Can spin chains describe colored d.o.f. in DIS? (I)

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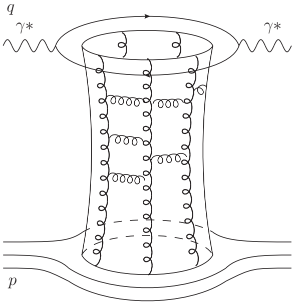
Lecture plan

The scattering of a lepton
on a hadron is a sum of Feynman diagrams. In leading logarithmic approximation ladder diagrams dominate. Virtual quarks exchange gluons with valence quarks. The BFKL Hamiltonian describes interactions of the gluons.



- The first lecture will
  be devoted to spin chains and quantum integrable models [Bethe Ansatz]. Some mathematics: quantum groups [Yangian symmetry].

- The second lecture will be about entanglement entropy production in deep inelastic scattering and small x asymptotic of gluon structure function.

Introduction

The small x behaviour of DIS structure functions is the high-energy (Regge) asymptotic of scattering. The first perturbative contribution from two reggeized gluons called BFKL pomeron [two gluon exchange] leads to an asymptotic for small x.

$$xG(x) \sim \frac{1}{x^{\Delta}}, \qquad \Delta = \frac{g^2 N_c 2 \log 2}{8\pi^2}$$

The applicability of the perturbative result depends essentially on the scale Q^2. This result was relevant in the kinematic regions accessible in the HERA experiments. The contribution of the multiple exchange of reggeized gluons becomes important at very small x. In the case of a heavy ion target this asymptotic regime sets in earlier. **The contribution of a very large number of reggeized gluons can be described by information spread in spin chains**. The argument of universality leads to the assertion that the critical behaviour of thermodynamic quantities, e.g. the entropy dependence under quenching, can be derived from perturbative results

The first spin chain

The Hamiltonian of the Heisenberg XXX spin-$\frac{1}{2}$ chain, described by a Hamiltonian with $\boldsymbol{\sigma}_{N+1} \equiv \boldsymbol{\sigma}_1$:

$$H = \sum_j H_{j\,j+1} = \frac{1}{2}\sum_j (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \sigma_j^z \sigma_{j+1}^z),$$

The σ^ξ are Pauli matrices, and $S^\alpha = \frac{1}{2}\sigma^\alpha$.

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Our main example will be infinite lattice $j = 0, \pm 1, \pm 2, \ldots$.
The chain has multiple applications: solid state, stat. mech, SYM.

The Hamiltonian of XXX spin-$\frac{1}{2}$ chain

We denote the Hamiltonian in terms of $su(2)$ spin variables at the sites j by S_j^{ab} with $a, \ b = 1, \ 2$ (standard basis).

$$H = \sum_j \sum_{ab} S_j^{ab} S_{j+1}^{ba}$$

They satisfy the commutation relations of the discretized $su(2)$ loop algebra:

$$\left[\ S_j^{ab} \ , \ S_k^{cd} \ \right] = \delta_{jk} \left(\ \delta^{cb} \ S_j^{ad} - \delta^{ad} \ S_j^{cb} \ \right)$$

The Hamiltonian is $su(2)$ invariant. The symmetry group is much bigger.

The discrete quantum charges Q^0 (spin generators) and Q^1 generators are:

$$Q^0_{ab} = \sum_k S^{ab}_k \,, \qquad Q^1_{ab} = \frac{h}{2} \sum_{j<k} \sum_d (S^{ad}_j S^{db}_k - S^{ad}_k S^{db}_j)$$

Here j and k are labels of the lattices cites. The lattice is infinite. The generators Q^0_{ab} commute with the Hamiltonian and the generators Q^1_{ab} formally commute for chains of infinite length. There are 3 of Q^0_{ab} generators and 3 of Q^1_{ab} generators, respectively. They satisfy the following commutations relations:

$$\left[Q^0_{ab}, Q^0_{cd}\right] = \delta_{cb} Q^0_{ad} - \delta_{ad} Q^0_{cb} \qquad \left[Q^0_{ab}, Q^1_{cd}\right] = \delta_{cb} Q^1_{ad} - \delta_{ad} Q^1_{cb}$$

$$\left[Q^1_{ab}, Q^1_{cd}\right] = \delta_{cb} Q^2_{ad} - \delta_{ad} Q^2_{cb} + \frac{h^2}{4} Q^0_{ad} (\sum_e Q^0_{ce} Q^0_{eb}) - \frac{h^2}{4} (\sum_e Q^0_{ae} Q^0_{ed}) Q^0_{cb}$$

The extra non-linear term in the last equation can be expressed only in terms of $su(2)$ generators Q^0_{ab}. Other generators of the Yangian are polynomials of these Qs. The Yangian is infinite dimensional linear algebra. The Yangian generators commute with the Hamiltonian.

This implies the following Serre relation involving only Q^0_{ab} and Q^1_{ab}:

$$\left[Q^0_{ab}, \left[Q^1_{cd}, Q^1_{ef}\right]\right] \quad - \quad \left[Q^1_{ab}, \left[Q^0_{cd}, Q^1_{ef}\right]\right]$$

$$= \frac{h^2}{4} \sum_{p\,q} \left(\left[Q^0_{ab}, \left[Q^0_{cp} Q^0_{pd}, Q^0_{eq} Q^0_{qf}\right]\right] \quad - \quad \left[Q^0_{ap} Q^0_{pb}, \left[Q^0_{cd}, Q^0_{eq} Q^0_{qf}\right]\right]\right)$$

The algebra generated by Q^0_{ab} and Q^1_{ab} is called the $su(2)$ Yangians (infinite dimensional quantum group). The Yangian is not a Lie algebra but a Hopf algebra.
D. Bernard, An Introduction to Yangian Symmetries, Int.J.Mod.Phys.B7:3517, (1993)
https://doi.org/10.1142/S0217979293003371

Remark

Niklas Beisert showd that Yangian is a symmetry of SSYM
https://www.youtube.com/watch?v=jIMPJCzBqXk

Except of Yangian the chain has many other conservation laws. We can say that the chain describes spin waves [excitations of color degrees of freedom]. These spin waves scatter on one another. Sometimes the scattering matrix is denoted by R. Individual momenta of these waves are conserved. The dynamics of several waves can be reduced to the dynamics of two waves. For consistency the two wave scattering matrix has to satisfy an algebraic equation: Yang-Baxter equation. This helps to diagonalize the Hamiltonian of the chain. The machinery of this diagonalization is called Bethe Ansatz. We shall present an algebraic form of Bethe Ansatz: it is equivalent to matrix product states: https://arxiv.org/pdf/1201.5627.pdf Special case of tensor networks. We need an auxiliary dimension: bond dimension. Details are in the Appendix in the end.

R matrix for Heisenberg XXX Spin Chain

Quantum spins will be assembled in to a two dimensional matrix in auxiliary dimension [bond dimension]. We shall denote it $L_n(\lambda)$. It also depends on rapidity [spectral parameter].

$$L(\lambda)_n = \lambda \mathbb{1} + i\boldsymbol{\sigma} \otimes \boldsymbol{S}_n = \begin{pmatrix} \lambda \mathbb{1}_n + \frac{i}{2}\sigma_n^z & \frac{i}{2}\sigma_n^- \\ \frac{i}{2}\sigma_n^+ & \lambda \mathbb{1}_n - \frac{i}{2}\sigma_n^z \end{pmatrix}$$

Commutation relations is given by R matrix:

$$R(\lambda, \mu) \left(L_n(\lambda) \bigotimes L_n(\mu) \right) = \left(L_n(\mu) \bigotimes L_n(\lambda) \right) R(\lambda, \mu) \qquad \dagger$$

The $R(\lambda, \mu)$ solves the Yang-Baxter equation:

$$R(\lambda, \mu) = i\mathbb{1} + (\lambda - \mu)\mathcal{P}, \qquad \mathcal{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Here $\mathcal{P}$ is permutation. The R matrix acts in the tensor product of two auxiliary spaces. In the eq $\dagger$ quantum spins of both L_n operators are on the same lattice site [same quantum space], but auxiliary spaces are different. A similar equations is valid, with both L_n operators in the same auxiliary space, but different quantum spaces.

V. Murg, V.E. Korepin, F. Verstraete, The Algebraic Bethe Ansatz and Tensor Networks, Phys. Rev. B 86, 045125 (2012) https://doi.org/10.1103/PhysRevB.86.045125
It is related to Matrix Product State and useful for inputting Bethe Ansatz (integrable models) into the quantum computer.

For XXX spin-$\frac{1}{2}$ chain, we can choose the Lax operator L

$$L(\lambda) = \lambda \mathbb{1} + i\boldsymbol{\sigma} \otimes \boldsymbol{S}$$

then L satisfies the fundamental algebraic relation (Yang-Baxter equation).
We may introduce the *monodromy* matrix, as

$$T_0(\lambda) = L_N(\lambda)\ L_{N-1}(\lambda) \cdots L_1(\lambda) = \begin{pmatrix} \mathcal{A}(\lambda) & \mathcal{B}(\lambda) \\ \mathcal{C}(\lambda) & \mathcal{D}(\lambda) \end{pmatrix}_0$$

We wrote it as the explicit matrix 2X2 in bond [auxiliary] dimention. It also satisfies the fundamental algebraic relation

$$R_{0\bar{0}}(\lambda, \mu)\ T_0(\lambda)\ T_{\bar{0}}(\mu) = T_{\bar{0}}(\mu)\ T_0(\lambda)\ R_{0\bar{0}}(\lambda, \mu)$$

The transfer matrix is given by

$$t(\lambda) = \mathrm{Tr}_0[L_N(\lambda)\cdots L_1(\lambda)] = \mathrm{Tr}_0\, T_0(\lambda),$$

It contains the Hamiltonian and all higher conservation laws. It constitutes a one-parameter family of commuting operators

$$[t(\lambda),\ t(\mu)] = 0.$$

In particular

$$H = \frac{1}{2}\sum_{j=1}^{N} \boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_{j+1} = \left.\frac{d \ln t(\lambda)}{d\lambda}\right|_{\lambda=0} - \frac{1}{2}N$$

The Hamiltonian of Heisenberg Chain with higher spin

A generalization of the model to spin $\mathbf{s} = 1$ was found by A.Zamolodchikov, V.Fateev, in Jadernaya Fizika 32, 581, (1980):

$$\mathbf{H}_1 = \sum_n \{X_n - X_n^2\}, \quad X_n = (S_n^x S_{n+1}^x + S_n^y S_{n+1}^y + S_n^z S_{n+1}^z)$$

It was solved by Takhatajan and Babujian. Generalization for higher spin $\mathbf{s}$ has the form

$$\mathbf{H}_s = \sum_n F(X_n), \qquad F(X) = 2 \sum_{l=0}^{2\mathbf{s}} \sum_{k=l+1}^{2\mathbf{s}} \frac{1}{k} \prod_{\substack{j=0 \\ j \neq l}}^{2\mathbf{s}} \frac{X - y_j}{y_l - y_j},$$

The function $F(X)$ is a polynomial of a degree $2\mathbf{s}$. Here $y_l = l(l+1)/2 - \mathbf{s}(\mathbf{s}+1)$. Spin s can be positive integer spin or half integer spin. The local Hilbert space is finite dimensional.

Lattice nonlinear Schrödinger equation (Lieb-Liniger model)

If we make the local Hilbert space infinite dimensional. Then we can construct Hamiltonian with fractional spin or negative spin.
Exact lattice Lax operator (all orders in Δ) of Lattice nonlinear Schrödinger model (NLS) was constructed by A. Izergin and V. Korepin in Doklady Akademii Nauk, 1981
https://arxiv.org/pdf/0910.0295.pdf
see also Nuclear Physics B 205 [FS5], 401, 1982

$$L_j(\lambda) = \begin{pmatrix} 1 - \frac{i\lambda\Delta}{2} + \frac{\kappa}{2}\chi_j^\dagger\chi_j & -i\sqrt{\kappa}\chi_j^\dagger\varrho_j \\ i\sqrt{\kappa}\varrho_j\chi_j & 1 + \frac{i\lambda\Delta}{2} + \frac{\kappa}{2}\chi_j^\dagger\chi_j \end{pmatrix}.$$

It is a chain of interacting harmonic oscillators. The χ_j is the quantum lattice Bose field, and Δ is lattice spacing.

$$[\chi_j, \chi_l^\dagger] = \Delta\delta_{j,l} \quad \text{and} \quad \varrho_j = (1 + \frac{\kappa}{4}\chi_j^\dagger\chi_j)^{\frac{1}{2}},$$

here $\kappa > 0$, and $\Delta > 0$. It has the same R matrix as the Heisenberg chain.

Equivalence of NLS and XXX chain with negative spin

The quantum lattice nonlinear Schrödinger equation is equivalent to XXX spin chain with negative spin.
We can rewrite the L operator as XXX Heisenberg spin chain

$$L_j^{XXX} = -\sigma^z L_j = i\lambda + S_j^k \otimes \sigma^k$$

$$S_j{}^+ = -i\sqrt{\kappa}\chi_j^\dagger \varrho_j, \quad S_j{}^- = i\sqrt{\kappa}\varrho_j\chi_j, \quad S_j{}^z = (1 + \frac{\kappa}{2}\chi_j^\dagger\chi_j).$$

The σ are Pauli matrices. The S_j form a representation of $su(2)$ algebra with negative spin $s = -\frac{2}{\kappa\Delta}$.
Main example will be $s = -1$ for Lipatov-Korchemsky chain. Its local Hamiltonian is the following

$$H_{jk} = \psi(-J_{jk}) + \psi(J_{jk} + 1) - 2\psi(1).$$

The ψ is the logarithmic derivative of the Gamma function:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

The Lipatov chain (spin $s = -1$ chain)

BFKL derived a linear integral equation for the sum of all Feynman diagrams describing interaction of gluons [frame 2]. The Fourier transform of the kernel of this equation gave the Hamiltonian of the 'spin chain'. It is equivalent to lattice nonlinear Schrödinger.

The holomorphic multicolor QCD Hamiltonian describes the nearest neighbor interactions of L particles (reggeized gluons):

$$\mathcal{H} = \sum_{k=1}^{L} H_{k,k+1},$$

with periodic boundary conditions $H_{L,L+1} = H_{L,1}$. The local Hamiltonians are given by the equivalent representations

$$\begin{aligned} H_{j,k} &= P_j^{-1} \ln(z_{jk}) P_j + P_k^{-1} \ln(z_{jk}) P_k + \ln(P_j P_k) + 2\gamma_E \\ &= 2 \ln(z_{jk}) + (z_{jk}) \ln(P_j P_k)(z_{jk})^{-1} + 2\gamma_E, \end{aligned}$$

where $P_j = i\partial/\partial z_j = i\partial_j$, $z_{jk} = z_j - z_k$, and γ_E is the Euler constant.

The Lipatov-Korchemsky chain (spin $s = -1$ chain)

Lipatov used holomorphic representation of $su(2)$

$$S_k^+ = z_k^2 \partial_k - 2sz_k, \qquad S_k^- = -\partial_k, \qquad S_k^z = z_k \partial_k - s$$

If we use another representation of $su(2)$: in terms of lattice Bose field [interactions harmonic oscillators] we shall see that the Lipatov's spin chain is equivalent to lattice nonlinear Schrödinger. BFKL mapped DIS to a spin chain. The definition of the chain is based on the fundamental matrix $R_{jk}^{(s,s)}(\lambda)$ which obeys the Yang-Baxter equation

$$R_{jk}^{(s,s)}(\lambda) = \frac{\Gamma(i\lambda - 2s)\Gamma(i\lambda + 2s + 1)}{\Gamma(i\lambda - J_{jk})\Gamma(i\lambda + J_{jk} + 1)}.$$

This is the second R matrix, it intertwines L operators in auxiliary [bond] dimension. The operator J_{jk} is defined in the space $V \otimes V$ as a solution of the operator equation,

$$J_{jk}(J_{jk} + 1) = 2\boldsymbol{S}_j \otimes \boldsymbol{S}_k + 2s(s + 1).$$

The Hamiltonian of the XXX model with spin $s = -1$ describes interaction of nearest neighbors

$$H_{jk} = \frac{-1}{i} \frac{d}{d\lambda} \ln R_{jk}(\lambda) \Big|_{\lambda=0}, \quad H_{jk} = \psi(-J_{jk}) + \psi(J_{jk} + 1) - 2\psi(1).$$

Here $\psi(x) = d \ln \Gamma(x)/dx$, $\psi(1) = -\gamma_E$, and γ_E is the Euler constant. This is the lattice nonlinear Schrödinger.

R. Kirschner derived the BFKL Hamiltonian from Yangian symmetry.
Yangian symmetry applied to Quantum chromodynamics, arXiv:2302.00449
https://arxiv.org/abs/2302.00449
https://www.worldscientific.com/doi/abs/10.1142/S0217751X23300065
The BFKL Hamiltonian is obtained up to normalization as the first non-trivial term,

$$H = \psi(\hat{m}) + \psi(1 - \hat{m}) - 2\psi(1)$$

directly related to the eigenvalues and the operators forms H mentioned before (here $\hat{m} = -J$). The ψ is the logarithmic derivative of the Gamma funciton. This Hamiltonian describes the nearest-neighbour interaction in the multiple exchange of gluon reggeons. It is also the Hamiltonian of lattice nonlinear Schrödinger.
The decomposition of the R matrix results in the set of commuting local observables of the corresponding spin chain.

Appendix: Bethe Ansatz for XXX spin-$\frac{1}{2}$ chain

Define the vacuum state of the system as $|\Omega\rangle$. It is the tensor product of N such states:

$$|\Omega\rangle = |\uparrow\rangle_1 \otimes \cdots \otimes |\uparrow\rangle_N, \quad \sigma_n^+ |\uparrow\rangle_n = \sigma_n^+ \begin{pmatrix} 1 \\ 0 \end{pmatrix}_n = 0.$$

where $|\uparrow\rangle_n$ is the local spin-up state of site n. It can be annihilated by σ_n^+.

$$\begin{aligned} \mathcal{A}(\lambda)|\Omega\rangle &= a(\lambda)|\Omega\rangle = (\lambda + i)^N |\Omega\rangle \\ \mathcal{D}(\lambda)|\Omega\rangle &= d(\lambda)|\Omega\rangle = \lambda^N |\Omega\rangle \\ \mathcal{C}(\lambda)|\Omega\rangle &= 0 \end{aligned}$$

The action of the transfer matrix on vacuum is then known

$$t(\lambda)|\Omega\rangle = (\mathcal{A}(\lambda) + \mathcal{D}(\lambda))|\Omega\rangle = (\lambda + i)^N + \lambda^N |\Omega\rangle.$$

The next step is to make the following Ansatz for a general Bethe state $|\Psi\rangle$:

$$|\Psi\rangle = \mathcal{B}(\lambda_1) \cdots \mathcal{B}(\lambda_M)|\Omega\rangle.$$

This is the matrix product state representation on Bethe Ansats wave function.

After some algebra, the fundamental algebraic relation gives the commutation relations between $\mathcal{A}, \mathcal{D}$ and $\mathcal{B}$. For example, one derives:

$$[\mathcal{A}(\lambda),\ \mathcal{A}(\mu)] = [\mathcal{B}(\lambda),\ \mathcal{B}(\mu)] = [\mathcal{C}(\lambda),\ \mathcal{C}(\mu)] = 0$$
$$\mathcal{A}(\lambda)\mathcal{B}(\mu) = \frac{\lambda-\mu-i}{\lambda-\mu}\mathcal{B}(\mu)\mathcal{A}(\lambda) + \frac{i}{\lambda-\mu}\mathcal{B}(\lambda)\mathcal{A}(\mu)$$
$$\mathcal{D}(\lambda)\mathcal{B}(\mu) = \frac{\lambda-\mu+i}{\lambda-\mu}\mathcal{B}(\mu)\mathcal{D}(\lambda) - \frac{i}{\lambda-\mu}\mathcal{B}(\lambda)\mathcal{D}(\mu)$$

Acting $t(\lambda)$ on $|\Psi\rangle$ we have

$$(\mathcal{A}(\lambda)+\mathcal{D}(\lambda))|\Psi\rangle = (\mathcal{A}(\lambda)+\mathcal{D}(\lambda))\mathcal{B}(\lambda_1)\cdots\mathcal{B}(\lambda_M)|\Omega\rangle$$
$$= \Lambda(\lambda)|\Psi\rangle + \sum_{j=1}^{M}\Lambda_j(\lambda)\mathcal{B}(\lambda_1)\cdots\mathcal{B}(\lambda_{j-1})\mathcal{B}(\lambda)\mathcal{B}(\lambda_{j+1})\cdots\mathcal{B}(\lambda_M)|\Omega\rangle$$

where the sum stand for the “unwanted” terms.

Appendix: Exact solution for XXX spin-$\frac{1}{2}$ chain

One sees that if the unwanted terms vanish, i.e., $\Lambda_j(\lambda) = 0$, then $|\Psi\rangle$ is an eigenstate of the transfer matrix, with known eigenvalues.

$$\Lambda(\lambda) = a(\lambda) \prod_{k=1}^{M} \frac{\lambda - \lambda_k - i}{\lambda - \lambda_k} + d(\lambda) \prod_{k=1}^{M} \frac{\lambda - \lambda_k + i}{\lambda - \lambda_k} = a(\lambda) \frac{Q(\lambda - i)}{Q(\lambda)} + d(\lambda) \frac{Q(\lambda + i)}{Q(\lambda)}$$

This induces the Bethe equations

$$\left(\frac{\lambda_j + i}{\lambda_j} \right)^N = \prod_{l \neq j}^{M} \frac{\lambda_j - \lambda_l + i}{\lambda_j - \lambda_l - i}, \quad j = 1, \cdots, M.$$

For consistency with arbitrary spin s, we put $\mu_j = \lambda_j + \frac{i}{2}$. The Bethe equations can be rewritten as

$$\left(\frac{\mu_j + \frac{i}{2}}{\mu_j - \frac{i}{2}} \right)^N = \prod_{l \neq j}^{M} \frac{\mu_j - \mu_l + i}{\mu_j - \mu_l - i}, \quad j = 1, \cdots, M.$$

The eigenvalue of the Hamiltonian in terms of the Bethe roots

$$E = \left. \frac{d \ln \Lambda(\lambda)}{d\lambda} \right|_{\lambda=0} - \frac{1}{2} N = - \sum_{j=1}^{M} \frac{1}{\mu_j^2 + \frac{1}{4}} + \frac{1}{2} N$$

References

- D. E. Kharzeev, E. Levin, Deep inelastic scattering as a probe of entanglement: confronting experimental data, Phys. Rev. D 104, 031503 (2021).
- M. Hentschinski, D. E. Kharzeev, K. Kutak, Z. Tu, Probing the onset of maximal entanglement inside the proton in diffractive DIS, arXiv:2305.03069
- V.O.Tarasov, L.A.Takhtajan and L.D.Faddeev, Local hamiltonians for integrable quantum models on a lattice, *Theor. Math. Phys.* **57** (1983) 163-181.
- L.A.Takhtajan and L.D.Faddeev, *Hamiltonian Methods in the Theory of Solitons*, *Springer 2007.*
- C. N. Yang, *Phys. Rev. Lett* **19** (1967) 1312-1314.
- V. E. Korepin and A.G. Izergin, Lattice model connected with nonlinear Schrodinger equation, *Sov. Phys. Doklady*, 26 (1981) 653-654; Lattice versions of quantum field theory models in two dimensions, *Nucl. Phys.* **B 205** (1982) 401-413.
- V. E. Korepin, N. M. Bogoliubov and A. G. Izergin *Quantum Inverse Scattering Method and Correlation Functions*, Cambridge University Press, 1993.
- C.N. Yang and C.P. Yang, *J. Math. Phys.* **10** (1969) 1115-1122.
- J. Bartels, E. Levin, V. Kim and A. Sabio-Vera, *From the Past to the Future: The Legacy of Lev Lipatov*, *World Scientific, 2021*