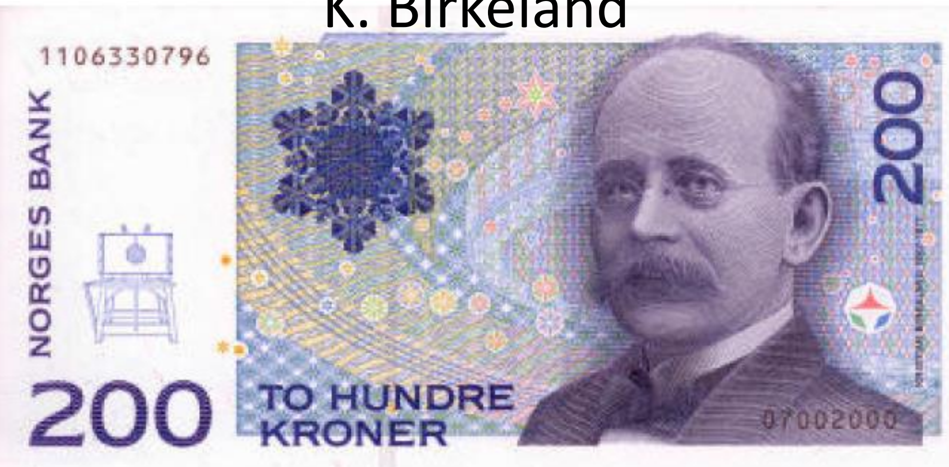
Integrable Dynamical Systems in Particle Accelerators

K. Birkeland



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2023 Workshop on Fixed Field Alternating Gradient Accelerators (FFA'23)

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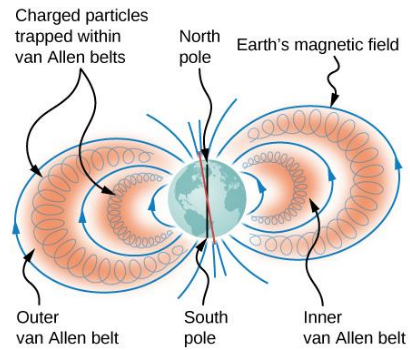

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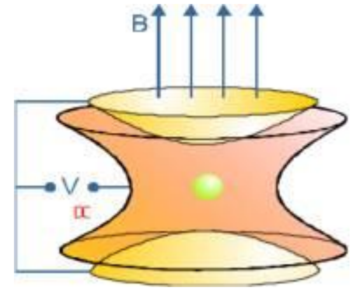
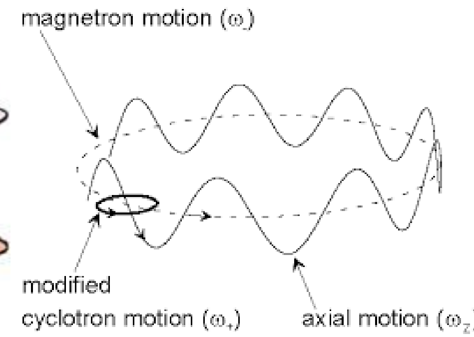
Hamiltonian systems and integrability

- There are many definitions of integrability in dynamical system theories
- Simplistic definition: Integrable means ‘Fully predictable, non-chaotic’
- For a 3D Hamiltonian system to be integrable, there must exist 3 nontrivial independent Poisson-communing invariants (including the Hamiltonian itself, if it is time-independent).
- Examples:
  - (1) All 3D harmonic oscillators are integrable.
    - An ideal Penning trap is an integrable system



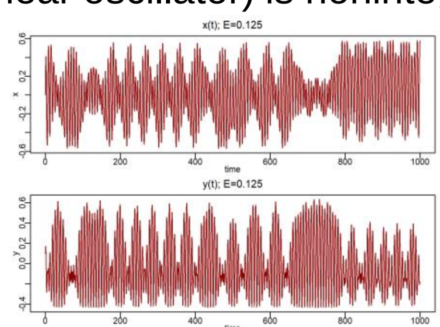


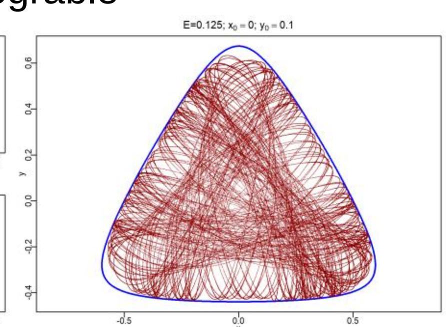
  - (2) An uncoupled 2D nonlinear oscillator is integrable

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + \frac{\alpha}{4}\left(x^4 + y^4\right)$$

  - (3) The Hénon–Heiles system (a coupled nonlinear oscillator) is nonintegrable

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + \lambda\left(x^2 y - \frac{y^3}{3}\right)$$





    - Chaos appears for $E > 0.125$

Modern accelerators (the LHC case)

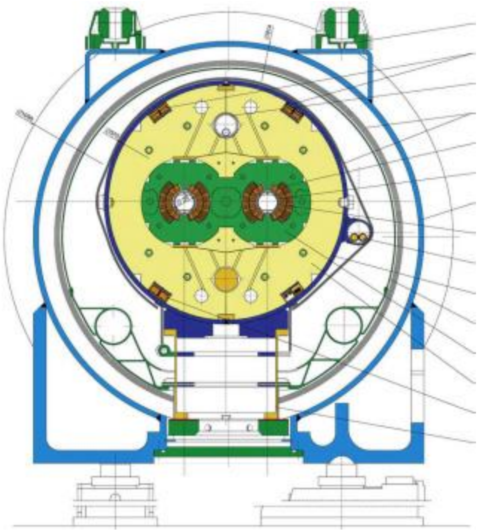
- LHC: 27 km, 13 TeV per beam
  - The total energy stored in the magnets is HUGE: 40 GJ (9,600 kilograms of TNT)
  - The total energy carried by the two beams reaches 1400 MJ (340 kilograms of TNT)
  - Loss of only one ten-millionth part (10^{-7}) of the beam is sufficient to quench a superconducting magnet
- LHC vacuum chamber diameter : ~40 mm
- LHC average rms beam size (at 13 TeV): ~0.1 mm
- LHC average rms beam angle spread: 2 μrad
  - Very large ratio of forward to transverse momenta
- LHC typical cycle duration: 10 hrs = 4×10^8 revolutions
  - Particles must have stable non-chaotic trajectories
- Kinetic energy of a typical semi truck at 60 mph: ~7 MJ

What keeps particles stable in an accelerator?

- Particles are confined (focused) by static magnetic fields in a vacuum.
  - Magnetic fields conserve the total energy
- An ideal focusing system in all modern accelerators is nearly integrable
  - There exist 3 conserved quantities (integrals of motion); the integrals are "simple" – polynomial in momentum.
  - The particle motion is confined by these integrals.

$$H \approx \omega_1 J_1 + \omega_2 J_2 + \omega_3 J_3$$

$$J = \frac{1}{2\pi} \oint pdq$$ -- particle's action

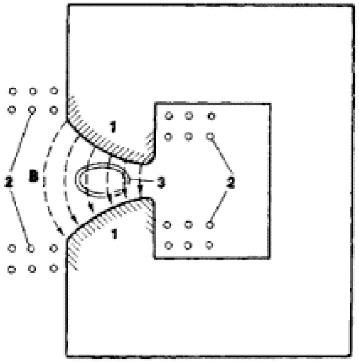


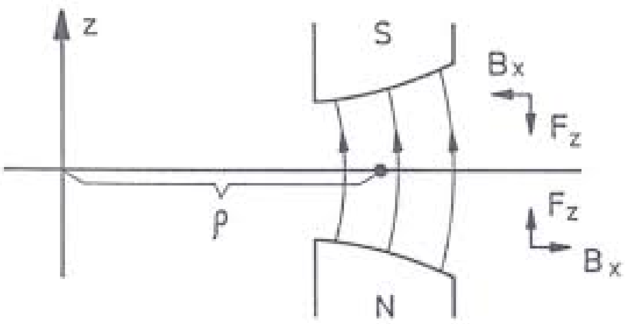
Particle motion in static magnetic fields

- For accelerators, there are no useful exactly integrable systems for axially symmetric magnetic fields in vacuum:

$$H = \frac{p_z^2 + p_r^2}{2m} + \frac{1}{2m}\left(\frac{p_\theta}{r} - \frac{eA_\theta(r,z)}{c}\right)^2$$

- Until 1959, all circular accelerators relied on approximate (adiabatic) integrability.
  - These are the so-called weakly-focusing accelerators
  - Required large magnets and vacuum chambers to confine particles;

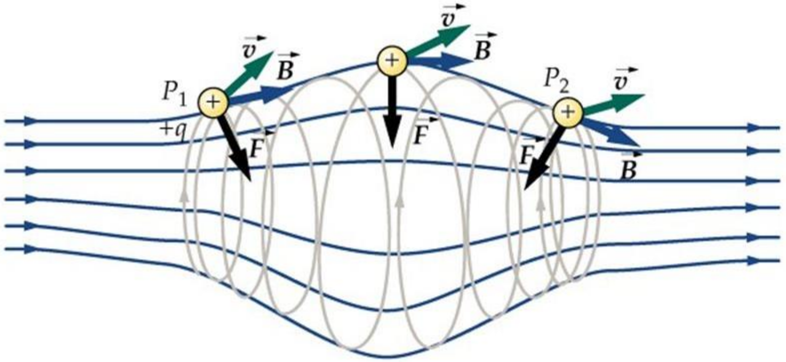




The magnetic fields can be approximated by the field of two magnetic monopoles of opposite polarity

Two magnetic monopoles ('ends' of a solenoid)

- One can imagine that the motion of an electric charge between two magnetic monopoles (of opposite polarity) would be integrable, but it is not.
  - Only approximate "adiabatic" integrals exist, when poles are far apart (as compared to the Larmour radius)
  - This is the principle of a magnetic "bottle" trap; also, the principle of "weak focusing in accelerators".



- The non-integrability in this case is somewhat surprising because the motion in the field of two Coulomb centers is integrable.
  - This has been know since Euler and was Poincare's starting point for the 3-body problem quest.

Strong Focusing with static magnetic fields

PHYSICAL REVIEW VOLUME 88, NUMBER 5 DECEMBER 1, 1952

The Strong-Focusing Synchroton—A New High Energy Accelerator*

ERNEST D. COURANT, M. STANLEY LIVINGSTON,† AND HARTLAND S. SNYDER
Brookhaven National Laboratory, Upton, New York

(Received August 21, 1952)

Strong focusing forces result from the alternation of large positive and negative *n*-values in successive sectors of the magnetic guide field in a synchrotron. This sequence of alternately converging and diverging magnetic lenses of equal strength is itself converging, and leads to significant reductions in oscillation amplitude, both for radial and axial displacements. The mechanism of phase-stable synchronous acceleration still applies, with a large reduction in the amplitude of the associated radial synchronous oscillations. To illustrate, a design is proposed for a 30-Bev proton accelerator with an orbit radius of 300 ft, and with a small magnet having an aperture of 1×2 inches. Tolerances on nearly all design parameters are less critical than for the equivalent uniform-*n* machine. A generalization of this focusing principle leads to small, efficient focusing magnets for ion and electron beams. Relations for the focal length of a double-focusing magnet are presented, from which the design parameters for such linear systems can be determined.

BETATRON OSCILLATIONS

RESTORING forces due to radially-decreasing magnetic fields lead to stable "betatron" and "synchrotron" oscillations in synchrotrons. The amplitudes of these oscillations are due to deviations from the equilibrium orbit caused by angular and energy spread in the injected beam, scattering by the residual gas, magnetic inhomogeneities, and frequency errors. The strength of the restoring forces is limited by the

* Work done under the auspices of the AEC.
† Massachusetts Institute of Technology, Cambridge, Massachusetts.

Strong focusing

Specifics of accelerator focusing:

- Focusing fields must satisfy Maxwell equations in vacuum

$$\Delta\varphi(x, y, z) = 0$$

- For stationary fields: focusing in one plane while defocusing in another
  - ➢ quadrupole: $$\varphi(x, y) \propto x^2 - y^2$$
  - ➢ However, alternating quadrupoles results in effective focusing in both planes

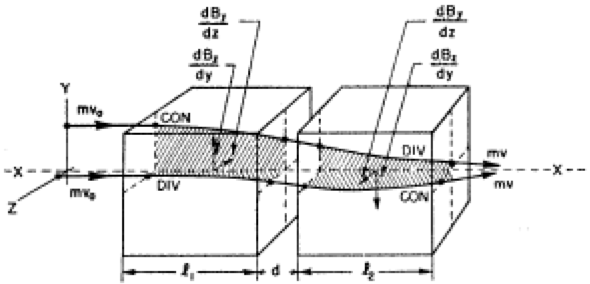


FIG. 8. Illustration of double-focusing in two magnetic lenses with field gradients in opposite directions, showing the alternately convergent and divergent forces and the net convergence of the system.

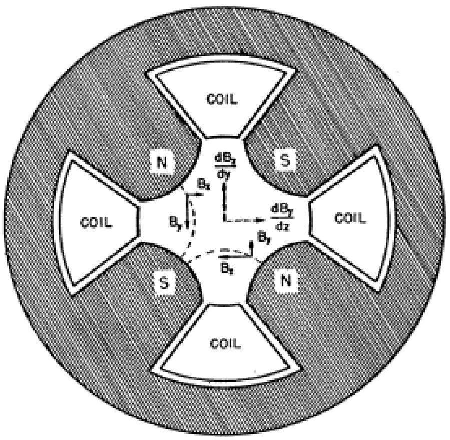
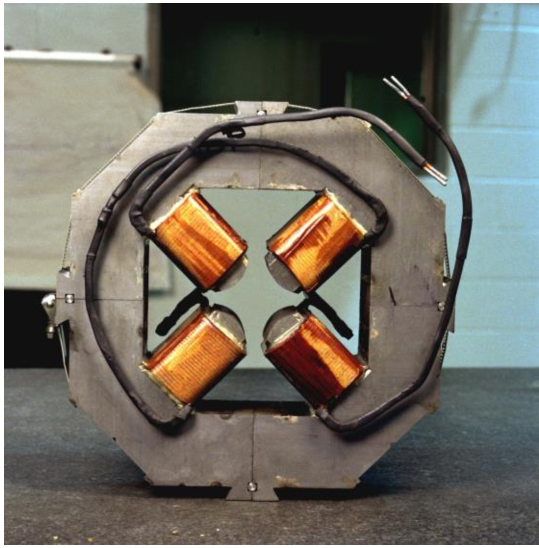


FIG. 9. Cross section of a 4-pole magnet with hyperbolic pole faces to produce uniform and equal field gradients dB_z/dy and dB_y/dz.



The accelerator Hamiltonian

$$H = c\left[m^2c^2 + \left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2\right]^{\frac{1}{2}}$$

- After some canonical transformations and in a small-angle approximation

$$H'(s) = \frac{p_x^2 + p_y^2}{2(1+\delta)} + \frac{x^2}{2\rho} + \frac{K(s)}{2}\left(x^2 - y^2\right) - \frac{x\delta}{\rho} + \ldots$$

where δ is the relative momentum deviation. For $\delta \ll 1$:

$$H' \approx \frac{p_x^2 + p_y^2}{2} + \frac{K_x(s)x^2}{2} + \frac{K_y(s)y^2}{2}$$

For a pure quadrupole magnet: $K_x(s) = - K_y(s)$

This Hamiltonian is separable and thus integrable!

Ideal linear equations of motion in an accelerator:

$$x'' + K_x(s)x = 0$$

$$y'' + K_y(s)y = 0$$

Courant and Snyder discovered in 1952 that these two uncoupled time-dependent equations can be transformed into two time-independent equations by introducing two betatron phase variables ψ_x and ψ_y instead of the time variable, *s*:

$$\frac{d^2 x_n}{d\psi_x^2} + \nu_x^2 x_n = 0 \qquad \frac{d^2 y_n}{d\psi_y^2} + \nu_y^2 y_n = 0 \qquad d\psi_{x,y} = \frac{ds}{\beta_{x,y}}$$

- All particles have the same frequencies: $\nu_x, \ \nu_y$
- There are two conserved quantities, corresponding to each degree of freedom – the so-called Courant-Snyder invariants

Theory of the Alternating-Gradient Synchrotron[1, 2]

E. D. Courant and H. S. Snyder

Brookhaven National Laboratory, Upton, New York

Received July 15, 1957

Non-linear focusing

- It became obvious very early on (~1960), that the use of nonlinear focusing elements in accelerators is necessary and some nonlinearities are **unavoidable** (magnet aberrations, space-charge forces, beam-beam forces)
  - Sexupoles appeared in 1960s for chromaticity corrections
  - Octupoles were installed in CERN PS in 1959 but not used until 1968. For example, the LHC has ~350 octupoles for Landau damping.
- It was also understood at the same time, that nonlinear focusing elements have both beneficial and detrimental effects, such as:
  - They drive nonlinear resonances (resulting in particle losses) and decrease the dynamic aperture (also particle losses).

Example: electron storage ring light sources

- Low beam emittance (size) is vital to light sources
  - Requires Strong Focusing
  - Strong Focusing leads to strong chromatic aberrations
  - To correct Chromatic Aberrations special nonlinear magnets (sextupoles) are added

dynamic aperture limitations lead to reduced beam lifetime

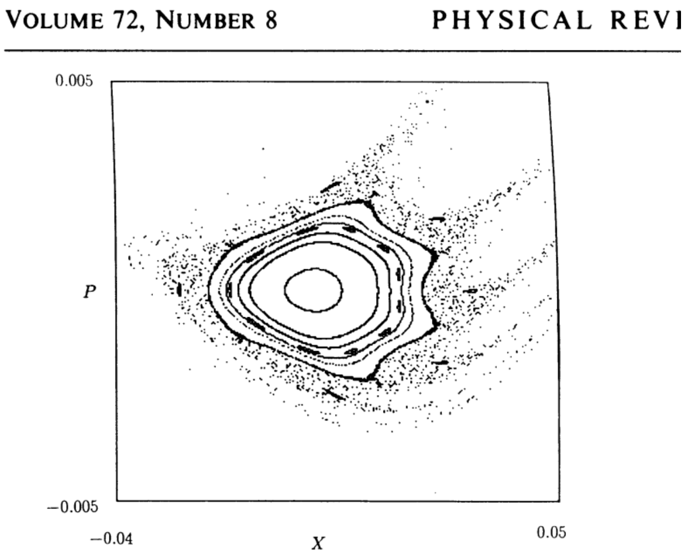


FIG. 1. Surface of section for the ALS.

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Example: Landau damping

COLLIDING BEAMS: PRESENT STATUS; AND THE SLAC PROJECT*

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The discovery in the early '60's at the Princeton-Stanford ring of what was thought to be the resistive wall instability brought the realization that circular accelerators are fundamentally unstable devices because of the interaction of the beam with its environment. Stability is achieved only through Landau damping and/or some external damping system.

Report at HEAC 1971

- Landau damping – the beam's "immune system". It is related to the spread of betatron oscillation frequencies. The larger the spread, the more stable the beam is against collective instabilities.
  - The spread is achieved by adding special magnets -- octupoles
- External damping (feed-back) system – presently the most commonly used mechanism to keep the beam stable.

Most accelerators rely on both

- LHC:
  - Has a transverse feedback system
  - Has 336 Landau Damping Octupoles
- Octupoles (an 8-pole magnet):

  - Octupole potential: $\varphi(x, y) \propto x^4 + y^4 - 6x^2y^2$
  - Results in a cubic nonlinearity (in force)

Note: this is a quartic Henon-Heiles potential, which is non-integrable

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + a\left(x^4 + y^4 - 6x^2y^2\right)$$

If we try to introduce a frequency spread (add nonlinearities):

$$x'' + K_x(s)x = S1(s)x^2 + S2(s)xy + S3(s)y^2 + O1(s)x^3 + \cdots$$

$$y'' + K_y(s)y = S4(s)x^2 + S5(s)xy + S6(s)y^2 + O2(s)y^3 + \cdots$$

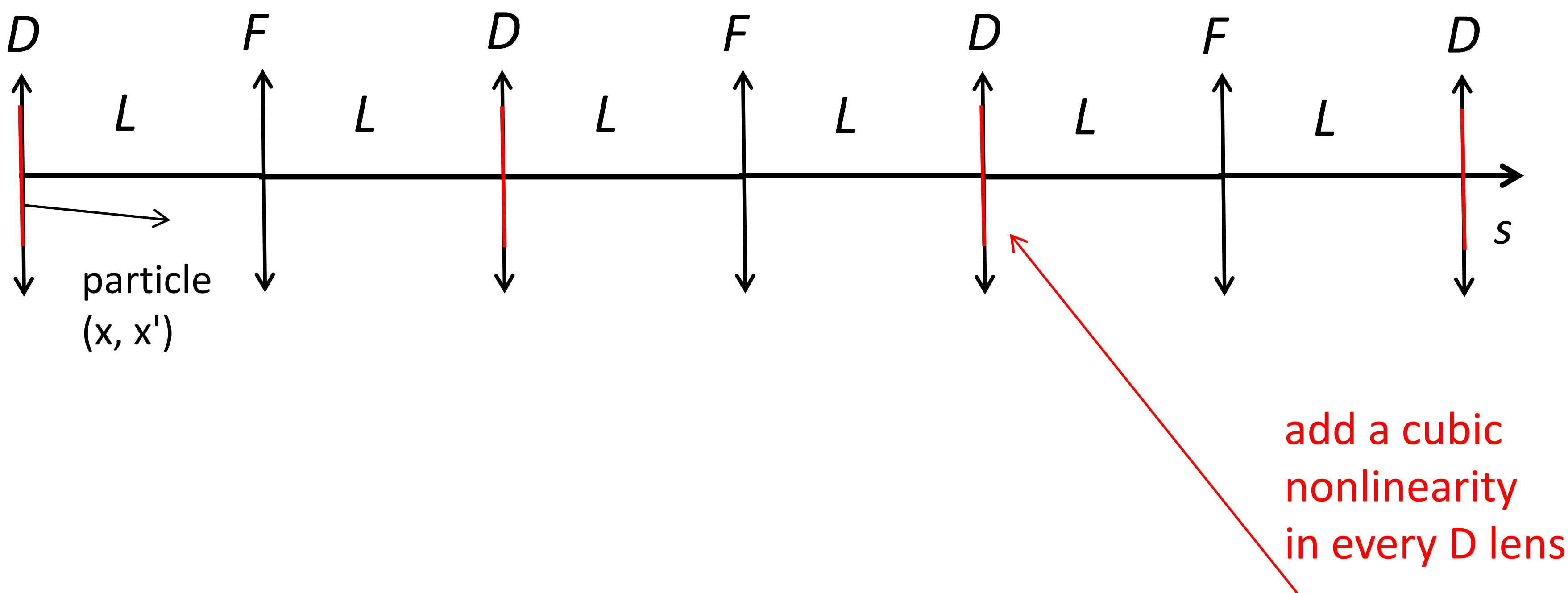
- Using the Courant-Snyder phase variables leads to:
  - Two coupled driven non-linear oscillators

$$\frac{d^2 x_n}{d\psi_x^2} + \nu_x^2 x_n = F_x\left(x_n^{\ 2}, y_n^{\ 2}, x_n y_n, \ldots, \psi_x, \psi_y\right)$$

$$\frac{d^2 y_n}{d\psi_y^2} + \nu_y^2 y = F_y\left(x_n^{\ 2}, y_n^{\ 2}, x_n y_n, \ldots, \psi_x, \psi_y\right)$$

  - These generally describe both regular, resonant and chaotic particle trajectories (depending on nonlinear terms and initial conditions)

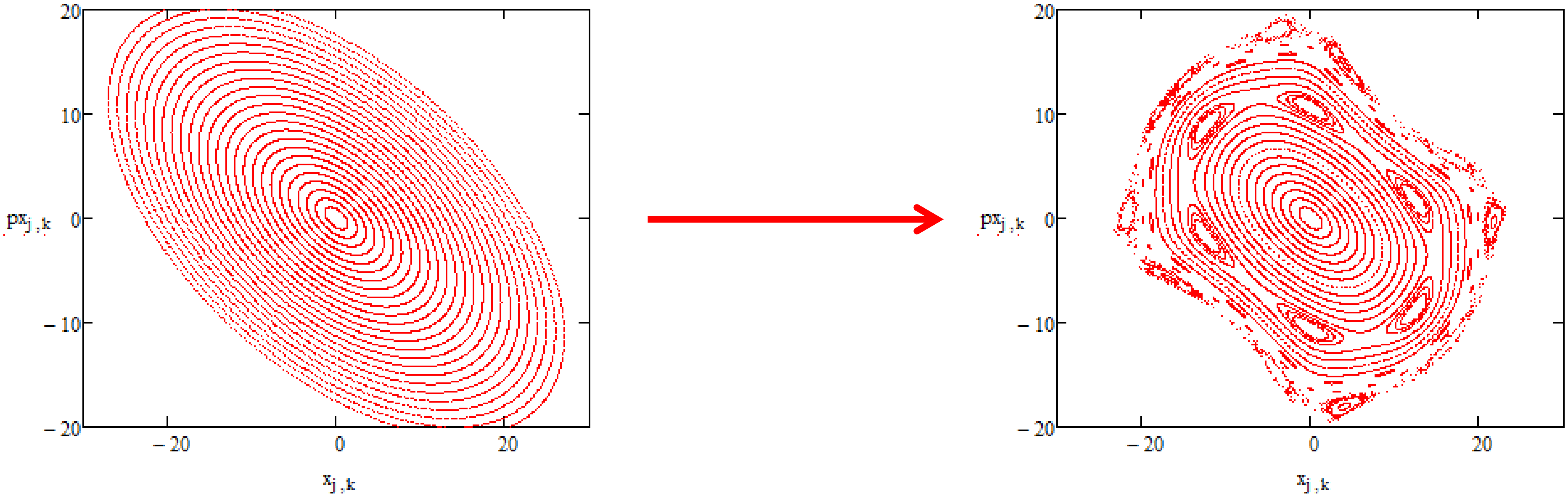
Let's add a cubic nonlinearity…



$$\begin{pmatrix} x \\ x' \end{pmatrix}_{i+1} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -F^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ F^{-1} & 1 \end{pmatrix} \begin{pmatrix} x \\ x' - \alpha x^3 \end{pmatrix}_i$$

The result of this nonlinearity:

- Betatron oscillations are no longer isochronous:
  - The frequency depends on particle amplitude (initial conditions)
- Stability depends on initial conditions
  - Regular trajectories for small amplitudes
  - Resonant islands (for larger amplitudes)
  - Chaos and loss of stability (for even larger amplitudes)



A 'model' accelerator mapping

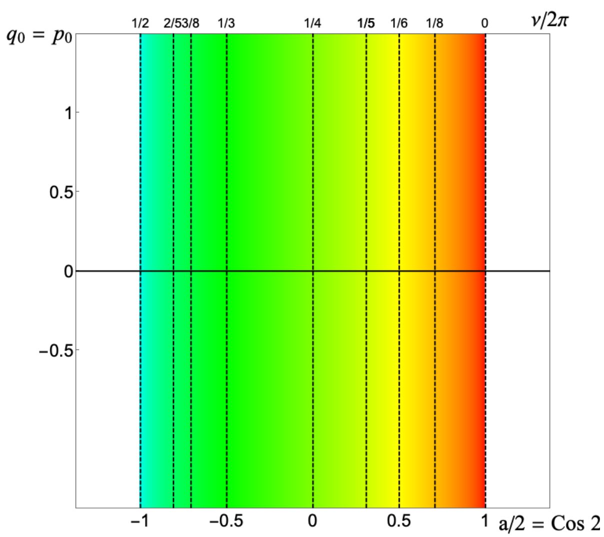
Dynamic aperture as a function of linear tune (betatron freq.)

$$2\pi\nu = \mathrm{acos}\left(\frac{a}{2}\right)$$

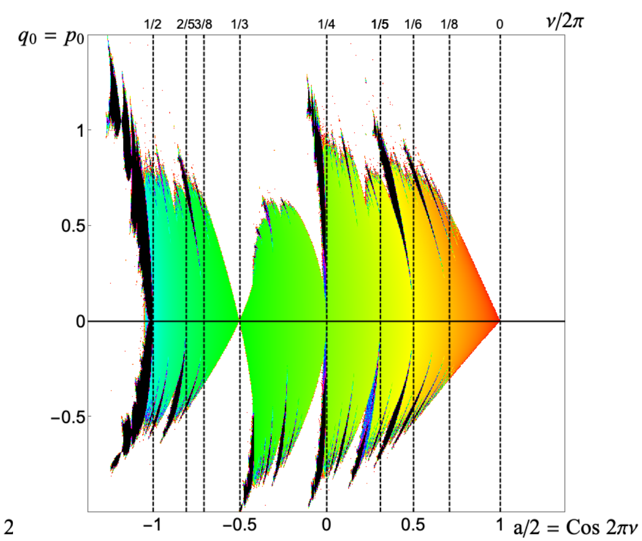
$$q_{n+1} = p_n$$
$$p_{n+1} = -q_n + a p_n$$

$$q_{n+1} = p_n$$
$$p_{n+1} = -q_n + a p_n + p_n^2$$

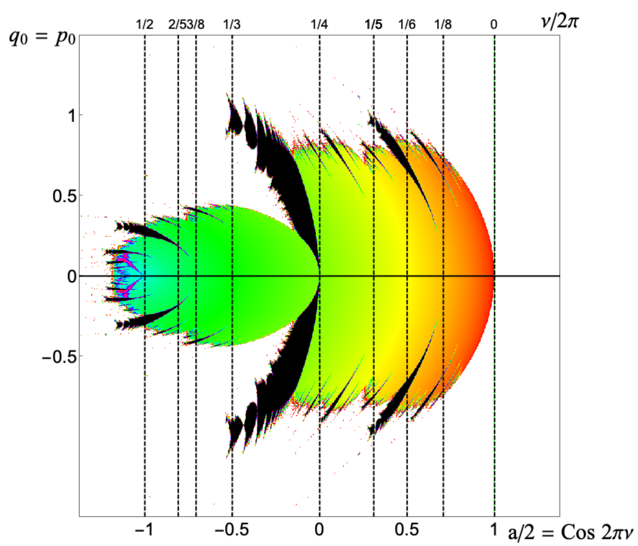
$$q_{n+1} = p_n$$
$$p_{n+1} = -q_n + a p_n + p_n^3$$



Linear



Sextupole

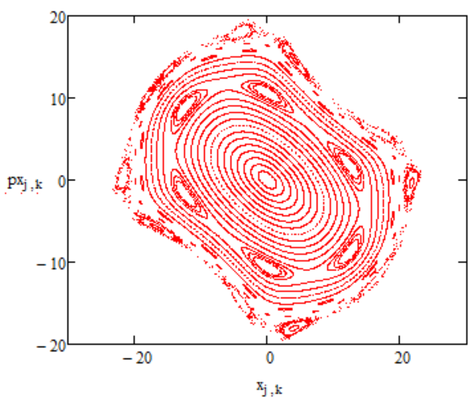


Octupole

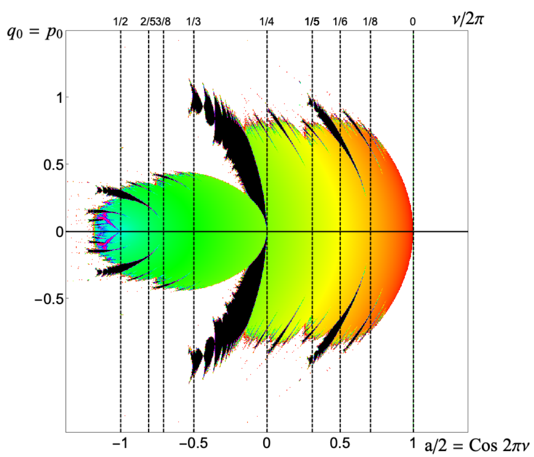
Are there "magic" nonlinearities with zero resonance strength?

- Yes, we call them "integrable"

Chaotic motion

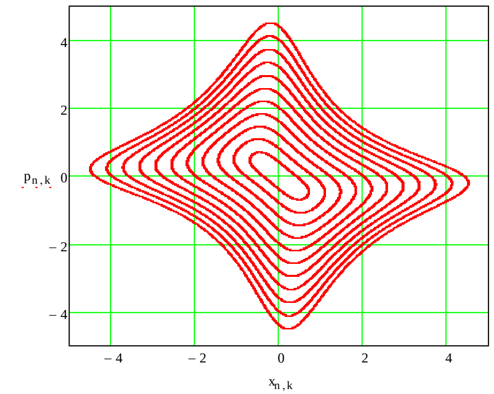


$$q_{n+1} = p_n$$
$$p_{n+1} = -q_n + ap_n + p_n^3$$

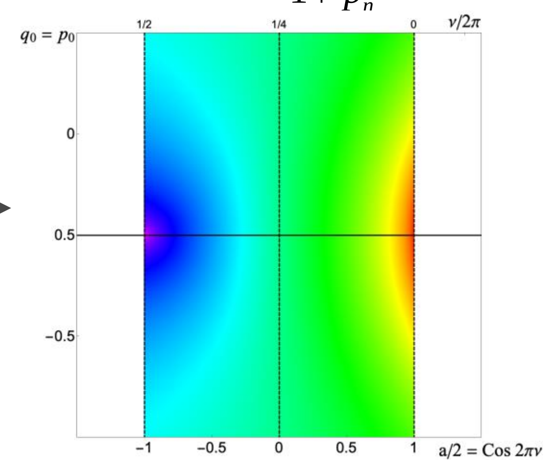


Octupole

Integrable motion



$$q_{n+1} = p_n$$
$$p_{n+1} = -q_n + \frac{ap_n}{1 + p_n^2}$$

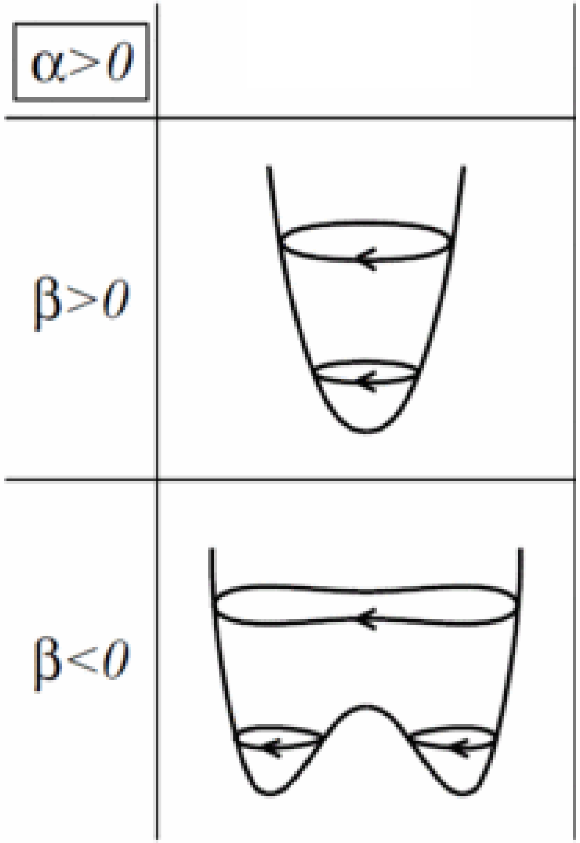


McMillan map

Non-accelerator example

- The unforced Duffing 1D oscillator

$$\ddot{x} + \beta x + \alpha x^3 = 0$$



Can we make something equivalent to this in accelerators in 2D and with *s*-dependent magnets that must also satisfy the Maxwell equations?

- Like the integrable Henon-Heiles potential

Accelerator research areas, where integrability would help

- Single particle dynamics:
    1. How to make the dynamical aperture larger? (light sources, colliders)
    2. How to make the tune spread larger? (Landau damping in high-intensity rings)
    3. How to reduce beam halo?
- Multi-particle dynamics:
    1. How to reduce detrimental beam-beam effects?
    2. How to compensate space-charge effects?
    3. How to suppress instabilities?
    4. How to reduce beam halo?

Specifics of accelerator focusing

- The transverse focusing system is effectively time-dependent
  - In a linear system (strong focusing), the time dependence can be transformed away by introducing 2 new “time” variables (the betatron phase advances). Thus, we have the Courant-Snyder invariant.

$$\frac{d^2 x_n}{d\psi_x^2} + \nu_x^2 x_n = 0 \qquad \frac{d^2 y_n}{d\psi_y^2} + \nu_y^2 y_n = 0 \qquad d\psi_{x,y} = \frac{ds}{\beta_{x,y}}$$

- The focusing elements we use in accelerator must satisfy:
  - The Laplace equation (for static fields in vacuum)
  - The Poisson equation (for devices based on charge distributions, such as electron lenses or beam-beam interaction)

Integrable nonlinearities in accelerators

- So far, we were able to find 2 classes of nonlinear accelerator-suitable systems;

1. Systems, where we are able to remove the time dependence, thus making it effectively autonomous.
   - This requires for the “time” variable to be the same in x and y. And then we can find some simple examples of autonomous “useful” integrable systems.
   - We know only a handful of examples in 4D
2. Systems, that are discrete integrable nonlinear mappings
   - This class originates from Edwin McMillan (the McMillan mapping).
   - We know only one example in 4D, suitable for accelerators.

Integrable Optics Concept Emerges

PHYSICAL REVIEW SPECIAL TOPICS - ACCELERATORS AND BEAMS **13**, 084002 (2010)

Nonlinear accelerator lattices with one and two analytic invariants

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(Received 3 March 2010; published 25 August 2010)

HB 2010 FERMILAB-CONF-10-390-AD

NONLINEAR OPTICS AS A PATH TO HIGH-INTENSITY CIRCULAR MACHINES*

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PAC'11 FERMILAB-CONF-11-114-AD-APC

RING FOR TEST OF NONLINEAR INTEGRABLE OPTICS

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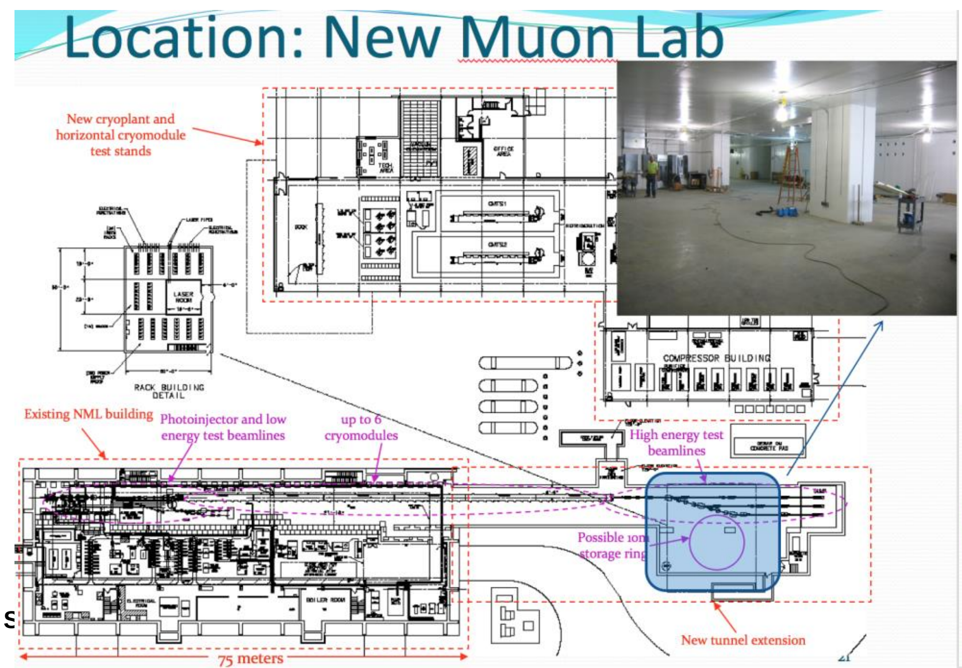
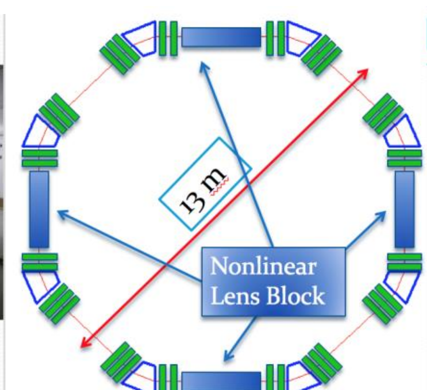
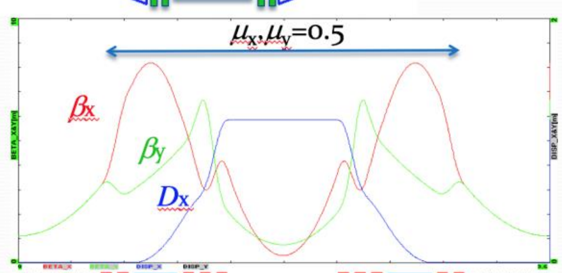




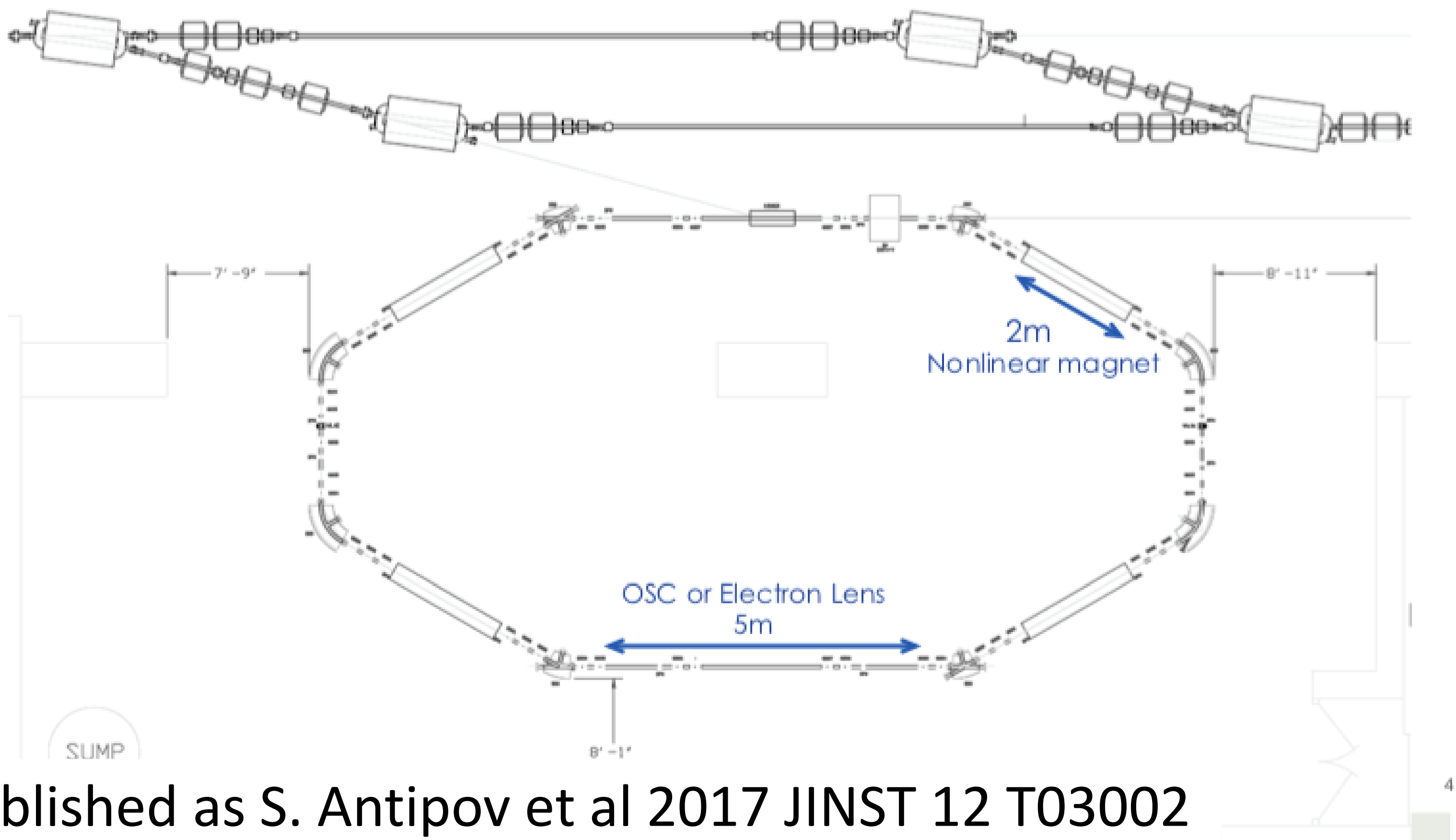
e- Energy	150 MeV
Circumference	38 m
Dipole field	0.5 T
Betatron tunes	Qx=Qy=3.2 (2.4 to 3.6)
Radiation damping time	1-2 s (10^7 turns)
Equilibrium emittance, rms, non-norm	0.06 μm



Nonlinear lens block	
Length	2.5 m
Number of elements	20
Element length	0.1 m
Max. gradient	1 T/m
Pole-to-pole distance (min)	~ 2 cm

IOTA Design – Layout

- In the ultimate integrable optics scenario 4 elements of periodicity (cells) with 4 2m-long drifts for nonlinear magnets
- 5m-long straight section for the Optical Stochastic Cooling experiment.



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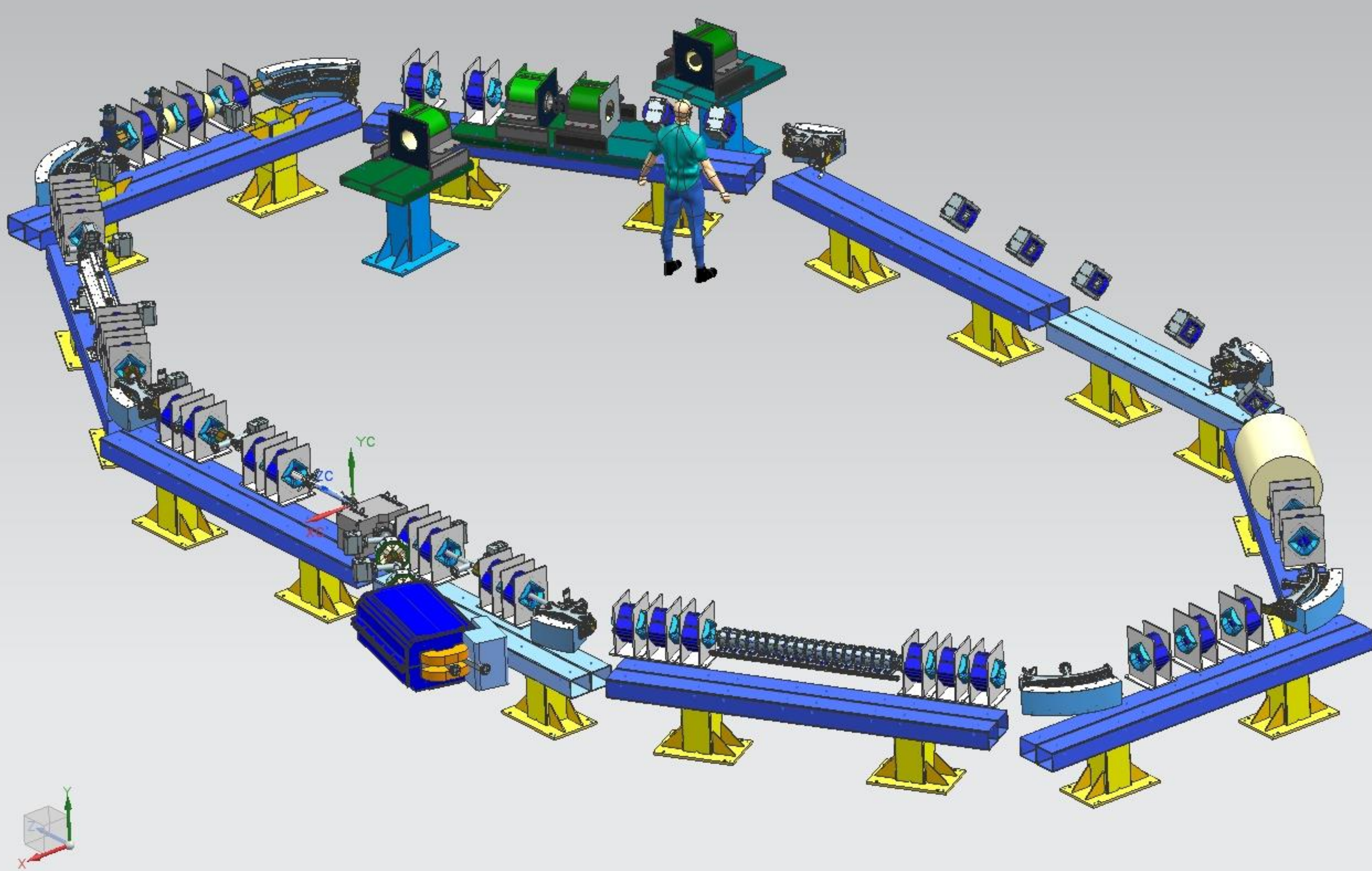
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IOTA Assembly Completed 7/29/2018

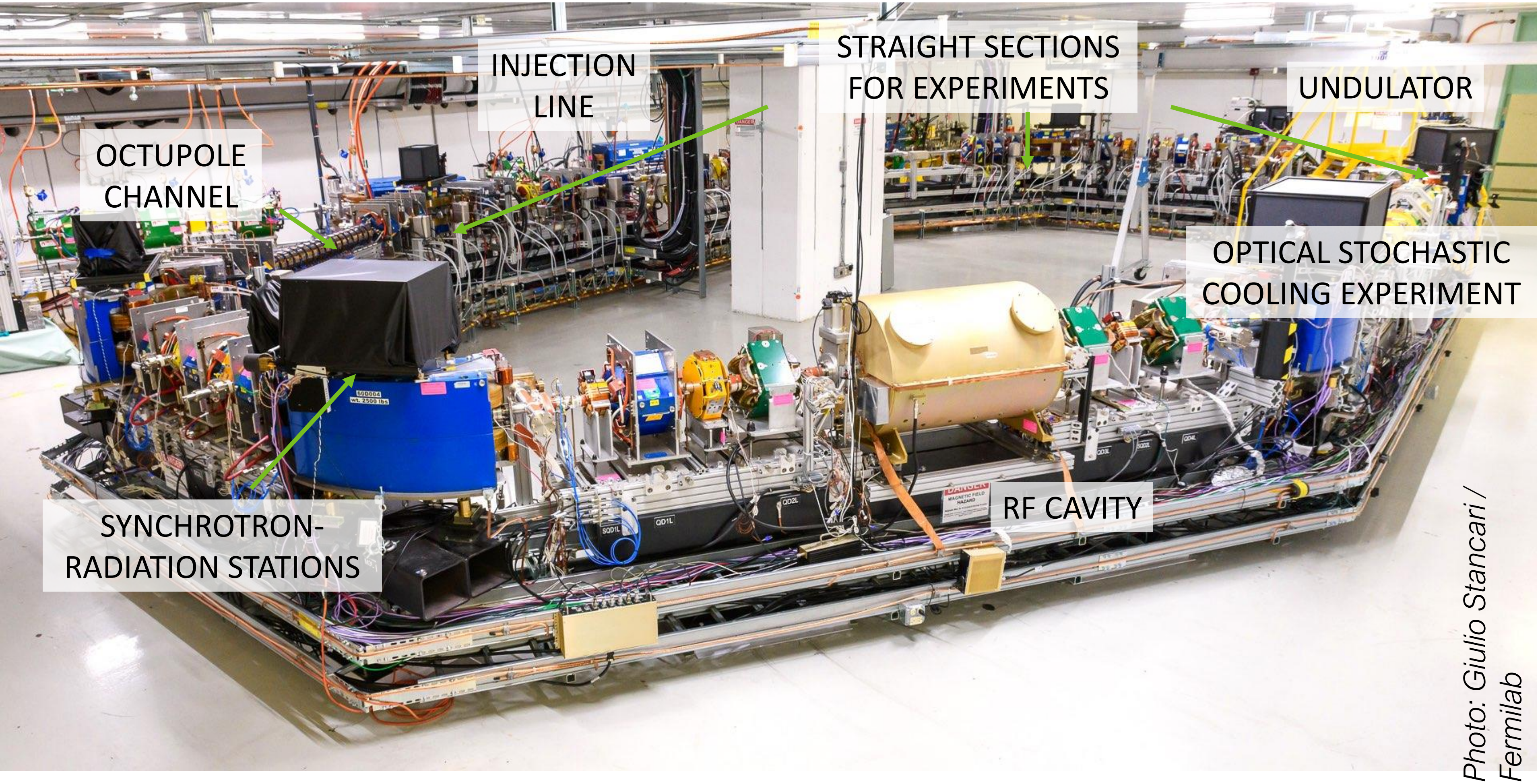


Photo: Giulio Stancari / Fermilab

The IOTA Storage Ring in May 2021

Example 1

- Conceptually, we now know how to make a focusing system (with quadrupoles and thin octupoles), which results in the following 2D integrable nonlinear Hamiltonian

$$H = \frac{1}{2}(p_{nx}^2 + p_{ny}^2) + \frac{1}{2}(x_n^2 + y_n^2) + \frac{\alpha}{4}\left(x_n^4 + y_n^4\right)$$

OR

$$H = \frac{1}{2}(p_{nx}^2 + p_{ny}^2) + \frac{1}{2}(x_n^2 + y_n^2) + \frac{\alpha}{4}\left(x_n^2 + y_n^2\right)^2$$

In normalized variables

$$x_n = \frac{x}{\sqrt{\beta(s)}},$$

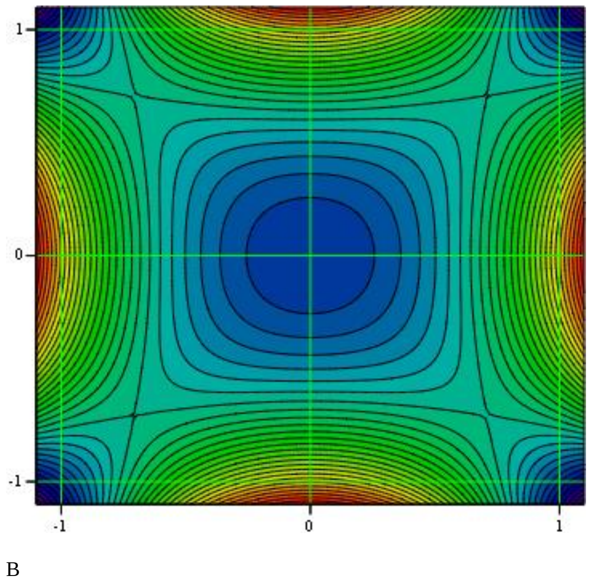
$$p_n = p\sqrt{\beta(s)} - \frac{\beta'(s)x}{2\sqrt{\beta(s)}},$$

- This concept is highly impractical but very important as it may serve as a model for modeling studies.

Example 2

- A nonlinear partially-integrable focusing system with one integral of motion. Can be implemented in practice (with octupoles). This system is being tested at Fermilab.
- A quartic Henon-Heiles system

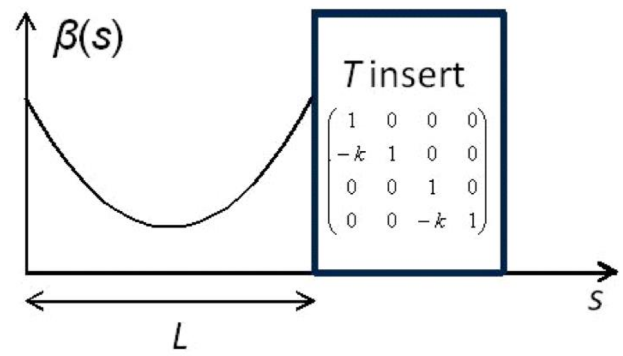
$$H = \frac{1}{2}(p_{nx}^2 + p_{ny}^2) + \frac{1}{2}(x_n^2 + y_n^2) + \frac{\alpha}{4}\left(x_n^4 + y_n^4 - 6x_n^2 y_n^2\right)$$



Implementation concept

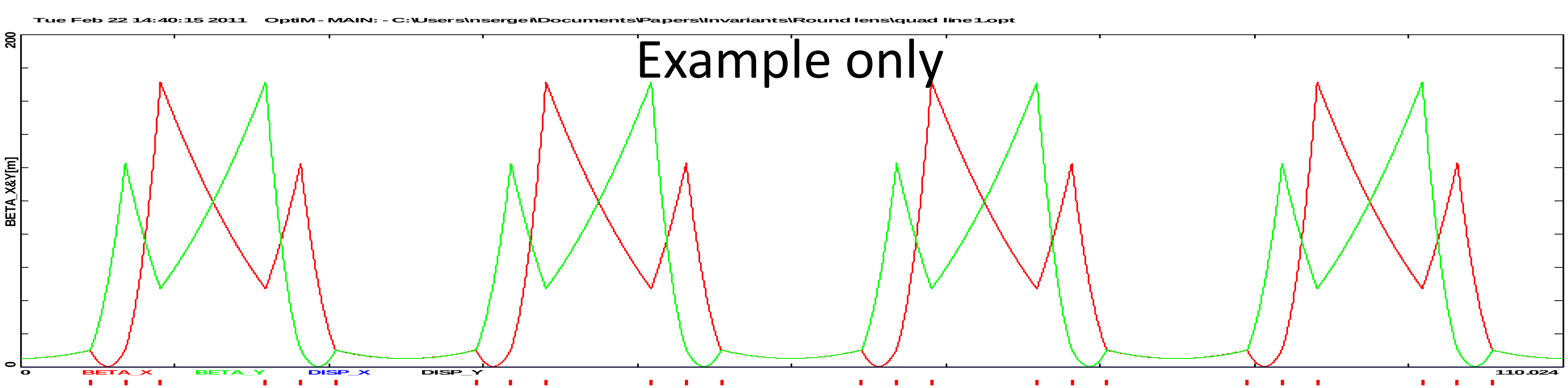
1 Start with a round axially-symmetric *linear* lattice (FOFO) with the element of periodicity consisting of

a. Drift L
b. Axially-symmetric focusing block “T-insert” with phase advance n×π

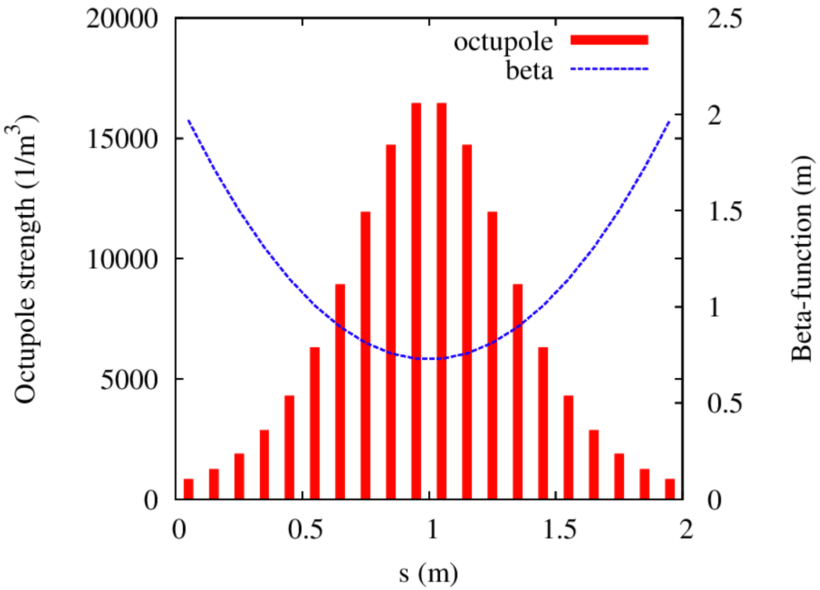


2 Add special nonlinear potential *V(x,y,s)* in the drift such that

$$\Delta V(x,y,s) \approx \Delta V(x,y) = 0$$

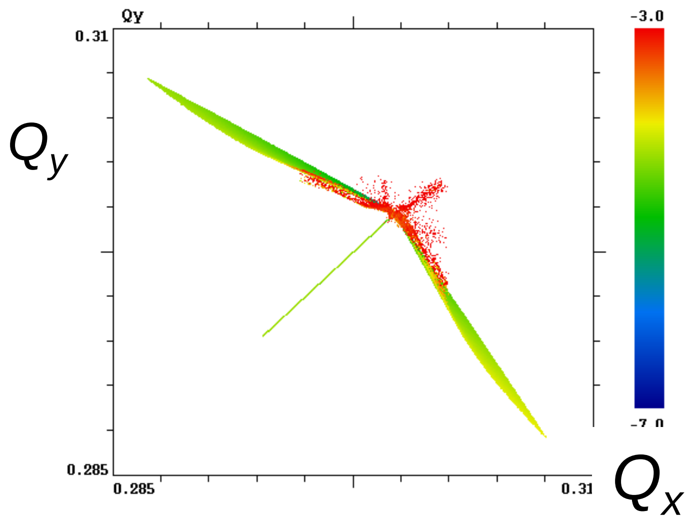


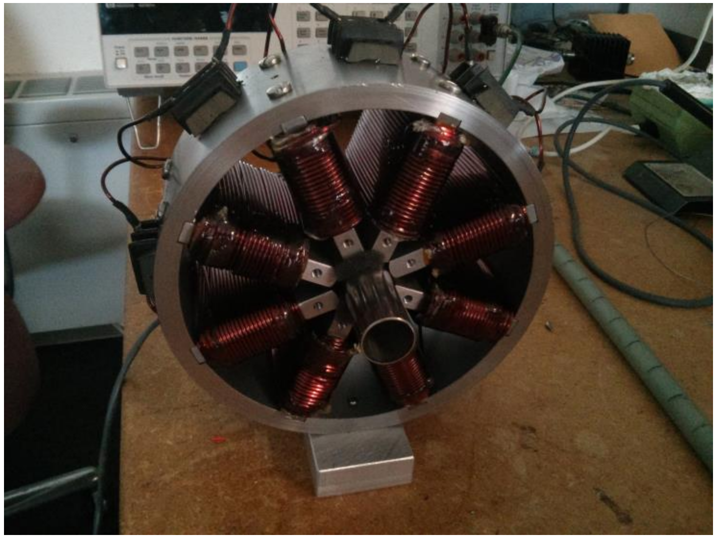
Implementation in IOTA

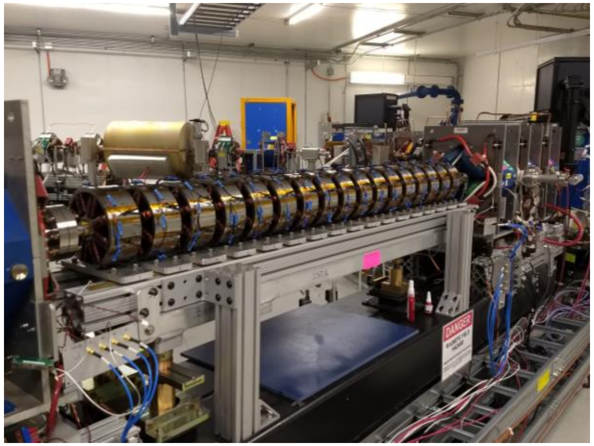


20 octupoles, scaled as $1/\beta(s)^3$

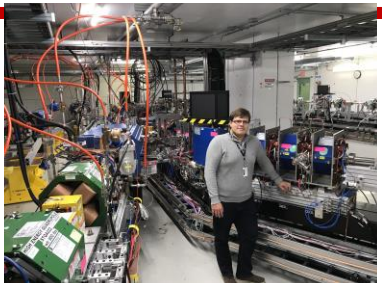
- While the dynamic aperture is limited, the attainable tune spread is large ~0.03 – compare to 0.001 created by LHC octupoles







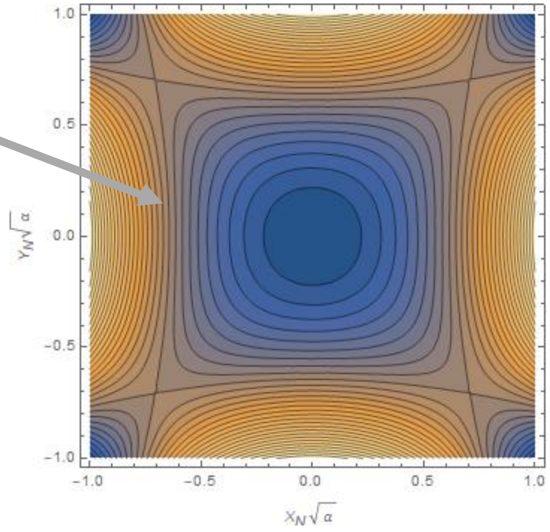
Henon-Heiles Type System with Octupoles (N.Kuklev (grad student 2021), U.Chicago)

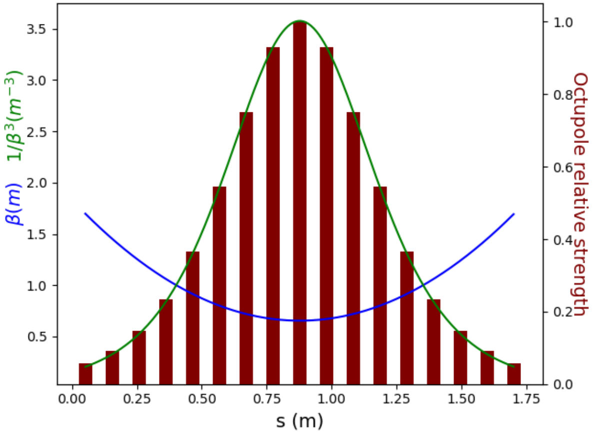


- One invariant of motion, ‘non-integrable’

$$H = H_0 + U = \frac{1}{2}\left(P_x^2 + P_y^2 + x_N^2 + y_N^2\right) + \alpha\left(\frac{x_N^4}{4} + \frac{y_N^4}{4} - \frac{3x_N^2 y_N^2}{2}\right)$$

- Theoretical stability limit – $1/\sqrt{2\alpha}$
  - (lower due to chaotic layer, $\sim 0.6/\sqrt{\alpha}$)
  - Tune spread = 0.4
- Implementation in IOTA with discrete elements
  - Imperfections complicate the dynamics
  - Performance predictions (at DA limit)
    - 0.12 ideal case
    - 0.08 for 18 octupoles

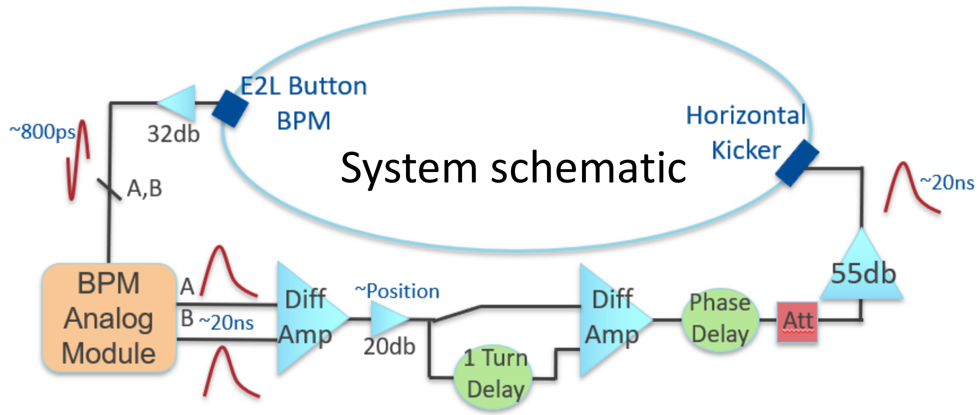


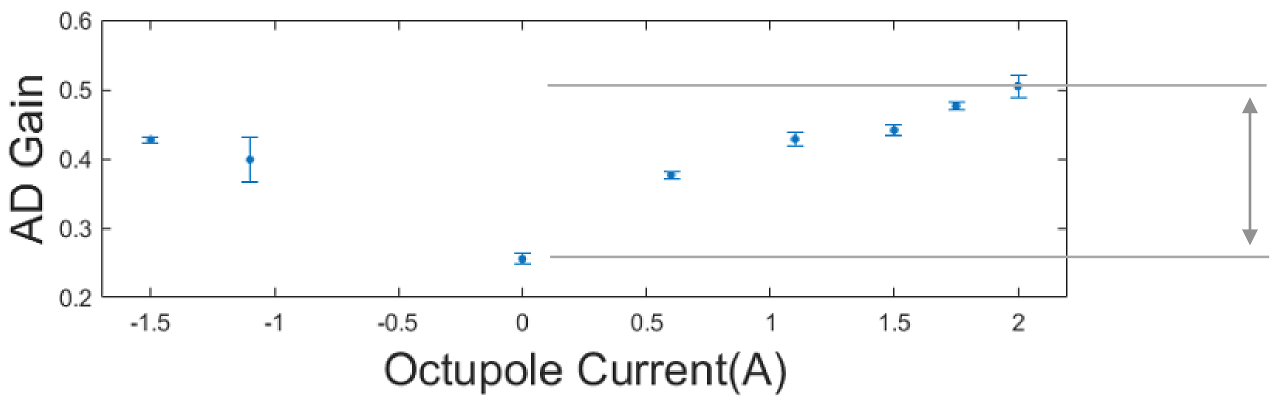


Suppression of Coherent Instability via Landau Damping

Experiment:

- Artificially induce controlled instability with a feedback system
- Study the effect of nonlinear optics on instability thresholds





2x increase of stability threshold !!!

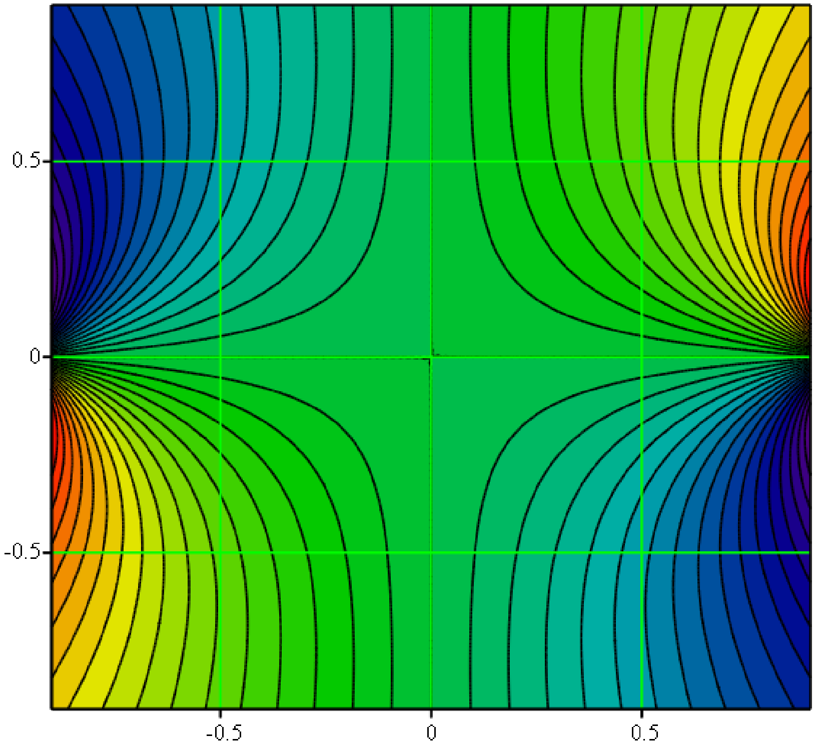
Example 3: (Danilov and Nagaitsev, Phys. Rev. Accel. Beams 13, 084002)

- An integrable nonlinear system with a special Darboux potential (separable in elliptic coordinates).

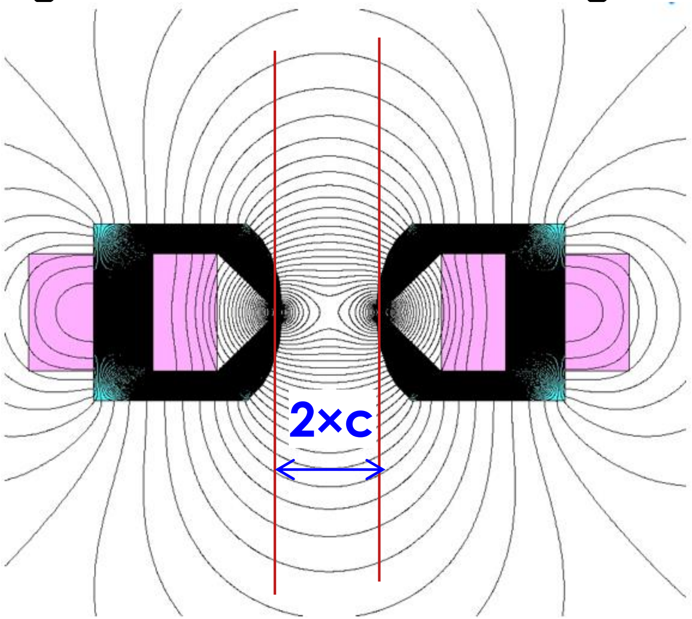
$$H = \frac{1}{2}(p_{nx}^2 + p_{ny}^2) + \frac{1}{2}(x_n^2 + y_n^2) + U(x_n, y_n)$$

$$U(x,y) \approx \frac{t}{c^2}\,\mathrm{Im}\left((x+iy)^2 + \frac{2}{3c^2}(x+iy)^4 + \frac{8}{15c^4}(x+iy)^6 + \frac{16}{35c^6}(x+iy)^8 + \ldots \right)$$

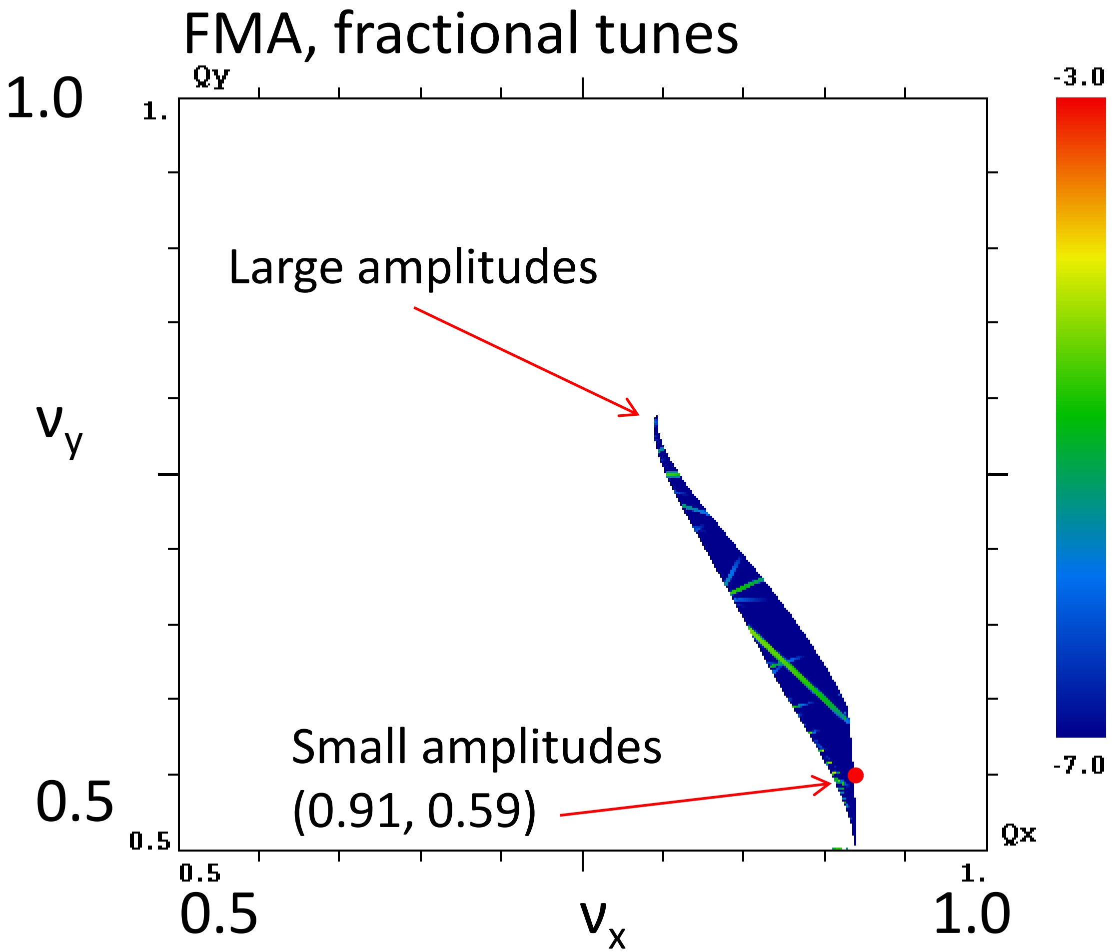
For $|z| < c$

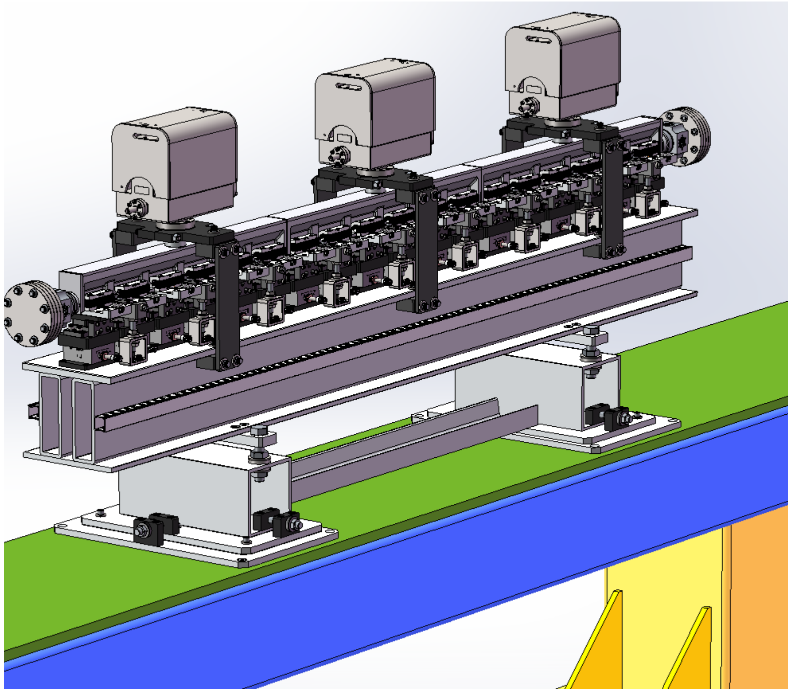


This potential has two adjustable parameters: t – strength and c – location of singularities



A single 2-m long nonlinear lens creates a tune spread of ~0.25.



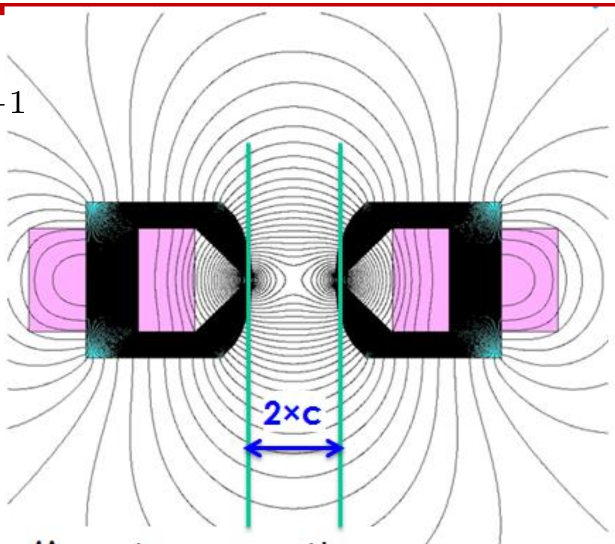


1.8-m long magnet to be delivered in 2016

Jefferson Lab

Two integrals of motion

$$B_y + iB_x = -t\frac{B\rho}{\beta(s)}\sum_{n=1}^{\infty}\frac{2^{2n-1}n!(n-1)!c}{(2n-1)!\sqrt{\beta(s)}}\left(\frac{x+iy}{c\sqrt{\beta(s)}}\right)^{2n-1}$$



Magnet cross section
V.Kashikhin

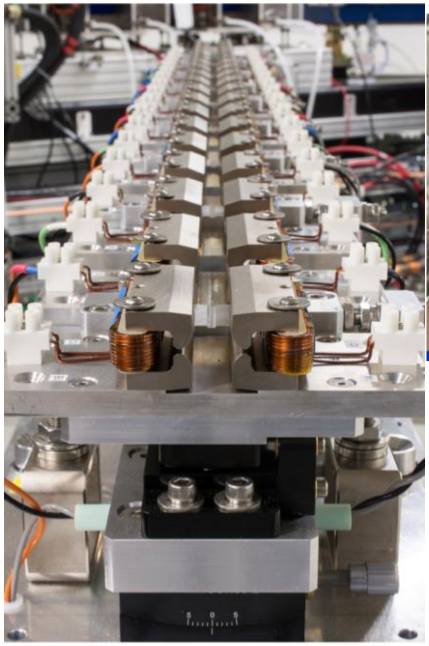
[1] V. Danilov and S. Nagaitsev, PRAB 13, 084002 (2010)

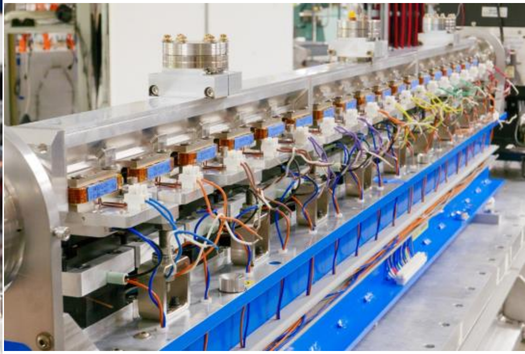
Two integrals of motion :

$$H_\perp = \frac{1}{2}(P_x^2 + P_y^2) - \frac{\tau c^2}{\beta(s)}U\left(\frac{X}{c\sqrt{\beta(s)}}, \frac{Y}{c\sqrt{\beta(s)}}\right)$$

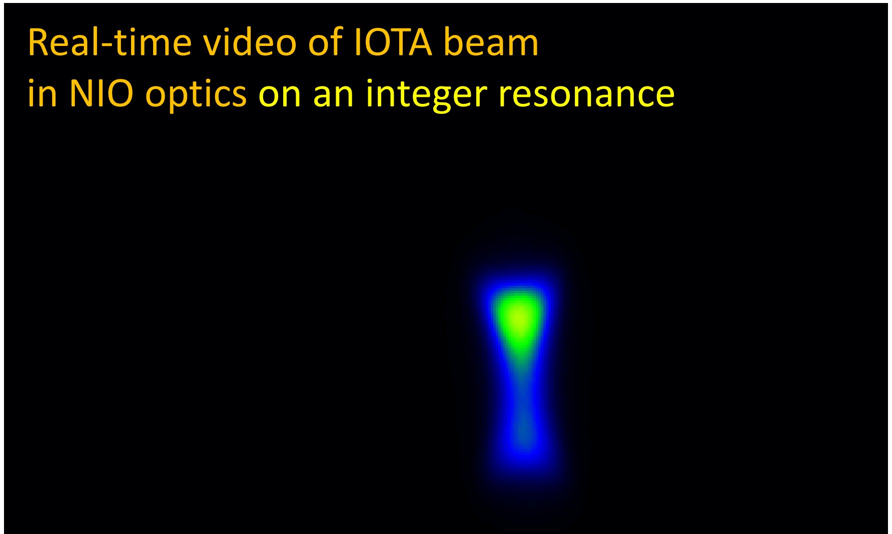
$$I = \left(xp_y - yp_x\right)^2 + c^2p_x^2 + \frac{2c^2t\cdot\xi\eta}{\xi^2-\eta^2} \times$$

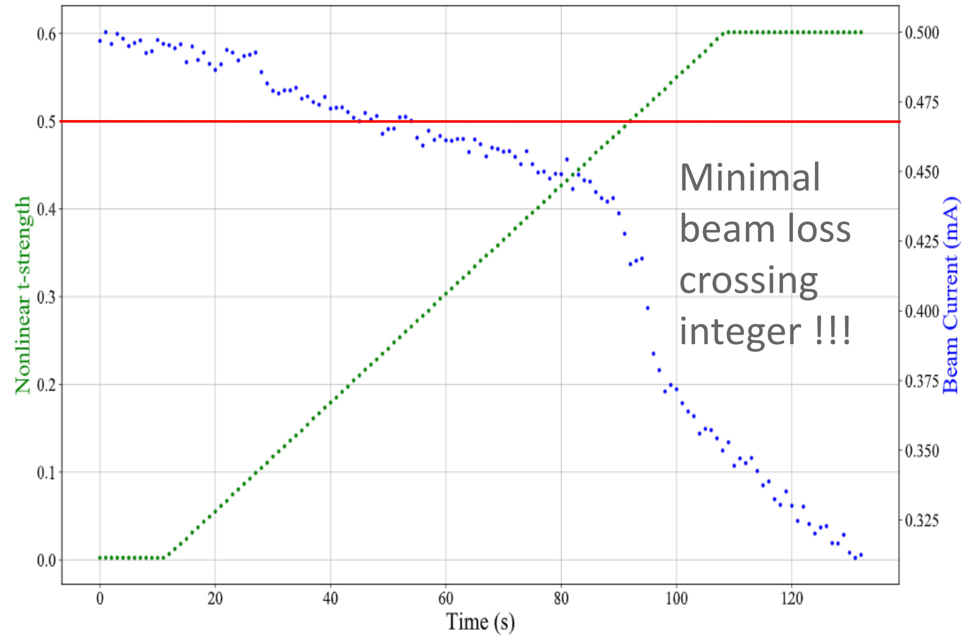
$$\left(\eta\sqrt{\xi^2-1}\cosh^{-1}(\xi) + \xi\sqrt{\eta^2-1}\left(\frac{\pi}{2}+\cosh^{-1}(\eta)\right)\right)$$

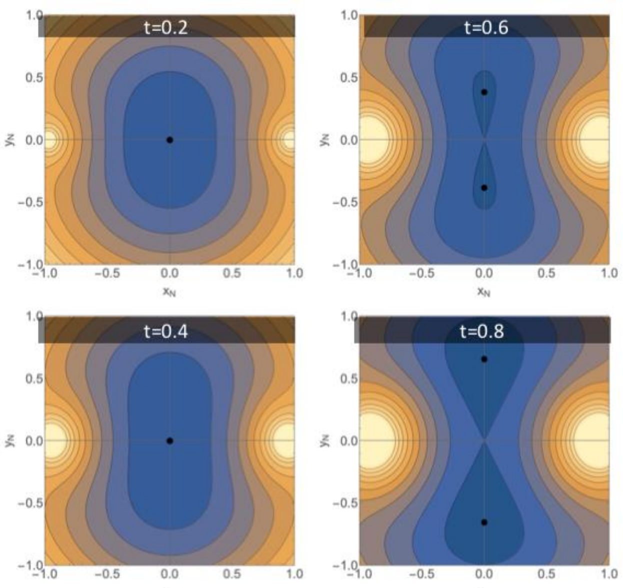




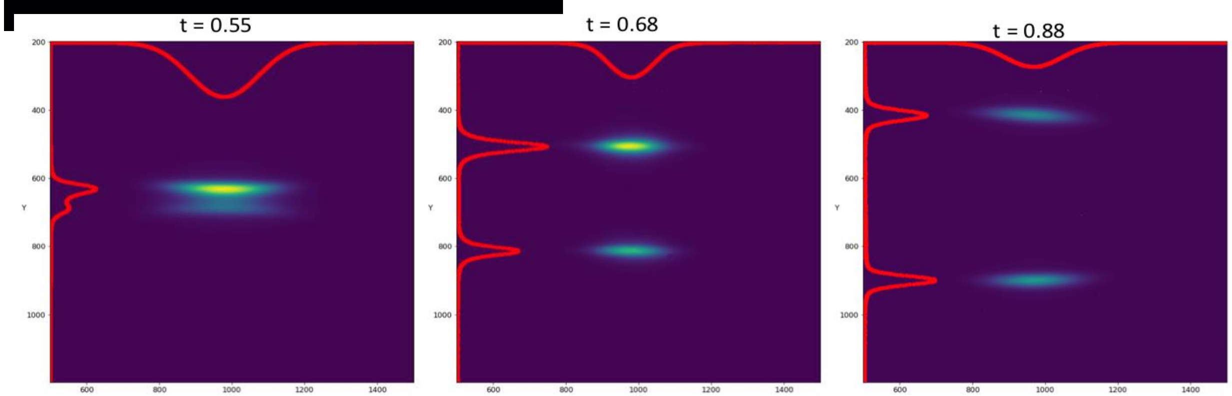
IOTA Beam on an integer resonance !!!







Theory / Model



IOTA Experiment

Example 4: McMillan mapping

- In 1967 E. McMillan published a paper

SOME THOUGHTS ON STABILITY
IN NONLINEAR PERIODIC FOCUSING SYSTEMS

Edwin M. McMillan

September 5, 1967

- Final report in 1971. This is what later became known as the "McMillan mapping":

$$x_i = p_{i-1}$$
$$p_i = -x_{i-1} + f(x_i)$$

$$f(x) = -\frac{Bx^2 + Dx}{Ax^2 + Bx + C}$$

$$Ax^2p^2 + B\left(x^2p + xp^2\right) + C\left(x^2 + p^2\right) + Dxp = \text{const}$$

If $A = B = 0$ one obtains the Courant-Snyder invariant

McMillan 1D mapping

- At small x: $f(x) \to -\frac{D}{C}x$

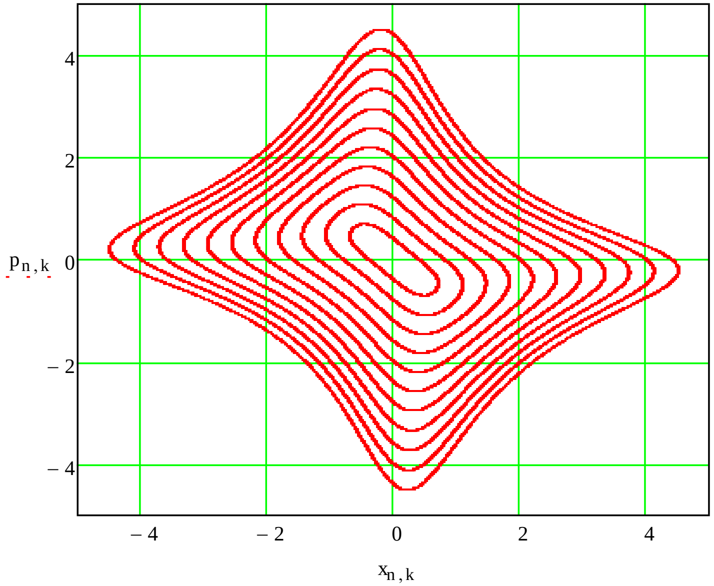
$$f(x) = -\frac{Bx^2 + Dx}{Ax^2 + Bx + C}$$

Linear matrix: $\begin{pmatrix} 0 & 1 \\ -1 & -\frac{D}{C} \end{pmatrix}$ Bare tune: $\frac{1}{2\pi}\mathrm{acos}\left(-\frac{D}{2C}\right)$

- At large x: $f(x) \to 0$

Linear matrix: $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ Tune: 0.25

A=1, *B* = 0, *C* = 1, D = 2



McMillan mapping in 2d

- We were unable to extend this mapping into 2d with magnets (Laplace equation).
- We have a solution on how to realize such a lens with a charge column (Poisson equation).

1. A ring with a transfer matrix

$$\begin{pmatrix} cI & sI \\ -sI & cI \end{pmatrix} \begin{pmatrix} 0 & \beta & 0 & 0 \\ -\frac{1}{\beta} & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & -\frac{1}{\beta} & 0 \end{pmatrix}$$

$$c = \cos(\phi)$$
$$s = \sin(\phi)$$
$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

2. An axially-symmetric kick

$$f(r) = \frac{kr}{ar^2 + 1}$$

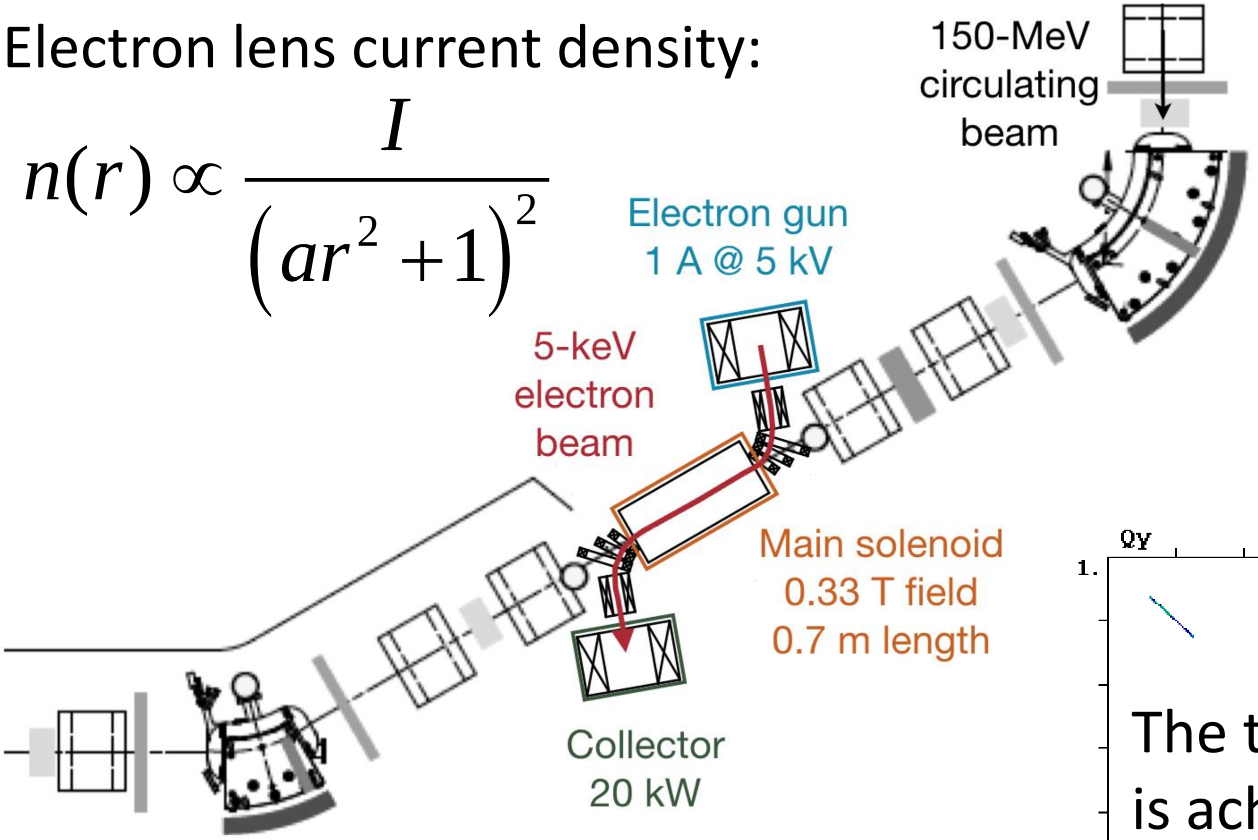
can be created with an electron lens

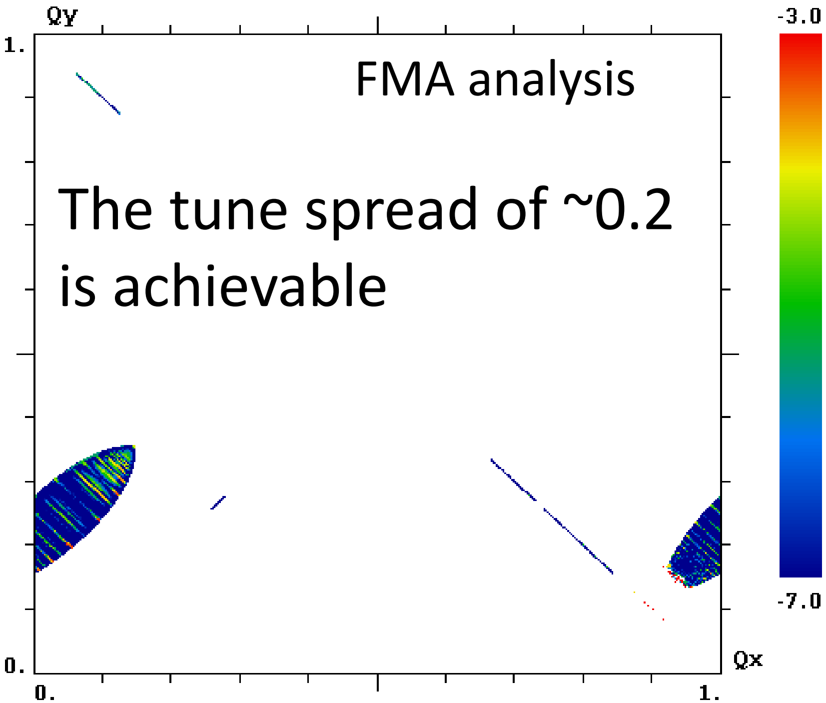
$$x_i = p_{i-1}$$
$$p_i = -x_{i-1} + f(x_i)$$

McMillan electron lens (future experiment at IOTA)

Electron lens current density:

$$n(r) \propto \frac{I}{\left(ar^2+1\right)^2}$$





Final thoughts and unanswered questions

1. Some nonlinear integrable systems are better than others. Which ones are most suitable for accelerators?

2. We need more examples of accelerator-suitable 4D integrable mappings.

3. How to “correct” the existing nonlinearities in a ring to improve integrability?

4. How to compensate a distributed nonlinear force from space charge of the beam itself with a localized nonlinear element?

Summary: Integrability in Accelerators

- All present machines are designed to be integrable: drifts, quadrupoles, dipoles-- can all be accommodated in the Courant-Snyder invariants.
  - These are all examples of linear systems (equivalent to a harmonic oscillator)
- The addition of nonlinear focusing elements to accelerators breaks the integrability, …but this additions are necessary and unavoidable in all modern machines – for chromatic corrections, Landau damping, strong beam-beam effects, space-charge, etc
- There are ‘magic’ nonlinearities that result in integrable dynamics. Such systems are now being explored experimentally at IOTA.