Longitudinal Beam Dynamics

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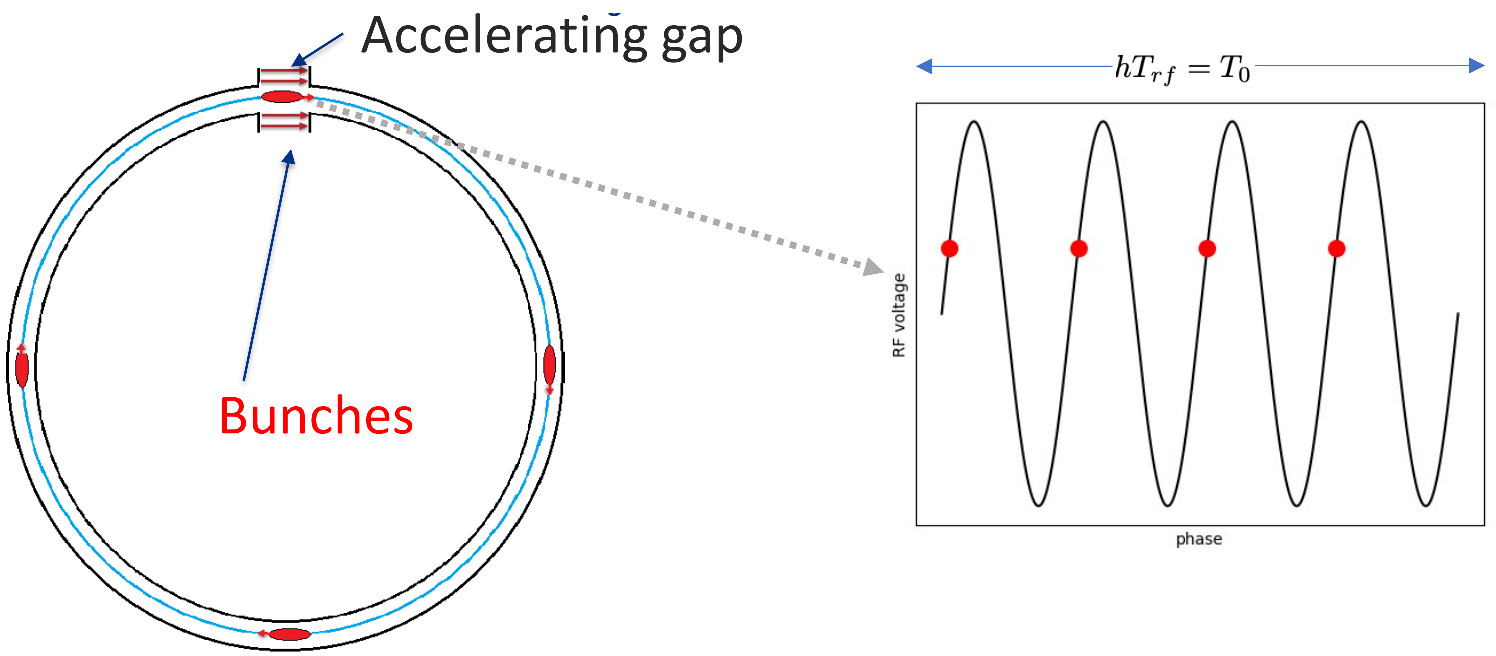
Contents

- Basics of longitudinal beam dynamics
- Beam stacking
- Fixed RF acceleration

Principles of RF acceleration (1)

- An RF cavity produces a time-varying electric field across an accelerating gap.
- The RF frequency must be in sync with the bunched beam in order to accelerate and ensure phase stability.
- In a ring, the RF frequency should be a multiple of the revolution frequency, $f_{rf} = hf_0$.



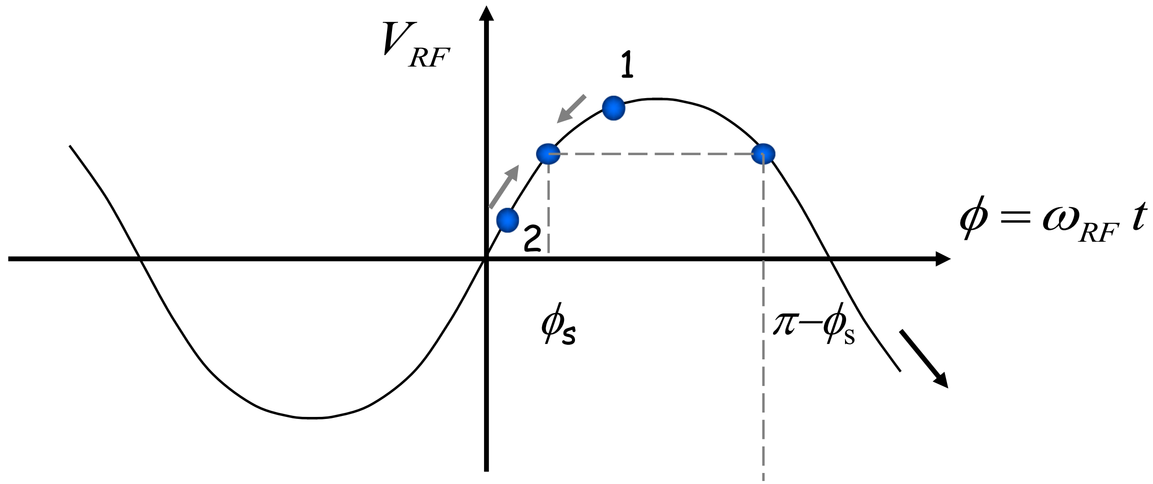
P. Ostroumov (2018)

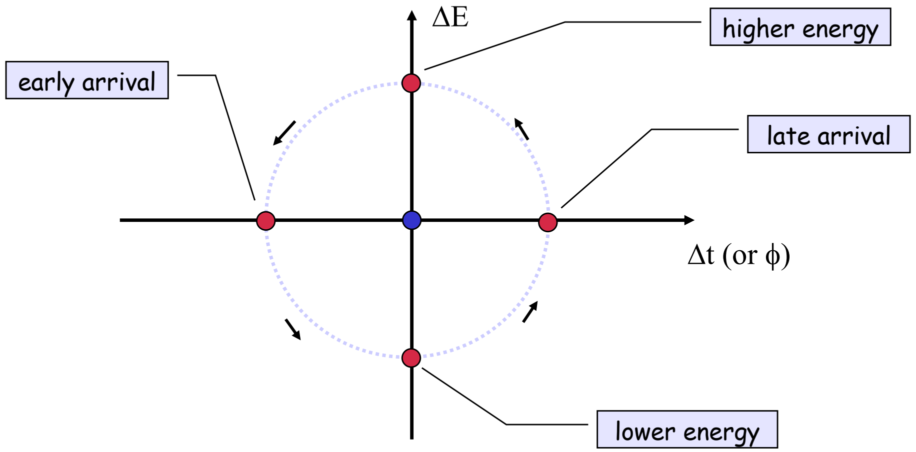
Principles of RF acceleration (2)

- Each particle gains energy according to

$$\Delta E = qV\sin\omega t$$

- Define a reference particle which arrives at the RF cavity at phase ϕ_s. For stability, want early and late arriving particles to move towards the synchronous phase.
- Particles that arrive at a different time (or phase) gain a different energy ΔE, and as a consequence, take a different time to reach the next cavity.





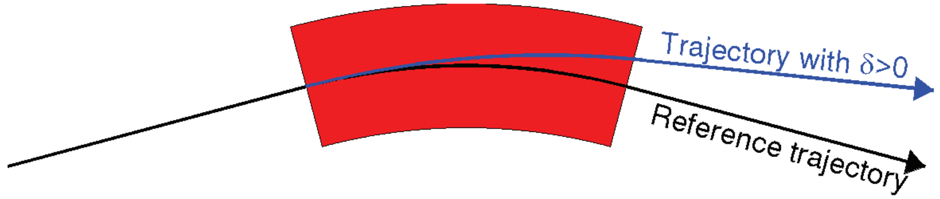
Revolution time for off-momentum particle

- Define the phase slip η and momentum compaction factor α

$$\frac{dT}{T} = \eta \frac{dp}{p} \qquad\qquad \alpha_p = \frac{dC/C}{dp/p}$$

where T is the revolution time (T=C/v), C is the path length and p the momentum. One can write

$$\frac{dT}{T} = \frac{dC}{C} - \frac{dv}{v} = \left(\alpha_c - \frac{1}{\gamma^2}\right)\frac{dp}{p}$$

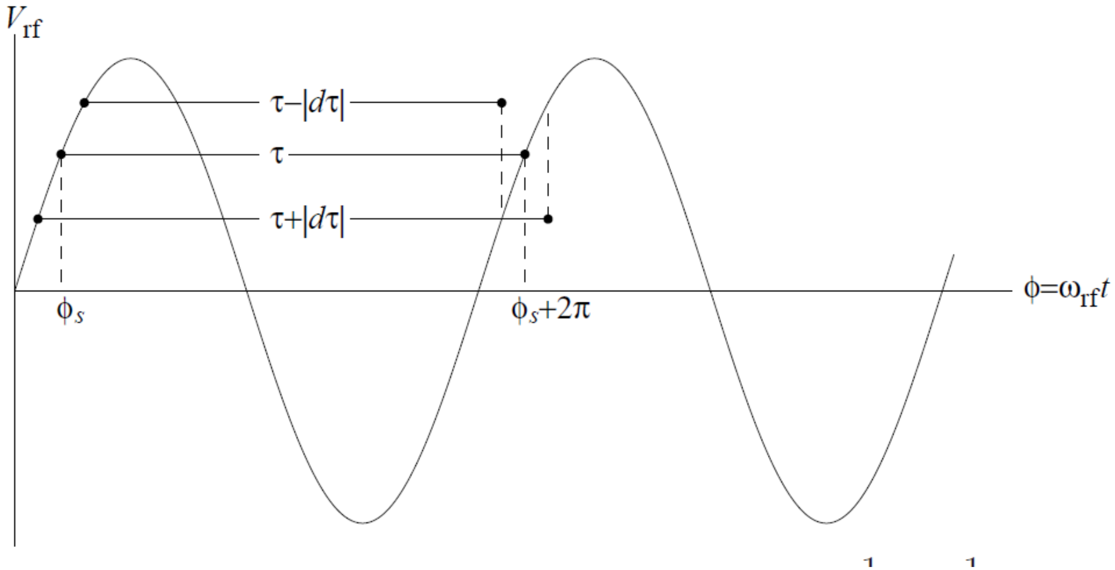


It follows

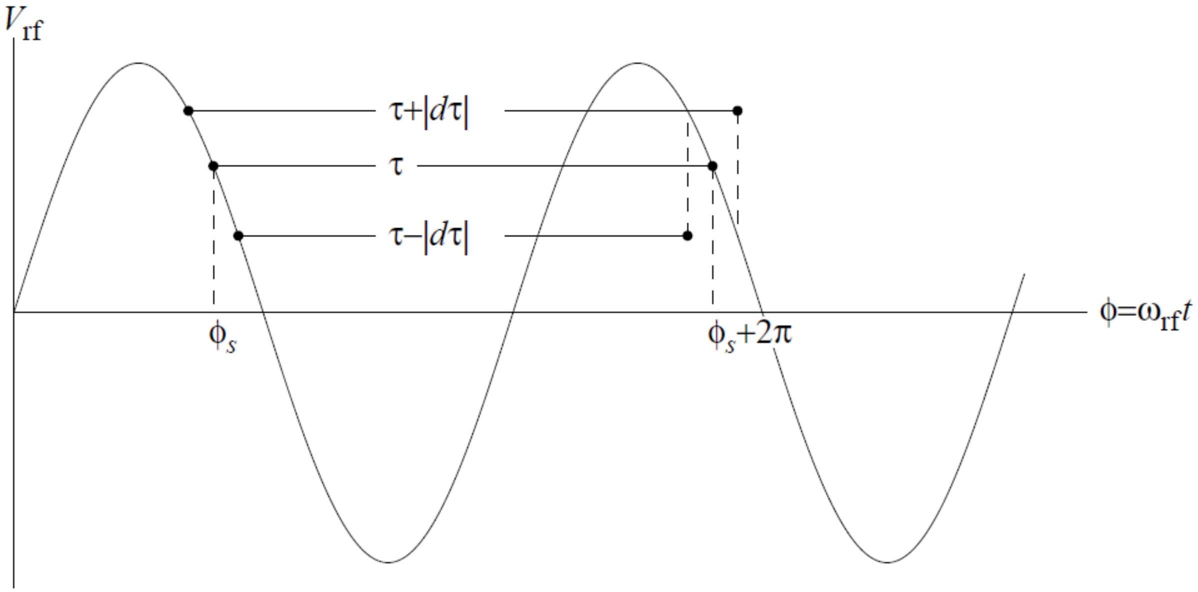
$$\eta = \left(\alpha_c - \frac{1}{\gamma^2}\right) \quad \text{or} \quad \eta = \left(\frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}\right)$$

Note: From relativistic momentum $p=\gamma m_0 v$, can show, $\frac{dv}{v} = \frac{1}{\gamma^2}\frac{dp}{p}$

Phase stability



Below transition



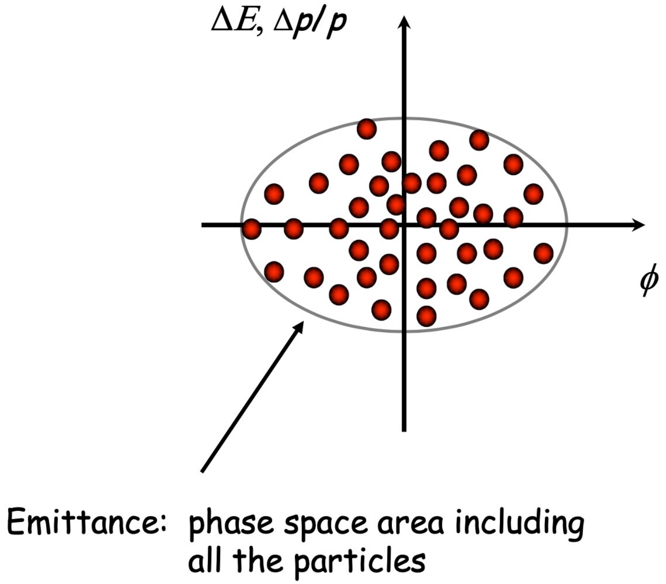
Above transition

$\eta_p < 0$	$\alpha_p < 0$ or $\gamma_0^2 < \frac{1}{\alpha_p}$	**Below Transition** revolution frequency *increases* with increasing energy
$\eta_p = 0$	$\gamma_0^2 = \frac{1}{\alpha_p}$	**At Transition** revolution frequency *independent* of energy
$\eta_p > 0$	$\gamma_0^2 > \frac{1}{\alpha_p}$	**Above Transition** revolution frequency *decreases* with increasing energy

A. Bogacz

Equations of motion

- Relative to the reference particle one can write



$$\delta\Delta E = \Delta E - \Delta E_s = qV\left(\sin(\phi_s + \Delta\phi) - \sin\phi_s\right)$$

$$\delta\Delta\phi = h\omega\Delta T = h\omega T_s \frac{\Delta T}{T_s} = h\omega T_s \eta \frac{\Delta p}{p_s} = h\omega T_s \frac{\eta}{\beta^2}\frac{\Delta E}{E_s}$$

Longitudinal Hamiltonian

$$H = \frac{1}{2}h\omega_0\eta\delta^2 + \frac{\omega_0 eV}{2\pi\beta^2 E_0}\left[\cos\phi - \cos\phi_s + (\phi - \phi_s)\sin\phi_s\right]$$

- The longitudinal equations of motion follow in terms of phase space coordinates (ϕ,δ).

$$\dot{\phi} = \frac{\partial H}{\partial \delta} = h\omega_0\eta\delta,\ \ \dot{\delta} = -\frac{\partial H}{\partial \phi} = \frac{\omega_0 eV}{2\pi\beta^2 E}\left[\sin\phi - \sin\phi_s\right]$$

- When $|\phi-\phi_s|$ is small,

$$\dot{\delta} \approx \frac{\omega_0 eV\cos\phi_s}{2\pi\beta^2 E_0}\left(\phi - \phi_s\right)$$

- Small amplitude synchrotron motion is simple harmonic with synchrotron tune Q_s

$$Q_s = \sqrt{-\frac{h\eta\cos\phi_s eV}{2\pi\beta^2 E}} \qquad \omega_s = Q_s\omega_0$$

Note: if $\eta<0$, require $0< \phi_s < \pi/2$ while if $\eta>0$ the stable fixed point moves to $\pi-\phi_s$.

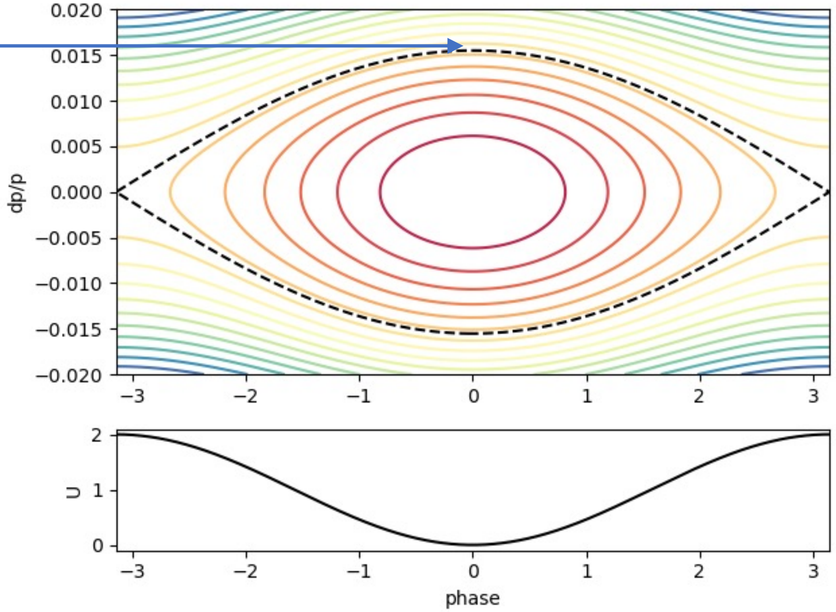
Note: δ=dp/p

$$Y(\phi_s) = \sqrt{|\cos\phi_s - 0.5 * (\pi - 2\phi_s)\sin\phi_s|}$$

$$B.H. = 2\sqrt{\frac{eV}{2\pi\beta^2 Eh|\eta|}}Y(\phi_s)$$

Hamiltonian contours

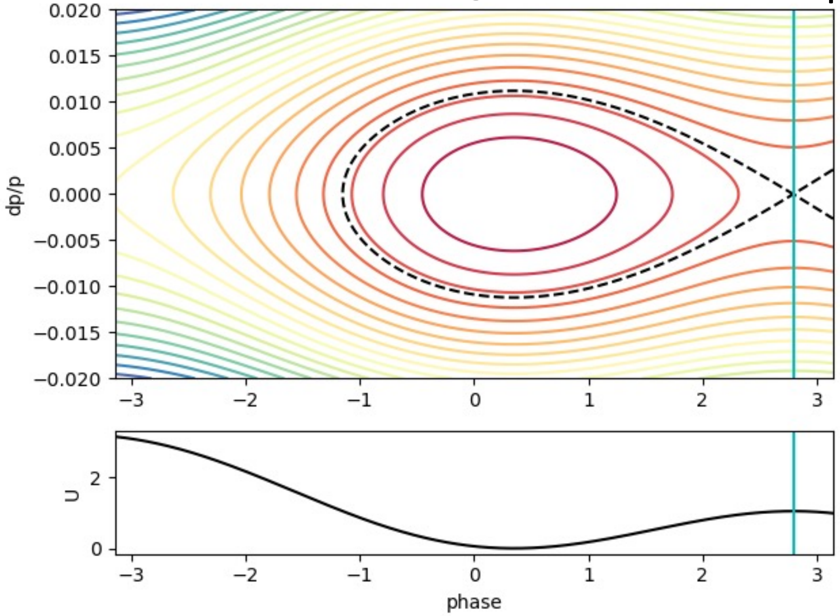
Stationary bucket



$\phi_s = 0°$

Moving bucket

$\pi - \phi_s$



$\phi_s = 20°$

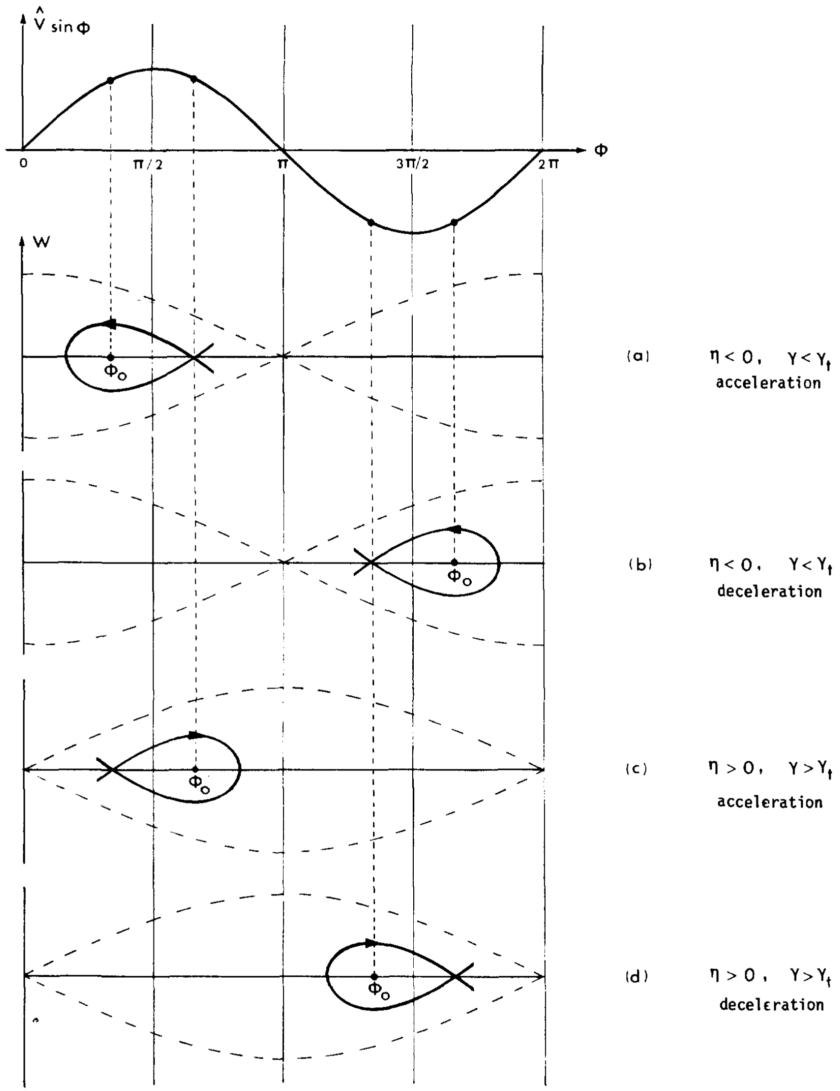
- Stable fixed point at $(\phi_s,0)$.
- Unstable fixed point at $(\pi-\phi_s,0)$
- Separatrix defined by contour $H(\pi-\phi_s,0)$.
- Bucket height and area maximum at $\phi_s = 0°$

- The area within the separatrix is known as a 'RF bucket'.
- Phase acceptance given by maxima of scaled potential term U.

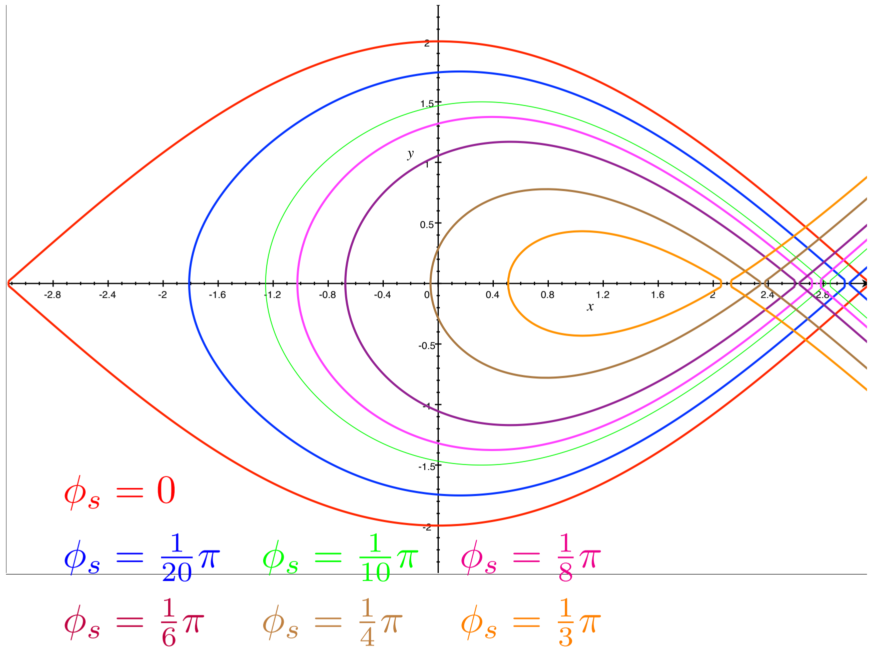
$$U = [\cos\phi - \cos\phi_s + (\phi - \phi_s)\sin\phi_s]$$

Assumed phase slip $\eta<0$ here.

RF buckets for various η,ϕ_s



C. Prior



- Bucket area greatest for ϕ_s=0 or π and falls to zero for ϕ_s= π/2.
- Synchronous phase is at ϕ_s or π- ϕ_s depending on sign of η.

Adiabaticity

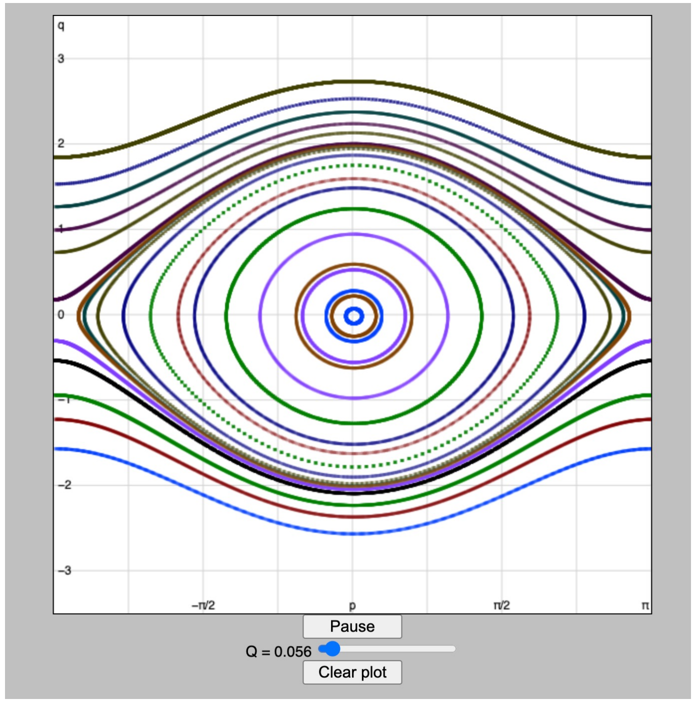
- If the RF parameters are changed at a slow rate with respect to the lowest frequency of oscillation, the process can be said to be adiabatic
- The adiabaticity parameter ε should be much lower than 1

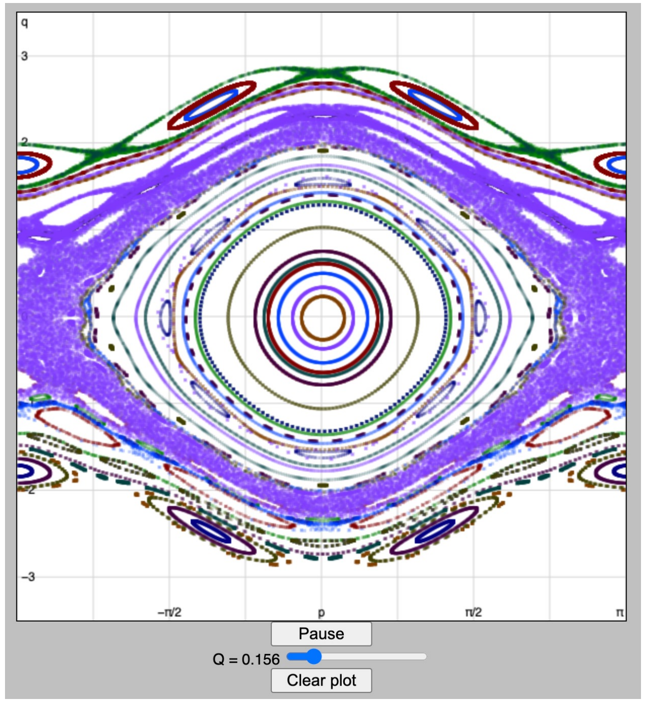
$$\epsilon = \frac{1}{\omega_s^2}\frac{\partial \omega_s}{dt} \ll 1$$

- The longitudinal motion is conservative (i.e., there is no energy dissipation effect like synchrotron radiation). Liouville’s theorem states that the local density of particles in the longitudinal phase plane is always constant is applicable. An implicit consequence is that any RF gymnastics is in principle reversible.

Standard map

- http://www.toddsatogata.net/2019-USPAS/lab/Standard.html





until done {
d = sqrt(2.0*(1.0 - cos(2*pi*Qs)))
q = q + 0.5*d*p
 p = p - d*sin(q) }

Synchrotron - Bdot

From the magnetic rigidity, write the time derivative

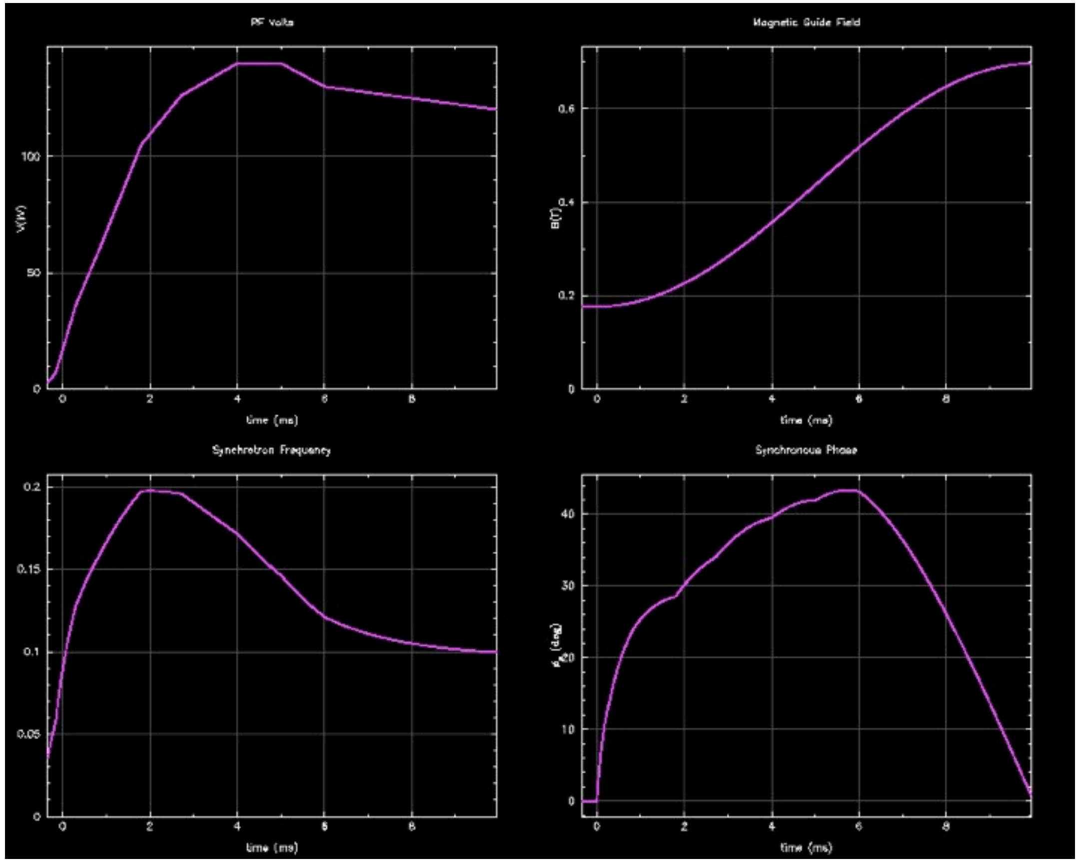
$$\frac{p}{q} = B\rho \qquad \frac{dp}{dt} = q\rho\dot{B}$$

Use following relativistic relation $\frac{dE}{dt} = \frac{pc^2}{E}\frac{dp}{dt}$

Energy gain per revolution $V\sin\phi_s = \frac{2\pi R}{\beta c}\frac{1}{q}\frac{dE}{dt} = 2\pi R\rho\dot{B}$

$$\sin\phi_s = \frac{2\pi R\rho}{V(t)}\dot{B} = \frac{2\pi R\rho}{V(t)} \times 2\pi f B_0 \sin 2\pi f t$$

Case of a sinusoid ramping field

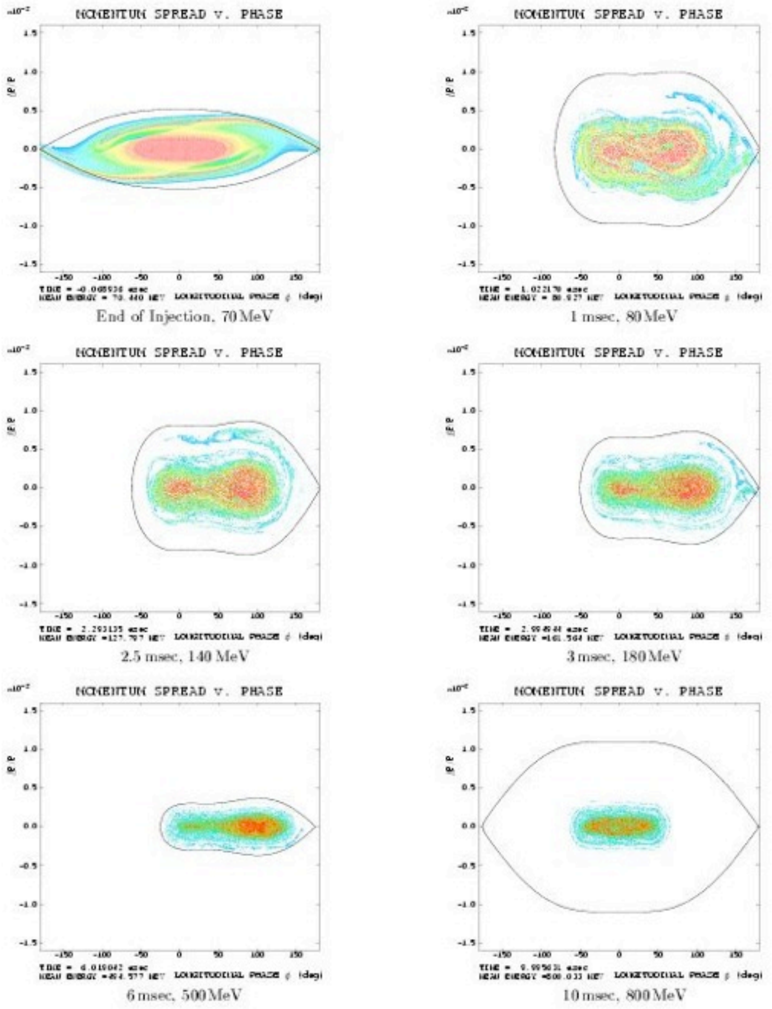


RF cycle in ISIS

f=50Hz, R=26m, B_0=0.26T, ρ=7m

Credit: C. Prior

Phase space during ISIS acceleration

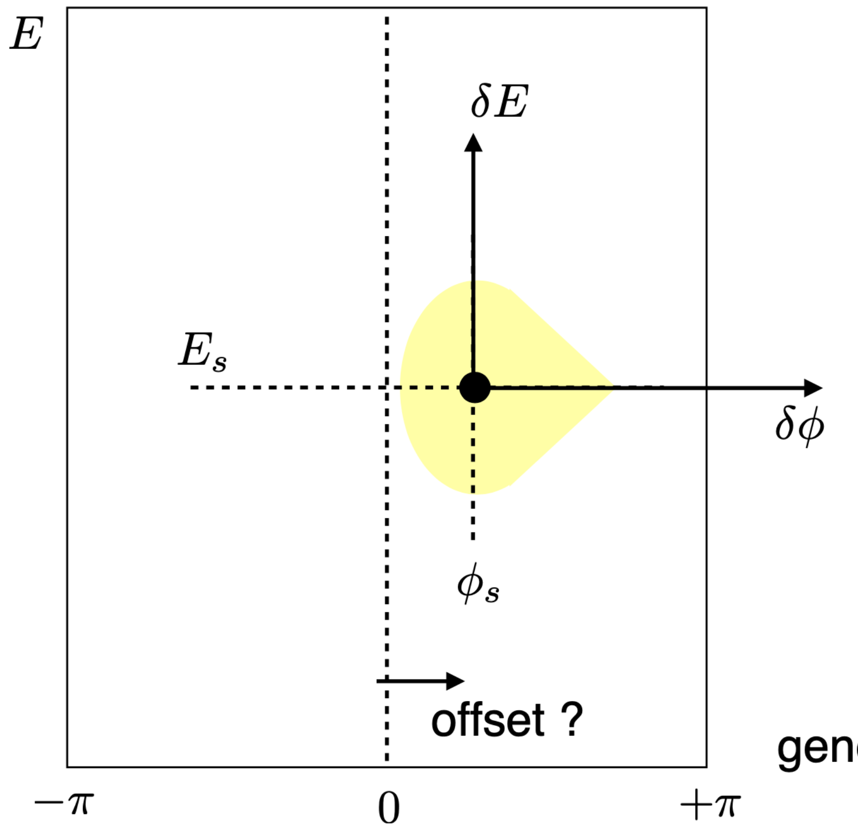


Longitudinal phase space plots for acceleration of 3×10^{13} protons in the ISIS synchrotron from 70 MeV to 800 MeV with dual harmonic RF system.

- RF waveform includes a 2nd harmonic.
- This flattens the line density, reduces peak current and so transverse space charge forces.

Credit: C. Prior

Acceleration in a scaling FFA



Rf frequency determines the *synchronous energy*.

$$h \cdot f(E_s) = f_{rf}$$

Synchronous phase is determined by the synchronous energy gain per turn, or, the ramping rate of rf frequency.

$$qV \sin\phi_s = \frac{\mathrm{d}E_s}{\mathrm{d}N} = \frac{1}{f}\frac{\mathrm{d}E_s}{\mathrm{d}t} = \frac{1}{f_{rf}}\frac{1}{(\mathrm{d}f/\mathrm{d}E_s)}\frac{\mathrm{d}f_{rf}}{\mathrm{d}t}$$

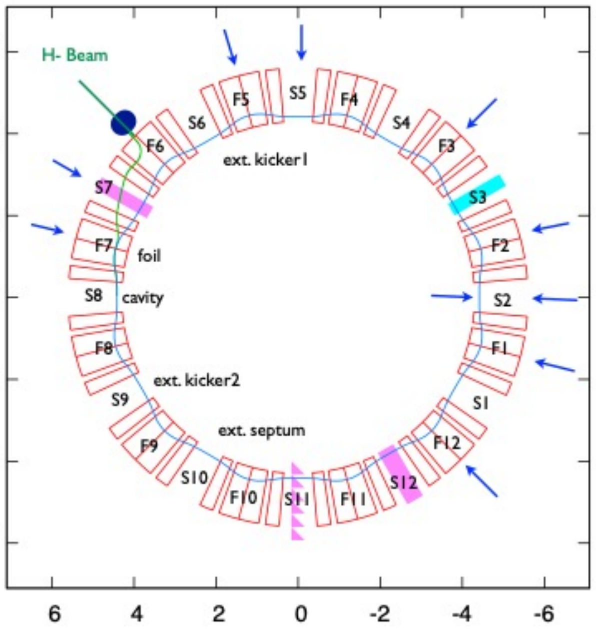
(ϕ_s, E_s) is a stable fixed point

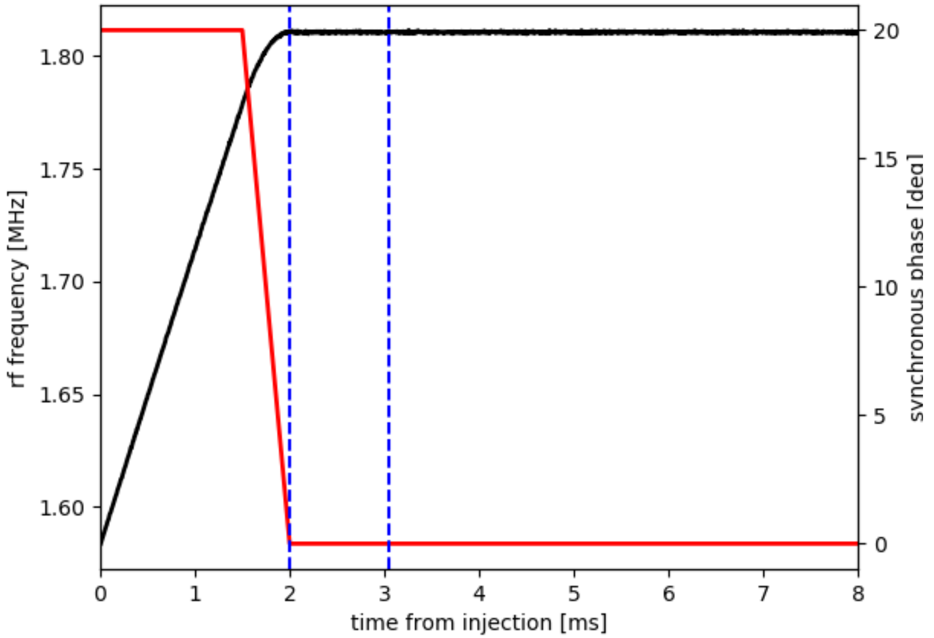
general particle

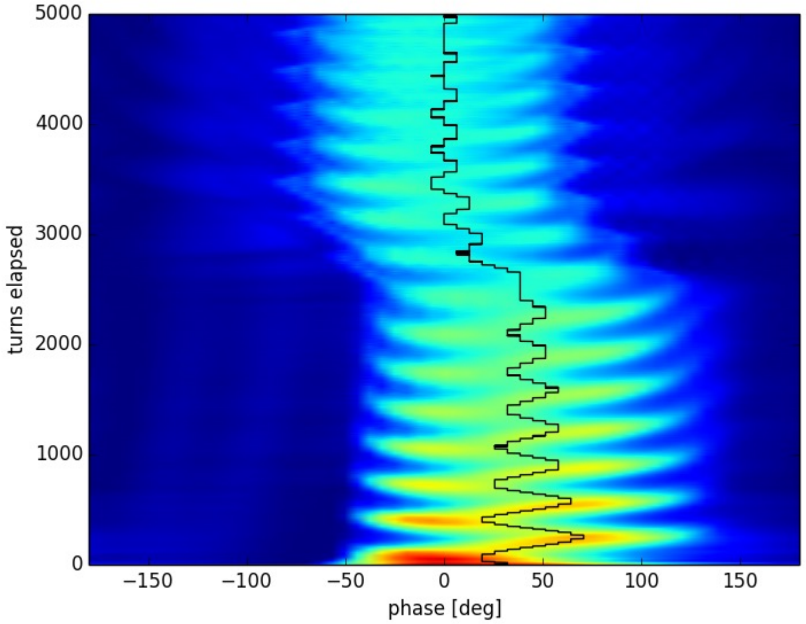
$$\phi = \phi_s + \delta\phi$$
$$E = E_s + \delta E$$

T. Uesugi

Acceleration in a scaling FFA - example



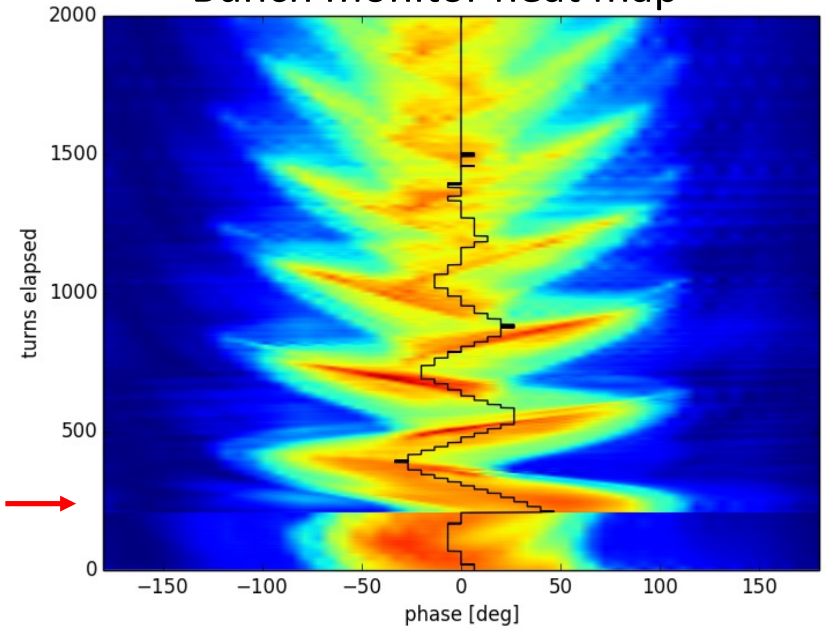


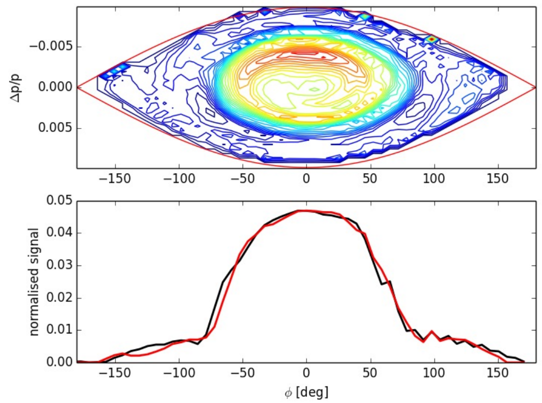


- 11-150 MeV KURNS FFA.
- The synchronous phase is normally 20°.
- In the example above, synchronous phase is brought to zero in 0.5ms to park the beam at some energy (and radius).

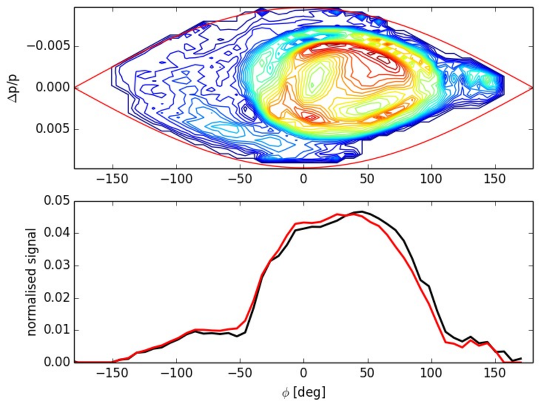
Phase jump in a scaling FFA

Bunch monitor heat map



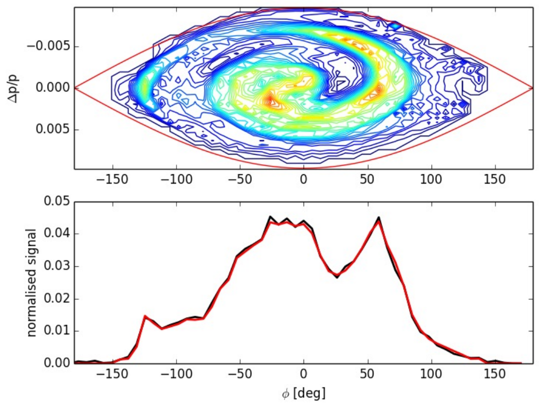


Initial distribution before jump

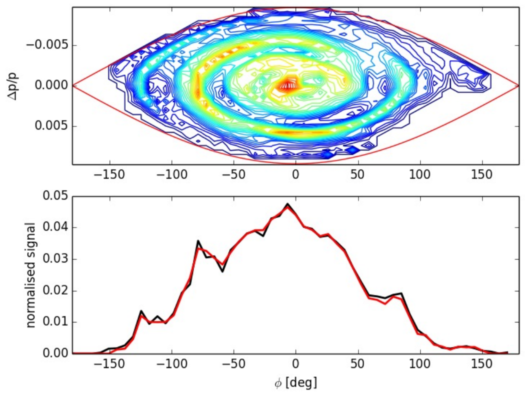


Immediately after jump

- 40 degree phase jump applied
- Longitudinal tomography reconstructs the distribution at various points.



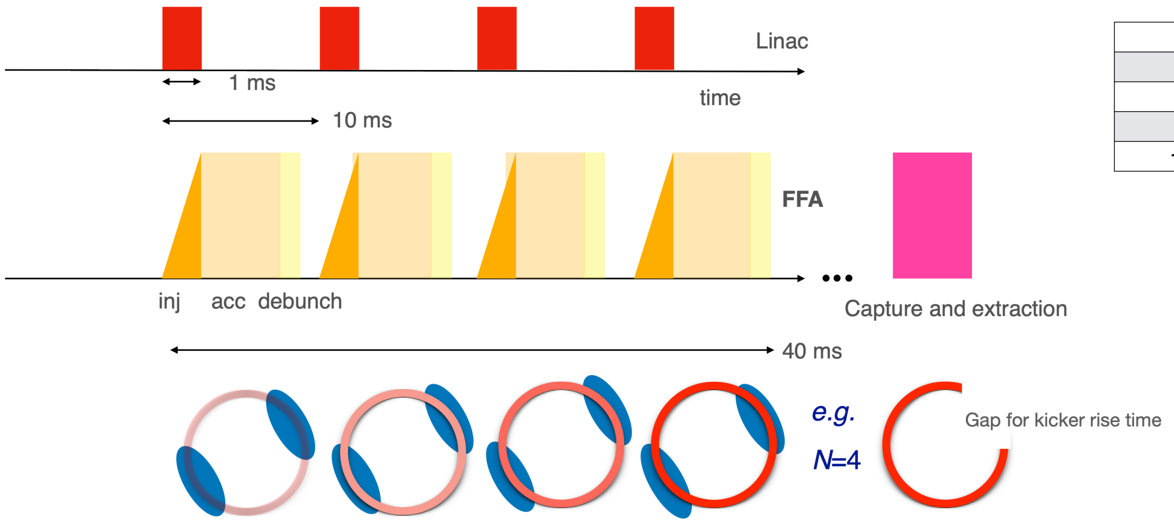
1000 turns after jump



1750 turns after jump

Beam stacking

- Successive beam pulses are stored in the ring. Coasting beams are stacked in terms of energy
- In order to minimize the final emittance, the process should be adiabatic.
- Stacked bunches are allowed to debunch and coast.
- In rings limited by transverse space charge at injection, beam stacking at higher energies allow a higher current beam to be accumulated and then extracted (but at lower repetition rate).



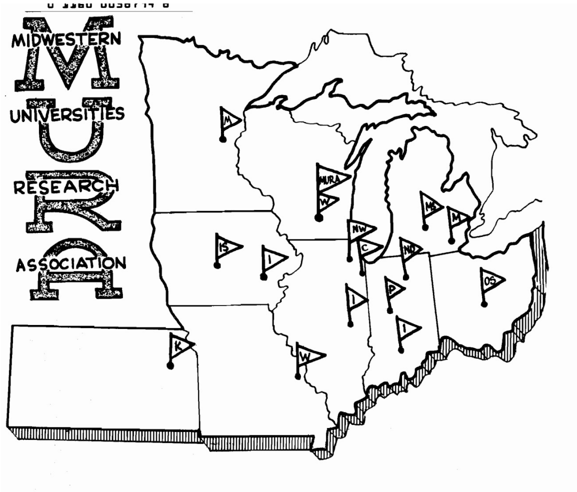
N	Rep rate
1	100 Hz
2	50 Hz
5	20 Hz
10	10 Hz

Phase displacement acceleration

- Accelerating bucket will cause, on average, a downward shift in the energy of the coasting beam it moves through (the opposite is true in the case of a decelerating bucket).
- In the adiabatic limit, the phase area moving downwards equals the bucket area moving up. This implies the following average shift in energy.

$$\Delta E_{shift} = \frac{\omega_0 A}{2\pi}$$

- The theory of phase displacement acceleration was developed at MURA in the 1950s. It was used at the CERN ISR to accelerate coasting proton beams from 26.6-31.1 GeV in 200 RF bucket sweeps.



Scattering & bucket lift

- Consider the statistical distribution of scattering of individual particles by RF modulation. First treatment by Symon & Sessler at MURA^. Further developed at the ISR*. The rms momentum spread caused by the passage of single bucket is given by#.

$$\sigma_{single} = \frac{16}{(2\pi)^{3/2}}\Gamma(\phi_s)\sqrt{\frac{eVE}{h|\eta|}}$$

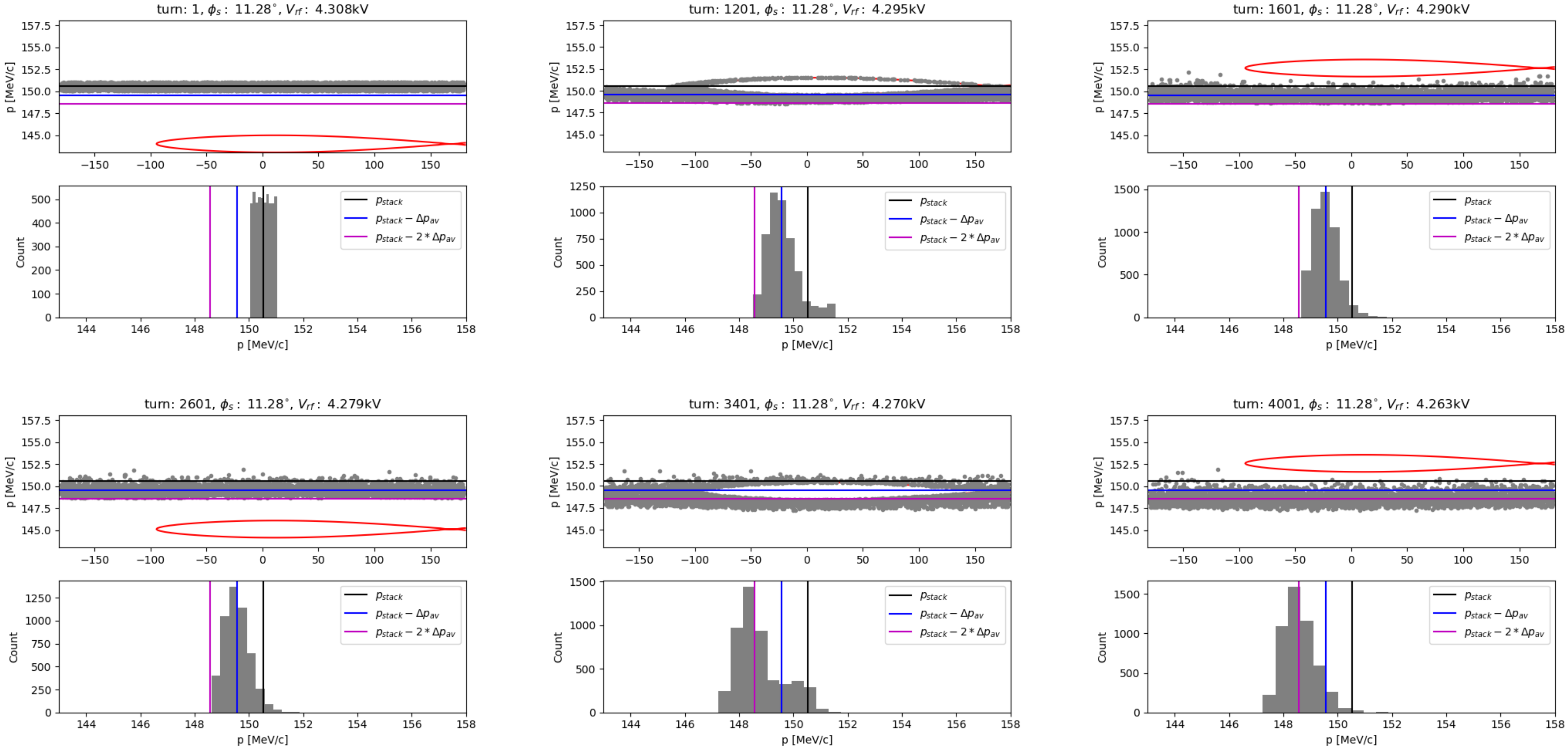
- Where $\Gamma = \sin\phi_s$. Note $\sigma_{single} = \Gamma A/(2\pi\,\alpha(\phi_s))$. For n stacked beams the total rms momentum spread is

$$\Sigma_n = \left(\sigma_0^2 + n\sigma_{single}^2\right)^{0.5}$$

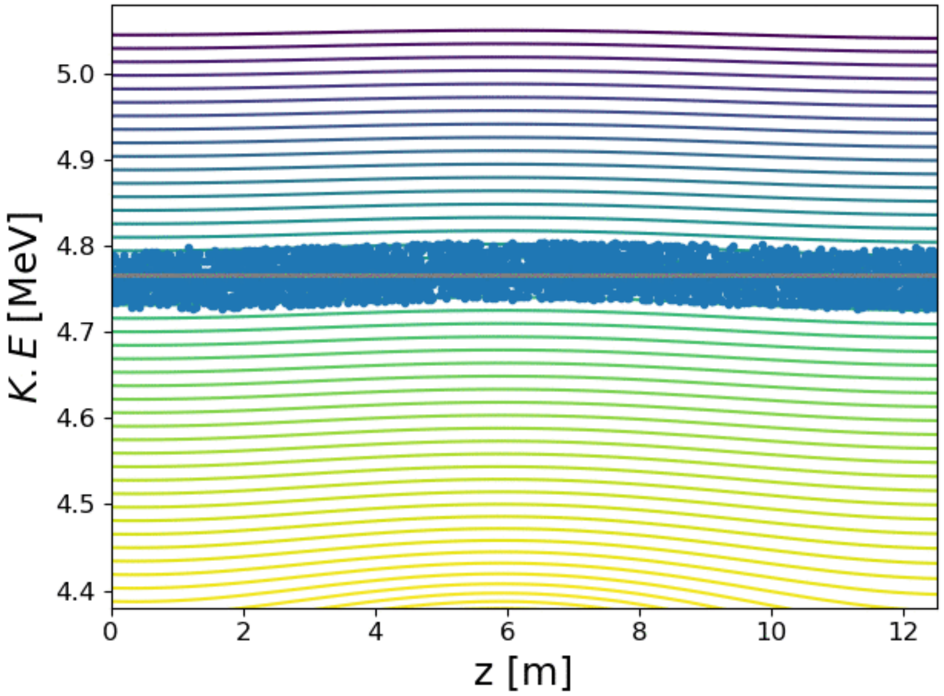
- If $f_{stack}/f_{rf} = m/n$ then the RF may affect the stacked beam. In the case of "bucket lift" some of the stacked beam is trapped and accelerated in a subharmonic bucket.

^ K. R. Symon and A. M. Sessler CERN Symposium on High-Energy Accelerators, 1956.
* E. W. Messerschmid, "Scattering of particles by phase displacement acceleration in storage rings", CERN/ISR-TH/73-31
# S. Watanabe et al, "Beam stacking experiments at the ion accumulation ring TARN", NIM A271 (1988) 359-374

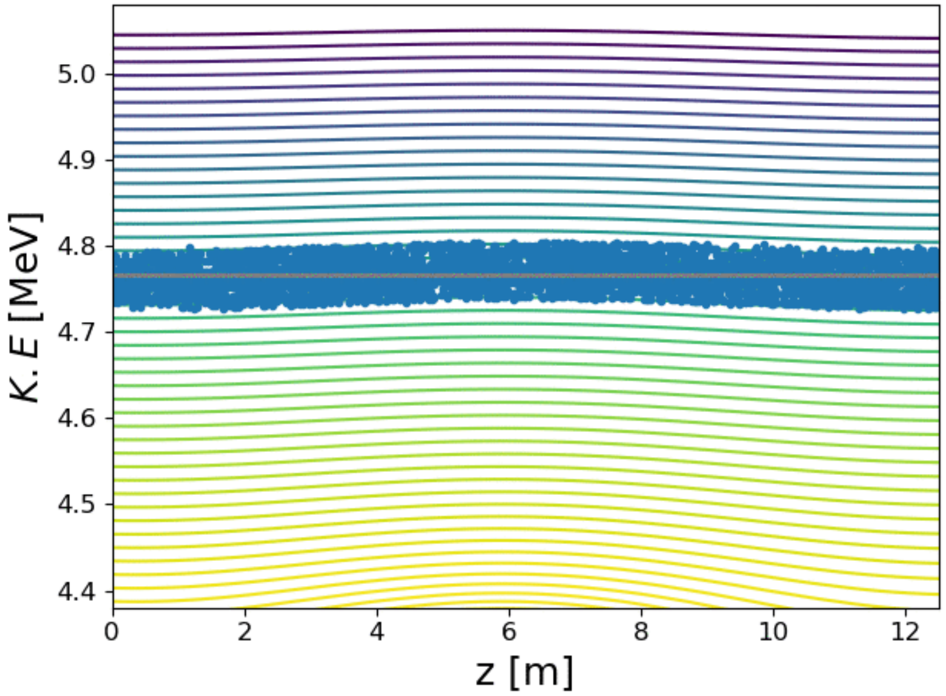
Phase displacement



Movies



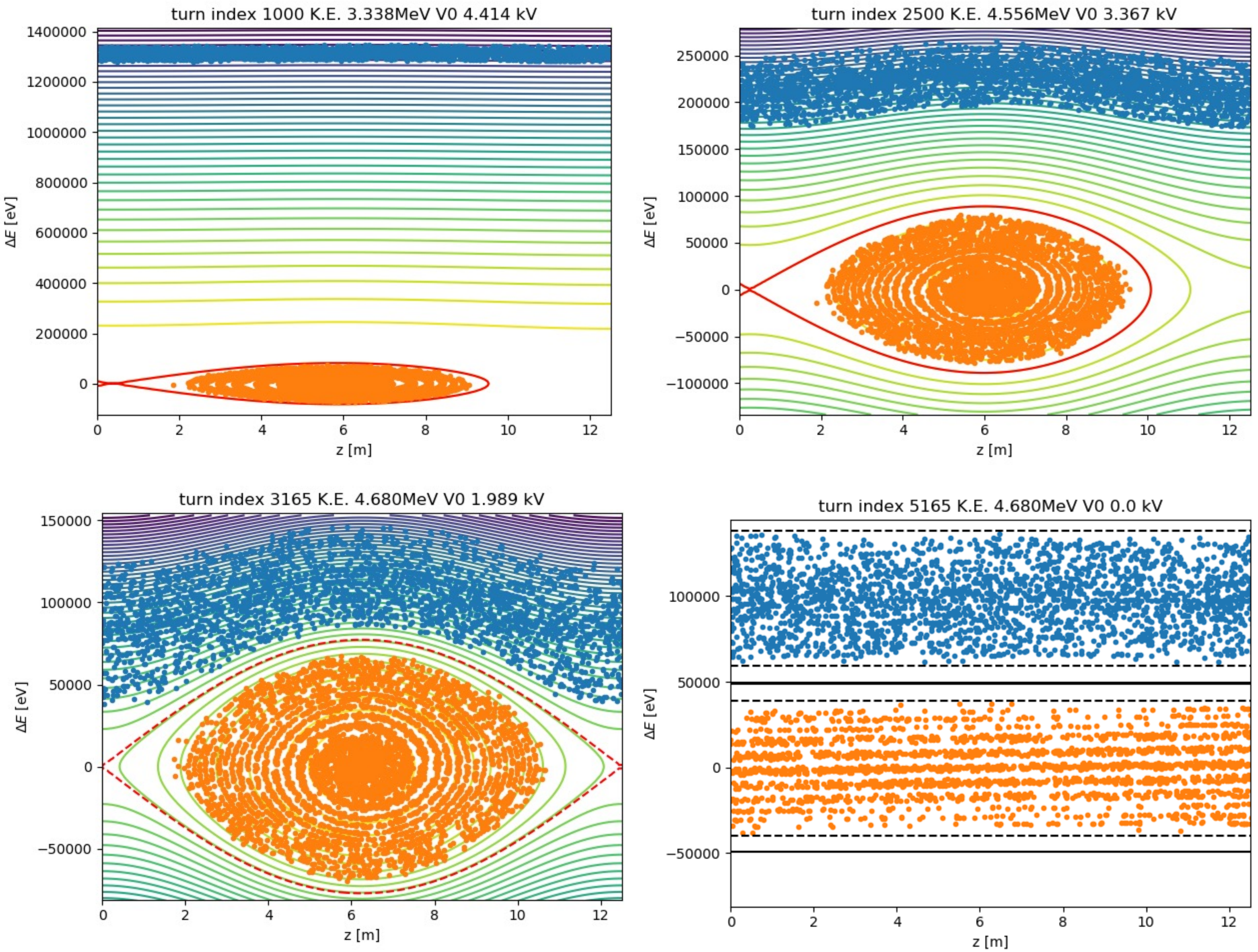
Φ_s=5°



Φ_s=10°

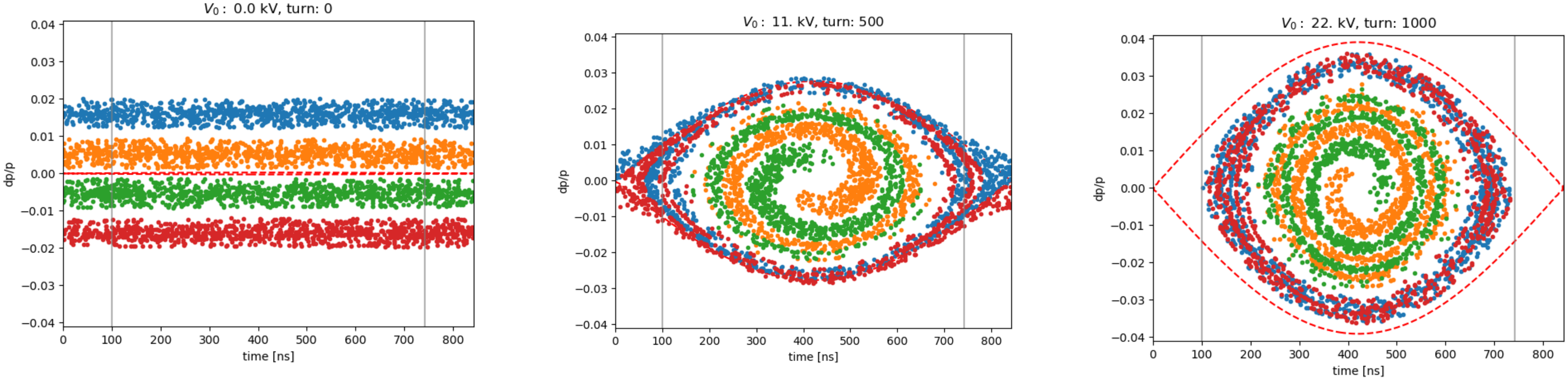
Distance between horizontal lines is $\Delta E_{shift} = \frac{\omega_0 A}{2\pi}$

Stacking process



- Assume a beam has already been stacked and is coasting (blue).
- Inject a second bunch (orange) and accelerate to just below the coasting beam.
- Ensure ϕ_s is zero at final energy.
- Debunch adiabatically.

Capture the stacked, coasting beam

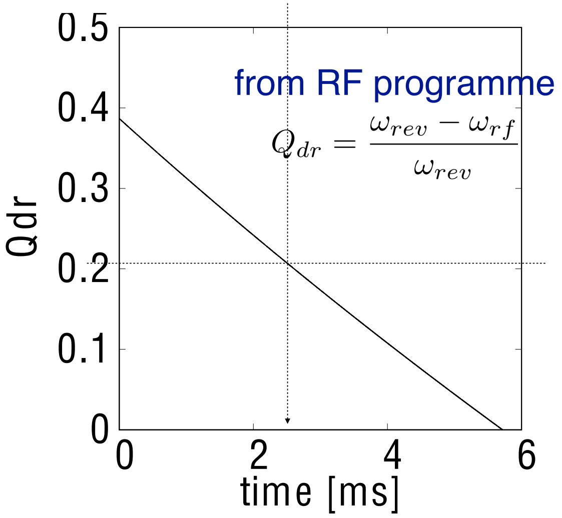


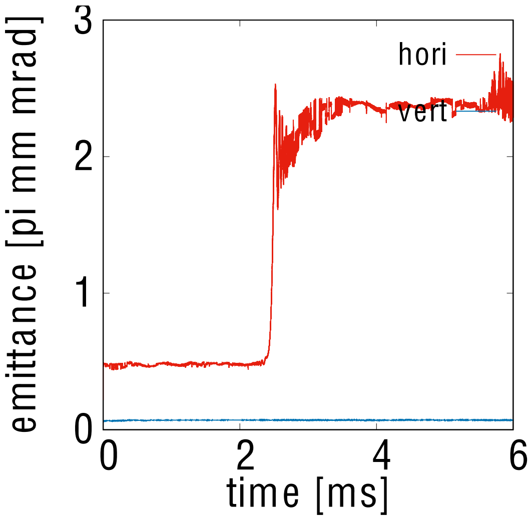
- Capture the beam by linearly increasing the RF voltage from zero to 22kV in 1000 turns.
- Beam free time created for extraction kicker.

RF Knockout

- Finite dispersion at the RF cavity results in an effective dipole kick.
- When the RF frequency and the betatron frequency satisfy a rational relationship, a resonance can occur.
- The effects was studied during the MURA years!

$$Q_x + mQ_{dr} = n \quad \text{where} \quad Q_{dr} = \frac{\omega_{rev} - \omega_{rf}}{\omega_{rev}}$$



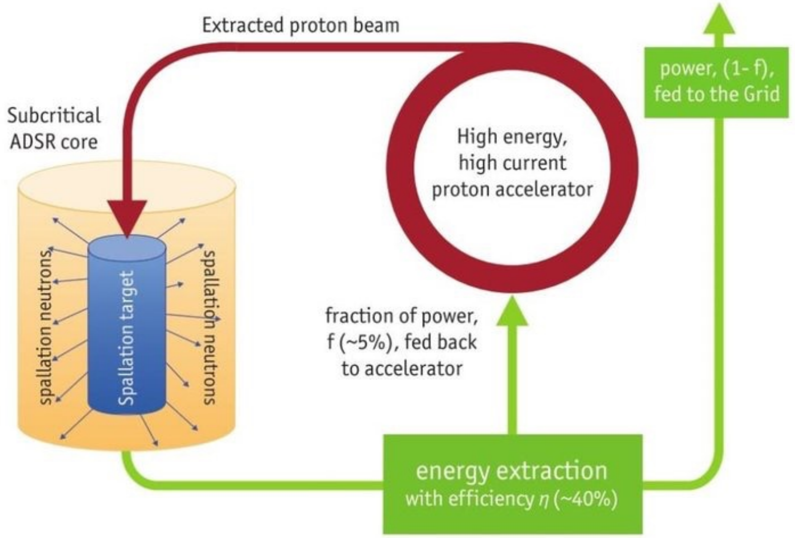


Simulation by S. Machida

Fixed RF Acceleration

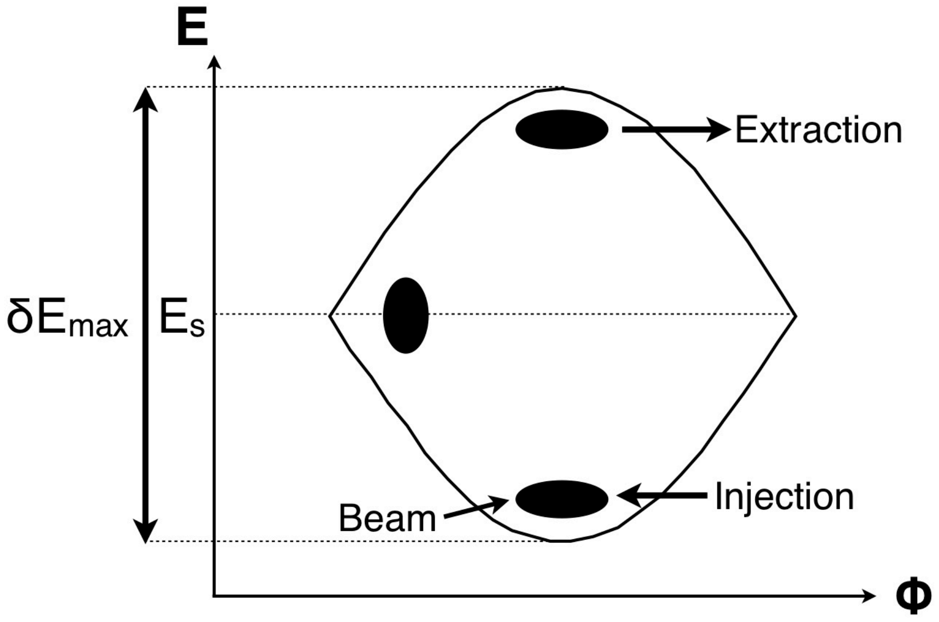
- Some applications require fixed RF parameters as well as DC magnetic fields.
- For example, to accelerate particles with a short lifetime (e.g. muons) there may be not enough time to change either the magnetic field or the RF voltage of frequency.
- Applications that require cw operation, for example an Accelerator-Driven Subcritical Reactor (ADSR) also need fixed RF acceleration.

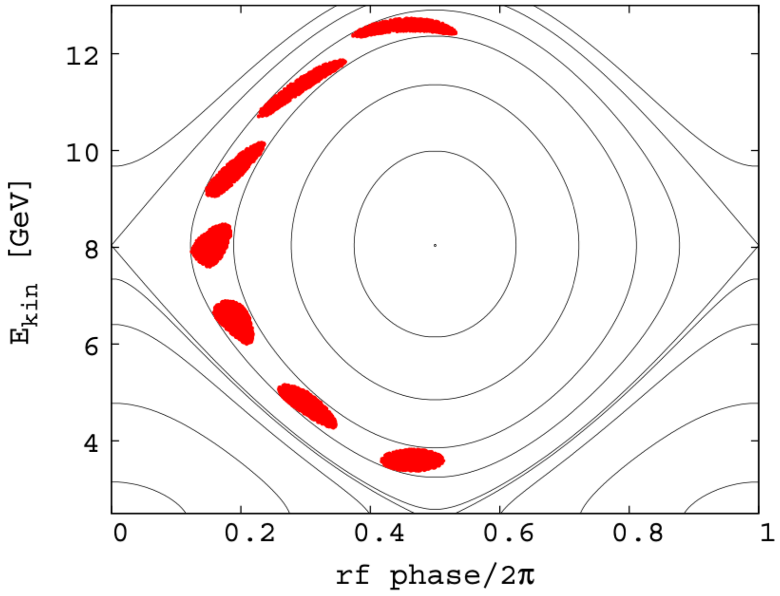
Methods
Acceleration in a stationary bucket
Acceleration in the serpentine channel
Harmonic number jump



Acceleration in a Stationary Bucket

- Inject beam into the bottom of bucket. Half a synchrotron oscillation later it will reach the extraction energy.
- The available energy gain is limited by the bucket height (proportional to $\sqrt{V/E}$).





6 turn acceleration in 3.6 – 12.6 GeV scaling FFA with 1.8GV RF voltage

Credit: E. Yamakawa, T. Planche

Hamiltonian in a scaling hFFA

- When we want to look at the longitudinal phase space spanning the transition energy, the linear approximation in dp/p no longer suffices.
- Instead write the Hamiltonian without referring to a reference energy.
- Note how the radius and hence the revolution frequency scales with momentum in a scaling hFFA

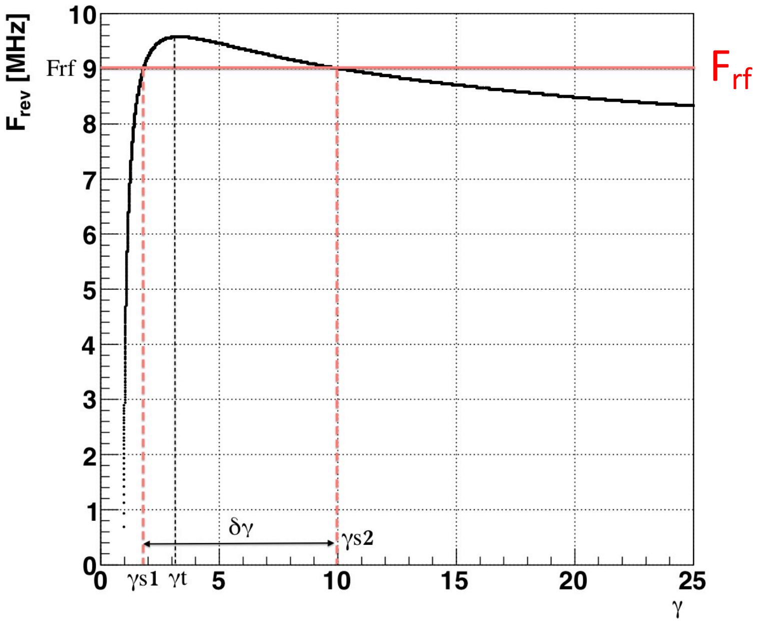
$$r = r_0 \left(\frac{p}{p_0}\right)^{\frac{1}{k+1}} \qquad \alpha = \frac{dr/r}{dp/p} = \frac{1}{k+1} \qquad \frac{T}{T_s} = \left(\frac{r}{r_s}\right) / \frac{p/E}{p_s/E_s} = \left(\frac{p}{p_s}\right)^{\alpha} / \frac{p/E}{p_s/E_s}$$

- The phase equation of motion follows from above leading to the Hamiltonian

$$H(E,\phi;\theta) = h\left[\frac{1}{\alpha+1}\frac{p^{\alpha+1}}{E_s p_s^{\alpha-1}} - E\right] + \frac{eV}{2\pi}\cos\phi$$

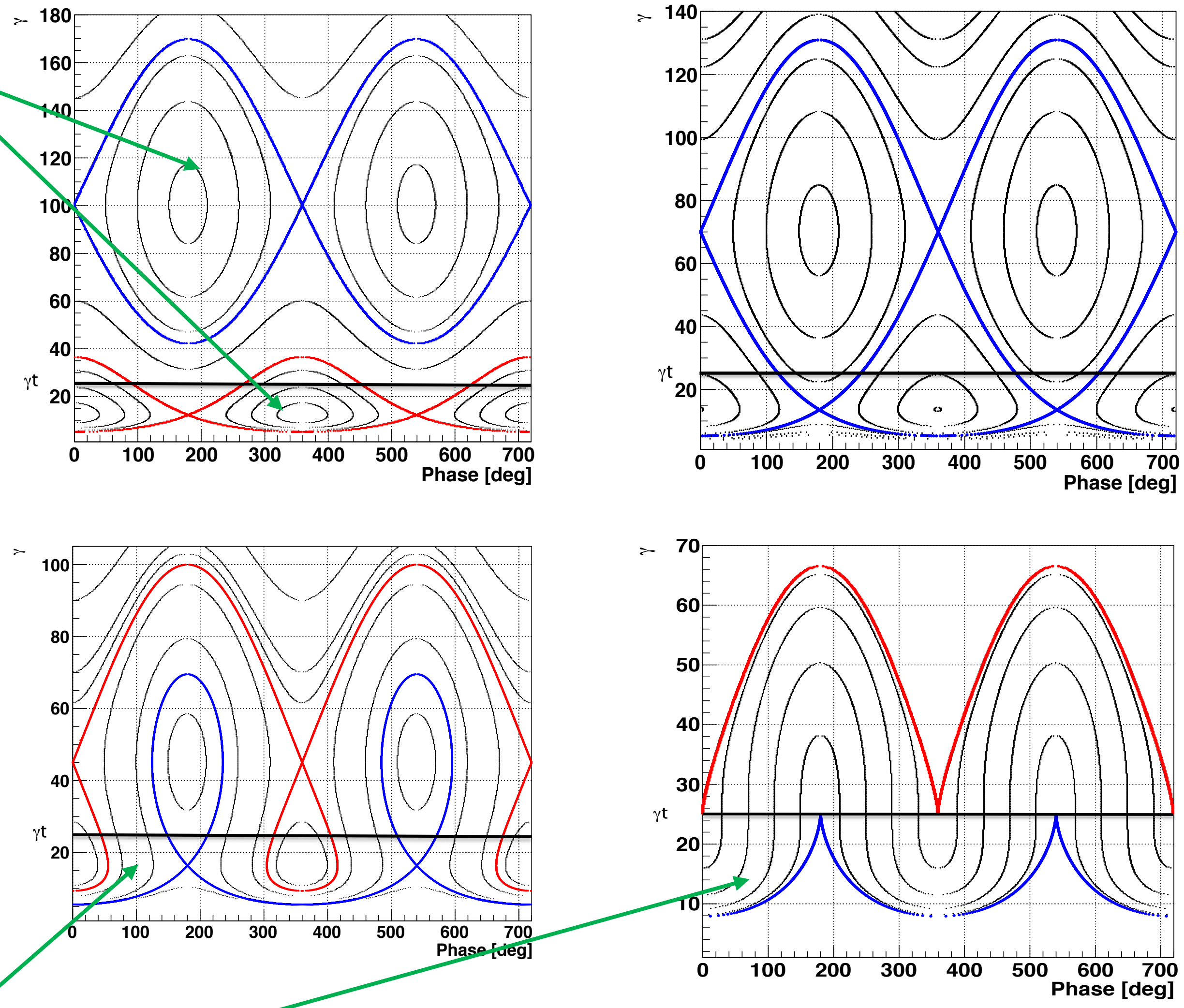
Credit: E. Yamakawa thesis

Longitudinal phase space close to transition



Set RF frequency to determine how close synchronous energies are to transition

Separate buckets



Serpentine channel

$\gamma_{s1} = \gamma_{s2} = \gamma_t$

Credit: E. Yamakawa thesis

Serpentine acceleration in non-relativistic regime

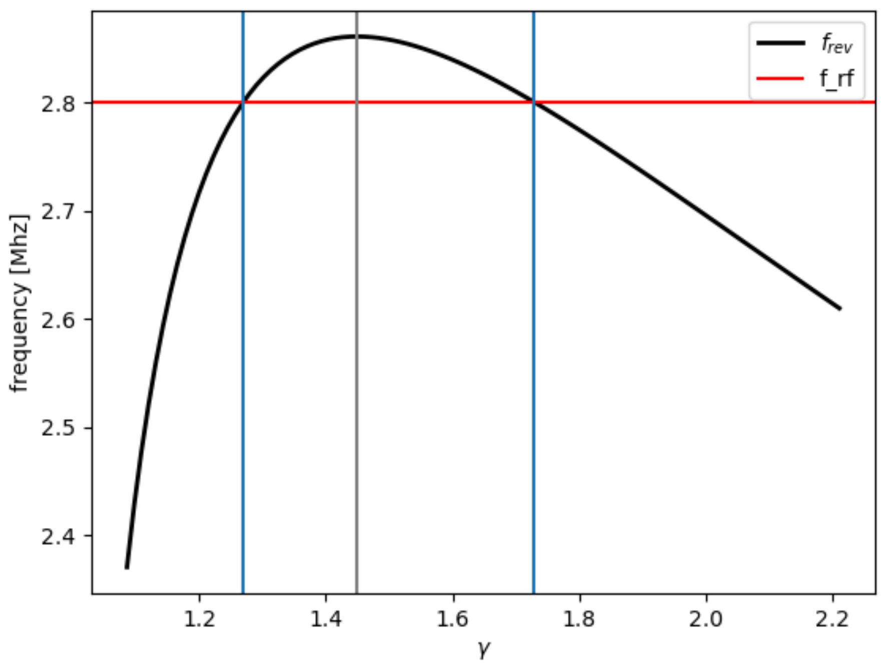
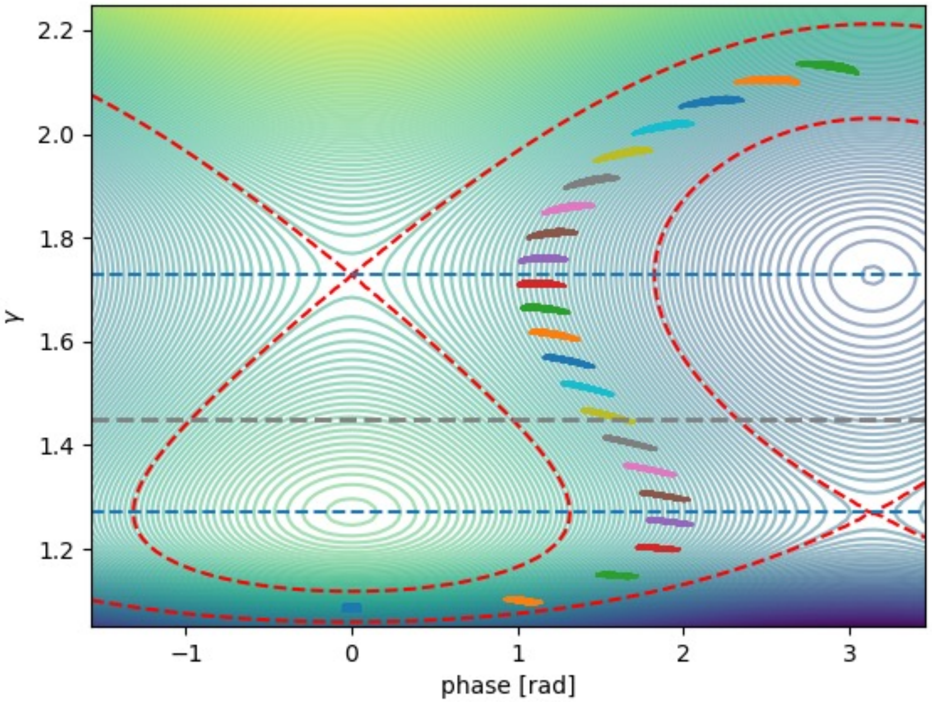


Table 3.1: Longitudinal parameters

k-value	1.1
Transition gamma	1.45
Equivalent mean radius at $\gamma = 1.21$ [m]	10
Stationary gamma below transition	1.27
Rf voltage [MV/turn]	50 (h=1)
Rf frequency [MHz]	2.8(h=1)



Inject a bunch at 80MeV. Track 22 turns to the top of the serpentine channel. Mean energy at extraction is 1 GeV.

Credit: E. Yamakawa thesis

Serpentine acceleration in linear non-scaling FFA

- Linear non-scaling FFAs consist of dipoles and quadrupoles only (or just shifted quadrupoles).
- Lattice is designed so that the revolution time is quadratic over the momentum range.
- Serpentine acceleration was experimentally demonstrated in EMMA.

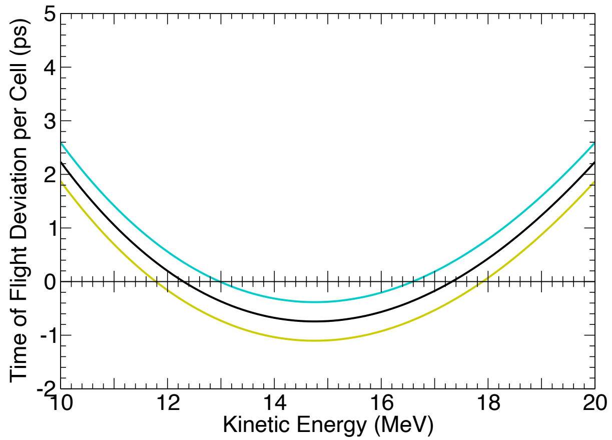


FIGURE 5. Time of flight as a function of energy for three different RF frequencies. Zero time of flight deviation is when the particle on the closed orbit at that energy is synchronized with the RF. The actual time of flight doesn't change between curves, only the RF frequency changes.

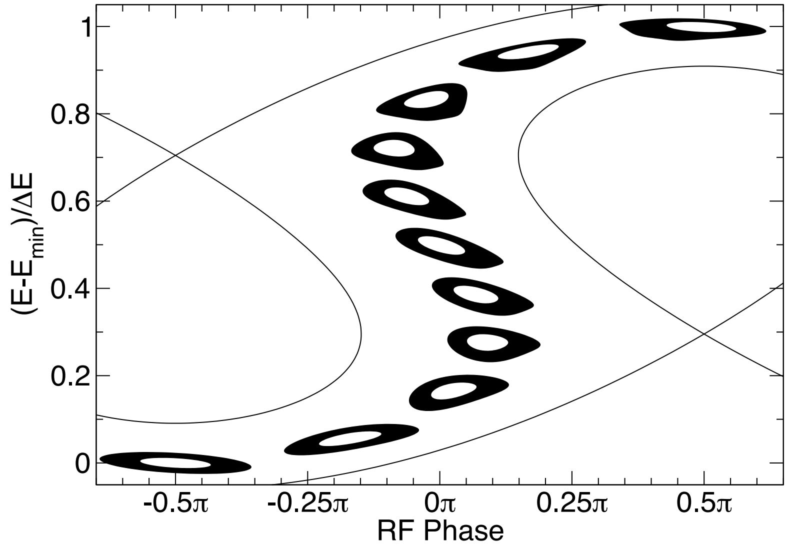
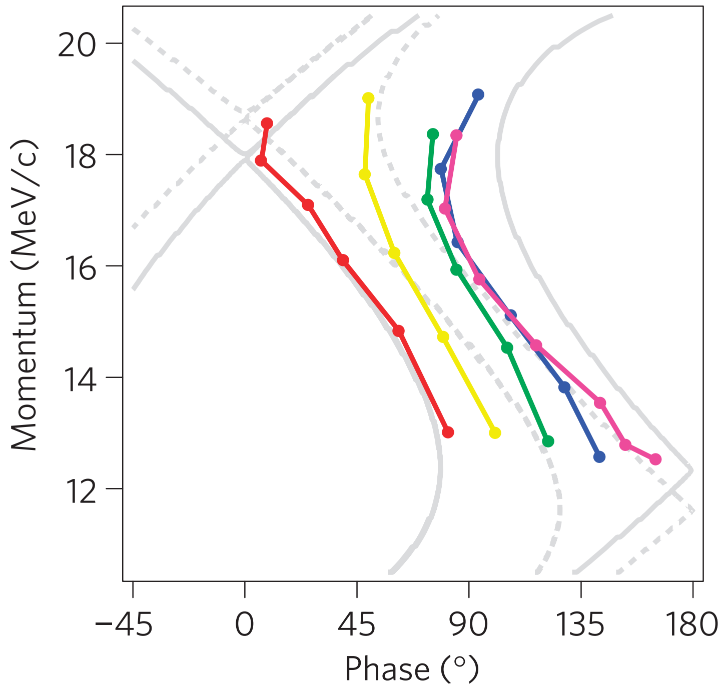
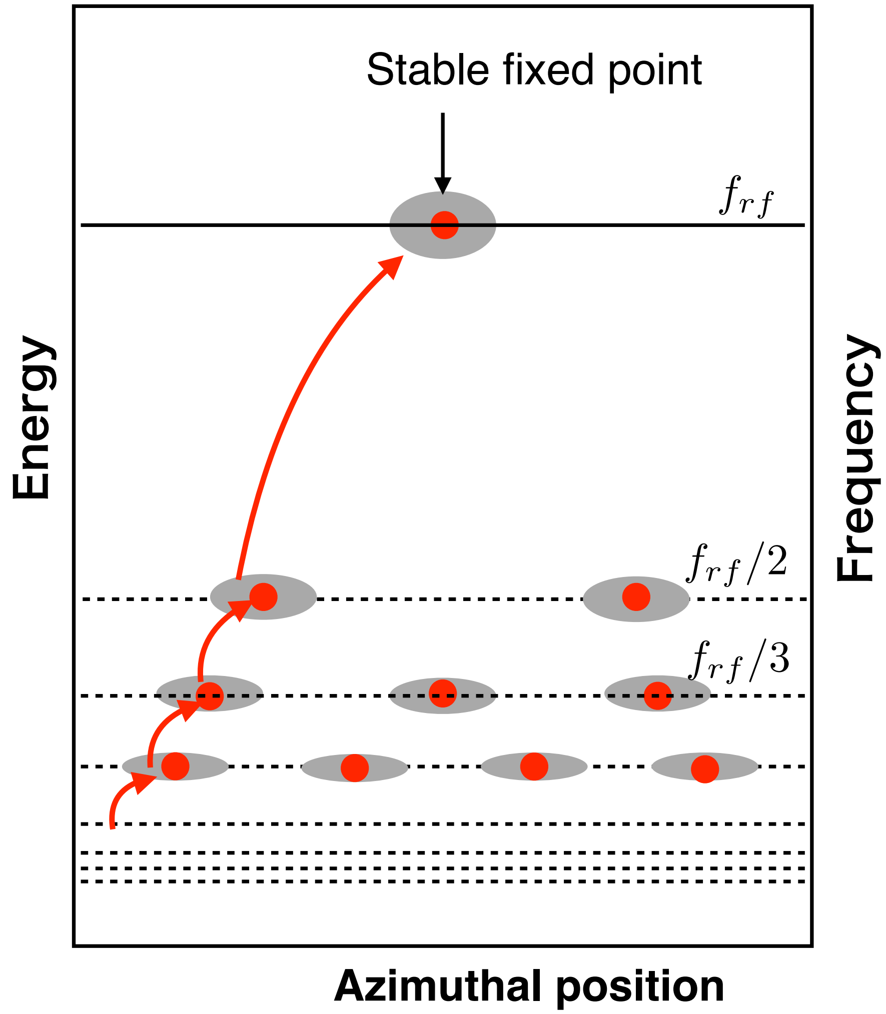


FIGURE 2. A bunch in longitudinal phase space for serpentine acceleration.



Credit: JS Berg

Harmonic Number Jump



Let us recall

$$f_{rf} = h \cdot f(E_s)$$

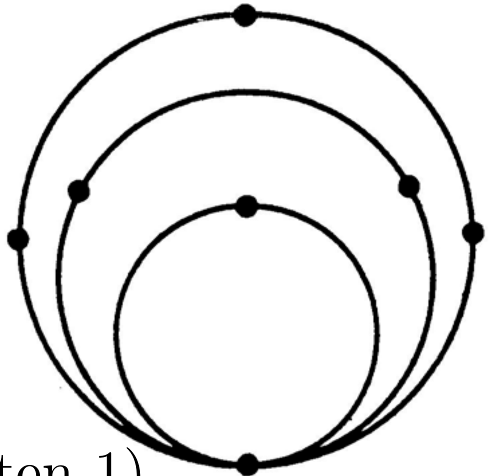
Synchronous energy E_s is not unique, because h can be any integer.

There are many stable fixed points, corresponding to h=1,2,3,4,…

Acceleration across different h's is possible,

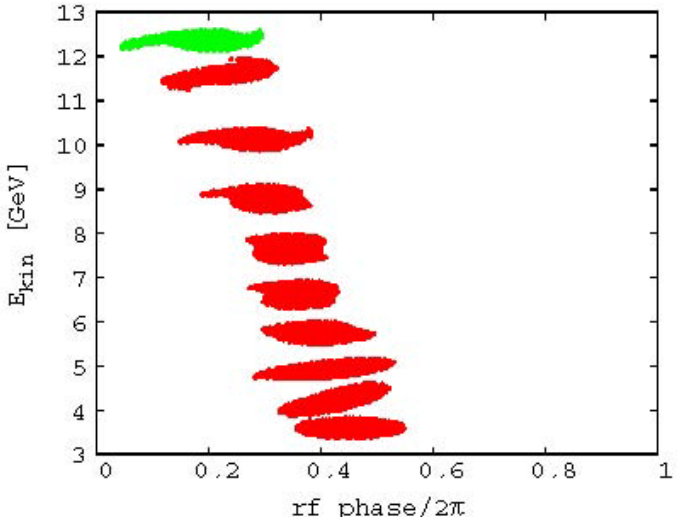
if **voltage is high enough** and **slippage is tuned well**.

- Harmonic number *h* is increased on each turn to keep particles synchronised with fixed-frequency RF

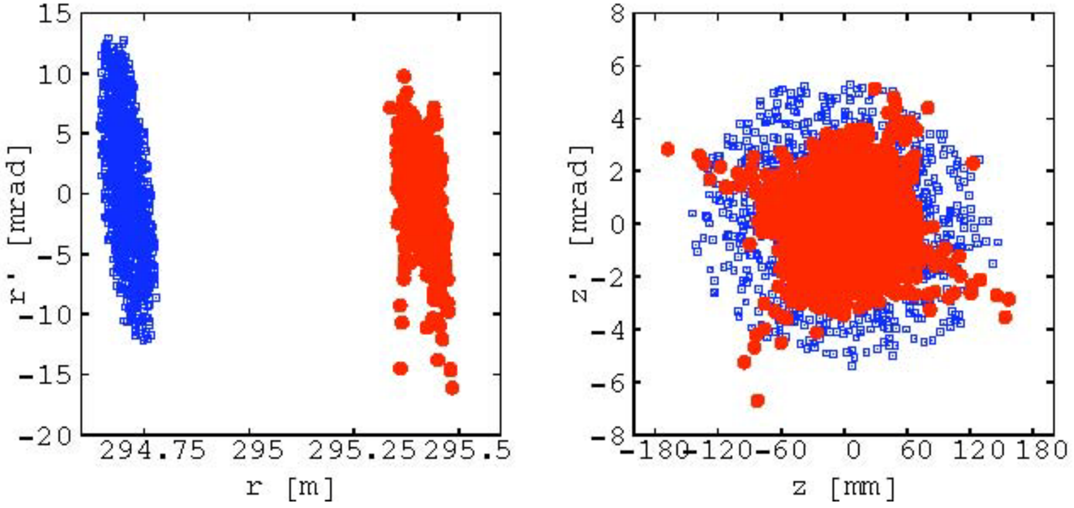


$$\tau_{\text{orbit}} = h\tau_{\text{rf}}$$

$$\left.\begin{array}{l} \tau_{\text{orbit}} \to \tau_{\text{orbit}} + \Delta\tau \\ h \to h + \Delta h \end{array}\right\} \implies \Delta\tau = \Delta h \tau_{\text{rf}} \quad (\Delta h \text{ often } 1)$$



Acceleration from 3-12 GeV in 8.5 turns



Initial (blue) and final (red) transverse emittances

C. Prior