(L)/L$$

$$F(z) = \xi(2L)/\xi(L)$$

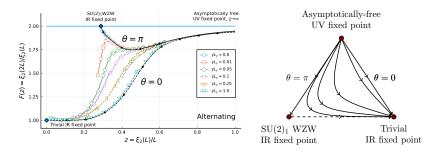
- IR physics: $z \to 0$
- UV physics: $z \to \infty$



Step-scaling function and the RG flow

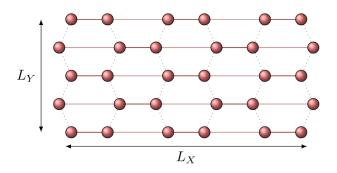


Step-scaling function and the RG flow



The step-scaling curves mimic the expected RG flow diagram beautifully!

On quantum simulators

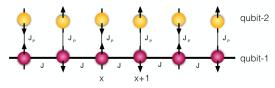


 On Rydberg systems with native Ising-type interactions, we can use Floquet engineering techniques to implement Heisenberg interactions²

²arXiv: 2207.09438 [Ciavarella, Caspar, HS, Savage, Lougovski, 2022]

A two-qubit regularization

- In another work ³, we showed that a two-qubit regularization of asymptotic freedom can also be obtained
- "Heisenberg Comb"



Hamiltonian

$$H = \sum_{i} J_{p} H_{(i,1),(i,2)} + J H_{(i,1),(i+1,1)}$$
 (17)

• Set $J_2=0,\ J_p=1.$ Continuum limit: $J\to\infty.$

³PRL 126, 172001 (2021) [Bhattacharya, Buser, Chandrasekharan, Gupta, HS]

Summary

- The 2d O(3) NLSM allows for a θ term, just like QCD.
- However, physics of θ is non-perturbative and therefore hard to study both
 - analytically (no small parameter, non-integrable),
 - and on the lattice (sign problem)
- We constructed a qubit regularization of the O(3) NLSM with a θ term
 - Completely solves the sign problem present in conventional approaches for the θ term, for the first time.
 - Allowed us to take the continuum limit and demonstrate asymptotic freedom for various θ
 - Step-scaling curves give a quantitative instantiation of the RG flow
 - Very natural for quantum simulators with qubit degrees of freedom
- Opens up many paths forward...
 - systematic understanding of the RG flow as a function of θ , comparison with analytical results from instanton calculations, ...

• We saw that there is a lattice regularization of the θ term where θ appears as the staggering of couplings

Staggering
$$\gamma \longleftrightarrow \theta$$
 term (18)

- But: why does such a regularization exist? Did we simply get lucky?
- Is there a way to systematically explore this space of lattice regularizations?

An interesting perspective comes from symmetries and anomalies

Lattice regularizations of θ vacua: Anomalies and qubit models

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[arXiv: 2209.12630]

Symmetries and Anomalies

- It is clear that symmetries play a huge role in constructing lattice regulators.
 - It is ideal if the lattice regulator explicitly preserves a symmetry of the continuum theory
- However, some symmetries have a subtle structure, which we call an anomaly

't Hooft Anomalies

- The word anomaly has many meanings..
- For us, anomaly = 't Hooft anomaly

"'t Hooft Anomaly"

G is a genuine global symmetry of the theory, but it cannot be gauged.

"Mixed anomaly"

 G_1 , G_2 are genuine global symmetries. They can be gauged individually, but gauging one breaks the other.

Classic example of a mixed 't Hooft anomaly

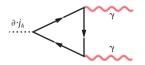
 $U(1) \times U(1)_A$ for a free Dirac fermion

• A free Dirac fermion has exact global (vector) U(1) and (chiral) $U(1)_A$ symmetries

$$\psi \stackrel{U(1)}{\longmapsto} e^{i\theta} \psi$$
 (19)

$$\psi \stackrel{U(1)_A}{\longmapsto} e^{i\theta\gamma_5}\psi$$
 (20)

• However, if you gauge U(1), then you lose $U(1)_A$!



• In other words, U(1) and $U(1)_A$ cannot be gauged simultaneously.

A mixed anomaly in the sigma models

• At $\theta=0,\pi$, the O(3) model has exact global SO(3) and charge conjugation C symmetries

$$\vec{\phi} \xrightarrow{SO(3)} e^{i\theta \hat{n} \cdot \vec{J}} \vec{\phi}$$
 (21)

$$\vec{\phi} \stackrel{C}{\longmapsto} -\vec{\phi} \tag{22}$$

(23)

- This is a mixed anomaly at $heta=\pi$ between SO(3) and C⁴
- If you gauge SO(3), then you lose C!

$$\langle \mathcal{O} \rangle \stackrel{C}{\longmapsto} e^{i \operatorname{Anomaly}} \langle \mathcal{O} \rangle$$
 (24)

⁴[Gaiotto, Kapustin, Komargodski, Seiberg, 2017]

Anomalies and lattice regularizations?

- We are interested in constructing new lattice regularizations
- The presence of an 't Hooft anomaly for *G* must be reflected by a lattice regularization.

Gauging a symmetry on the lattice

- Assume a lattice regulator, with
 - same spacetime dimensionality
 - exact symmetry on the lattice
 - locality
 - symmetry implemented "onsite"
- ullet Onsite \Longrightarrow we can gauge it on the lattice by introducing link variables
- But if there is an anomaly, there must be some obstruction to this procedure!

QFT Lore

"There are no anomalies on the lattice"

 Example: attempts to put chiral fermions on the lattice result in doublers [Nielsen, Ninomiya]

Obstructions to gauging on the lattice

- A way in which it is impossible to gauge a symmetry is that the symmetry is not onsite
 - Well appreciated in cond-mat 5
- If the symmetry is offsite, there is no obvious way to gauge it on the lattice
- Indeed, in the spin-chain regularization, the charge conjugation symmetry was offsite

$$C: \vec{S}_i \mapsto \vec{S}_{i+1} \tag{25}$$

- The spin-1/2 chain naturally realizes the $\theta=\pi$ model, which has a SO(3) imes C anomaly
- Now, we see that the offsite-ness of the symmetry was no accident it
 is almost forced by the anomaly!

⁵For example: Jian, Bi, Xu (2018); Cho, Hsieh, Ruy (2017)

Obstructions to gauging on the lattice

- But what if we insist that an anomalous symmetry is onsite on the lattice?
- Consider the standard lattice action at $\theta = 0$

$$S_0 = -\frac{1}{g^2} \sum_{\langle xy \rangle} \vec{n}_x \cdot \vec{n}_y \tag{26}$$

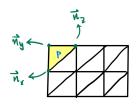
- Both SO(3) and C are onsite
- Indeed, a topological θ term on the lattice was defined by [Berg, Lüscher, 1981] which maintains this property

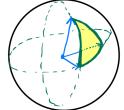
A topological definition of the θ term on the lattice

• Topological θ term on the lattice [Berg, Lüscher, 1981]

$$S[\vec{n}] = S_0[\vec{n}] + i\theta Q[\vec{n}] \tag{27}$$

$$Q[\vec{n}] = \sum_{\langle xyz \rangle} q_{\langle xyz \rangle} \tag{28}$$





Obstruction?

- So if the symmetry is both exact and onsite on the lattice, where is the obstruction? What prevents us from gauging it on the lattice?
- In [arXiv: 2209.12630, Nguyen, HS] we find that this rather well-known lattice model **explicitly** reproduces the exact anomaly on the lattice!

How to detect the anomaly?

 $SO(3) \times C$ anomaly

• We turn on a background gauge field (A,B) for the SO(3) symmetry and then perform a C transformation

$$Z \xrightarrow{\mathsf{Gauging PSU}(N)} \tilde{Z}[A, B]$$
 (29)



• We then check whether Z is invariant under C.

C:
$$\begin{cases} \tilde{Z}[A,B,\theta=0] & \mapsto \tilde{Z}[A,B,\theta=0], \\ \tilde{Z}[A,B,\theta=\pi] & \mapsto \tilde{Z}[A,B,\theta=\pi] & \underbrace{e^{-ik\int B}}_{\text{anomaly}} \end{cases}$$
(30)

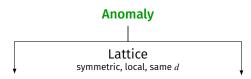
• Analogy: Chiral anomaly \implies we turn on background field for U(1) and find that the partition function is not invariant under $U(1)_{\chi}$

"No anomalies on the lattice"

- The lore "no anomalies on the lattice" is incomplete!
- Indeed there can be anomalies on the lattice
- Other examples of anomalies on the lattice: Kahler-Dirac fermions [Catterall, 2022], Ginsparg-Wilson realization of chiral fermions⁶ [Lüscher, Neuberger, Kaplan, ...]

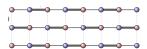
⁶for a modified chiral symmetry

Anomalies and Lattice Regularizations of θ theta vacua



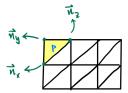
Offsite symmetry

- Qubit regularization
- Staggered couplings
- No sign problem!
- Natural for quantum computers



Exact anomaly

- Berg–Lüscher θ term
- Manifestly topological
- Sign problems
- Infinite-dimensional Hilbert space



Guidance from anomalies

- These arguments seem general. Do all models with mixed 't Hooft anomalies have such a dichotomy of lattice regularizations?
- We were able to generalize the O(3) constructions to a wider class of 2d asymptotically free theories, called the Grassmannian nonlinear sigma models.
- Here, instead of unit vectors on \mathbb{R}^3 , the fields P live on

$$P_x \in \operatorname{Gr}_k(N) = \frac{U(N)}{U(N-k) \times U(k)}$$
 (31)

with the action

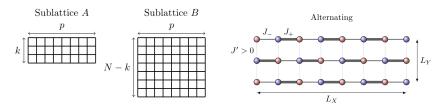
$$S = \frac{1}{g^2} \int d^2x \operatorname{Tr}(\partial_{\mu} P)^2 + \frac{\theta}{4\pi} \int d^2x \, \epsilon^{\mu\nu} \operatorname{Tr} P \, \partial_{\mu} P \, \partial_{\nu} P \tag{32}$$

• These $\operatorname{Gr}_k(N)$ models also have an anomaly at $\theta=\pi$ (N) between $\operatorname{PSU}(N)$ and C for (N,k)=(even, odd)⁷.

⁷for other cases, we have a more subtle scenario called "global inconsistency"

Lattice regularization for Grassmannian models

- Qubit regularization
 - Now, we have SU(N) spins at each site in certain conjugate representations⁸



- Again, we can argue that a continuum limit at a fixed θ arises in the $L_y \to \infty$ limit if you keep γL_y fixed.
- Conventional regularization
 - The geometric Berg-Lüscher construction can also be generalized

⁸[Read, Sachdev, 1989]

Summary

- In the quest to find new regulators with unique advantages for quantum and classical simulation, anomalies can be a strong guide.
- The importance of anomalies been long appreciated for chiral fermions on the lattice.
- For the O(3) model (and Grassmannian $\operatorname{Gr}_k(N)$ models), we saw a dichotomy of regularizations: qubit and conventional, which reflect how the $\operatorname{SO}(3) \times \operatorname{C}$ anomaly manifests.
- They have quite different advantages! Which one is useful depends on the hardware and the question.
- There are very suggestive parallels with 4d nonabelian gauge theories
 - Indeed, pure SU(N) Yang-Mills has a very similar anomaly at $\theta=\pi$, between time reversal and \mathbb{Z}_N center symmetry 9
 - What does this say for lattice/qubit regularizations of QCD?

⁹[Gaiotto et al, 2017]

Conclusions

- "Qubit" regularization: continuum limit of a desired QFT with a finite-dim local Hilbert space.
- Such regularizations seem very natural for quantum computers, unlike traditional lattice regularizations
- Question: Can the Standard Model be obtained in such a way?
- We have demonstrated that a qubit regularization of the ${\cal O}(3)$ NLSM with arbitrary θ can be constructed
 - solved a sign problem along the way
- Lattice regularizations where anomalies are manifested differently seem to have quite different properties. What does this imply for lattice QCD?
 - We found that anomalies can indeed be present on the lattice, invalidating an old lore in QFT. Implications?
- The space of such non-traditional formulations of lattice QFTs is quite rich and important for near-term quantum computers
- Developing new tools for simulations of Standard Model, occasionally finding new perspectives on old problems

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