Analytic Model of the Vertical FFA

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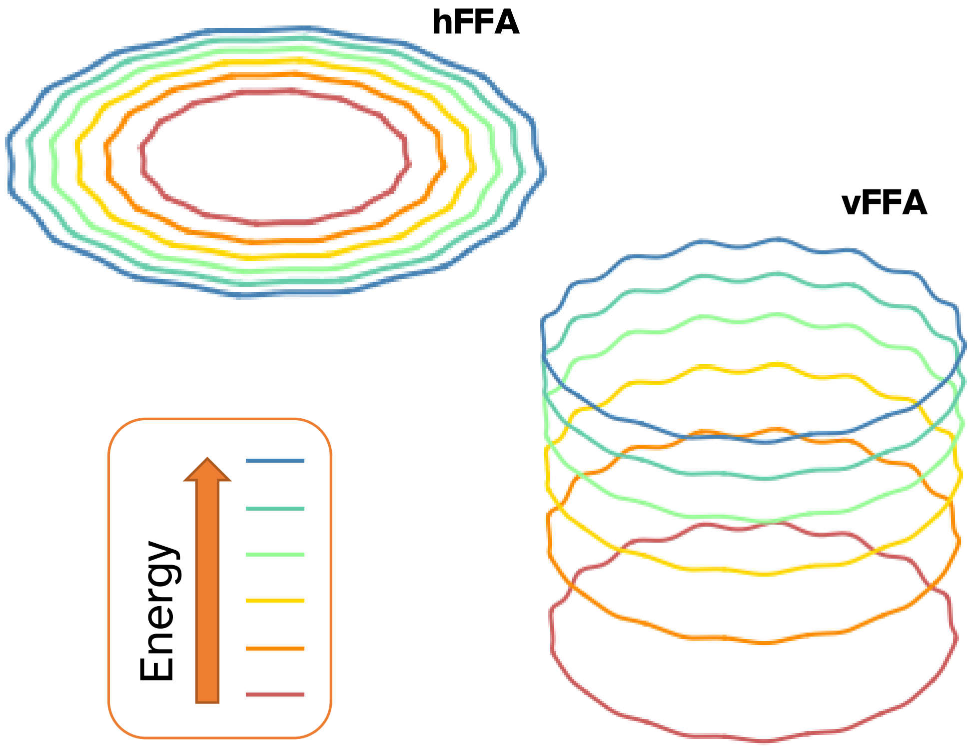
Contents

Introduction to the vFFA



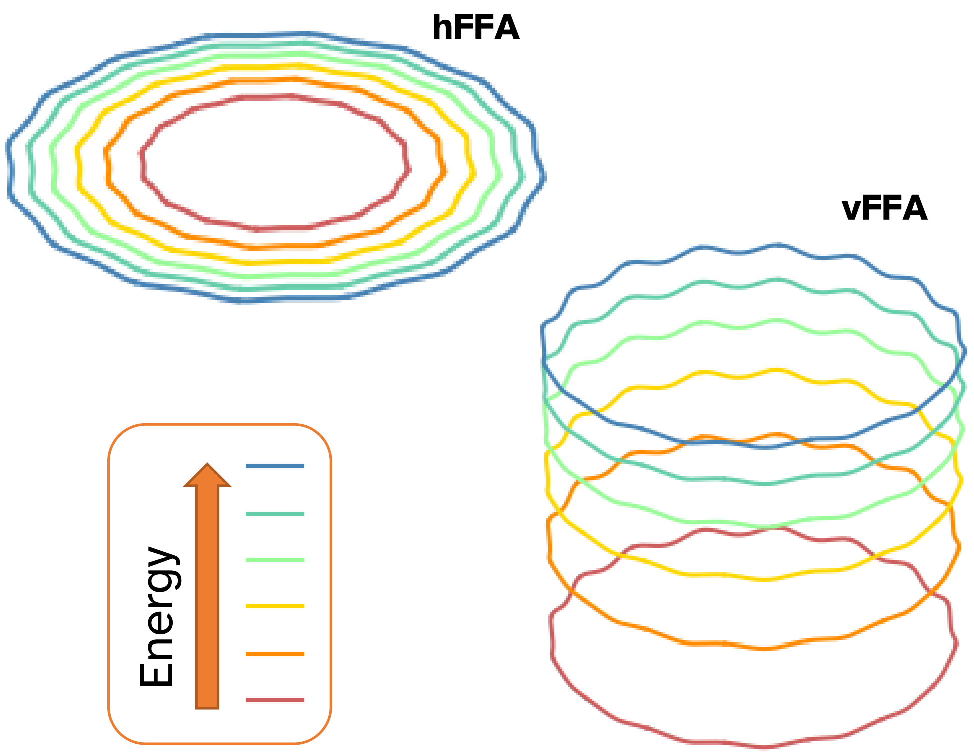
- Horizontal excursion FFA (hFFA):
  - Orbits move outwards with increasing energy
  - Fields increase radially
  - Zero chromaticity if fields follow scaling law

$$B = B_0 \left({}^{r}/_{r_0} \right)^k$$

- Vertical excursion FFA (vFFA):
  - Higher energy orbits are vertically translated copies of lower energy orbits
  - Zero chromaticity if fields increase with vertical coordinate (Z) following scaling law

$$B = B_0 e^{mZ}$$

vFFA continued



- Higher energy orbits are vertically translated copies of lower energy orbits
  - Zero path length difference
    → Zero momentum compaction factor α_c
- Zero chromatic
- Quasi-isochronous for relativistic acceleration

$$\frac{df_r}{f_r} = \eta \frac{dp}{dp} = \left(\alpha_c - \frac{1}{\gamma^2}\right)\frac{dp}{dp}$$

vFFA has complicated behaviour and so far has only been modelled using numerical approaches

Modelling accelerators

- Linear optics requires knowledge of:
  - Closed orbit
  - Local magnetic fields along closed orbit
- How can we determine these analytically?

Closed orbit in an hFFA FODO

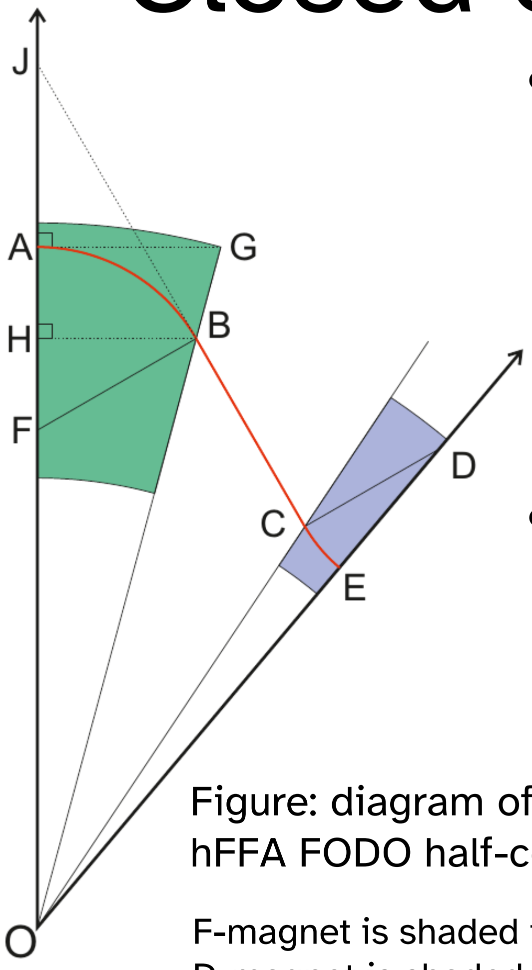


Figure: diagram of hFFA FODO half-cell

F-magnet is shaded in green; D-magnet is shaded in blue.

- Assumptions:
  - Orbit crosses perpendicularly to half-cell boundary
  - Constant field (-> const. radius of curvature) in magnets
  - Zero curvature outside magnets
  - Orbit exists in horizontal plane
- System has an exact, unique solution for all variables given a set of input parameters

Parameter	Definition
β_F	$\angle AOB$
β_D	$\angle COE$
$\frac{\pi}{N}$	$\angle AOE$
r_0	$\overline{OA}$
θ_F	$\angle AFB$

Variable	Definition
θ_D	$\angle CDE$
ρ_F	$\overline{AF} = \overline{BF}$
ρ_D	$\overline{CD} = \overline{ED}$
r_1	$\overline{OB}$
r_2	$\overline{OC}$
r_3	$\overline{OE}$
L_s	$\overline{BC}$

hFFA FODO solution

$$\theta_F - \theta_D = \pi/N$$

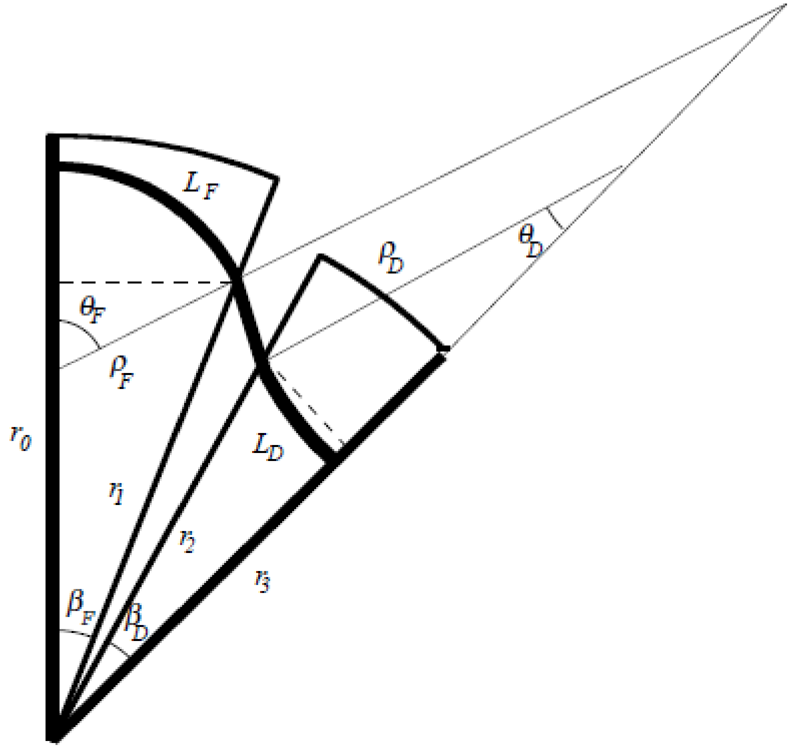
$$\frac{\rho_F}{r_0} = \frac{\tan\beta_F}{\sin\theta_F + (1 - \cos\theta_F)\tan\beta_F}$$

$$\frac{\rho_D}{r_2} = \frac{\sin\beta_D}{\sin\theta_D}$$

$$\frac{r_2}{r_1} = \frac{\cos\beta_F + \tan\theta_F \sin\beta_F}{\cos(\pi/N - \beta_D) + \tan\theta_F \sin(\pi/N - \beta_D)}$$

$$r_1 = \frac{\rho_F \sin\theta_F}{\sin\beta_F}.$$

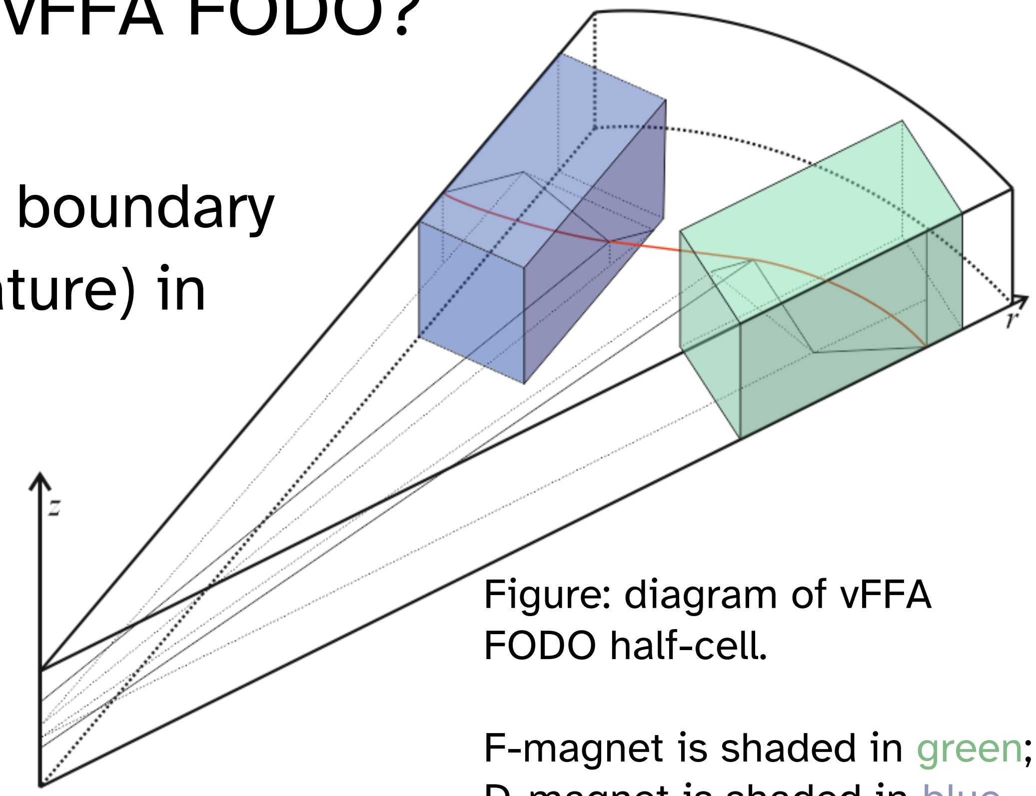
$$r_3 = r_2 \cos\beta_D - \rho_D(1 - \cos\theta_D)$$



S. Machida / Nuclear Instruments and Methods in Physics Research A 503 (2003) 322–327

Equivalent approach for vFFA FODO

- Can we take the same approach for a vFFA FODO?
- Similar set of approximations:
  1. Orbit crosses perpendicular to half-cell boundary
  2. Constant field (-> const. radius of curvature) in magnets
  3. Zero curvature outside magnets
  4. **Orbit can be non-planar**



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vFFA FODO

- Const. field implies planar arc within magnet
- Conditions (1), (2), and (4) can only be satisfied by a rotation of the plane of curvature within the magnet about an axis perpendicular to the cell boundary
- This introduces a new parameter, γ, known as the inclination angle
  - Angle between plane of curvature and horizontal plane
  - Defined fully with geometry on next slide

Geometry of a vFFA FODO lattice

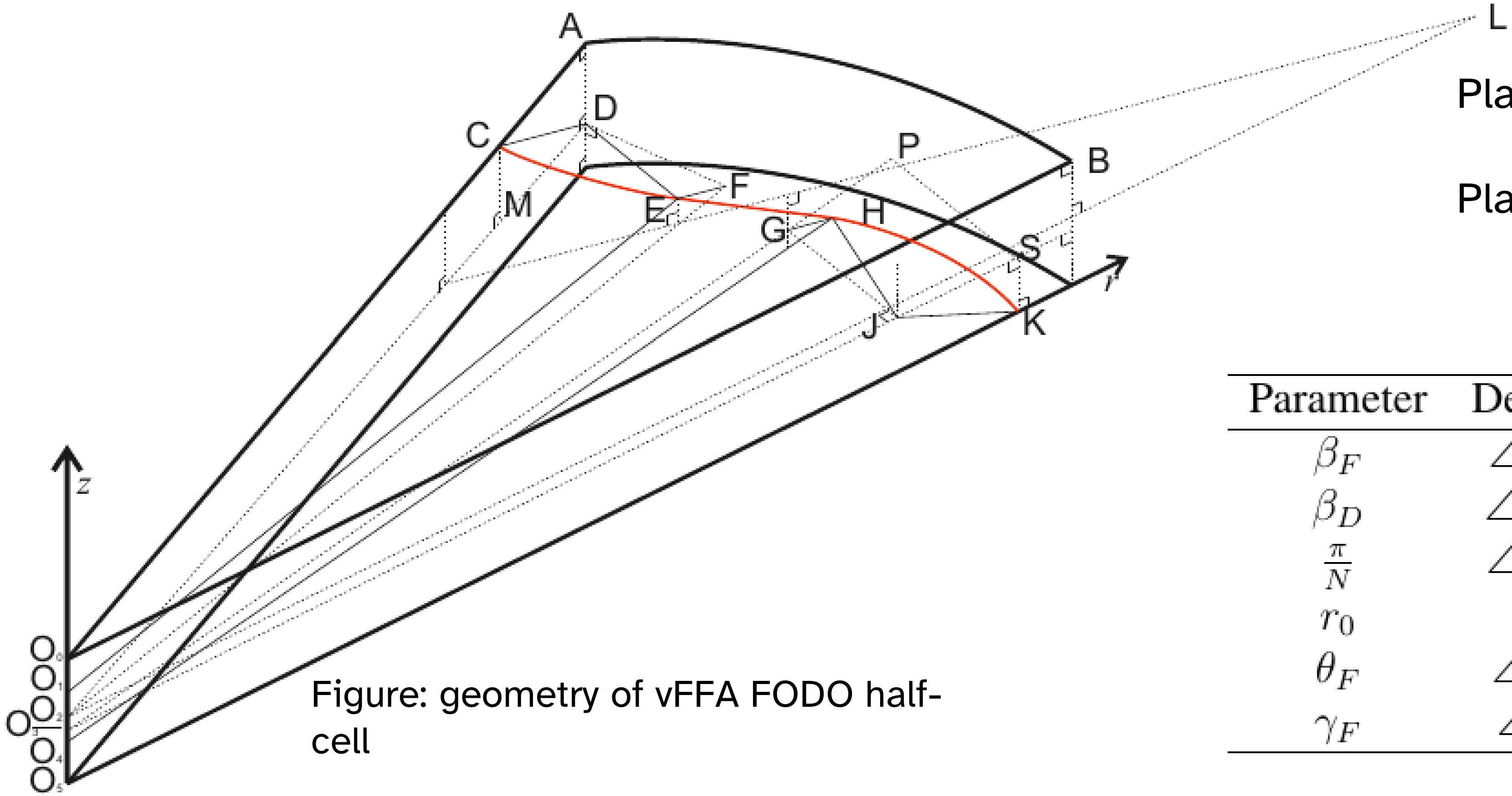


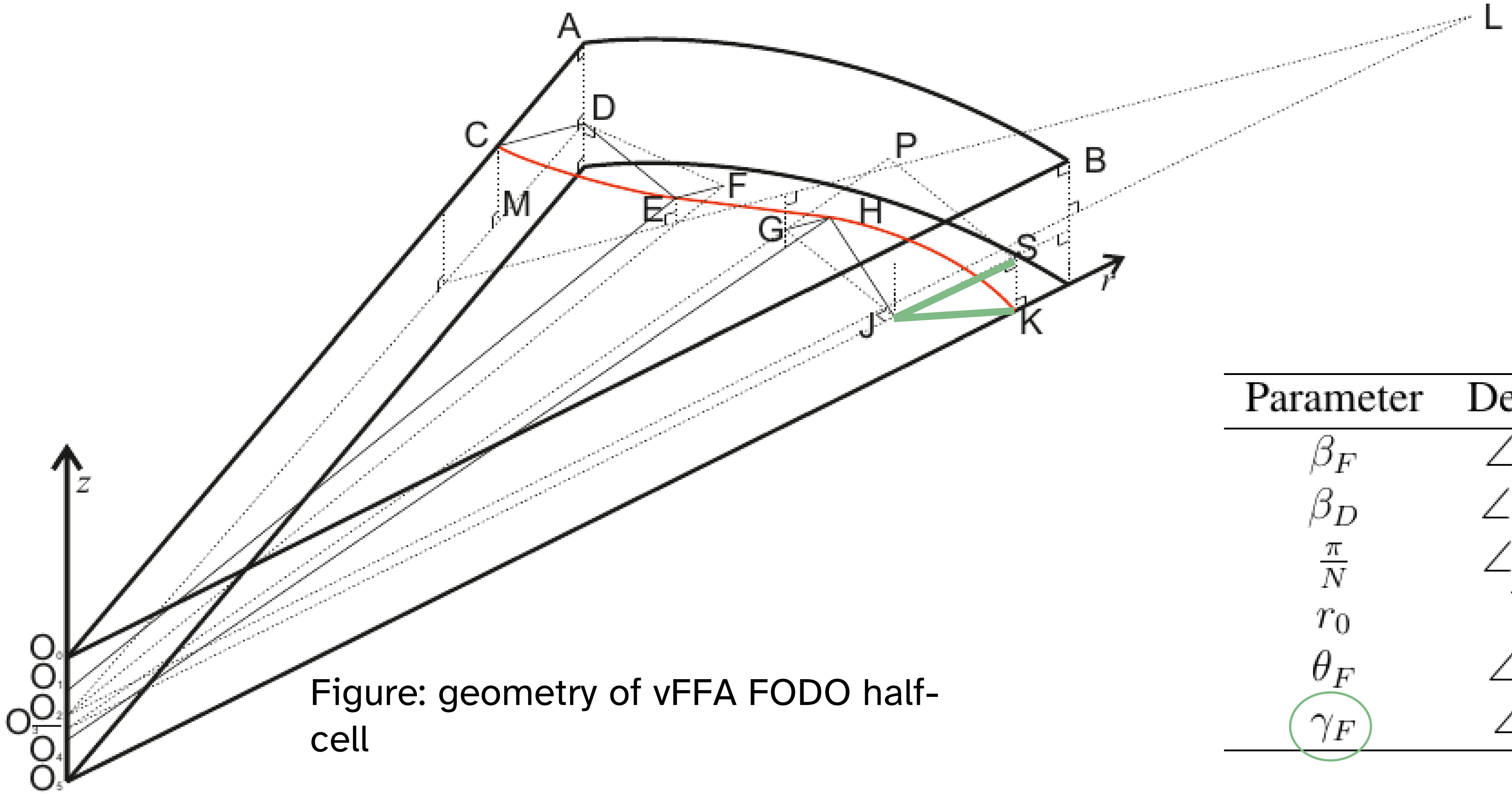
Figure: geometry of vFFA FODO half-cell

Plane of curvature in F - magnet:
HJK
Plane of curvature in D - magnet:
CDE

Parameter	Definition
β_F	$\angle SO_3P$
β_D	$\angle DO_2F$
$\frac{\pi}{N}$	$\angle AO_0B$
r_0	$\overline{O_5K}$
θ_F	$\angle HJK$
γ_F	$\angle SJK$

Variable	Definition
θ_D	$\angle CDE$
ρ_F	$\overline{HJ} = \overline{JK}$
ρ_D	$\overline{CD} = \overline{ED}$
r_1	$\overline{O_4H}$
r_2	$\overline{O_1E}$
r_3	$\overline{O_0C}$
L_s	$\overline{EH}$
γ_D	$\angle CDM$

Geometry of a vFFA FODO lattice



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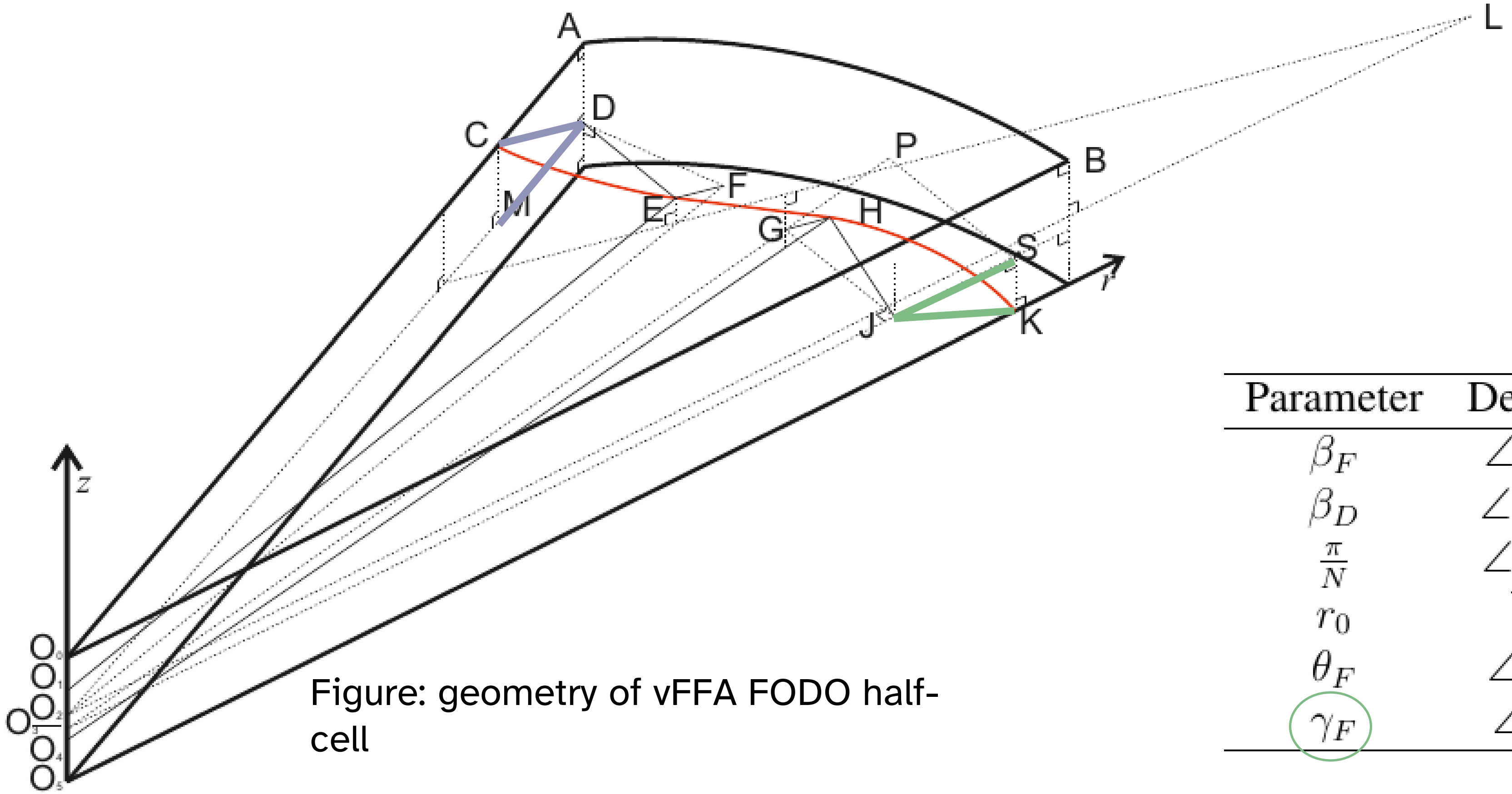

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γ_D	$\angle CDM$

$$\frac{\tan\theta_F\cos\gamma_F - \tan\theta_D\cos\gamma_D}{1+\tan\theta_F\cos\gamma_F\tan\theta_D\cos\gamma_D} = \tan\left(\frac{\pi}{N}\right)$$

$$\sin\theta_F\sin\gamma_F = \sin\theta_D\sin\gamma_D.$$

$$\rho_f[\sin\theta_F + (1-\cos\theta_F)\cos\gamma_F\tan\beta_F] = r_0\tan\beta_F,$$

$$r_1\sin\beta_F = \rho_F\sin\theta_F.$$

$$r_1[\cos\beta_F + \tan\theta_F\cos\gamma_F\sin\beta_F] = r_2[\cos\left(\frac{\pi}{N}-\beta_D\right) + \tan\theta_F\cos\gamma_F\sin\left(\frac{\pi}{N}-\beta_D\right)],$$

$$r_2\sin\beta_D = \rho_D\sin\theta_D,$$

$$r_3 = r_2\cos\beta_D - \rho_D(1-\cos\theta_D\cos\gamma_D).$$

hFFA equation:	vFFA equation:
$\theta_F - \theta_D = \pi/N$	$\begin{cases} \dfrac{\tan\theta_F\cos\gamma_F - \tan\theta_D\cos\gamma_D}{1+\tan\theta_F\cos\gamma_F\tan\theta_D\cos\gamma_D} = \tan\left(\dfrac{\pi}{N}\right) \\ \sin\theta_F\sin\gamma_F = \sin\theta_D\sin\gamma_D. \end{cases}$
$\dfrac{\rho_F}{r_0} = \dfrac{\tan\beta_F}{\sin\theta_F + (1-\cos\theta_F)\tan\beta_F}$ →	$\rho_f[\sin\theta_F + (1-\cos\theta_F)\cos\gamma_F\tan\beta_F] = r_0\tan\beta_F,$
$\dfrac{r_2}{r_1} = \dfrac{\cos\beta_F + \tan\theta_F\sin\beta_F}{\cos(\pi/N-\beta_D) + \tan\theta_F\sin(\pi/N-\beta_D)}$ →	$r_1[\cos\beta_F + \tan\theta_F\cos\gamma_F\sin\beta_F] = r_2[\cos\left(\dfrac{\pi}{N}-\beta_D\right) + \tan\theta_F\cos\gamma_F\sin\left(\dfrac{\pi}{N}-\beta_D\right)],$
$r_1 = \dfrac{\rho_F\sin\theta_F}{\sin\beta_F}.$ →	$r_1\sin\beta_F = \rho_F\sin\theta_F.$
$\dfrac{\rho_D}{r_2} = \dfrac{\sin\beta_D}{\sin\theta_D}$ →	$r_2\sin\beta_D = \rho_D\sin\theta_D,$
$r_3 = r_2\cos\beta_D - \rho_D(1-\cos\theta_D)$ →	$r_3 = r_2\cos\beta_D - \rho_D(1-\cos\theta_D\cos\gamma_D).$

Relationship between inclination angle and magnet midplane positions

- Plane of curvature is always perpendicular to dipole field
  - Therefore angle between dipole field and vertical equals inclination γ
- For vFFA magnet with no longitudinal variation, with midplane at X=0, scaling law and Maxwell equations dictate

$$B(z) = B_0 e^{mz}$$

$$B_X = -B_0 e^{mZ} \sin mX,$$
$$B_Y = 0,$$
$$B_Z = B_0 e^{mZ} \cos mX.$$

X = Horizontal transverse
Y = Horizontal longitudinal
Z = Vertical
m = Normalised field gradient

- Therefore: $\frac{\mathbf{B} \cdot \hat{\mathbf{Z}}}{|\mathbf{B}|} = \cos\gamma$ → $\cos\gamma = \cos mX$

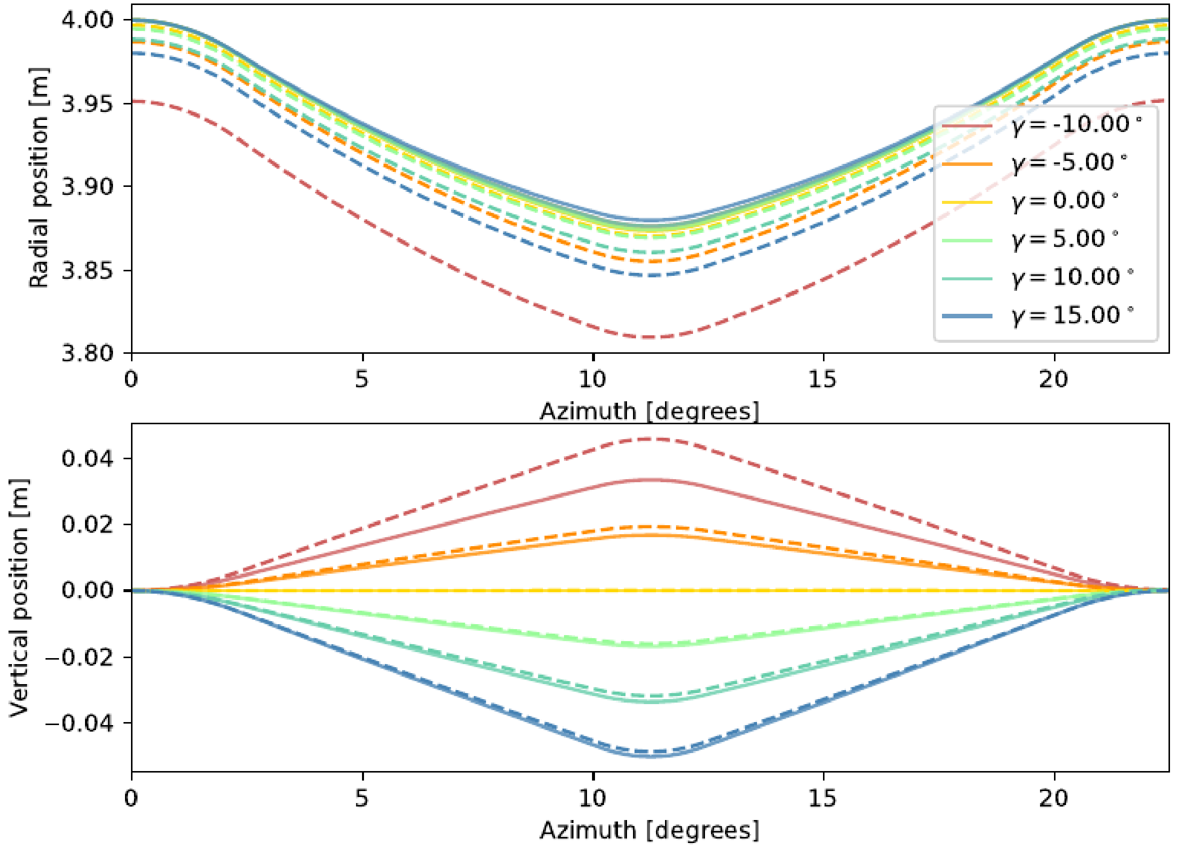
Performance of closed orbit model

- Numerical testing:

Parameter	Value
N	16
β_F	$2.25°$
β_D	$1.15°$
r_0	4m
θ_F	$45°$
m	1m^{-1}

- Solid line: analytic model
- Dotted line: simulated model

  (magnet body only – no fringe fields!)
- For $\gamma \geq -5^{\circ}$ in this example
  - Radial discrepancy $< 1.5\%$
  - Vertical discrepancy $< 3\%$



In this test case, $\gamma = -10°$ shows a large discrepancy between analytic and simulated results. This is because of a large displacement of the closed orbit from the magnet midplane in the D-magnet – less significant for larger rings or higher m-values.

vFFA triplet model

- Same approach can be taken for modelling a triplet lattice
  - Works for both FDF and DFD triplets – just change sign of θ_F, ρ_F, etc...

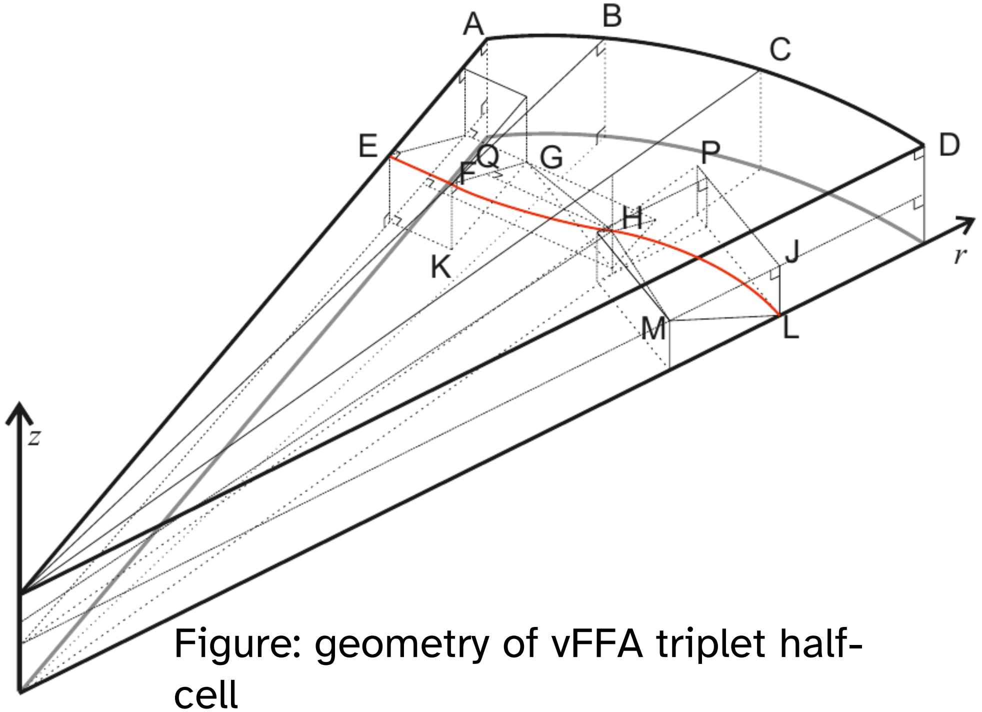
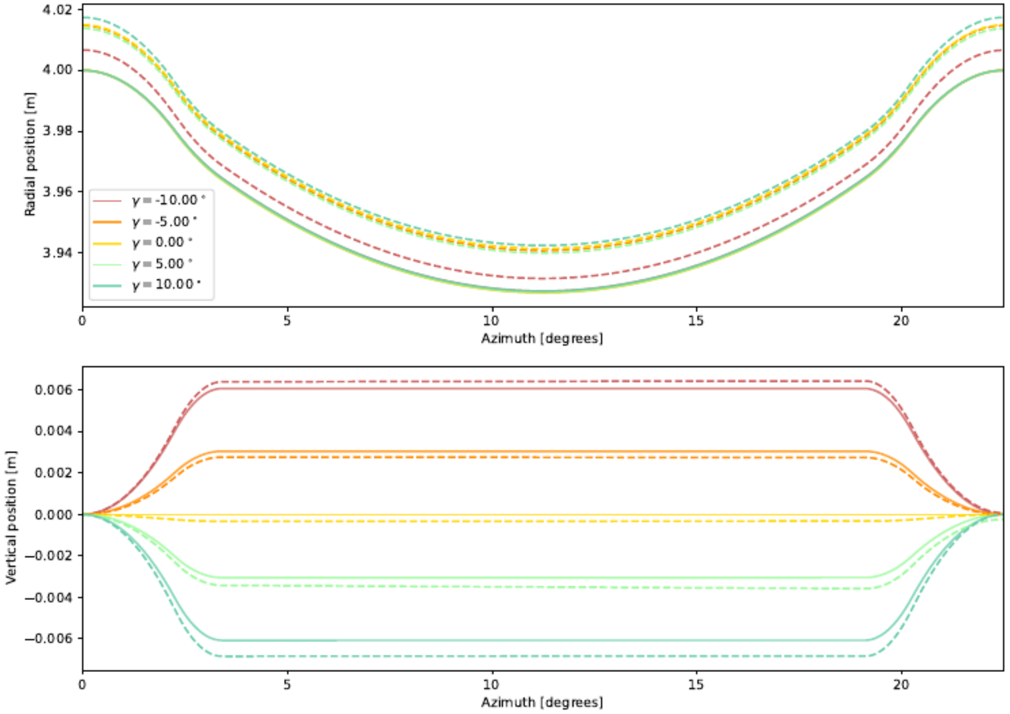


Figure: geometry of vFFA triplet half-cell



- Closed orbit can be modelled analytically
  - Similar performance to hFFA analytic models
  - Limited to cases where fringe fields have small effects on closed orbit
- Now the closed orbit is known, we can use it as a reference trajectory to analyse the optics of the system

vFFA optics

- Vector potential for vFFA magnet with no longitudinal dependence is

$$A_X = 0,$$
$$A_Y = \frac{B_0}{m} e^{mZ} \sin mX,$$
$$A_Z = 0,$$

(see slide 10 – this just comes from combining the scaling law and Maxwell's Equations!)

- Substitute into generalised Frenet-Serret Hamiltonian
  - Magnet is positioned at X = 0; reference particle orbit passes through magnet at $X = x_0$
  - Make coordinate transform $x \to X - x_0$
  - $\gamma = m\, x_0$ (slide 10)

$$\mathcal{H} = \left(1 + \frac{1}{\rho}[x\cos\gamma + z\sin\gamma]\right)\sqrt{1 + p_x^2 + p_z^2}$$
$$- \left(1 + \frac{1}{\rho}[x\cos\gamma + z\sin\gamma]\right)\frac{B_0 q}{m P_0} e^{mz}\left(\cos\gamma\sin mx + \cos mx\sin\gamma\right)$$

Linearise and define: $\frac{1}{\rho + \frac{\sin\gamma}{m}} = \frac{B_0 q}{P_0}$

$$\mathcal{H} \simeq \frac{p_x^2}{2} + \frac{p_z^2}{2}$$
$$- \frac{1}{\rho + \frac{\sin\gamma}{m}}\left[\cos\gamma\left(m + \frac{2\sin\gamma}{\rho}\right)xz - \frac{1}{2}m\left(x^2 - z^2\right)\sin\gamma\right]$$
$$+ \frac{1}{\rho\left(\rho + \frac{\sin\gamma}{m}\right)}\left(x^2\cos^2\gamma + z^2\sin^2\gamma\right)$$

- Hamiltonian has normal quad + skew quad + geometric terms!

$$\mathcal{H} \simeq \frac{p_x^2}{2} + \frac{p_z^2}{2} - \frac{1}{\rho + \frac{\sin\gamma}{m}} \left[\cos\gamma \left(m + \frac{2\sin\gamma}{\rho} \right) xz - \frac{1}{2} m \left(x^2 - z^2 \right) \sin\gamma \right] + \frac{1}{\rho \left(\rho + \frac{\sin\gamma}{m} \right)} \left(x^2 \cos^2\gamma + z^2 \sin^2\gamma \right)$$

- Thick lens transfer matrix from integrating equations of motion derived from this Hamiltonian needed almost two full pages to write out in full – not reproduced here due to spatial constraints!

vFFA linear optics recipe

- Choose set of input parameters
- Compute closed orbit geometry for lattice type (FODO, triplet, …)
- Input geometric parameters into transfer matrices derived from Hamiltonian
- Compute transfer matrix for each element of the cell

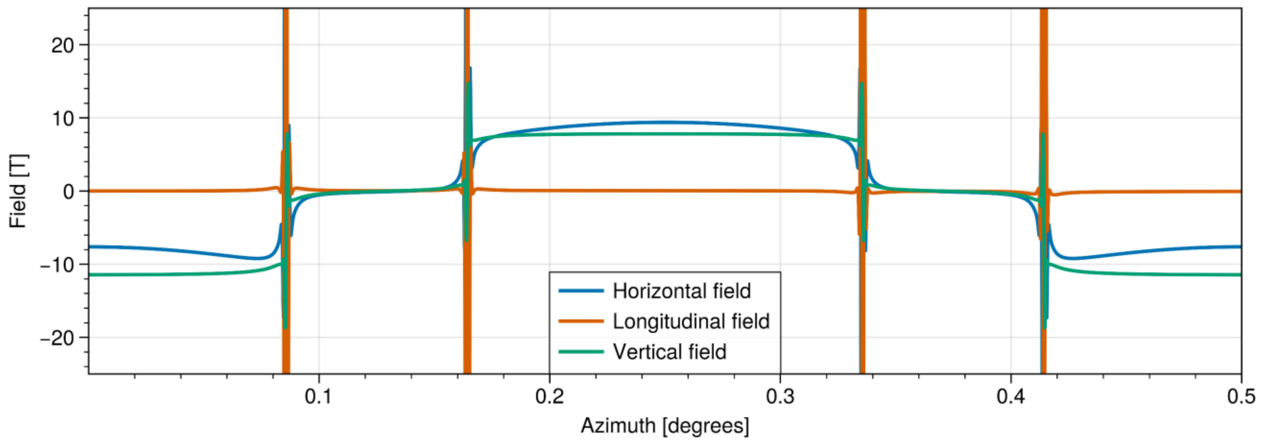
Testing the model

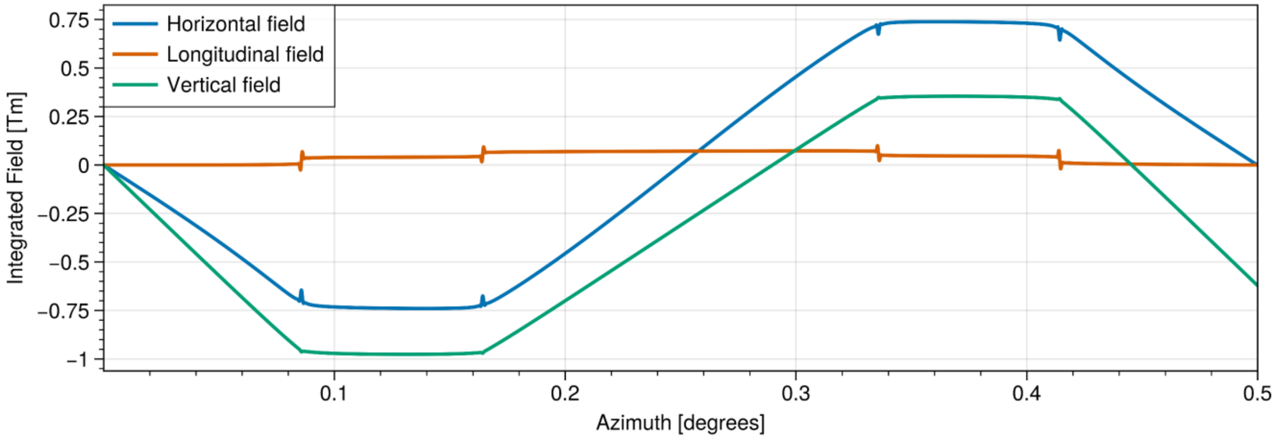
- Choose parameters of FODO lattice
- Assemble analytic model using recipe on previous slide and compute tunes for range of inputs using transfer matrix approach
- Use simulation code FIXFIELD to generate equivalent lattices
  - Magnet strengths B_{0F}, B_{0D} computed from radii of curvature ρ_F, ρ_D
    - Have to account for height change of orbit between magnets!
  - Magnet positions with respect to orbit radius computed from γ_F, γ_D
- Use numerical integration to compute optics of simulated lattices

Test lattice

Input parameter	Value
Number of cells	720
r_0	4010.651232m
β_F	0.085714°
β_D	0.085714°
θ_F	0.875°
γ_F	-25.0°
m-value	4.0/m

Table 1: Input parameters of test lattice based on proposal for a 27km circumference muon accelerator lattice by S. Machida





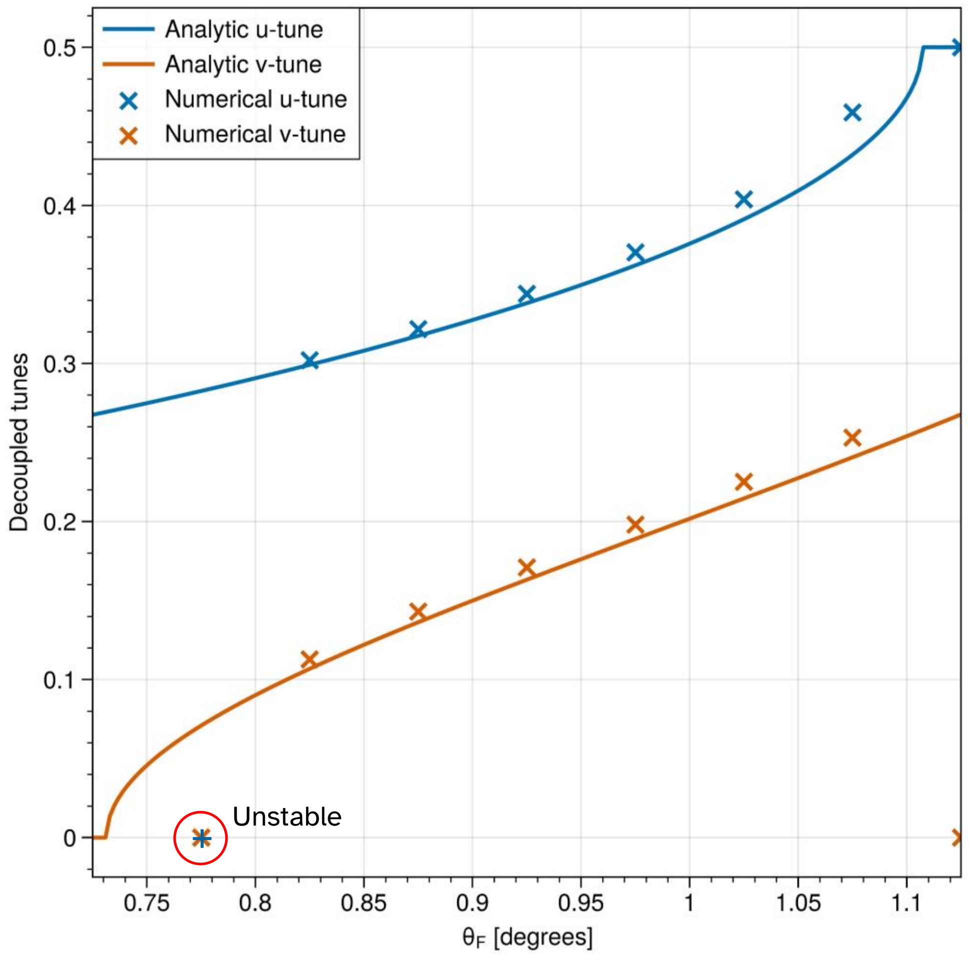
Example data from numerical simulation:

We can see field is approx. const. within magnets and zero outside.

Fringe fields are short compared to length of magnet

- longitudinal field contribution small compared to transverse fields

Results of optics test



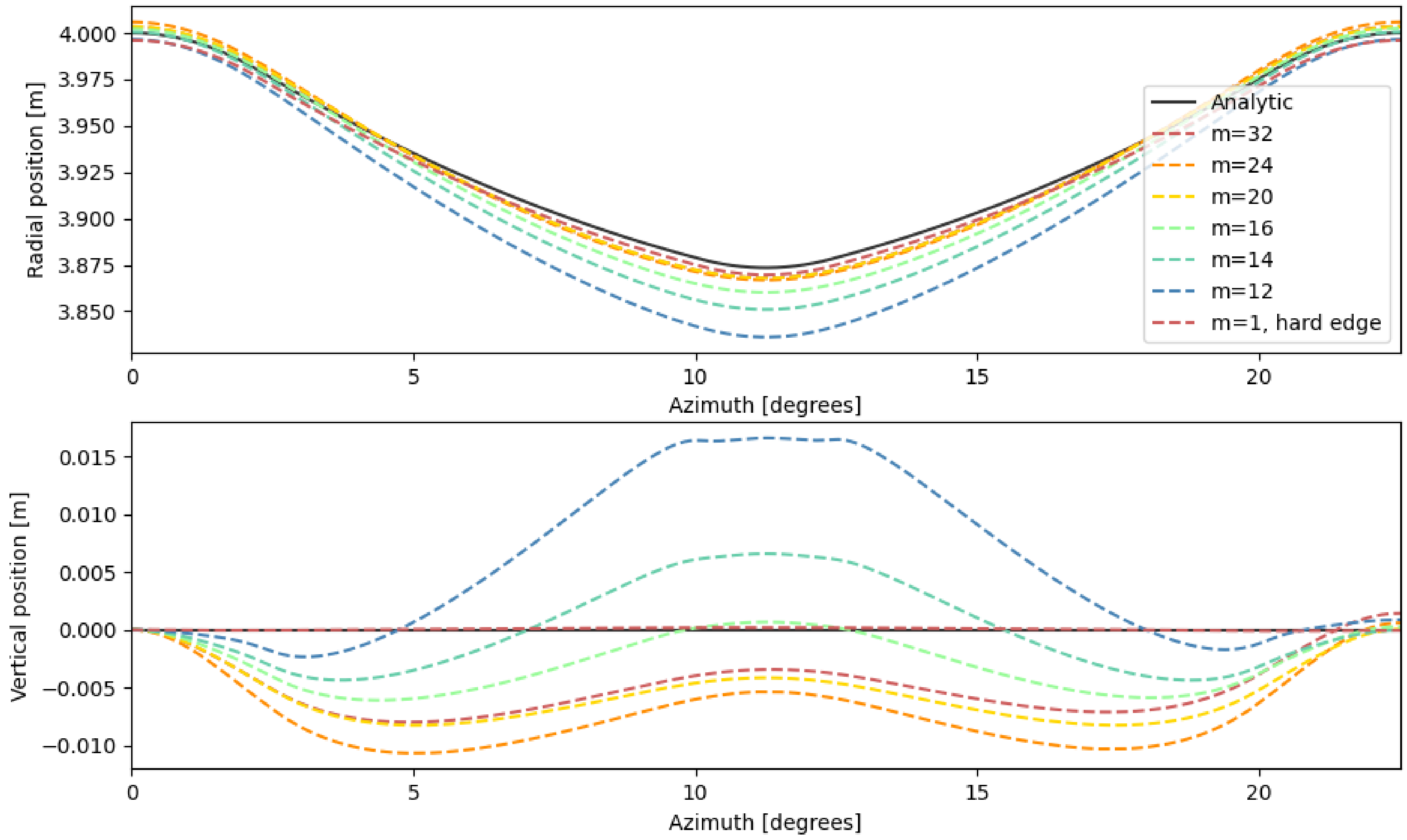
- Decoupled tunes as a function of θ_F are plotted on the left for an example FODO lattice
  - Continuous line shows the predictions of the analytic model
  - Discrete points plot the tune computed using numerical integration in Fixfield
- Analytic model is able to predict tune to within 2 decimal places for most values of θ_F

Conclusions

- First working analytic model of vFFA has been demonstrated
  - Prediction of closed orbit performs similarly to existing techniques for hFFA
    - Limited to cases where effect of fringe fields on closed orbit is small
  - Prediction of tune demonstrated for small bending angle per cell, large radius of curvature, short-fringe model
- Further testing to follow on smaller-scale rings with more significant fringe field effects

Additional material

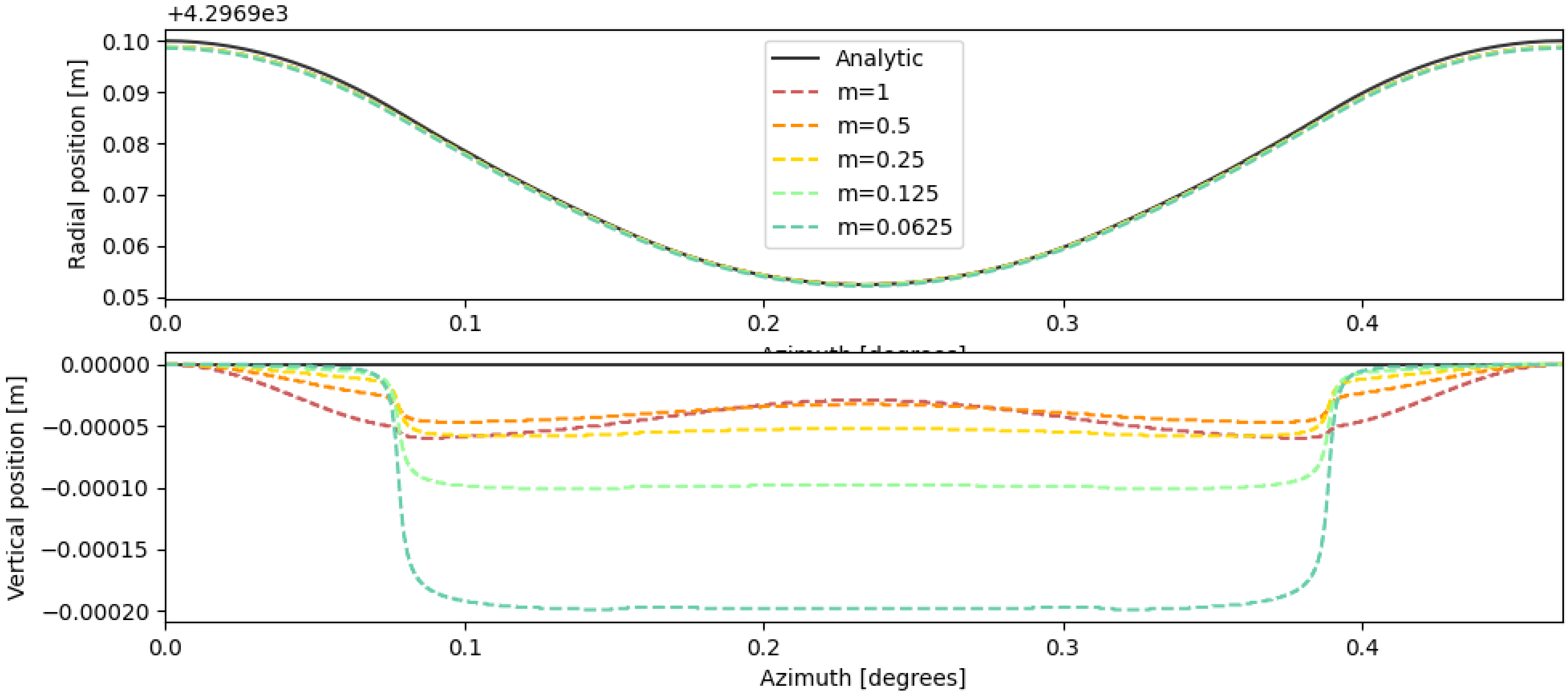
Limitations of closed orbit model



Parameter	Value
N	16
β_F	$2.25°$
β_D	$1.15°$
r_0	4m
θ_F	$45°$
m	1m^{-1}
γ_F	$0°$

- Fringe field has longitudinal component proportional to 1/m
- When crossed at an angle, this has a large effect on the closed orbit
- Not accounted for in analytic model

- Effect is much less significant for large rings!
    - Tune results in main presentation were generated with fringe fields included.
    - Even low values of m show good performance for large enough rings.



Parameter	Value
N	771
β_F	$0.07782°$
β_D	$0.07782°$
r_0	4297m
θ_F	$0.3502°$
m	1m^{-1}
c_1	0.05m

vFFA edge focussing + vFFA fringe focussing

- vFFA edge focussing (geometric effect due to extra ‘wedge’ of magnet seen by beam crossing at an angle) transfer matrix has been derived, but further testing is needed!

$$\mathcal{M}_{\text{v-edge}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{qB_0}{P_0}\cos\gamma\tan\alpha\cos mx_0 & 1 & \frac{qB_0}{P_0}\sin\gamma\tan\alpha\cos mx_0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{qB_0}{P_0}\cos\gamma\tan\alpha\sin mx_0 & 0 & -\frac{qB_0}{P_0}\sin\gamma\tan\alpha\sin mx_0 & 1 \end{pmatrix}.$$

- vFFA thin lens fringe field model has also been derived; must now be tested