Analytic Tools for FFA Design

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Aims of the session

- Motivate why analytic models are useful
- Derive step-by-step an analytic model of an FFA system
  - Closed orbits
  - Optics
- Demonstrate the performance of this analytic model with some examples comparing analytic results and numerical simulations
- Interactive demo
- Design your own lattice from an analytic starting point!

Why use an analytic model?

Analytic model	Numerical model
Offers insight into behaviour of system	‘black-box’ approach
Allows rapid exploration of parameter space	Many CPU-hours needed to fully explore large, multidimensional parameter spaces
Needs to be developed specifically for an individual system	Can be constructed as a generalised tool with application to many different systems
May require approximations to find exact solutions	Can numerically solve problems without need for approximations

Modelling an accelerator

- Understanding the behaviour of a machine requires two things

  1. Knowledge of where the beam is – the closed orbit

  2. Knowledge of how a particle will behave around the closed orbit

- Life is easy when you work with synchrotrons and linacs*
  - Single closed orbit determined by magnets
  - Simple reference orbit
  - Simple transfer matrix model about reference orbit
- How can we do the same for an FFA?
  - What's our reference orbit? Let's look at the simplest example…

*this opinion may not be shared by people who actually do work with synchrotrons and linacs.

hFFA FODO lattices

- Two magnets per cell
  - One normal bend (F)
  - One reverse bend (D)
- Magnet centroids evenly spaced azimuthally
  - Lattice is symmetric about middle of F and middle of D
  - Orbit must be perpendicular to the radius with respect to machine centre at the middle of the F and middle of the D-magnets

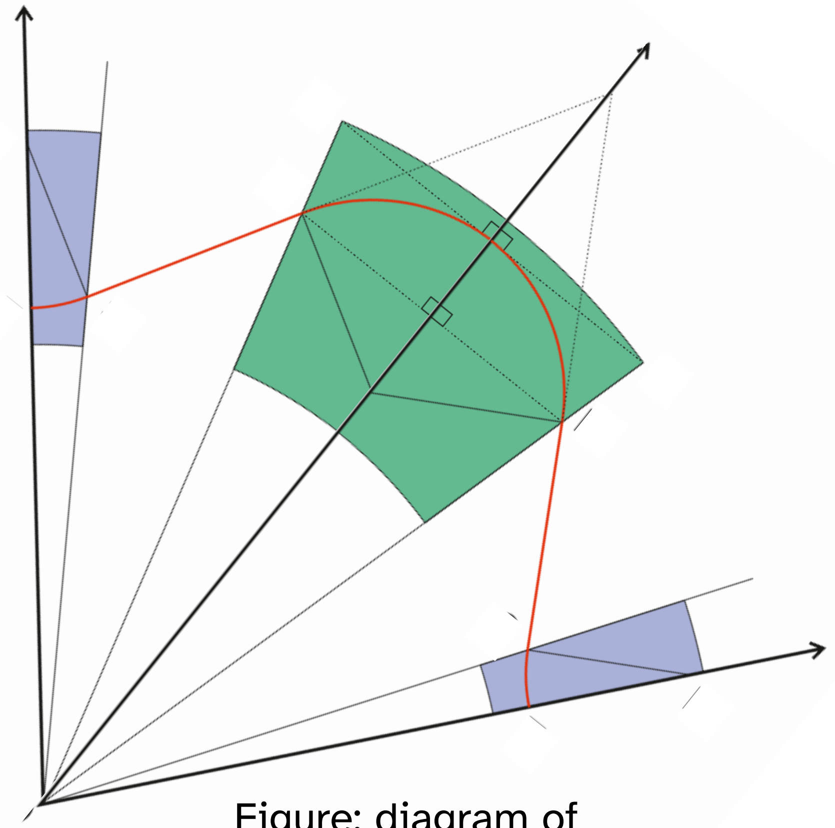


Figure: diagram of hFFA FODO cell

F-magnet is shaded in green; D-magnet is shaded in blue.

hFFA FODO lattices

- Approximations:
  - Field is zero outside magnets
  - Field is constant inside magnets
  - Orbit is planar

→ Orbit consists of plane circular arcs and straight sections in the horizontal plane

- What do we need to know about this lattice’s geometry to understand its properties?

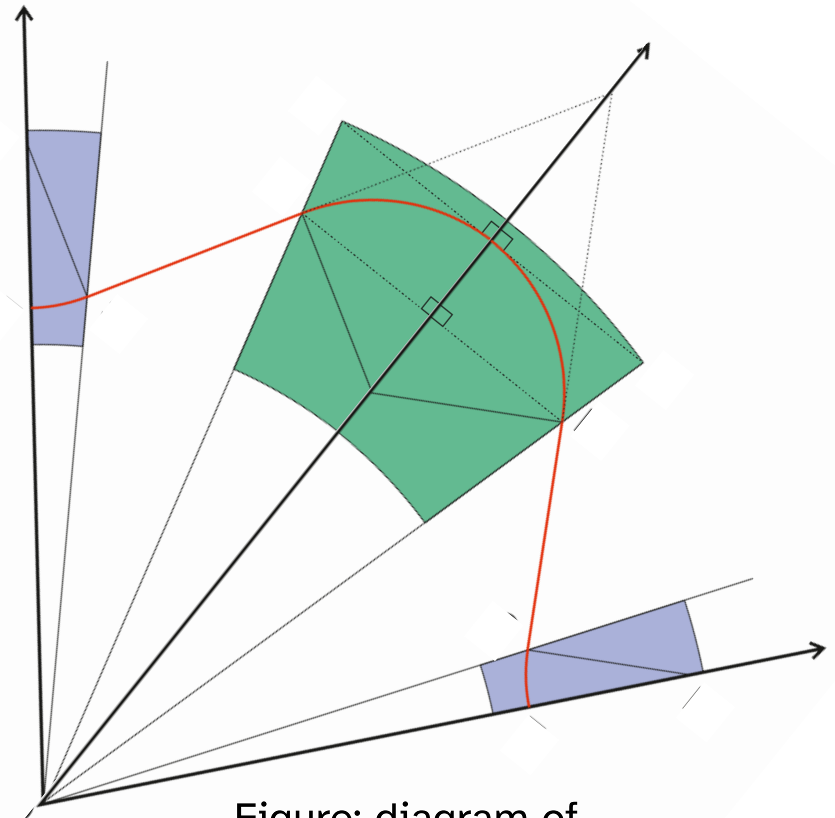
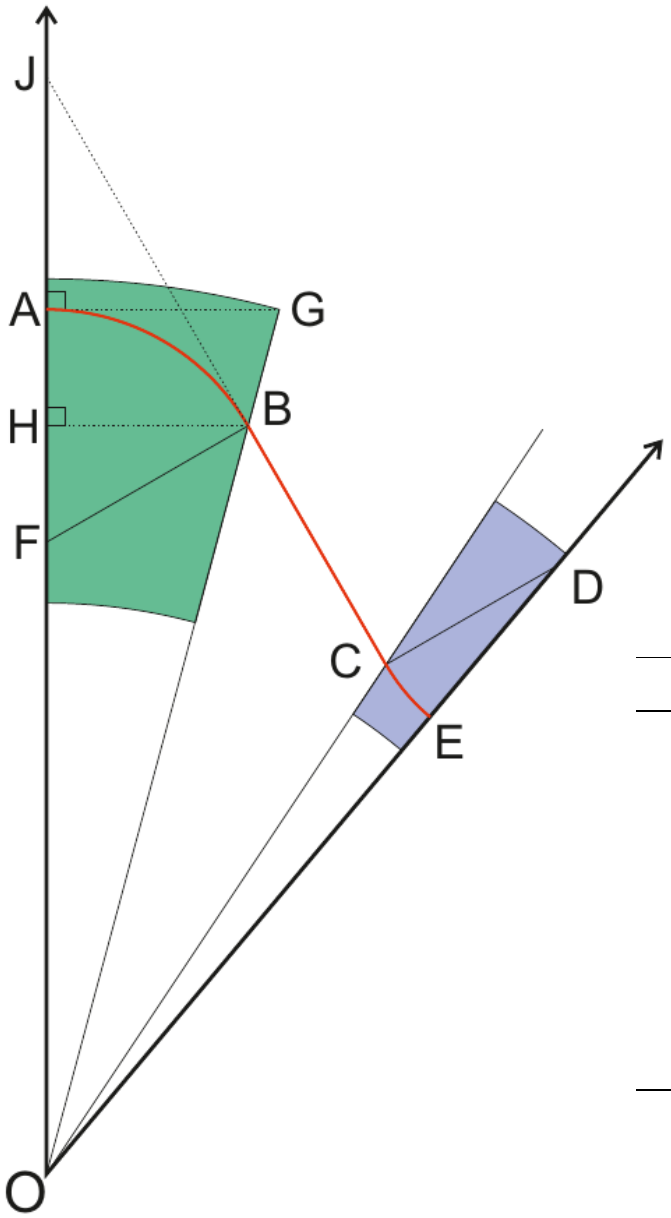


Figure: diagram of hFFA FODO cell

F-magnet is shaded in green; D-magnet is shaded in blue.

- Length of drift space
  L_S
- Bending radius of magnets
  ρ_F, ρ_D
- Length travelled in magnets
  $= \rho\theta$
- Orbit radius at
  - Middle of magnets r_0, r_3
  - Edge of magnets r_1, r_2



Variable	Definition
θ_D	$\angle CDE$
ρ_F	$\overline{AF} = \overline{BF}$
ρ_D	$\overline{CD} = \overline{ED}$
r_1	$\overline{OB}$
r_2	$\overline{OC}$
r_3	$\overline{OE}$
L_s	$\overline{BC}$

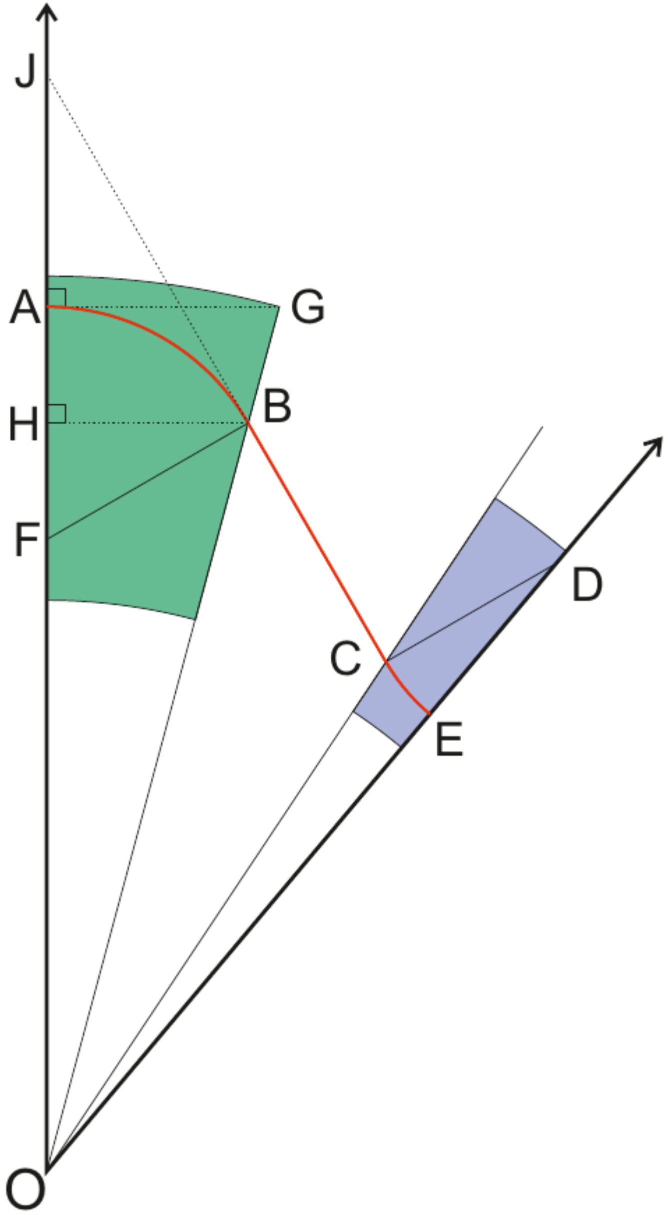
- What do we already know about the lattice?
  - Or what can we specify?

- Number of cells N
- Angular size of magnets

  β_F, β_D
- Reference radius of orbit

  r_0
- *Bending angle in F-magnet*

  θ_F



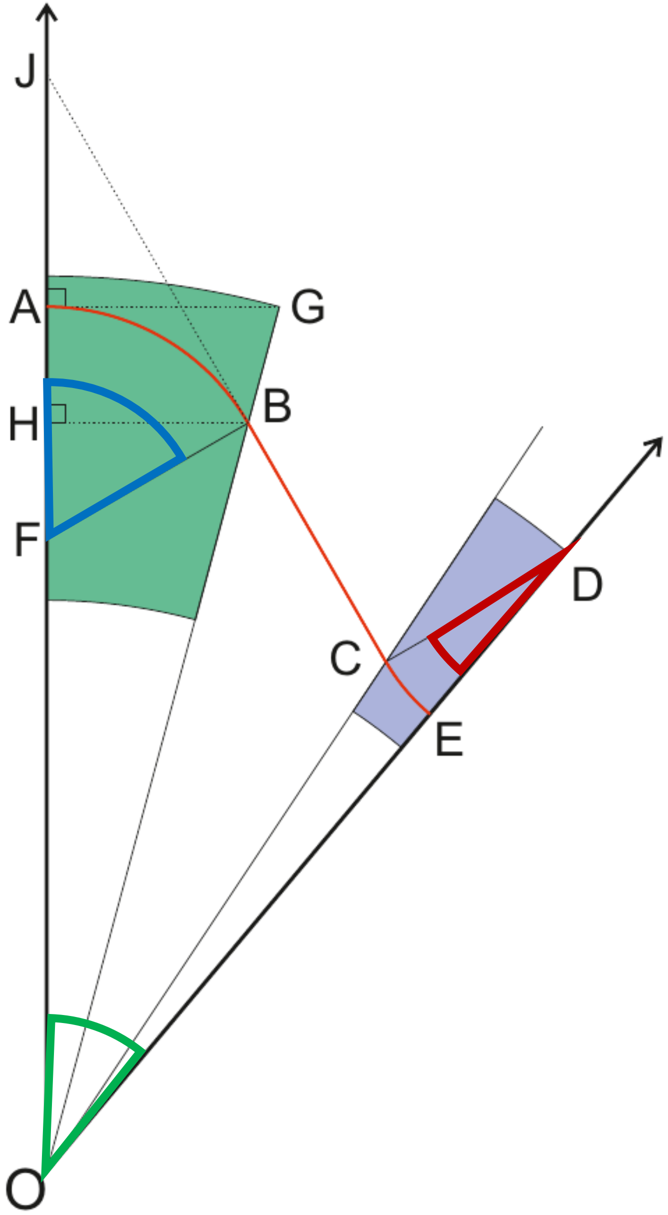
Parameter	Definition
β_F	$\angle AOB$
β_D	$\angle COE$
$\frac{\pi}{N}$	$\angle AOE$
r_0	$\overline{OA}$
θ_F	$\angle AFB$

- How can we work out the unknown variables from our input parameters?

- Identify geometric constraints

Example:

$$\theta_F - \theta_D = \pi/N \qquad (1)$$



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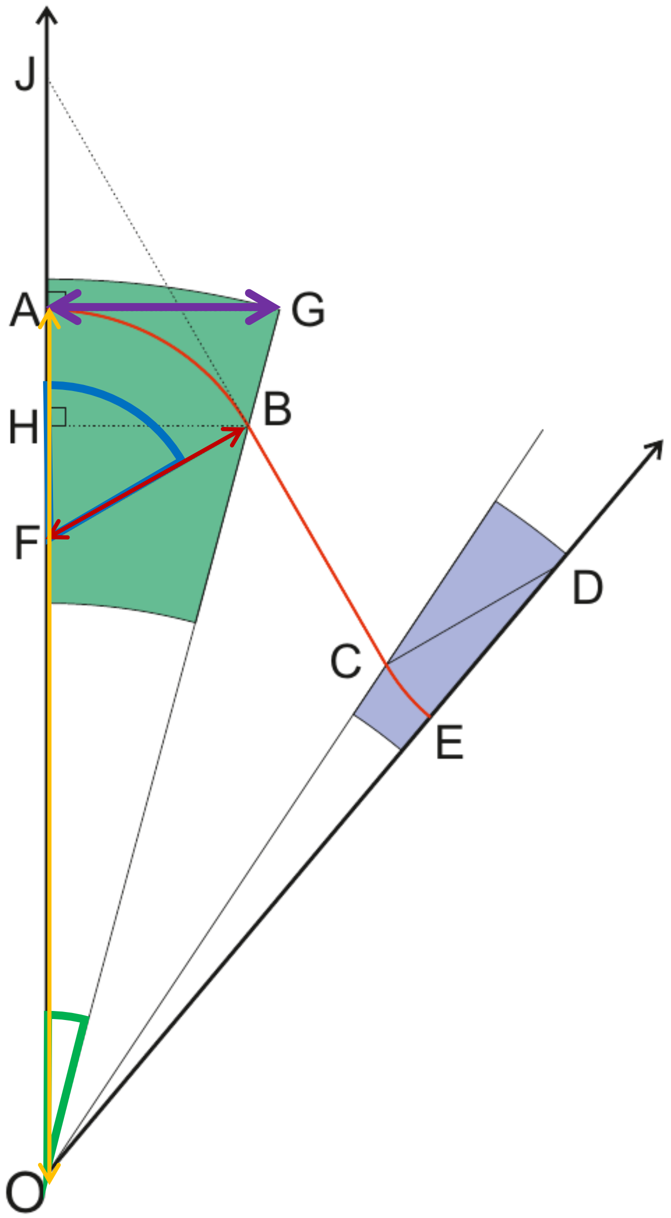
Constraint 2:

$$r_0 \tan \beta_F = \overline{AG}$$

$$\rho_F (\sin \theta_F + [1 - \cos \theta_F] \tan \beta_F) = \overline{AG} \quad (2)$$



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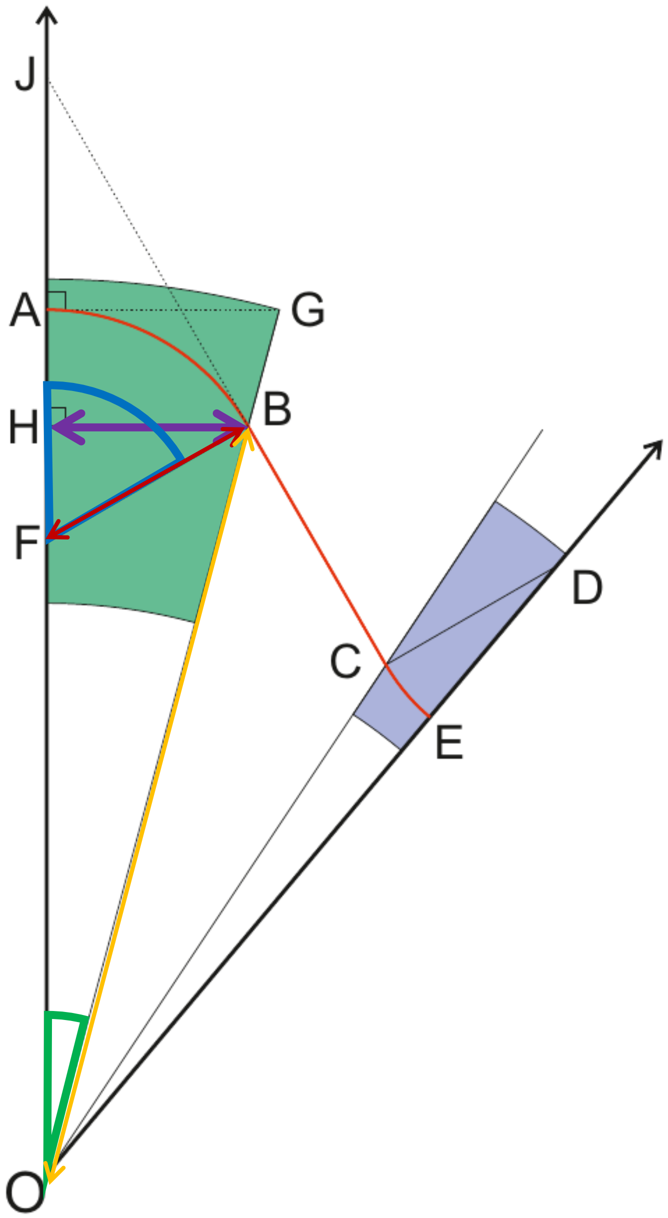
Constraint 3:

$$\rho_F \sin\theta_F = \overline{HB} = r_1 \sin\beta_F \qquad (3)$$

Constraint 4:

$$\rho_D \sin\theta_D = r_2 \sin\beta_D \qquad (4)$$



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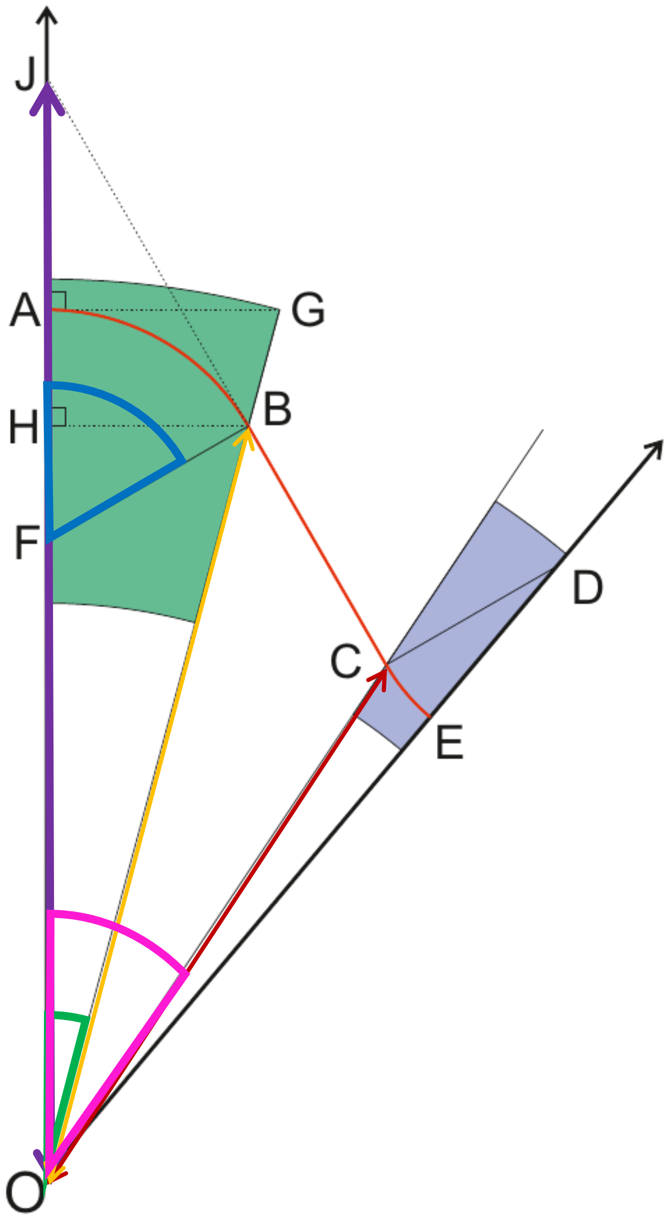
Constraint 5:

$$r_2\left(\cos\left[\frac{\pi}{N}-\beta_D\right]+\tan\theta_F\sin\left[\frac{\pi}{N}-\beta_D\right]\right)$$
$$=\overline{OJ}$$
$$=r_1(\cos\beta_F+\tan\theta_F\sin\beta_F) \quad (5)$$



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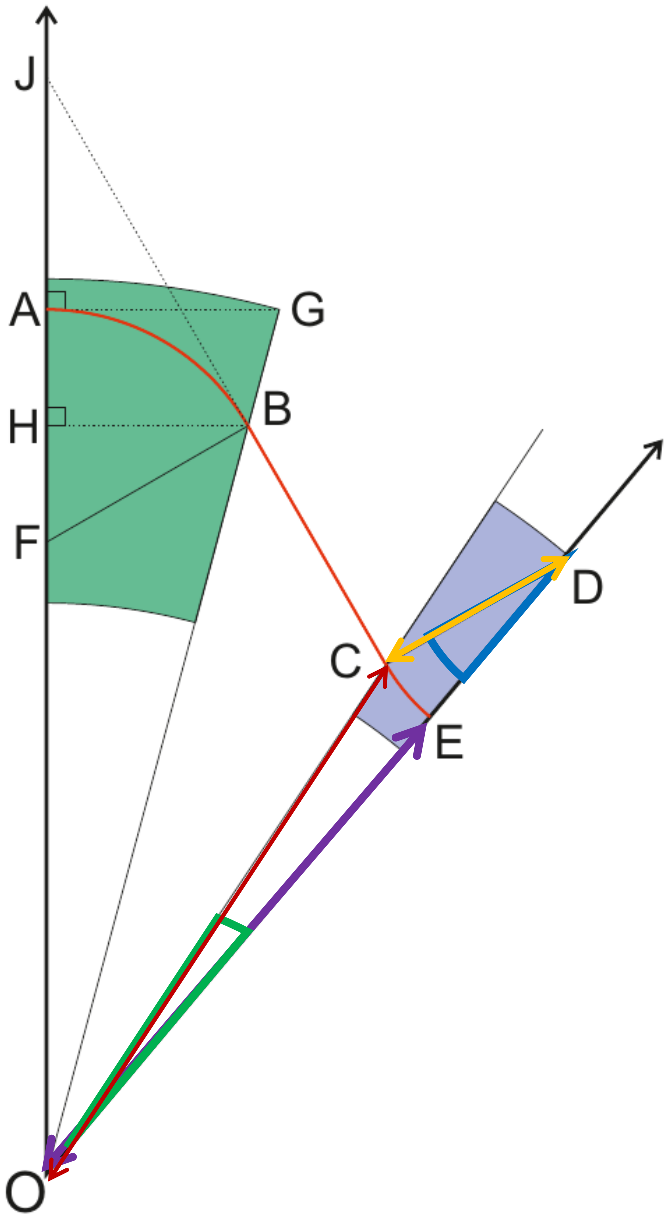
Constraint 6:

$$r_3 = r_2 \cos \beta_D - \rho_D (1 - \cos \theta_D) \qquad (6)$$



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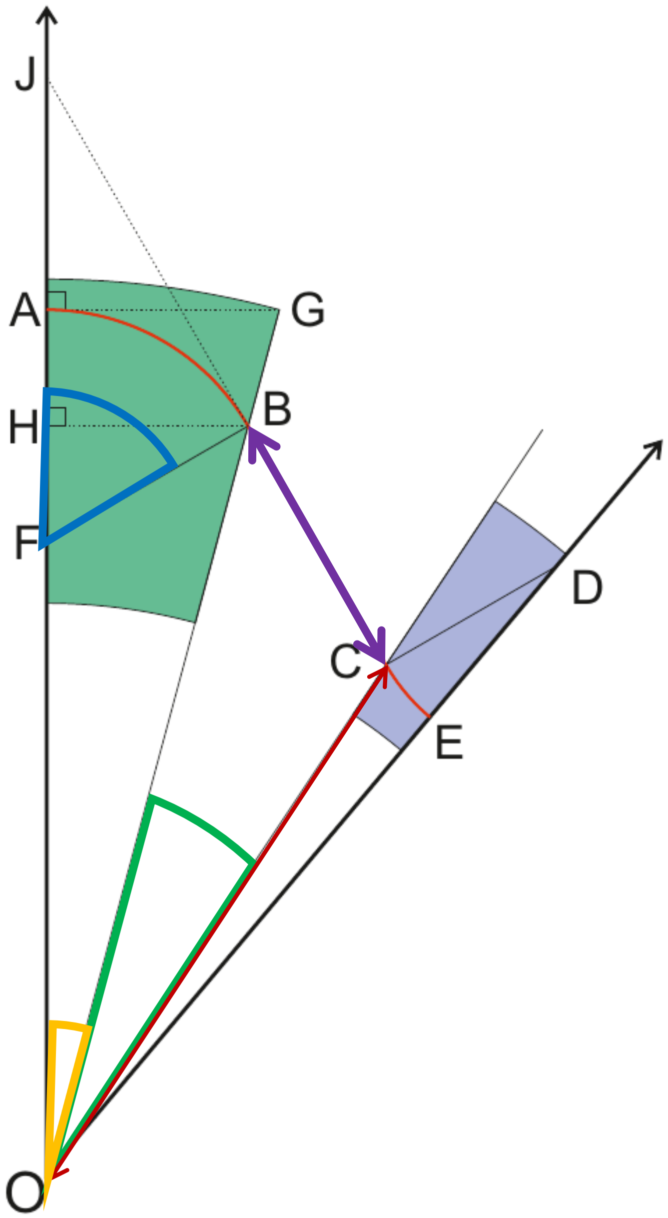
Drift length:

$$L_S = \frac{r_2 \sin\left(\frac{\pi}{N} - \beta_D - \beta_F\right)}{\cos(\theta_F - \beta_F)} \quad (7)$$



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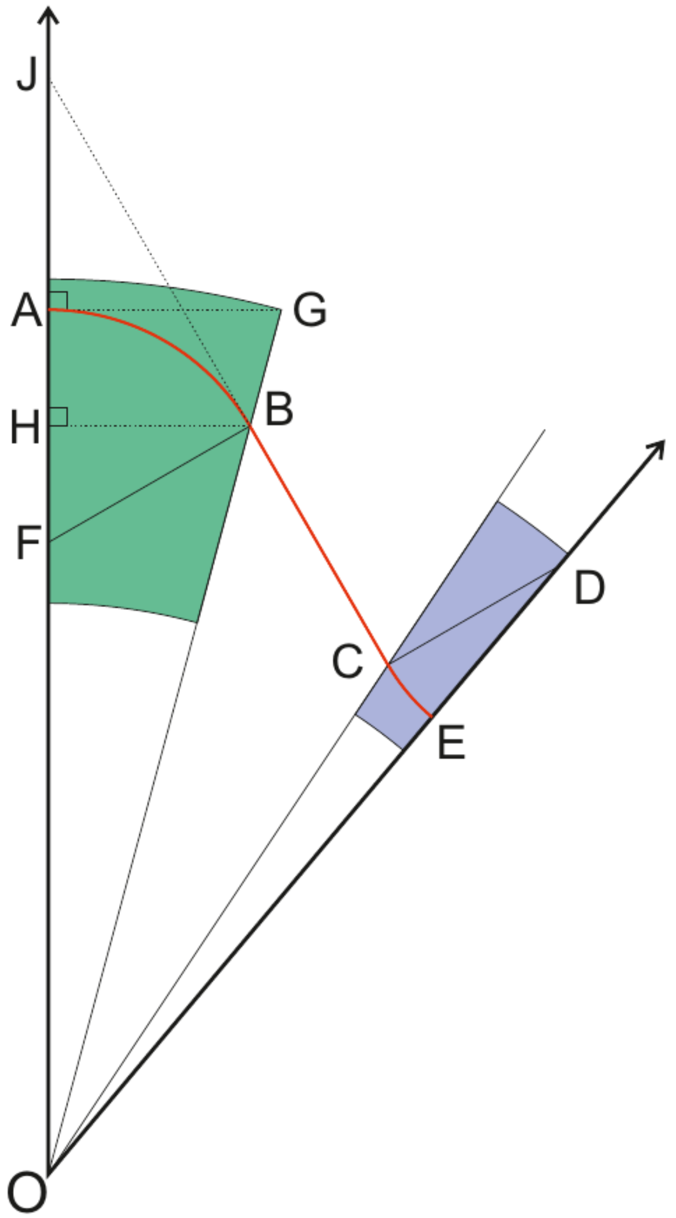
$$\theta_F - \theta_D = \frac{\pi}{N}$$
$$r_0 \tan\beta_F = \rho_F(\sin\theta_F + [1 - \cos\theta_F]\tan\beta_F)$$
$$\rho_F \sin\theta_F = \overline{HB} = r_1 \sin\beta_F$$
$$r_2\left(\cos\left[\frac{\pi}{N} - \beta_D\right] + \tan\theta_F \sin\left[\frac{\pi}{N} - \beta_D\right]\right) = r_1(\cos\beta_F + \tan\theta_F \sin\beta_F)$$
$$\rho_D \sin\theta_D = r_2 \sin\beta_D$$
$$r_3 = r_2\cos\beta_D - \rho_D(1 - \cos\theta_D)$$
$$L_S = \frac{r_2 \sin\left(\frac{\pi}{N} - \beta_D - \beta_F\right)}{\cos(\theta_F - \beta_F)}$$



FODO closed orbit analytic model

- How well does this model predict the closed orbit of a given ring?

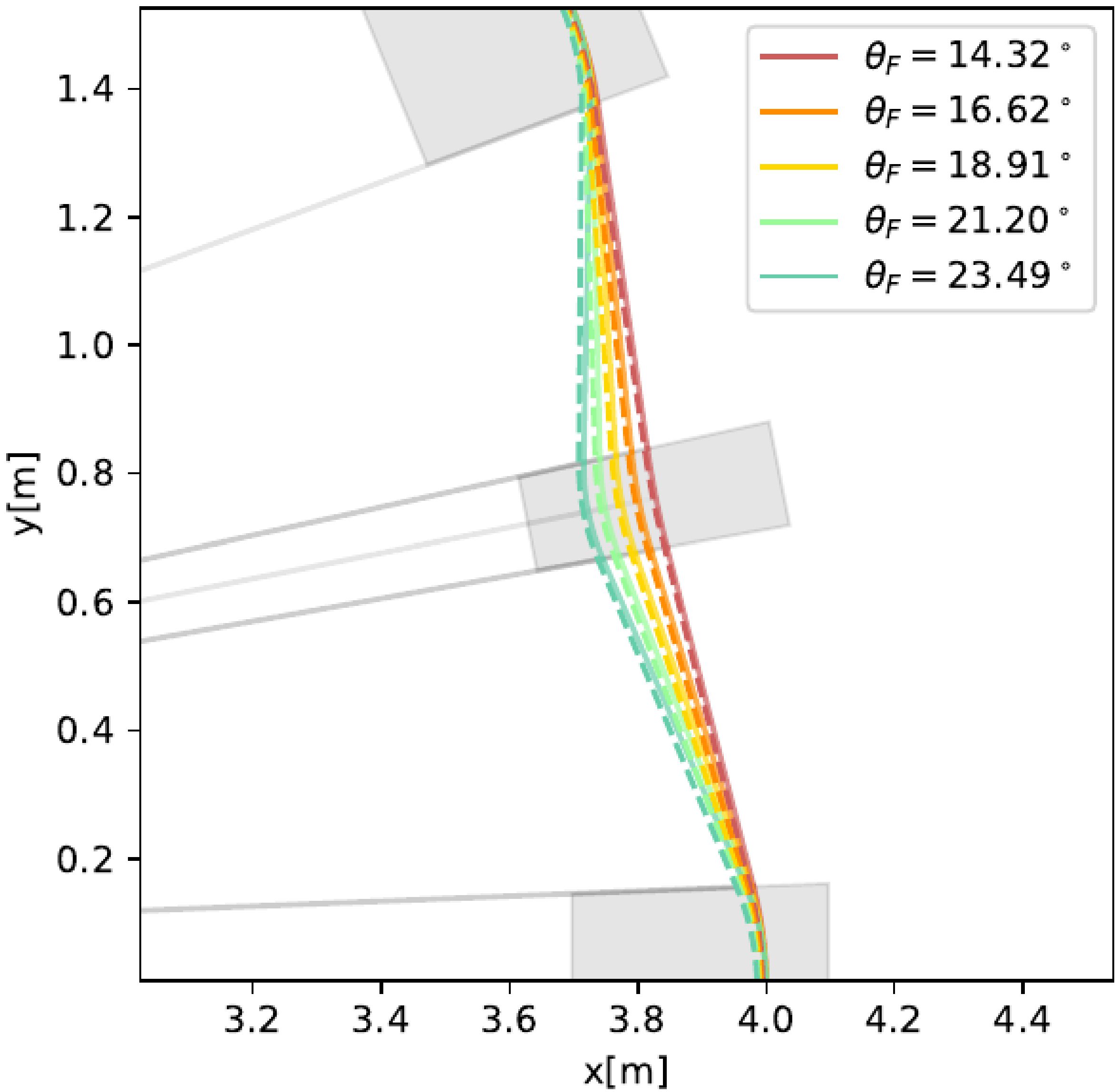
Let's see!

- Assemble example lattice using analytic model

- Construct equivalent in numerical code
  - Compute magnet strengths from radii of curvature

    (You have to account for radial scaling of fields!)

Parameter	Value
N	16
β_F	$2.25°$
β_D	$1.15°$
r_0	4m
θ_F	$45°$
k	8.0095
c_1	0.07m



- Solid line – analytic
- Dashed line – numerical

- Simplified analytic model is really good at predicting position and properties of closed orbit
  - Even when we consider transverse field gradients and fringe fields in our numerical simulations!

Linear optics

- Now we understand the closed orbit, we can start to look at the optics and focussing structure. Let's examine the behaviour inside the magnet body for a scaling hFFA:

Magnetic field in a scaling FFA:

$$B_z(x) = B_0 \left(\frac{r_1}{r_0}\right)^k \left[1 + \frac{kx}{r_1} + \mathcal{O}\left(x^2\right) + \ldots\right]$$

X = Horizontal transverse
Y = Horizontal longitudinal
Z = Vertical
r_0 = reference radius of magnet where $B = B_0$

(scaling law expanded in terms of $x = (r - r_1)$)

[where r_1 is the reference orbit position]

$$B_z(x) = B_0 \left(\frac{r_1}{r_0}\right)^k \left[1 + \frac{kx}{r_1} + \mathcal{O}(x^2) + ...\right]$$

- Find a horizontal field that satisfies Maxwell eqns

$$\nabla \times B = 0$$

$$B_x(z) = B_0 \left(\frac{r_1}{r_0}\right)^k \left[\frac{kz}{r_1} + \mathcal{O}(z^2) + ...\right],$$

- Field (locally) looks like quadrupole + dipole!

- Find vector potential and substitute into curved Frenet-Serret Hamiltonian

- Expand Hamiltonian

$$\mathcal{H} = \frac{1}{2}p_x^2 + \frac{1}{2}p_z^2 + \left[\frac{q}{P_0}B_0\left(\frac{r_1}{r_0}\right)^k - \frac{1}{\rho}\right]x + \frac{1}{2}\frac{q}{P_0}B_0\left(\frac{r_1}{r_0}\right)^k\left(\frac{1}{\rho} + \frac{k}{r_1}\right)x^2 - \frac{k}{2r_1}\frac{q}{P_0}B_0\left(\frac{r_1}{r_0}\right)^k z^2.$$

- Linearise!
  - Note that a linear map needs to map the zero vector to itself – we need to eliminate first order terms in x by choosing $\rho = \frac{P_0}{qB_0}\left(\frac{r_0}{r_1}\right)^k$

$$\mathcal{H} = \frac{1}{2}p_x^2 + \frac{1}{2}p_z^2 + \frac{1}{2\rho}\left(\frac{1}{\rho} + \frac{k}{r_1}\right)x^2 - \frac{k}{2\rho r_1}z^2,$$

- Apply Hamilton's equations to get equations of motion

$$\frac{dx}{ds} = p_x,$$
$$\frac{dp_x}{ds} = -\frac{1}{\rho}\left(\frac{1}{\rho} + \frac{k}{r_1}\right)x,$$
$$\frac{dz}{ds} = p_z,$$
$$\frac{dp_z}{ds} = \frac{1}{\rho}\frac{k}{r_1}z.$$

Solve 2nd order linear differential equation

General solution

$$x\left(s\right) = C_x e^{i\omega_x s} + D_x e^{-i\omega_x s},$$
$$p_x\left(s\right) = i\omega_x C_x e^{\omega_x s} - i\omega_x D_x e^{-\omega_x s}$$

Apply boundary conditions to find C and D

Repeat for vertical plane

Transfer matrix!

4x4 transfer matrix for scaling hFFA magnet

$$\mathcal{M} = \begin{pmatrix} \cos\omega_x L & \frac{\sin\omega_x L}{\omega_x} & 0 & 0 \\ -\omega_x \sin\omega_x L & \cos\omega_x L & 0 & 0 \\ 0 & 0 & \cos\omega_z L & \frac{\sin\omega_z L}{\omega_z} \\ 0 & 0 & -\omega_z \sin\omega_z L & \cos\omega_z L \end{pmatrix}$$

$$\omega_x = \sqrt{\frac{1}{\rho}\left(\frac{1}{\rho} + \frac{k}{r_1}\right)}, \omega_z = \sqrt{-\frac{1}{\rho}\frac{k}{r_1}},$$

- Analogous to combined function quadrupole/dipole
- Sign of ρ determines F or D magnet
- How do we set these coefficients?

- r_1, ρ and L are computed from the lattice geometry we worked out earlier!
- k is a design parameter

N.b. r_1 has a different definition here compared to when we computed the geometry!

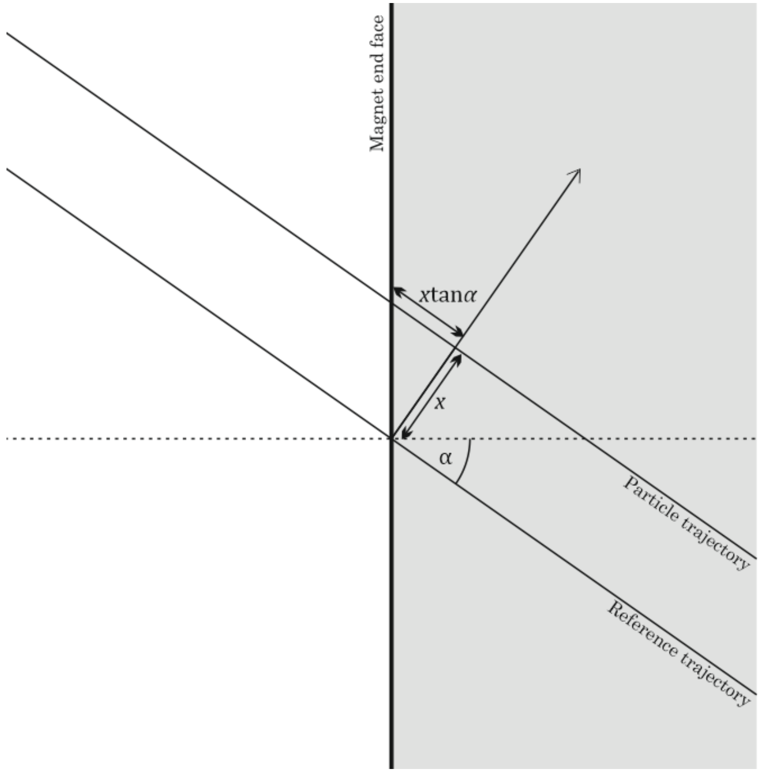
- Now we have a transfer matrix for the magnet body of a scaling FFA magnet…

- What else do we need to include?

Edge focussing:

Particle sees a small ‘wedge’ of extra magnet depending on horizontal position

$$\mathcal{M}_{\text{edge}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{\tan\alpha}{\rho} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$



- Now we have a transfer matrix for the magnet body of a scaling FFA magnet…

- What else do we need to include?

  Fringe fields:

  Field can't instantaneously change value! We need to account for this in our model.

  Maxwell's equations also say that there must be a nonzero longitudinal field when the field is varying longitudinally. When we cross this the longitudinal field at an angle it has effects in the transverse plane.

$$\mathcal{M}_{\text{fringe}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2}\frac{kL}{r_1\rho} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2}\frac{kL}{r_1\rho} - \frac{1}{\rho}\tan\alpha & 1 \end{pmatrix}$$

In an hFFA we need to account for these effects in order to have vertical stability!

- Now we have a transfer matrix for the magnet body of a scaling FFA magnet…

- What else do we need to include?

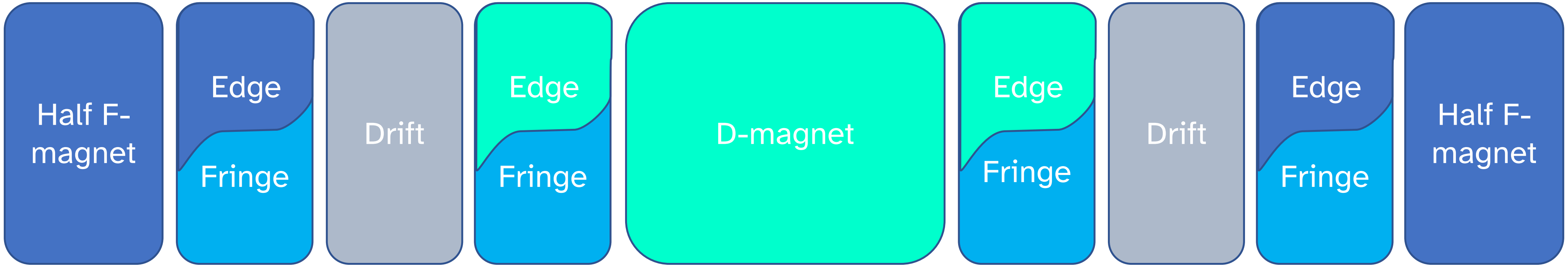
Drift spaces:

The spaces between magnets! As there are no magnetic fields here, the particle's momentum remains constant whilst its position changes…

$$\begin{bmatrix} 1 & L & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

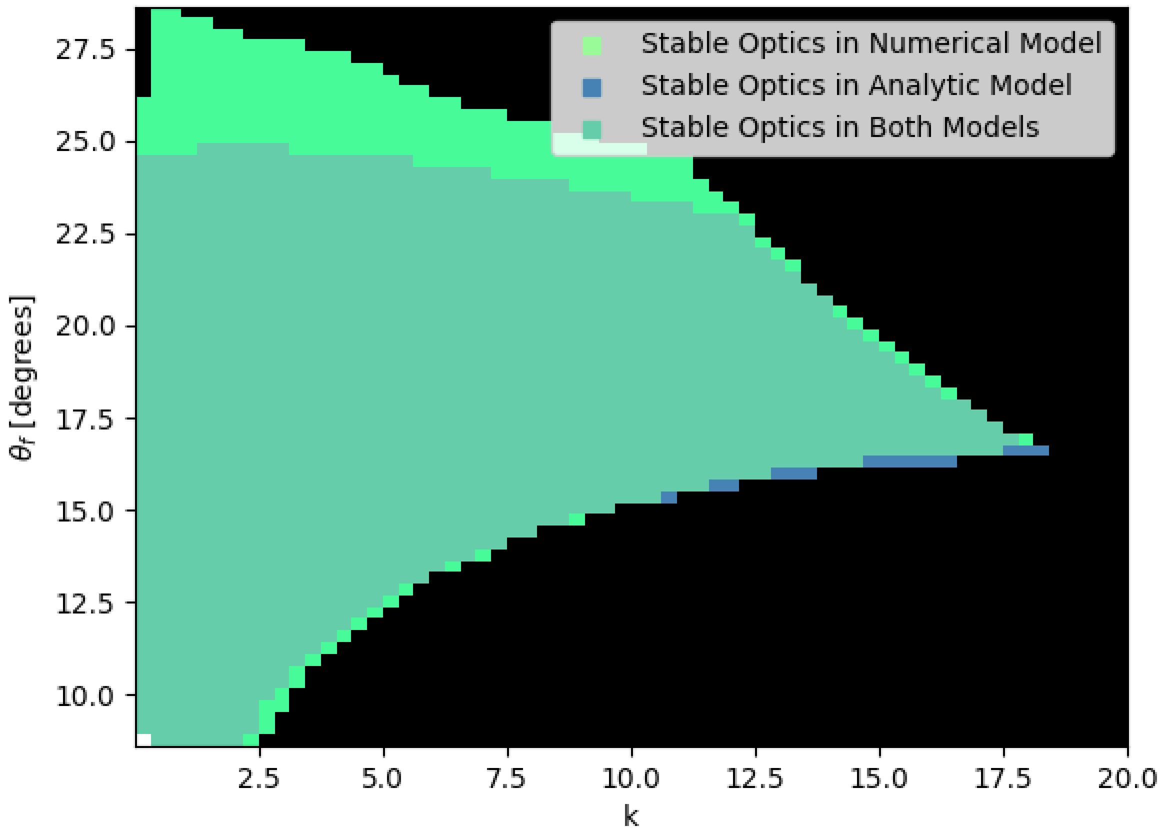
Full analytic recipe

- Specify input parameters
- Compute lattice geometry
- Use lattice geometry to work out coefficients of transfer matrices
- Assemble map of cell from transfer matrices



- Compute properties of lattice!

Performance of analytic model



- Able to predict stability footprint of given lattice
- Analytic model performs worse at higher θ_F - why?
- This data takes several hours to compute numerically – analytic model runs instantly!

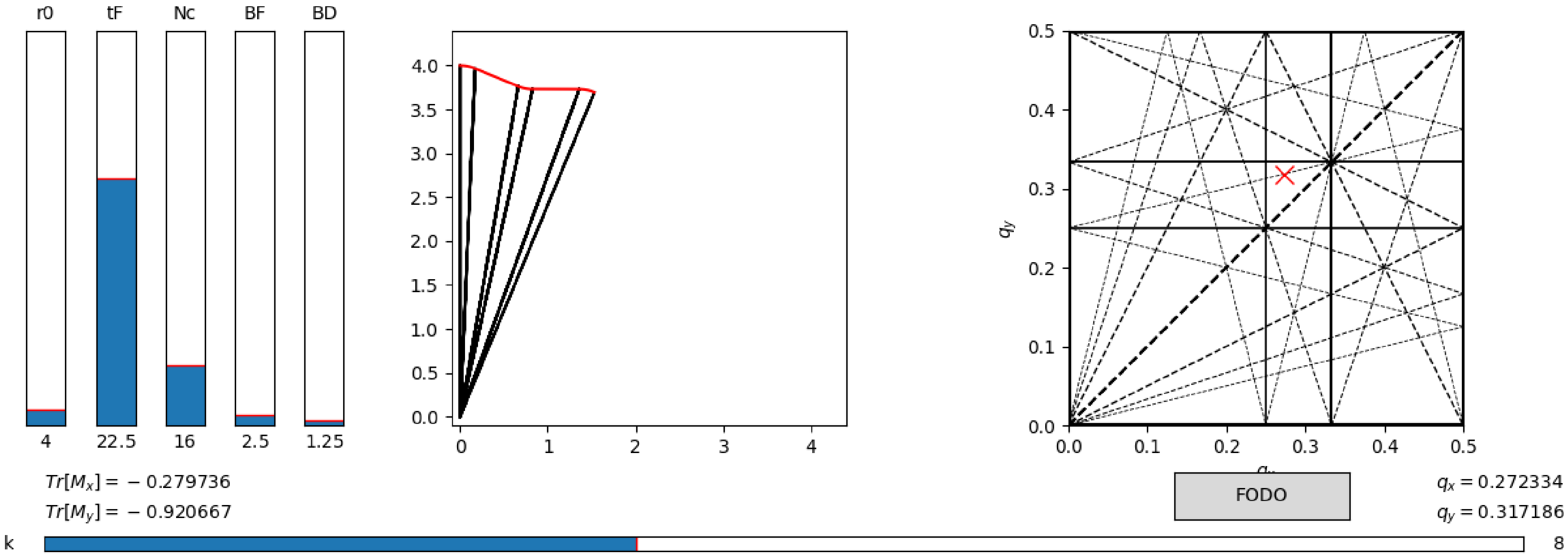
The ‘FFA-designer’ code

https://github.com/maxtopp-mugglestone/FFA-designer

- Contains Python module with implementation of analytic solutions for hFFA FODO and triplet
  - Added as custom classes – can be flexibly imported and used for your own code, parameter scans, etc
  - Functions to compute tune, plot closed orbits, etc

Interactive demo

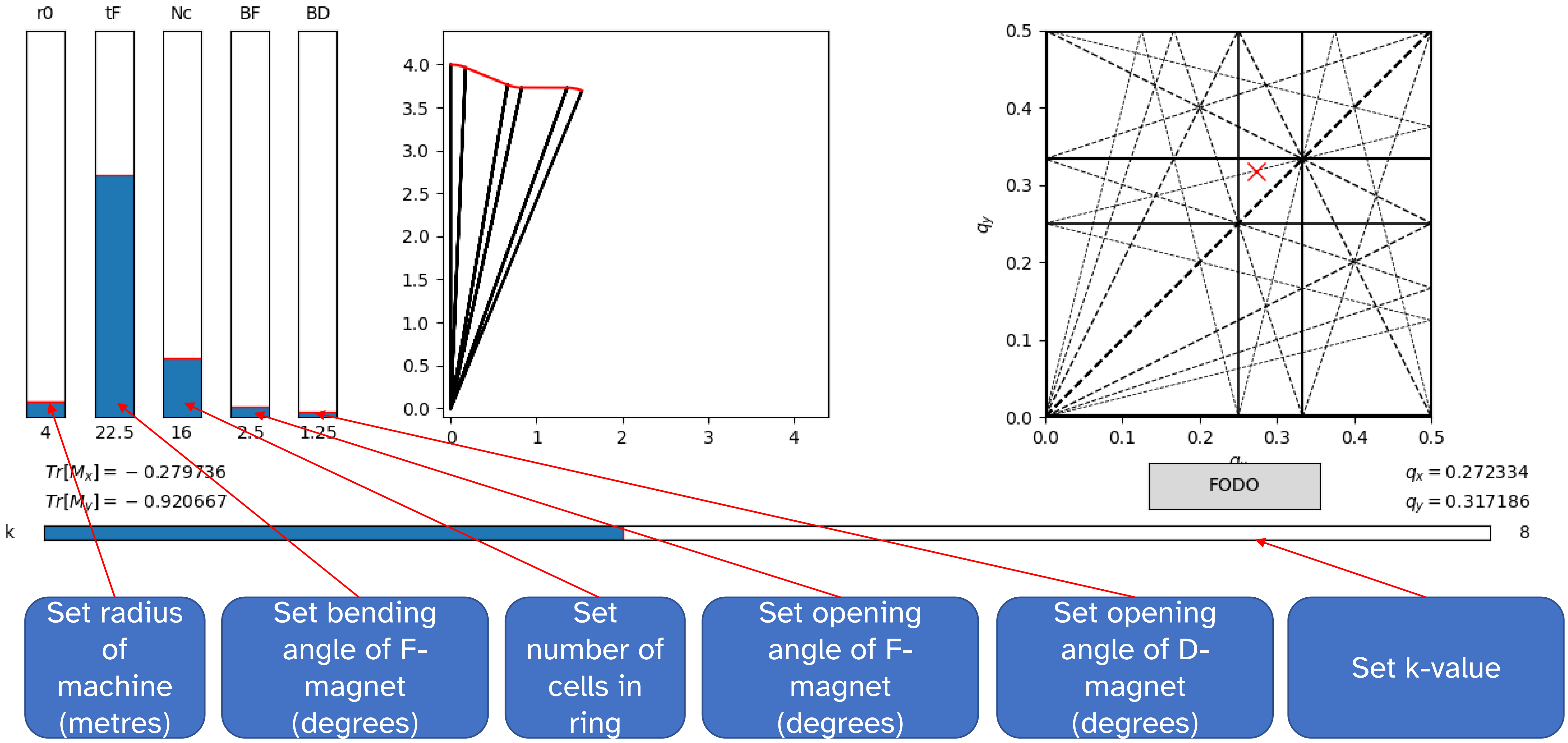
- 'FFA-designer' github also contains a GUI script for interacting with FFA parameters in realtime and seeing how they affect lattice properties



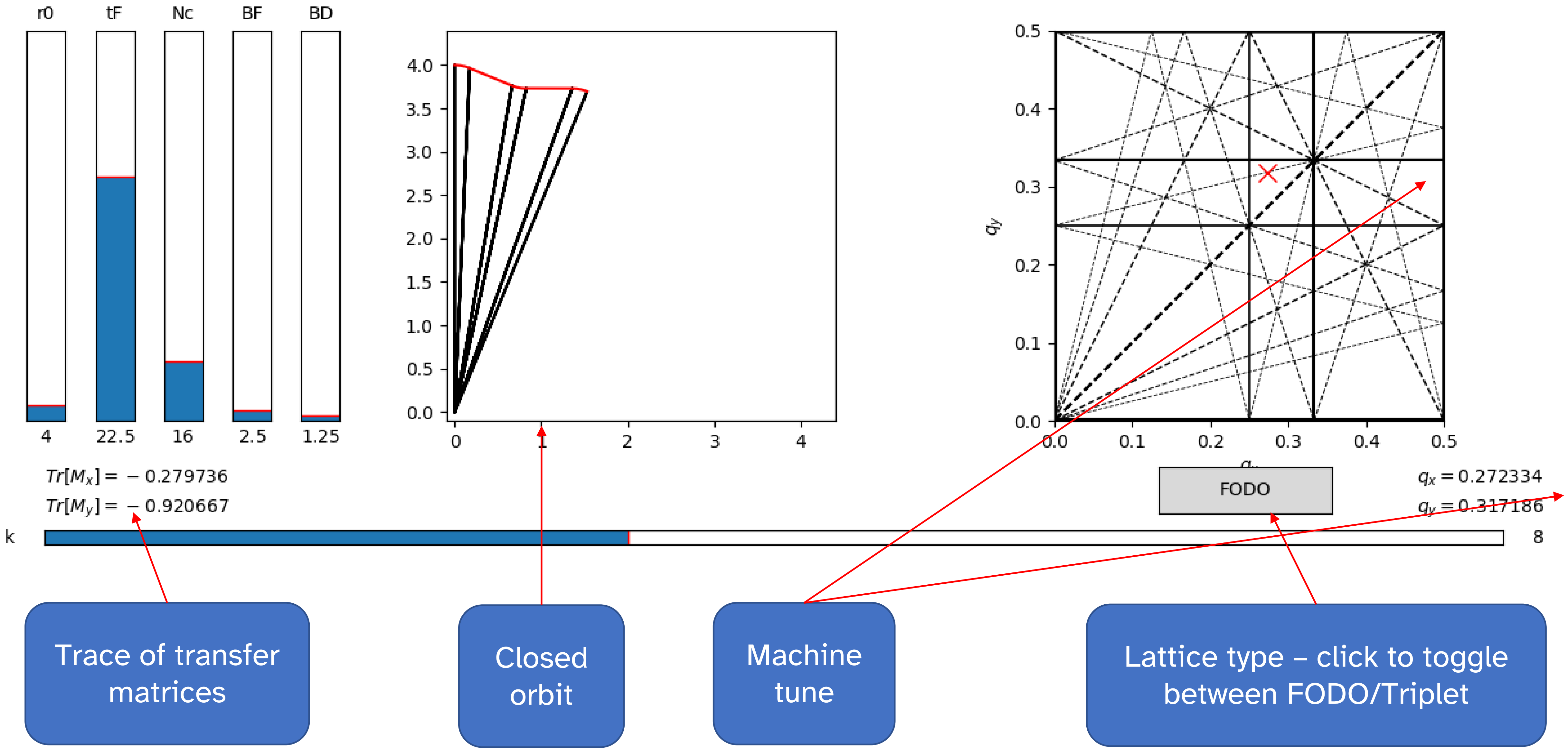
Interactive demo

- Install prerequisites
  - Python 3
  - Numpy
  - Matplotlib

- Navigate to FFA-designer git and download all files into new folder
- Run interactive-gui.py



r0
tF
Nc
BF
BD
4
22.5
16
2.5
1.25
$Tr[M_x] = -0.279736$
$Tr[M_y] = -0.920667$
k
8
FODO
$q_x = 0.272334$
$q_y = 0.317186$
q_y
Set radius of machine (metres)
Set bending angle of F-magnet (degrees)
Set number of cells in ring
Set opening angle of F-magnet (degrees)
Set opening angle of D-magnet (degrees)
Set k-value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q_y
Trace of transfer matrices
Closed orbit
Machine tune
Lattice type – click to toggle between FODO/Triplet

Interactive demo

- Experiment with different inputs – get an impression of how each input affects the tune

- Design your own FFA lattice:

  - Make a note of the parameters – they may come in useful in a later session

- Thank you for listening!

- Any questions?