CTEQ / CFNS 2023
Stony Brook
George Sterman

Lecture III (June 7)

- **A taste of our experimental knowledge of g_1**
- **As time permits . . .**

Factorization and Evolution in More Detail

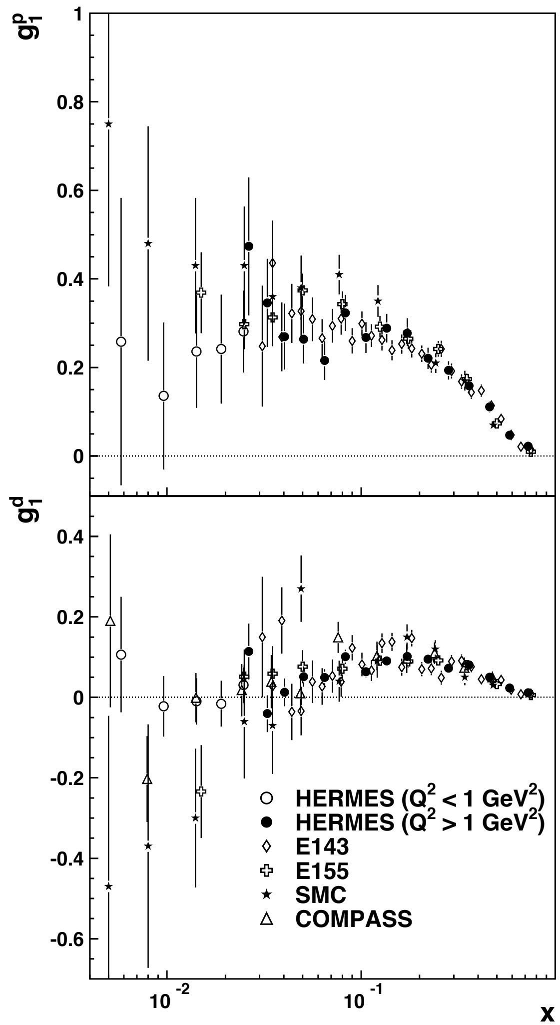
A. Factorization in DIS

B. DIS at one loop

C. (DGLAP) Evolution

Appendix: Factorization in hadron-hadron scattering

- **Measurement of g_1 (Proton and deuteron, from various experiments)**



- **Unified figure from Hermes Collaboration: Phys.Rev.D 75 (2007) 012007 e-Print: hep-ex/0609039 [hep-ex]**

- **A focus of spin physics: the Bjorken Sum Rule:**

$$\int_0^1 d\xi \, \left(g_1^p(\xi, Q) - g_1^n(\xi, Q)\right) \; = \; \frac{1}{6} g_A \, (1 + \mathcal{O}(\alpha_s(Q)))$$

  - $g_1^{p,n}$ **on LHS from parton distributions** $\Delta f_{a/n,p}$ **for** u, d, s, Δg ..., **but we expect** s **and** g **parts to cancel in the "non-singlet" difference – only** Δu **and** Δd **left.**

  - **RHS from neutron beta decay,** $n \rightarrow p + e + \bar{\nu}_e$ **with QCD corrections from factorization (the** C_a **for** g_1**s).**

  - **Fairly well confirmed by experiment, although there seems to be a need of contributions from** x **too small to be measured by experiments so far (we'll probably have to wait for the EIC).**

– The Bjorken sum rule is considered a firm prediction of QCD as we understand it. Analogous relations for $g_1^{p,n}$ separately require more knowledge of Δg, Δs etc. Classic predictions are from *Ellis-Jaffe sum rules*, which require input from hyperon (Λ, Σ) decays – applications require further assumptions, less well understood.

– Historically, working backwards from g_1 measurements using hyperon decay information suggested that Δu and Δd were much less that 1 – often referred to as "spin crisis". By now new measurements, lattice results and interpretation suggest a more balanced sharing of spin between quarks, gluons and orbital angular momentum.

– Understanding orbital angular momentum requires going beyond collinear PDFs . . .

– **From these and other data, recent fits (DSSV and NNPDF, shown in de Florian et al., Phys.Rev.D 100 (2019) 11, 114027 e-Print: 1902.10548 [hep-ph])**

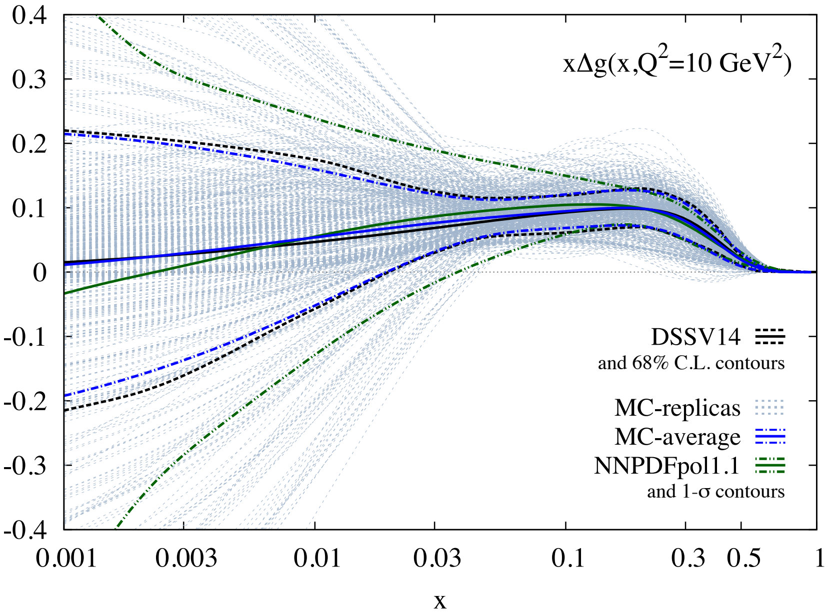


FIG. 1: The ensemble of replicas (dotted blue lines) for the NLO gluon helicity density $\Delta g(x, Q^2)$ at $Q^2 = 10\,\mathrm{GeV}^2$ shown along with its statistical average (solid blue line) and variance (dot-dashed blue lines). The corresponding results from the DSSV14 fit (black lines) and the NNPDFpol1.1 analysis (green lines) are shown for comparison; see text.

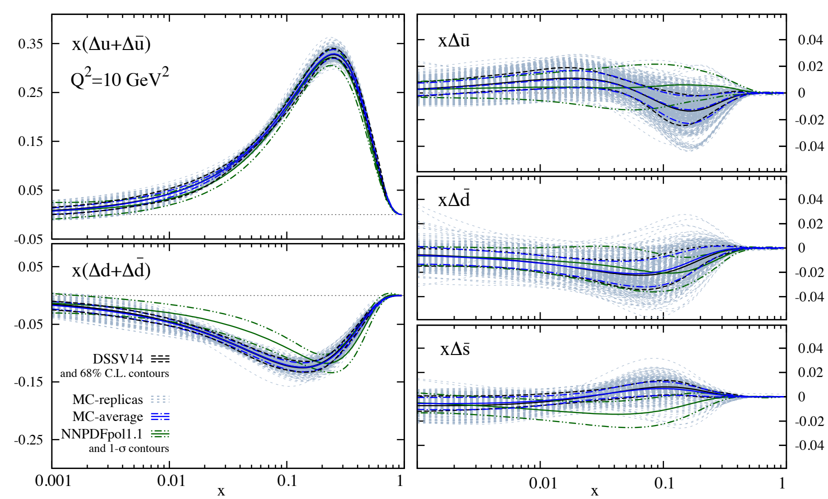


FIG. 2: Same as Fig. 1 but now showing our results for the quark and antiquark helicity PDFs at $Q^2 = 10\,\mathrm{GeV}^2$ in comparison to the analyses of DSSV14 and NNPDFpol1.1.

A. Factorization in DIS

- **Challenge: use AF in observables σ (cross sections, also some amplitudes) that are not infrared safe**

- **Possible if: σ has a short-distance subprocess. Separate *IR Safe* from IR: this is factorization**

- **IR Safe part (short-distance) is calculable in pQCD**

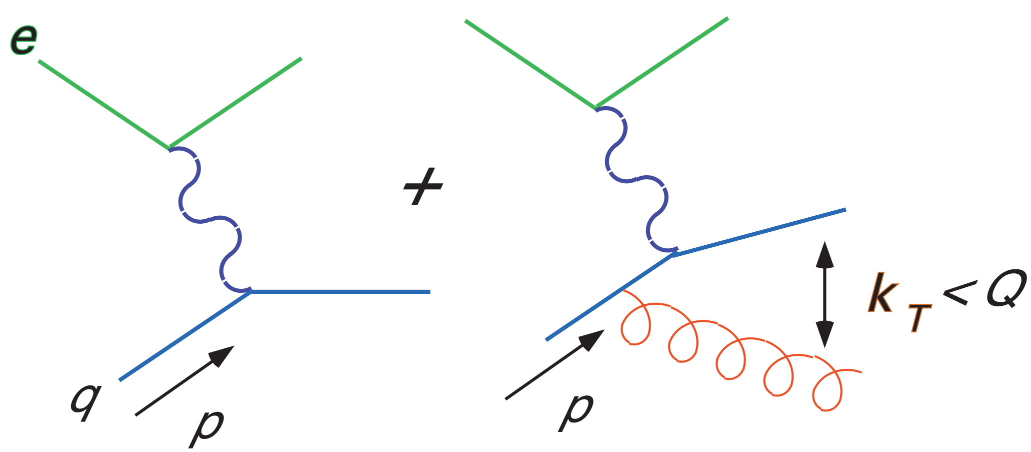
- **Infrared part – example: parton distribution – measureable and universal**

- **Infrared safety – insensitive to soft gluon emission collinear rearrangements**

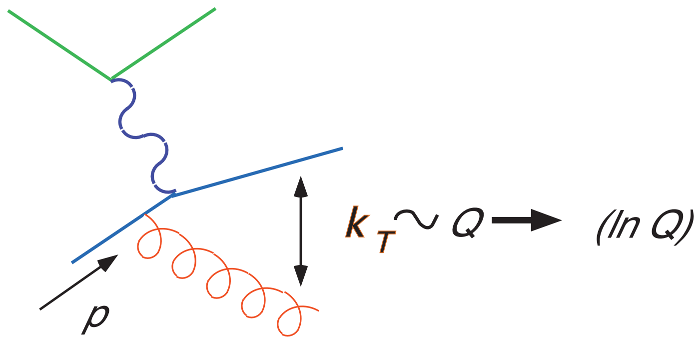
- For DIS, find a result . . .
- Just like Parton Model except in Parton Model the infrared safe part is $\sigma_{\mathrm{LO}} \Rightarrow \phi(x)$ normalized uniquely

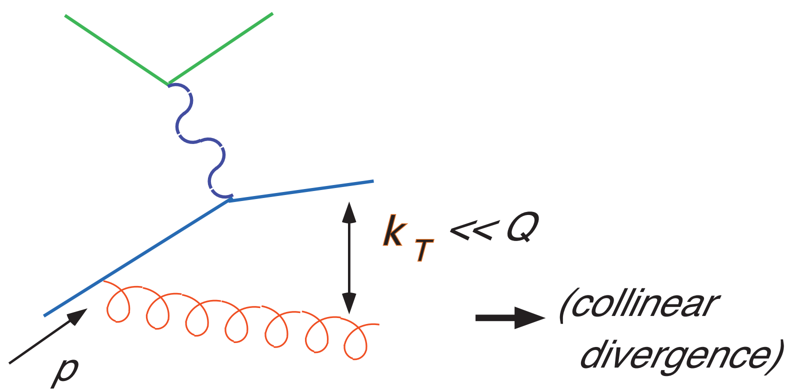
- In pQCD must define parton distributions more carefully: the factorization scheme

- Basic observation: virtual states are not truly frozen. Some states fluctuate on scale $1/Q$ . . .



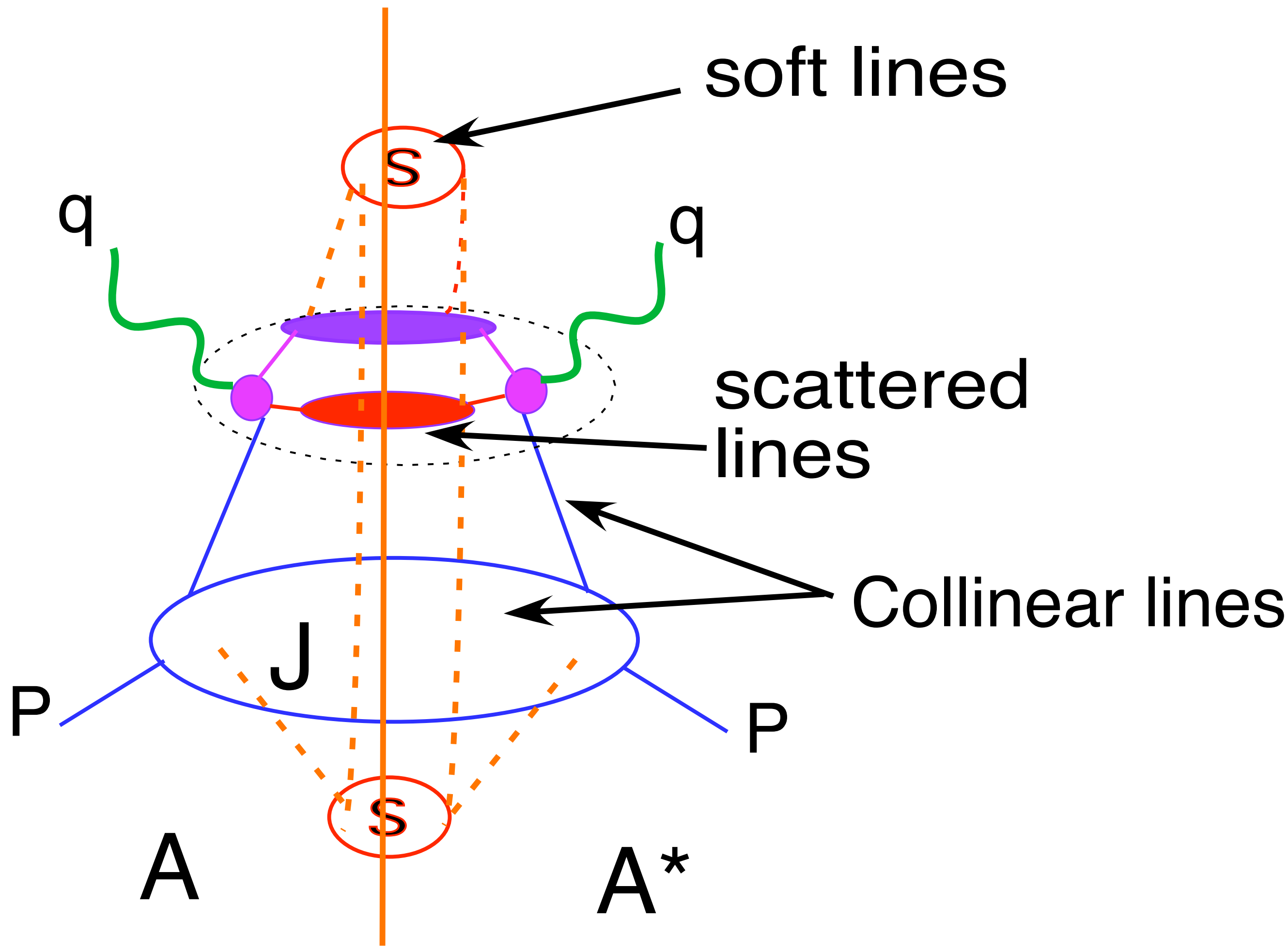
Short-lived states, which give $\ln(Q)$





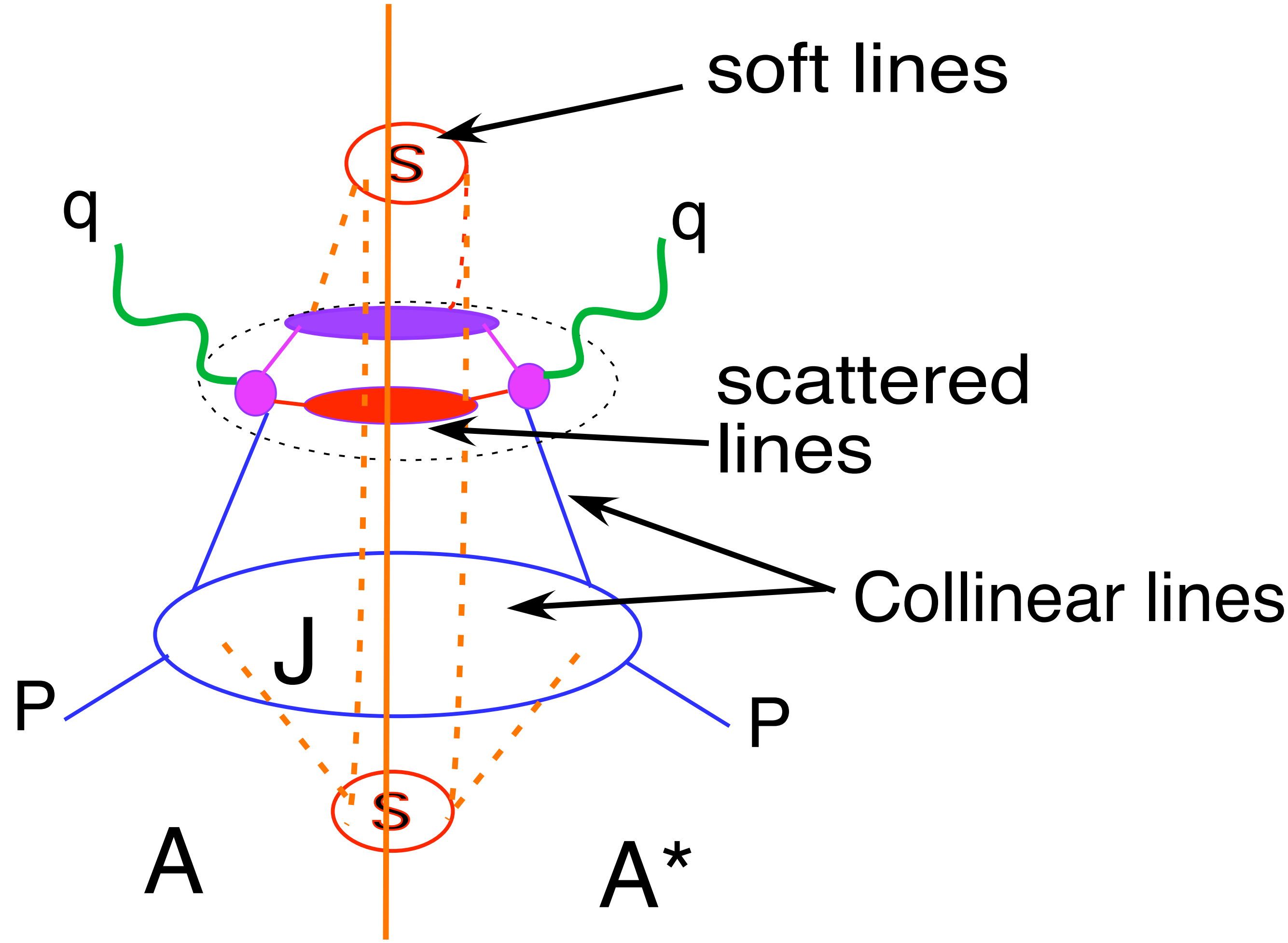
- **Longer-lived states $\Rightarrow$ Collinear Singularity (IR)**
- **How we systematize to all orders in perturbation theory . . . a taste of "all-orders" proofs in pQCD.**

- **We can generalize to all IR singularities (logarithms).** **"Rule": only classical processes with on-shell particles.**



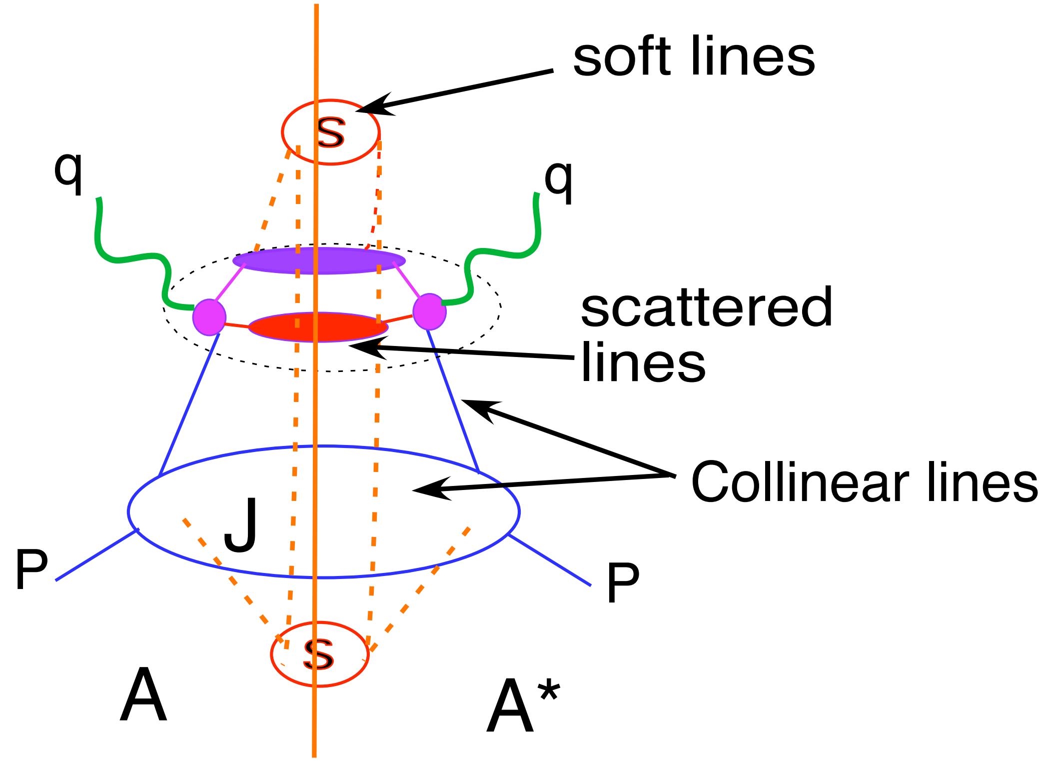
- **This is "Cut diagram notation", representing the amplitude and complex conjugate. Adding up all cut diagrams is the same as summing diagrams of A and then taking $|A|^2$.**

- **Again, the "rule": to produce a singularity, the on-shell lines of a cut diagram have to tell a classical story.**



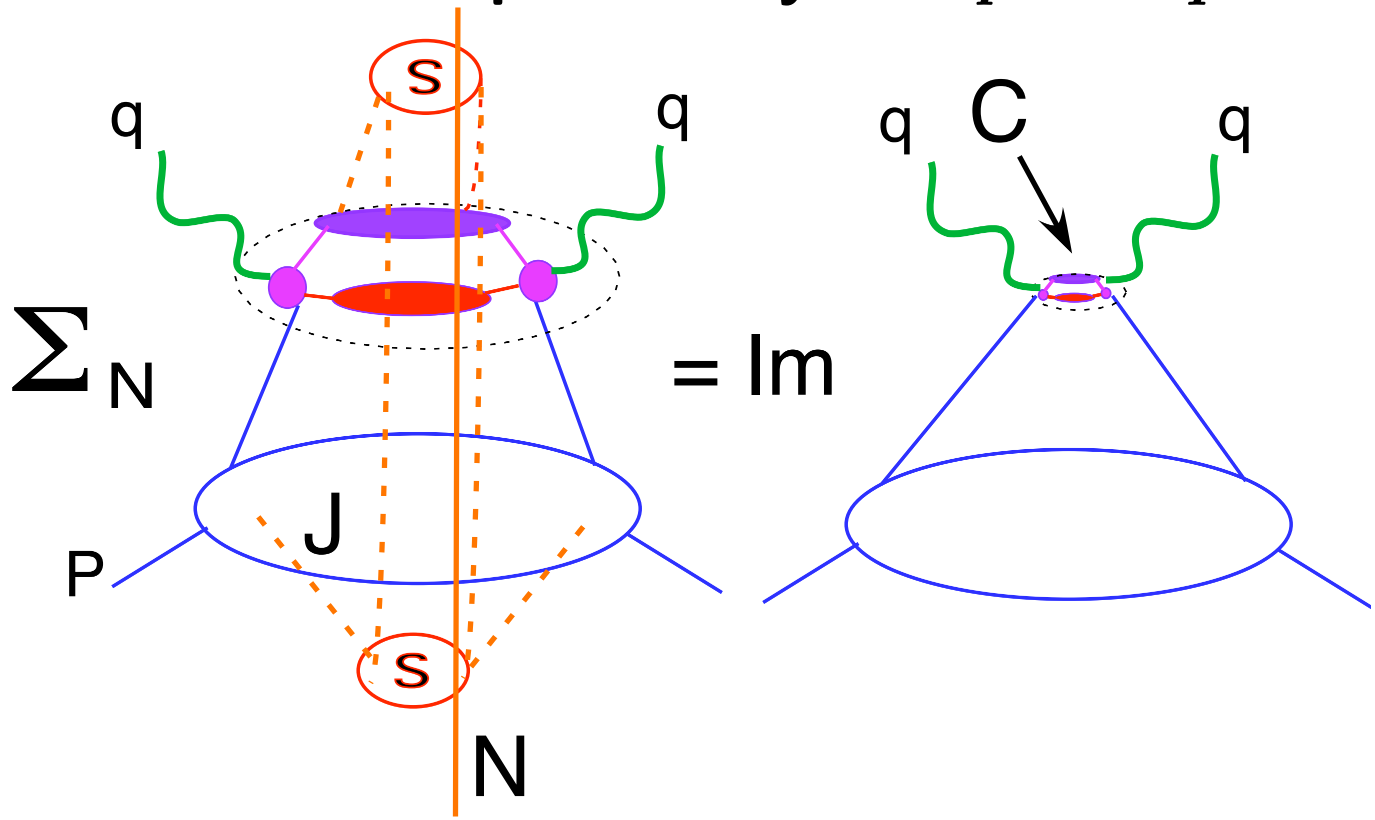
- **The classical story: h splits into collinear partons, then one of them scatters, producing jets that recede at speed of light, connected only by "infinite wavelength soft" quanta.**

- **One more time: the structure of on-shell lines in an arbitrary cut diagram. For massless partons, this is the only kind of classical story DIS has to tell.**



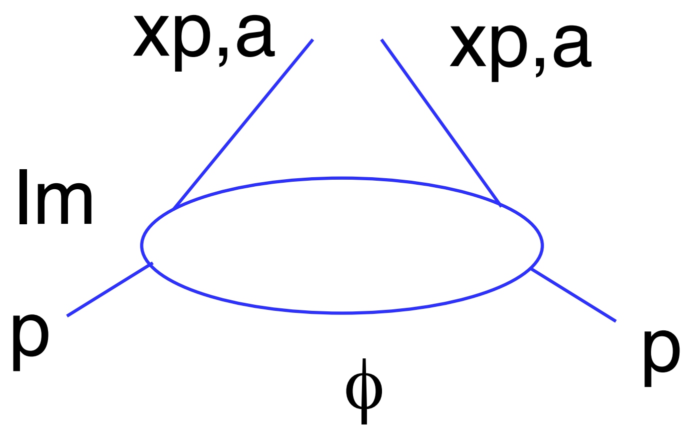
- **"Soft collinear effective theory (SCET)" builds this structure into calculations by isolating the parts of the full QCD Lagrangian that give S, J and the "scattered jet". SCET organizes calculations that are equivalent to full QCD when factorization applies.**

- **Use of the optical theorem – relate the inclusive cut diagram to forward scattering. No classical processes are possible, because the scattered quarks must re-scatter, and all interactions after the hard scattering collapse to a "short-distance" function C, that depends only on xp and q:**



- **All long-distance logs cancels because of the inclusive sum over states. Soft gluons in S can't see the "tiny" final state.**

- The partons on each side of the short distance function $C(p,q)$ must have the same flavor and momentum fraction.



- Definition of parton distribution generates all the same long-distance behavior left in the original diagrams (quark case) after the sum over hadronic final states:

$$\phi_{a/h}(x,\mu_F) = \sum_{\text{spins } \sigma} \int \frac{dy^-}{2\pi} e^{-ixp^+y^-} \langle p,\sigma|\bar{q}(y^-)\gamma^+ q(0)|p,\sigma\rangle$$

- This matrix element requires renormalization: thus the 'μ_F'.
- Here in $A^+ = 0$ gauge – more generally with Wilson lines (see I. Stewart lectures).

- **The result: factorized DIS**

$$F_2^{\gamma h}(x,Q^2) = \int_x^1 d\xi \; C_2^{\gamma q}\left(\frac{x}{\xi},\frac{Q}{\mu_R},\frac{\mu_F}{\mu_R},\alpha_s(\mu_R)\right) \times \phi_{q/h}(\xi,\mu_F,\alpha_s(\mu_F))$$

$$\equiv C_2^{\gamma q}\left(\frac{x}{\xi},\frac{Q}{\mu_R},\frac{\mu_F}{\mu_R},\alpha_s(\mu_R)\right) \otimes \phi_{q/h}(\xi,\mu_F,\alpha_s(\mu_F))$$

- $\phi_{q/h}$ **has** $\ln(\mu_F/\Lambda_{\rm QCD})$ **. . .** **with** μ_F **its independent renormalization scale.**

- C **has** $\ln(Q/\mu_R),\ \ln(\mu_F/\mu_R)$

- Often pick $\mu_R = \mu_F$ and often pick $\mu_F = Q$. So often see:

$$F_2^{\gamma h}(x, Q^2) = C_2^{\gamma q}\left(\frac{x}{\xi}, \alpha_s(Q)\right) \otimes \phi_{q/h}(\xi, Q^2)$$

B. DIS at one loop

- But we still need to specify what we *really* mean by factorization: *scheme* as well as *scale*.

- For this, compute $F_2^{\gamma q}(x, Q)$, i.e. the hadron $h = q_f$, a quark say flavor f.

- Keep $\mu = \mu_F$ for simplicity.

- **“Compute quark-photon scattering”** – *What does this mean?*

  Must use an *IR-regulated* **theory**

  Extract the *IR Safe part* **then take away the regularization**

- **Let’s see how it works . . .**

- **At** *zeroth order – no interactions:*

  $C^{\gamma q_f(0)} = e_f^2\, \delta(1 - x/\xi)$
  (LO cross section; parton model)

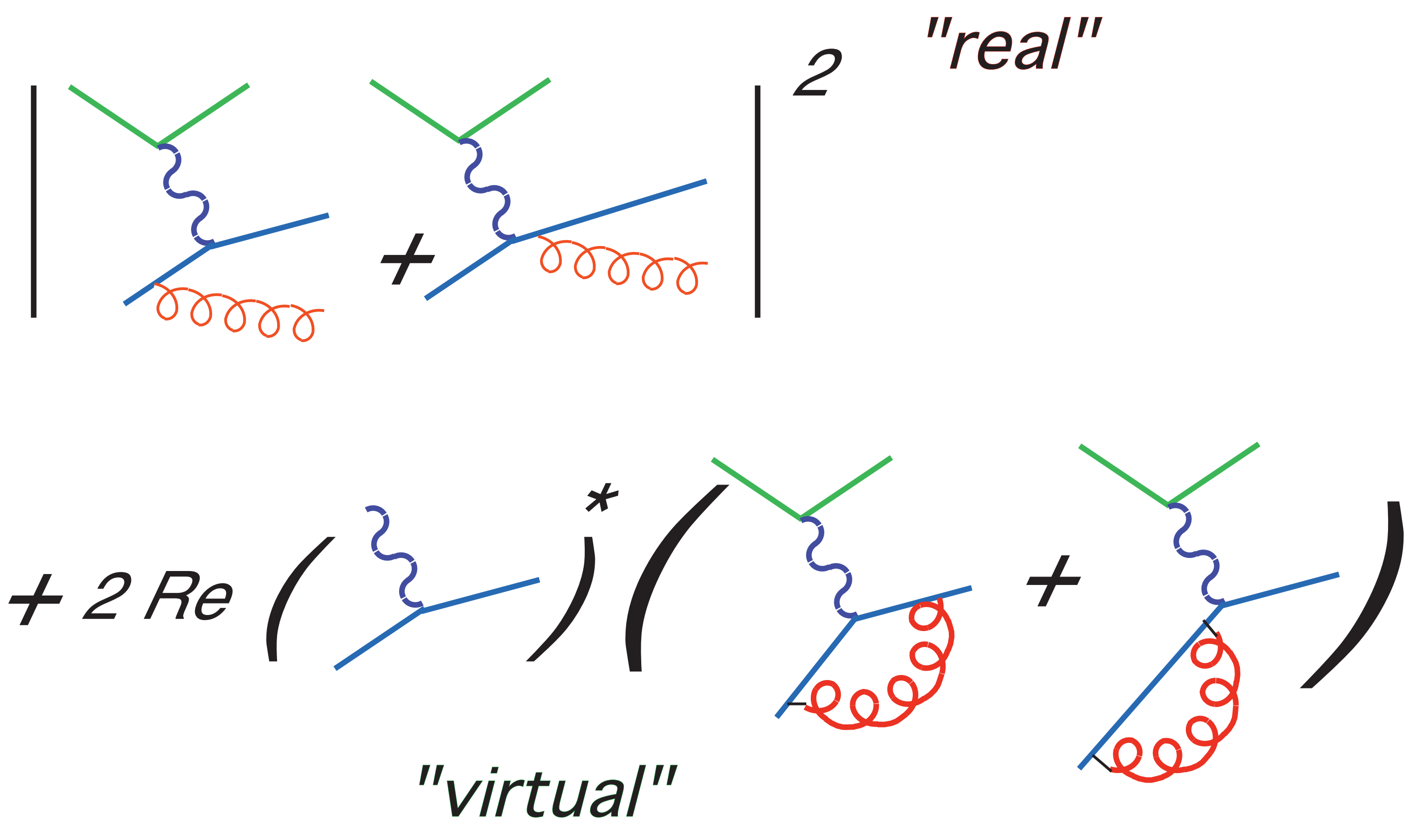
  $\phi^{(0)}_{q_f/q_{f'}}(\xi) = \delta_{ff'}\, \delta(1 - \xi)$
  (at zeroth order, momentum fraction conserved)

$$
\begin{aligned}
F_2^{\gamma q_f\,(0)}(x,Q^2) &= \int_x^1 d\xi\; C_2^{\gamma q_f\,(0)}\left(\frac{x}{\xi},\frac{Q}{\mu_R},\frac{\mu_F}{\mu_R},\alpha_s(\mu_R)\right) \\
&\quad\times\; \phi^{(0)}_{q_f/q_f}(\xi,\mu_F,\alpha_s(\mu_F)) \\
&= e_f^2\; \int_x^1 d\xi\; \delta(1-x/\xi)\;\delta(1-\xi) \\
&= e_f^2\; x\,\delta(1-x)
\end{aligned}
$$

- **On to one loop . . .**

- $F^{\gamma q}$ **at one loop: factorization schemes**

- **Start with** F_2 **for a** $quark$**:**



Have to combine final states with different phase space . . .

- "Plus Distributions":

$$\int_0^1 dx\, \frac{f(x)}{(1-x)_+} \equiv \int_0^1 dx\, \frac{f(x)-f(1)}{(1-x)}$$

$$\int_0^1 dx\, f(x) \left(\frac{\ln(1-x)}{1-x}\right)_+ \equiv \int_0^1 dx\, \left(f(x) - f(1) \right) \frac{\ln(1-x)}{(1-x)}$$

and so on . . . . In DIS:

- $f(x)$ will be parton distributions (not constant!)
- $f(x)$ term: real gluon, with momentum fraction $1-x$
- $f(1)$ term: virtual, with elastic kinematics

- DGLAP "evolution kernel" = "splitting function"

$$P_{qq}^{(1)}(x) = C_F\, \frac{\alpha_s}{\pi} \left[\frac{1+x^2}{1-x}\right]_+$$

Important note: with f constant,

$$\int_0^1 dx \left[\frac{\ln^n(1-x)}{1-x}\right]_+ = 0\,.$$

But for us, $f(x)$ is a parton distribution, and hence not a constant.

- α_s **Expansion:**

$$F_2^{\gamma q}(x,Q^2) = \int_x^1 d\xi\; C_2^{\gamma q}\left(\frac{x}{\xi},\frac{Q}{\mu_R},\frac{\mu_F}{\mu_R},\alpha_s(\mu_R)\right) \times \phi_{q/q}(\xi,\mu_F,\alpha_s(\mu_F))$$

$$F_2^{\gamma q}(x,Q^2) = C_2^{(0)}\;\phi^{(0)} + \frac{\alpha_s}{2\pi}\,C^{(1)}\;\phi^{(0)} + \frac{\alpha_s}{2\pi}\,C^{(0)}\;\phi^{(1)} + \ldots$$

- **And result:**

$$F_2^{\gamma q}(x,Q^2) = e_f^2 \; \{ \; x \; \delta(1-x)$$

$$+ \frac{\alpha_s}{2\pi} \, C_F \left[\frac{1+x^2}{1-x} \left(\frac{\ln(1-x)}{x} \right) + \frac{1}{4} \, (9-5x) \right]_+$$

$$+ \frac{\alpha_s}{2\pi} \, C_F \int_0^{Q^2} \frac{dk_T^2}{k_T^2} \; \left[\frac{1+x^2}{1-x} \right]_+ \; \} \; + \ldots$$

$$F_1^{\gamma q}(x,Q^2) = \frac{1}{2x} \; \left\{ F_2^{\gamma q_f}(x,Q^2) - C_F \, \alpha \, \frac{\alpha_s}{\pi^2} \, 2x \; \right\}$$

Note: to compare to $\mathrm{e^+e^-}$ integrals:
$k_T^2 \leftrightarrow k^2(1-\cos^2\theta)$, $k \leftrightarrow Q(1-x)$. Real and virtual would cancel here too, if we just integrated over x, but we don't – we multiply times $\phi_{q_f/h}$, which depends on x.

- **Factorization Schemes**

$\overline{\mathrm{MS}}$ (Corresponds to matrix element above.)

$$\phi^{(1)}_{q/q}(x,\mu^2) = \frac{\alpha_s}{\pi^2}\, P_{qq}(x)\ \int_0^{\mu^2} \frac{dk_T^2}{k_T^2}$$

With k_T-integral "IR regulated".

Advantage: technical simplicity; not tied to process.

$C^{(1)}(x)_{\overline{\mathrm{MS}}} = (\alpha_s/2\pi)\, P_{qq}(x) \ln(Q^2/\mu^2) + \mu$-independent

DIS:

$$\phi_{q/q}(x,\mu^2) = \frac{\alpha_s}{\pi^2}\, F^{\gamma q_f}(x,\mu^2)$$

Absorbs all uncertainties in DIS into a PDF.

Closer to experiment for DIS.

$C^{(1)}(x)_{\overline{DIS}} = (\alpha_s/2\pi)\ P_{qq}(x) \ln(Q^2/\mu^2) + 0$

- **Using the Regulated Theory to Get Parton Distributions for Real Hadrons . . .**

IR-regulated QCD is not $REAL$ **QCD**

BUT it only differs at low momenta

THUS we can use it for IR Safe functions: $C_2^{\gamma q}$**, etc.**

THIS enables us to get PDFs from experiment.

- **Compute $F_2^{\gamma q}$, $F_2^{\gamma G}$ ...**

  Define factorization scheme; find IR Safe C's

  Use factorization in the full theory

$$F_2^{\gamma h} = \sum_{a=q_f,\bar{q}_f,G} C^{\gamma a} \otimes \phi_{a/h}$$

  Measure F_2 ($h = n, p$); then use the known C's to derive $\phi_{a/h}$

  NOW HAVE $\phi_{a/h}(\xi, \mu^2)$ AND CAN USE IT IN ANY OTHER PROCESS THAT FACTORIZES.

- **Multiple flavors and cross sections complicate technicalities; not logic (Global Fits)**

- **C. Evolution: Q^2-dependence**

- **In general, Q^2/μ^2 dependence still in $C_a\left(x/\xi, Q^2/\mu^2, \alpha_s(\mu)\right)$**

  Choose $\mu = Q$

$$F_2^{\gamma h}(x, Q^2) = \sum_a \int_x^1 d\xi\; C_2^{\gamma a}\left(\frac{x}{\xi}, 1, \alpha_s(Q)\right)\; \phi_{a/h}(\xi, Q^2)$$

  $Q \gg \Lambda_{\rm QCD} \rightarrow$ *compute* C*'s in PT*.

$$C_2^{\gamma a}\left(\frac{x}{\xi}, 1, \alpha_s(Q)\right) = \sum_n \left(\frac{\alpha_s(Q)}{\pi}\right)^n C_2^{\gamma a(n)}\left(\frac{x}{\xi}\right)$$

  But still need PDFs at $\mu = Q$: $\phi_{a/A}(\xi, Q^2)$ for different Q's.

- **How evolution works . . .**

- **A remarkable consequence of factorization.**

- *Can use* $\boldsymbol{\phi_{a/A}(x, Q_0^2)}$ **to determine**

  $\boldsymbol{\phi_{a/A}(x, Q^2)}$ *and hence* $\boldsymbol{F_{1,2,3}(x, Q^2)}$ *for any* $\boldsymbol{Q}$

- **So long at $\alpha_s(Q)$ is still small.**

- **Let's see how it works explicitly in an example.**

- **The 'nonsinglet' distribution (recall Bjorken SR: $g_1^p - g_1^n$)**

$$F_a^{\gamma \mathrm{NS}} = F_a^{\gamma p} - F_a^{\gamma n}$$

$$F_2^{\gamma \mathrm{NS}}(x, Q^2) = \int_x^1 d\xi \; C_2^{\gamma \mathrm{NS}}\left(\frac{x}{\xi}, \frac{Q}{\mu}, \alpha_s(\mu)\right) \phi_{\mathrm{NS}}(\xi, \mu^2)$$

Gluons, antiquarks cancel

At one loop: $C_2^{\mathrm{NS}} = C_2^{\gamma N}$

- **Basic tool:**

- **'Mellin' Moments and Anomalous Dimensions**

$$\bar{f}(N) = \int_0^1 dx\ x^{N-1}\ f(x)$$

- **Reduces convolution to a product**

$$f(x) = \int_x^1 dy\ g\left(\frac{x}{y}\right)\ h(y) \rightarrow \bar{f}(N) = \bar{g}(N)\ \bar{h}(N+1)$$

- **Moments applied to NS structure function $\mu_F = \mu_R = \mu$:**

$$\bar{F}_2^{\gamma\mathrm{NS}}(N, Q^2) = \bar{C}_2^{\gamma\mathrm{NS}}\left(N, \frac{Q}{\mu}, \alpha_s(\mu)\right) \bar{\phi}_{\mathrm{NS}}(N, \mu^2)$$

(Note $\phi_{\mathrm{NS}}(N, \mu^2) \equiv \int_0^1 d\xi \xi^N f(\xi, \mu^2)$ here.)

- $\bar{F}_2^{\gamma\mathrm{NS}}(N, Q^2)$ **is Physical**

$$\Rightarrow \quad \mu \frac{d}{d\mu} \bar{F}_2^{\gamma\mathrm{NS}}(N, Q^2) = 0$$

- ‘Separation of variables’

$$\mu\frac{d}{d\mu}\ln\bar{\phi}_{\rm NS}(N,\mu^2) = -\gamma_{\rm NS}(N,\alpha_s(\mu))$$

$$\gamma_{\rm NS}(N,\alpha_s(\mu)) = \mu\frac{d}{d\mu}\ln\bar{C}_2^{\gamma{\rm NS}}(N,\alpha_s(\mu))$$

- Because α_s is the only variable held in common.

- $\gamma_{\rm NS}$ an “anomalous dimension”, which controls the logarithmic μ dependence.

$$\mu \frac{d}{d\mu} \ln \bar{\phi}_{\mathrm{NS}}(N, \mu^2) = -\gamma_{\mathrm{NS}}(N, \alpha_s(\mu))$$

$$\gamma_{\mathrm{NS}}(N, \alpha_s(\mu)) = \mu \frac{d}{d\mu} \ln \bar{C}_2^{\gamma \mathrm{NS}}(N, \alpha_s(\mu))$$

- **Only need to know C's $\Rightarrow \gamma_N$ from IR regulated theory!**

$\Downarrow$

Q-DEPENDENCE DETERMINED BY PT

EVOLUTION

THIS WAS HOW WE FOUND OUT QCD IS 'RIGHT'

AND THIS IS HOW QCD PREDICTS PHYSICS AT NEW SCALES

- $\gamma_{\rm NS}$ **at one loop** **(5th line is an exercise.)**

$$
\begin{aligned}
\gamma_{\rm NS}(N,\alpha_s) &= \mu\frac{d}{d\mu}\ \ln \bar{C}_2^{\gamma{\rm NS}}(N,\alpha_s(Q)) \\
&= \mu\frac{d}{d\mu}\ \left\{ (\alpha_s/2\pi)\ \bar{P}_{qq}(N)\ln(Q^2/\mu^2) + \mu \text{ indep.} \right\} \\
&= -\frac{\alpha_s}{\pi}\ \int_0^1 dx\ x^{N-1}\ P_{qq}(x) \\
&= -\frac{\alpha_s}{\pi} C_F\ \int_0^1 dx\ \left[\left(x^{N-1}-1\right)\ \frac{1+x^2}{1-x}\right] \\
&= -\frac{\alpha_s}{\pi} C_F\ \left[4\sum_{m=2}^{N}\frac{1}{m} - 2\frac{2}{N(N+1)} + 1 \right] \\
&\equiv -\frac{\alpha_s}{\pi}\ \gamma_{\rm NS}^{(1)}
\end{aligned}
$$

Hint: $(1-x^2)/(1-x) = 1 + x \ldots (1-x^k)/(1-x) = \sum_{i=0}^{k-1} x^k$

- **Solution and scale breaking.**

$$\mu\frac{d}{d\mu}\,\bar{\phi}_{\rm NS}(N,\mu^2) = -\gamma_{\rm NS}(N,\alpha_s(\mu))\;\bar{\phi}_{\rm NS}(N,\mu^2)$$

$$\bar{\phi}_{\rm NS}(N,\mu^2) = \bar{\phi}_{\rm NS}(N,\mu_0^2)\times\exp\left[-\frac{1}{2}\int_{\mu_0^2}^{\mu^2}\frac{d\mu'^2}{\mu'^2}\,\gamma_{\rm NS}(N,\alpha_s(\mu))\right]$$

$$\Downarrow$$

$$\bar{\phi}_{\rm NS}(N,Q^2) = \bar{\phi}_{\rm NS}(N,Q_0^2)\left(\frac{\ln(Q^2/\Lambda_{\rm QCD}^2)}{\ln(Q_0^2/\Lambda_{\rm QCD}^2)}\right)^{-2\gamma_N^{(1)}/\beta_0}$$

Hint:

$$\alpha_s(Q) = \frac{4\pi}{\beta_0\,\ln(Q^2/\Lambda_{\rm QCD}^2)}$$

So also: $\bar{\phi}_{\rm NS}(N,Q^2) = \bar{\phi}_{\rm NS}(N,Q_0^2)\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{-2\gamma_N^{(1)}/\beta_0}$

Qualitatively,

$$\bar{\phi}_{\mathrm{NS}}(N,Q^2) \;=\; \bar{\phi}_{\mathrm{NS}}(N,Q_0^2)\,\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{-2\gamma_N^{(1)}/\beta_0}$$

- Is ‘mild’ scale breaking, to be contrasted to

- Case of $\alpha_s \to \alpha_0 \neq 0$, get a power Q-dependence:

$$\left(Q^2\right)^{\gamma^{(1)}\frac{\alpha_s}{2\pi}}$$

- $\Rightarrow$ QCD’s consistency with the Parton Model (73-74)

- **Inverting the Moments.**

$$\mu \frac{d}{d\mu}\, \bar{\phi}_{\rm NS}(N, \mu^2) = -\gamma_N(\alpha_s(\mu))\, \bar{\phi}_{\rm NS}(N, \mu^2)$$

$$\Downarrow$$

$$\mu \frac{d}{d\mu}\, \phi_{qq}(x, \mu^2) = \int_x^1 \frac{d\xi}{\xi}\, P_{\rm NS}(x/\xi, \alpha_s(\mu))\, \phi_{\rm NS}(\xi, \mu^2)$$

Splitting function $\leftrightarrow$ Anomalous dimensions

$$\int_0^1 dx\, x^{N-1}\, P_{qq}(x, \alpha_s) = \gamma_{NS}(N, \alpha_s)$$

- **Singlet (Full) Evolution**

$$\mu \frac{d}{d\mu}\, \phi_{b/A}(x,\mu^2) = \sum_{b=q,\bar{q},G} \int_x^1 \frac{d\xi}{\xi}\, P_{ab}(x/\xi, \alpha_s(\mu))\, \phi_{b/A}(\xi,\mu^2)$$

- **The Physical Context of Evolution**

  – **Parton Model:** $\phi_{a/A}(x)$ **density of parton** a **with momentum fraction** x**, assumed independent of** Q

  – **PQCD:** $\phi_{a/A}(x,\mu)$**: same density, but with transverse momentum** $\leq \mu$

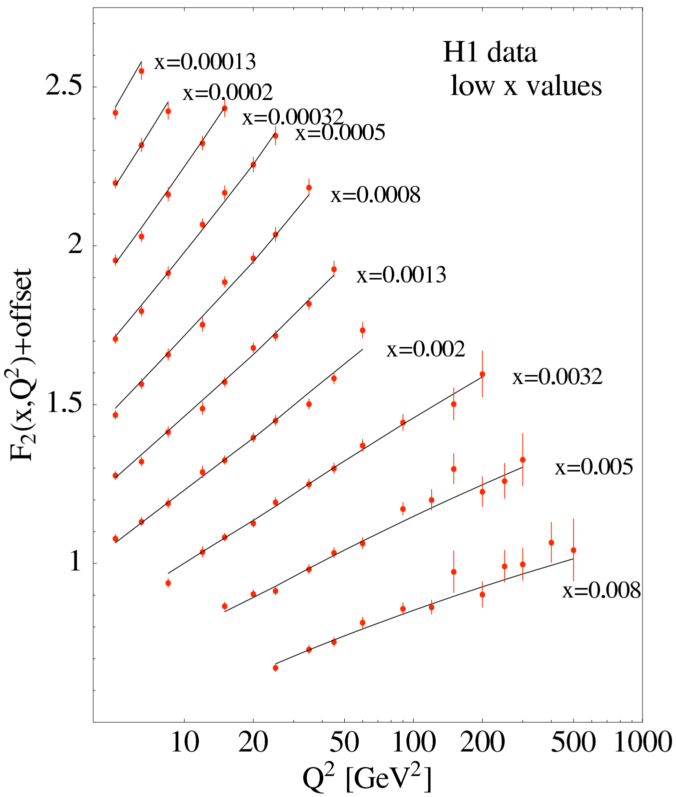
- **If there** *were* **a maximum transverse momentum Q_0, each $\phi_{a/h}(x, Q_0)$ would freeze for $\mu \geq Q_0$.**

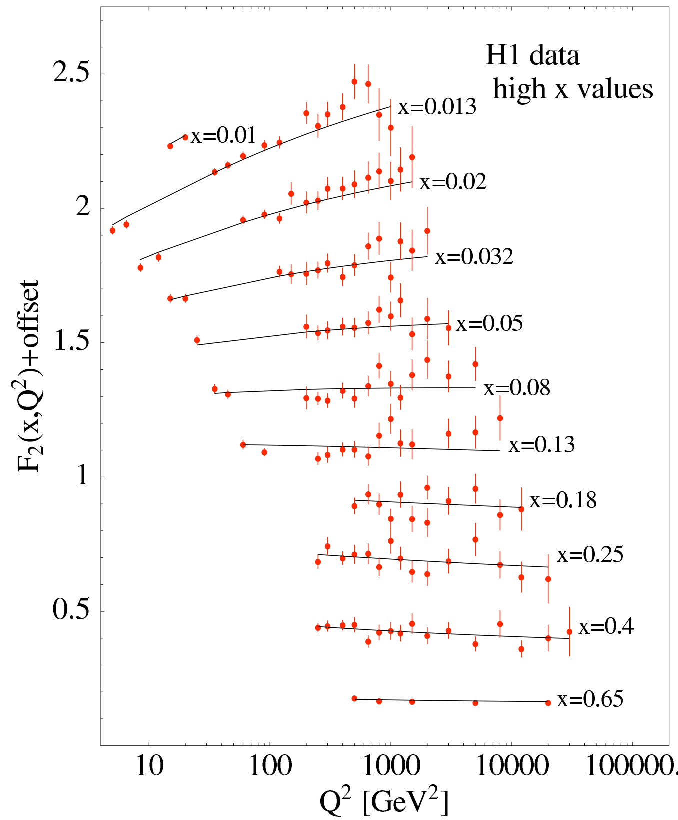
- *Not so* **in renormalized PT.**

- **Scale breaking measures the change in the density as maximum transverse momentum increases.**

- **Cross sections we compute still depend on our choice of μ through uncomputed "higher orders" in C and evolution.**

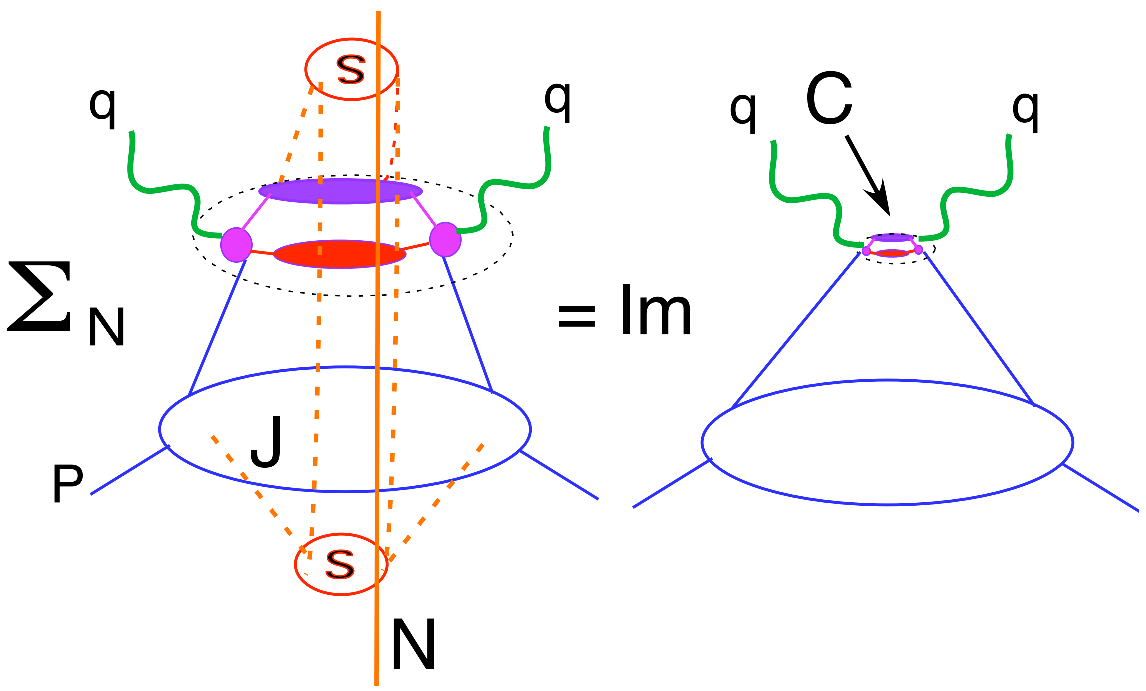
• Evolution in DIS (with nice, old CTEQ6 fits)





Conclude with a few comments . . .

- **Factorization, although powerful, is brittle. To apply it, we must define our cross sections to be "sufficiently inclusive". We have to be able to apply an analog of the optical theorem as in DIS, recall:**



- **Event generators for showering depend on the physics of factorization: each sequential branching (gluon emission, pair creation) is independent. A series of "mini-factorizations".**

- **The key to applications of perturbative QCD is to avoid uncontrolled dependence of long-distance physics. It must either cancel or be factorized from calculable quantities.**

- **Once factorized, we can learn about long-distance parts by experiment, and bring other methods to bear on them.**

- **In its own terms, pQCD will give sensible answers if you ask the right questions.**

Appendix: Intuitive description of factorization in hadron-hadron scattering

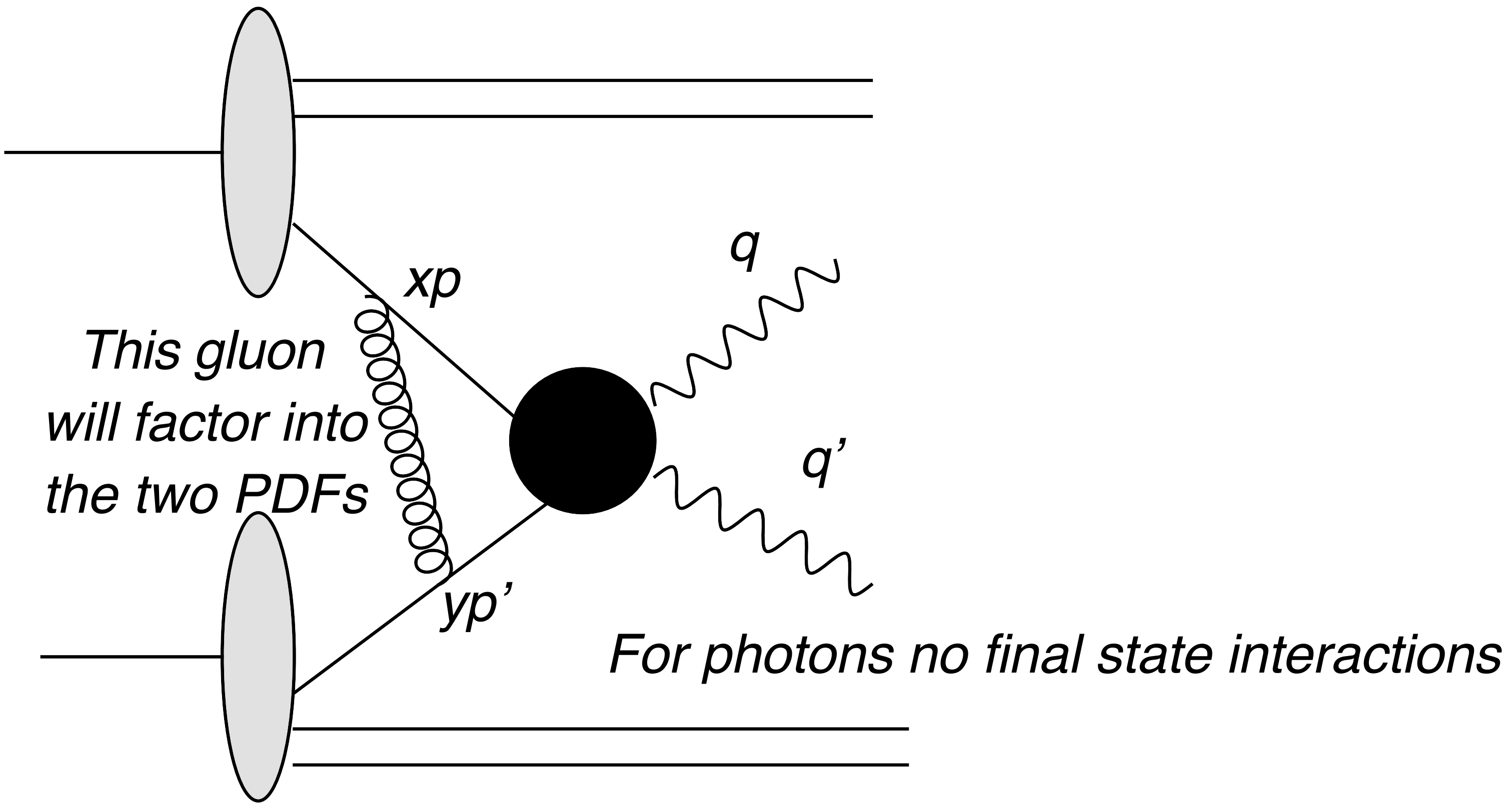
- General relation for hadron-hadron scattering for a hard, inclusive process with momentum transfer M to produce final state $F + X$: $(\mu_F = \mu_R = \mu)$

$$
\begin{aligned}
d\sigma_{\mathrm{H_1H_2}}(p_1, p_2, M) = \\
&\sum_{a,b} \int_0^1 d\xi_a \, d\xi_b d\hat{\sigma}_{ab \to F+X} \, (\xi_a p_1, \xi_b p_2, M, \mu) \\
&\times \phi_{a/H_1}(\xi_a, \mu) \, \phi_{b/H_2}(\xi_b, \mu),
\end{aligned}
$$

- Factorization proofs justify of the universality of the parton distributions.
- Also underly a range of generalizations of evolution: resummations (see I. Stewart lectures!).

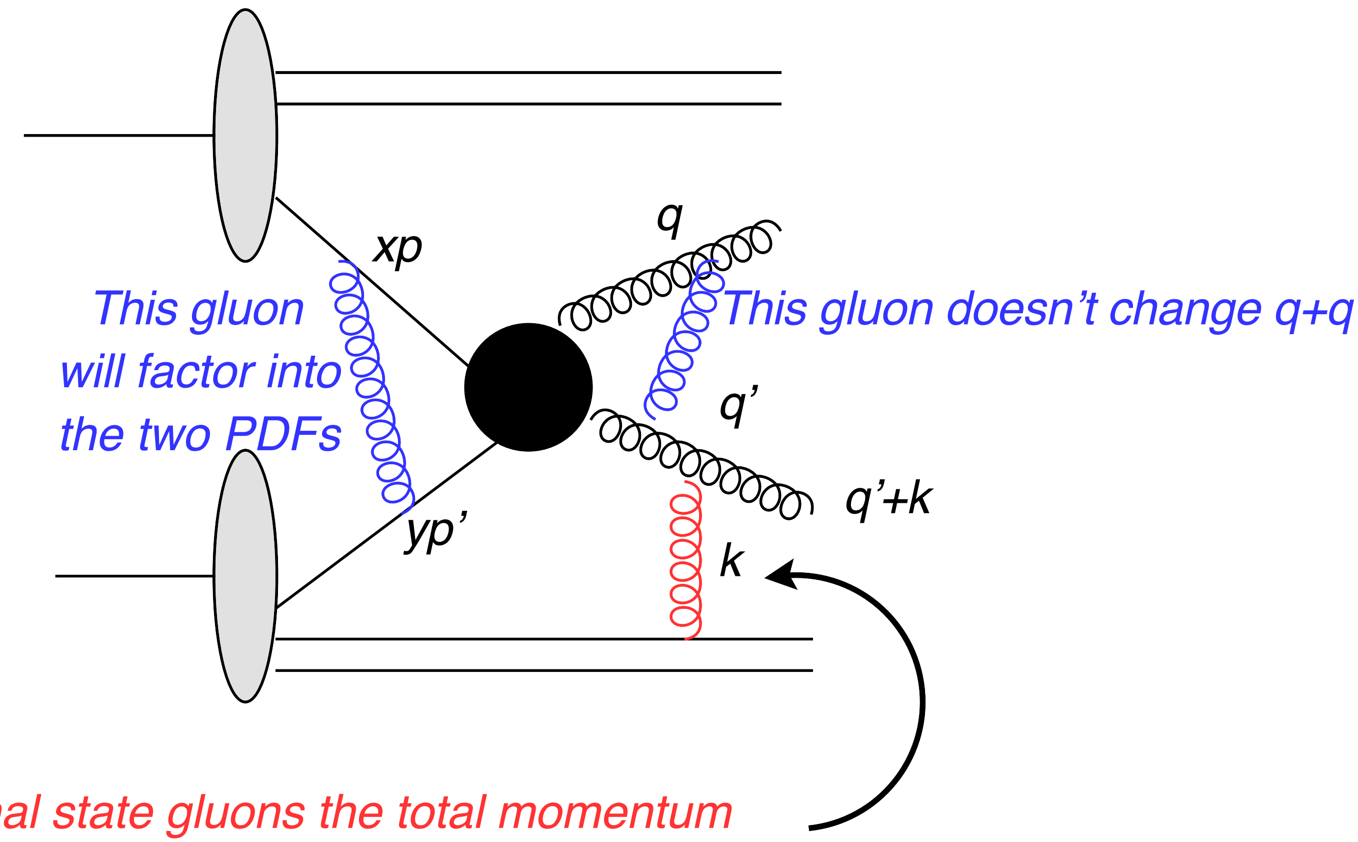
- **Two examples that illustrate the application and limitations of factorization in hadron-hadron scattering.**

1. $p + p \to \gamma + \gamma$: (similar to Drell-Yan – Q_T factorization)



Factorization for measured (q+q')^2 and (q+q')T

2. $p + p \rightarrow g(jet) + g(jet)$: (TMD factorization doesn't apply)

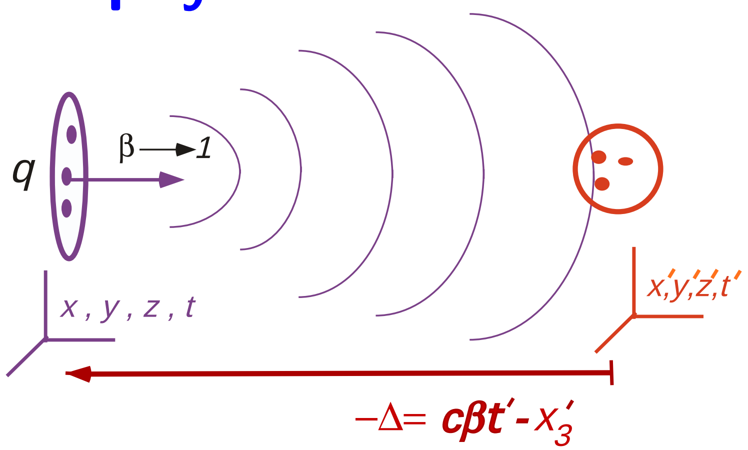


For final state gluons the total momentum
may be changed by momentum transfers
of order 1/(proton size)

Doesn't matter much for (q+q')^2

But really affects (q+q')T

- **The physical basis of factorization in classical fields**



$$\Delta \equiv x_3' - \beta c t'$$

- **Why a classical picture isn't far-fetched . . .**

  The correspondence principle is the key to to IR divergences.

  An accelerated charge must produce classical radiation,

  and an infinite numbers of soft gluons are required to make a classical field.

Transformation of a scalar field:

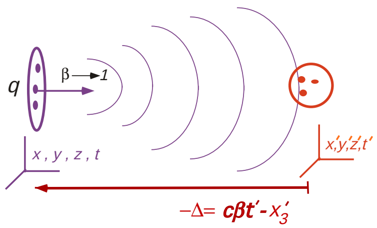
$$\phi(x) \;=\; \frac{q}{(x_T^2 + x_3^2)^{1/2}} \;=\; \phi'(x') \;=\; \frac{q}{(x_T^2 + \gamma^2 \Delta^2)^{1/2}}$$

From the Lorentz transformation:

$$x_3 = -\gamma(\beta c t' - x_3') \equiv \gamma \Delta.$$

Closest approach is at $\Delta = 0$, i.e. $t' = \frac{1}{\beta c} x_3'$.

The scalar field transforms "like a ruler": At any fixed $\Delta \neq 0$, the field decreases like $1/\gamma = \sqrt{1 - \beta^2}$.



field	x frame	x' frame
scalar	$\frac{q}{\lvert\vec{x}\rvert}$	$\frac{q}{(x_T^2+\gamma^2\Delta^2)^{1/2}}$
gauge (0)	$A^0(x) = \frac{q}{\lvert\vec{x}\rvert}$	$A'^0(x') = \frac{-q\gamma}{(x_T^2+\gamma^2\Delta^2)^{1/2}}$
field strength	$E_3(x) = \frac{q}{\lvert\vec{x}\rvert^2}$	$E_3'(x') = \frac{-q\gamma\Delta}{(x_T^2+\gamma^2\Delta^2)^{3/2}}$
Gauge fields :	$E_3 \sim \gamma^0,$	$E_3 \sim \gamma^{-2}$

- **The “gluon” $\vec{A}$ is enhanced, yet is a total derivative:**

$$A^\mu = q\frac{\partial}{\partial x'_\mu}\ \ln\left(\Delta(t', x'_3)\right) + \mathcal{O}(1-\beta) \sim A^-$$

- **The “large” part of A^μ can be removed by a gauge transformation!**

- **The "force" $\vec{\mathrm{E}}$ field of the incident particle does not overlap the "target" until the moment of the scattering.**

- **"Advanced" effects are corrections to the total derivative:**

$$1-\beta \sim \frac{1}{2}\left[\sqrt{1-\beta^2}\right]^2 \sim \frac{m^2}{2E^2}$$

- **Power-suppressed! These are corrections to factorization.**

- **At the same time, a gauge transformation also induces a phase on charged fields:**

$$q(x) \Rightarrow q(x)\ e^{i\,\ln(\Delta)}$$

Cancelled if the fields are well-localized $\Leftrightarrow$ σ inclusive

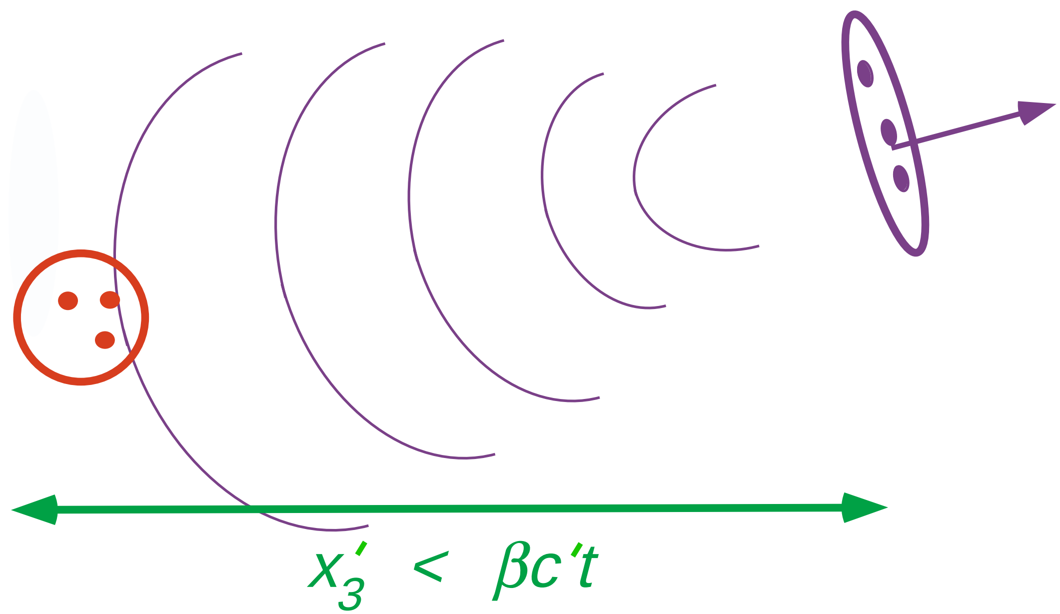
- **Initial-state interactions decouple from hard scattering**
- **Summarized by multiplicative factors: the parton distributions.**

  $\Rightarrow$ Cross section for inclusive hard scattering is IR safe, with power-suppressed corrections.

- **Factorizing dynamics at short and long distance can be built into effective field theories based on the QCD Lagrangian: in particular "soft-collinear effective field theory" (SCET) can streamline many applications.**

- **What about cross sections where we observe specific particles in the final state? Single hadrons, dihadron correlations, etc?**

- **Much of the same reasoning holds:**



- **For single-particle inclusive . . .**

  Interactions after the scattering are too late to affect large momentum transfer, creation of heavy particle, etc.

  The fragmentation of partons to jets is too slow to know details of the hard scattering: factorization of fragmentation functions.