Small x physics:
from HERA, through LHC to EIC

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Outline

Lecture 1

- DIS paradigm: collinear factorization and DGLAP evolution
- Why small x ? A bit of Pomeron history
- BFKL evolution at small x
- NLL BFKL and the problems with convergence
- Collinear resummation at small x
- Parton saturation
- Nonlinear evolution equation. Saturation scale
- Impact parameter dependence(*)

Outline

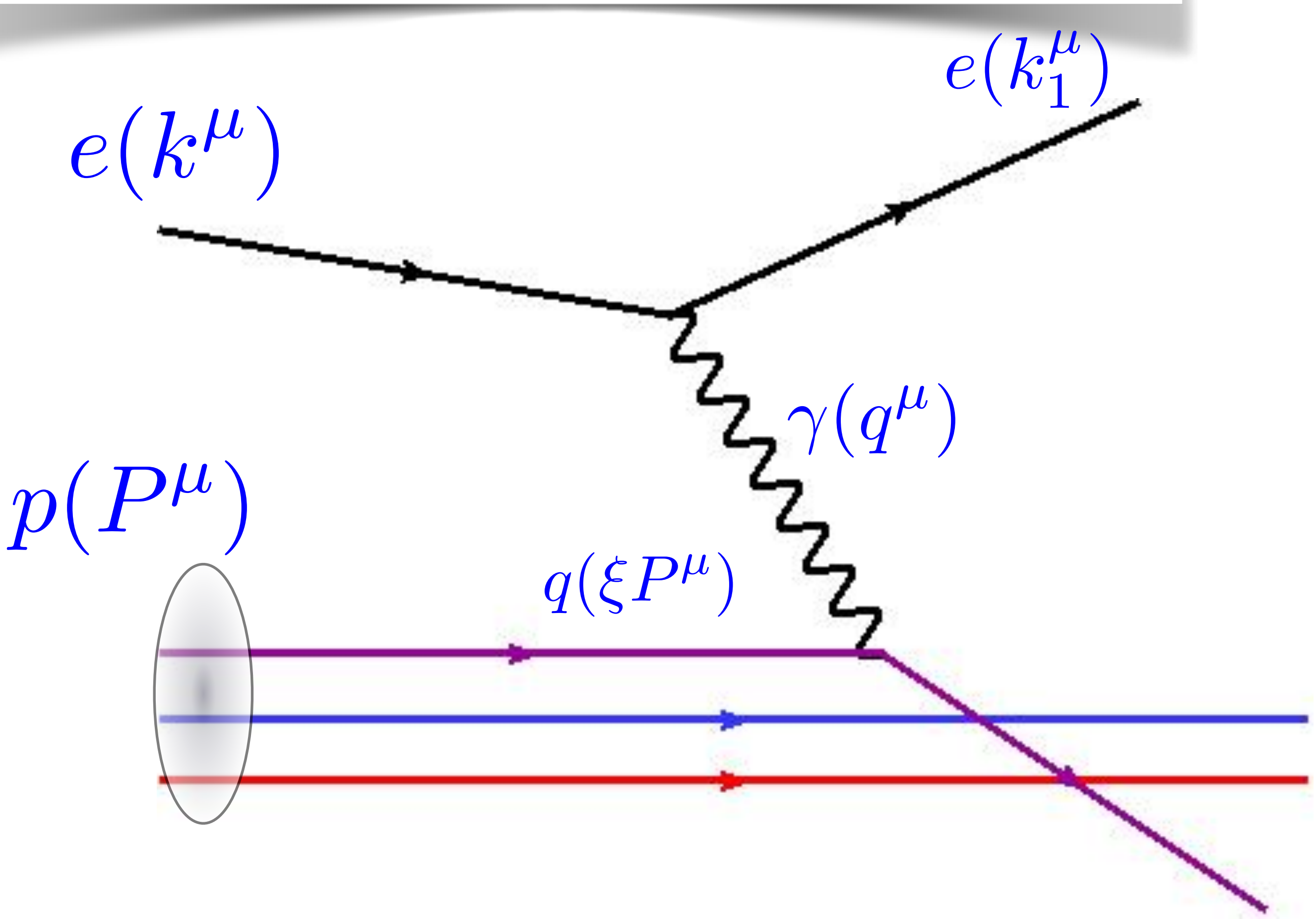
Lecture 2

- Is BFKL needed ? DGLAP success
- Hints of small x physics in the structure function data
- Two-scales processes
  - Forward jet in DIS
  - $\gamma^*\gamma^*$ at LEP
  - Mueller-Navelet jets at pp collider
- Searching for saturation: small x and/or large A
- Diffraction at small x and nuclei

Deep Inelastic Scattering

Inelastic scattering off proton

Elastic scattering off parton (quark)



$$Q^2 = -q^2$$ Photon virtuality resolving power

$$x = \frac{Q^2}{2P\cdot q} \simeq \frac{Q^2}{Q^2+W^2}$$ Bjorken x

$$s_{\gamma^* p} \equiv W^2 = (p+q)^2$$ total energy of photon-proton system

$$s_{ep} = (p+k)^2$$ total energy of electron-proton system

x has the interpretation of the longitudinal momentum fraction of the proton carried by the struck quark (in the frame where proton is fast)

$$x \simeq \xi$$

Deep Inelastic Scattering: structure functions

Inclusive DIS cross section for $lp \to lX$ (l charged lepton, $Q^2 \ll M_Z^2$, $s \gg M_p^2$)

$$\frac{d^2\sigma}{dx dQ^2} = \frac{2\pi\alpha_{\rm em}^2}{Q^4 x}\left[(1+(1-y)^2)\underbrace{F_2(x,Q^2)}_{} - y^2 \underbrace{F_L(x,Q^2)}_{}\right]$$

structure functions

$$y = \frac{p\cdot q}{p \cdot k} = Q^2/(sx)$$ inelasticity

Structure functions encode all the information about the **proton(hadron) structure**

$F_T(x,Q^2) = F_2(x,Q^2) - F_L(x,Q^2)$ transversely polarized photons

$F_L(x,Q^2)$ longitudinally polarized photons

Often experiments give **reduced cross section**

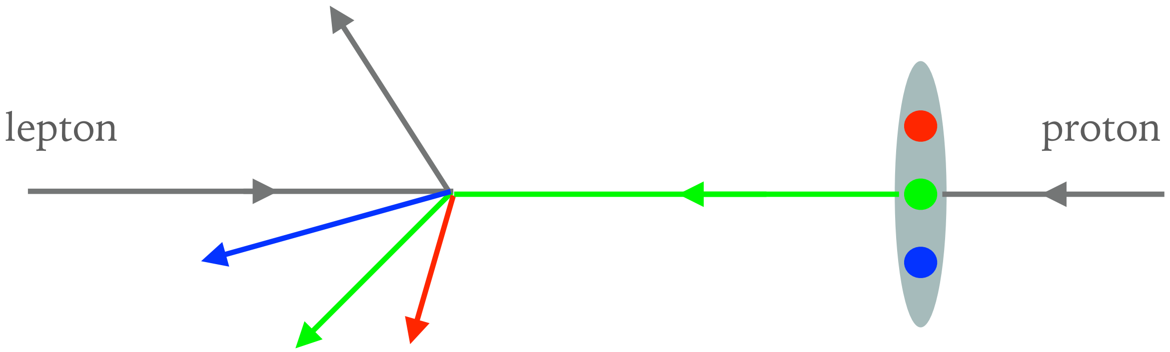
$Y_+ = 1 + (1-y)^2$

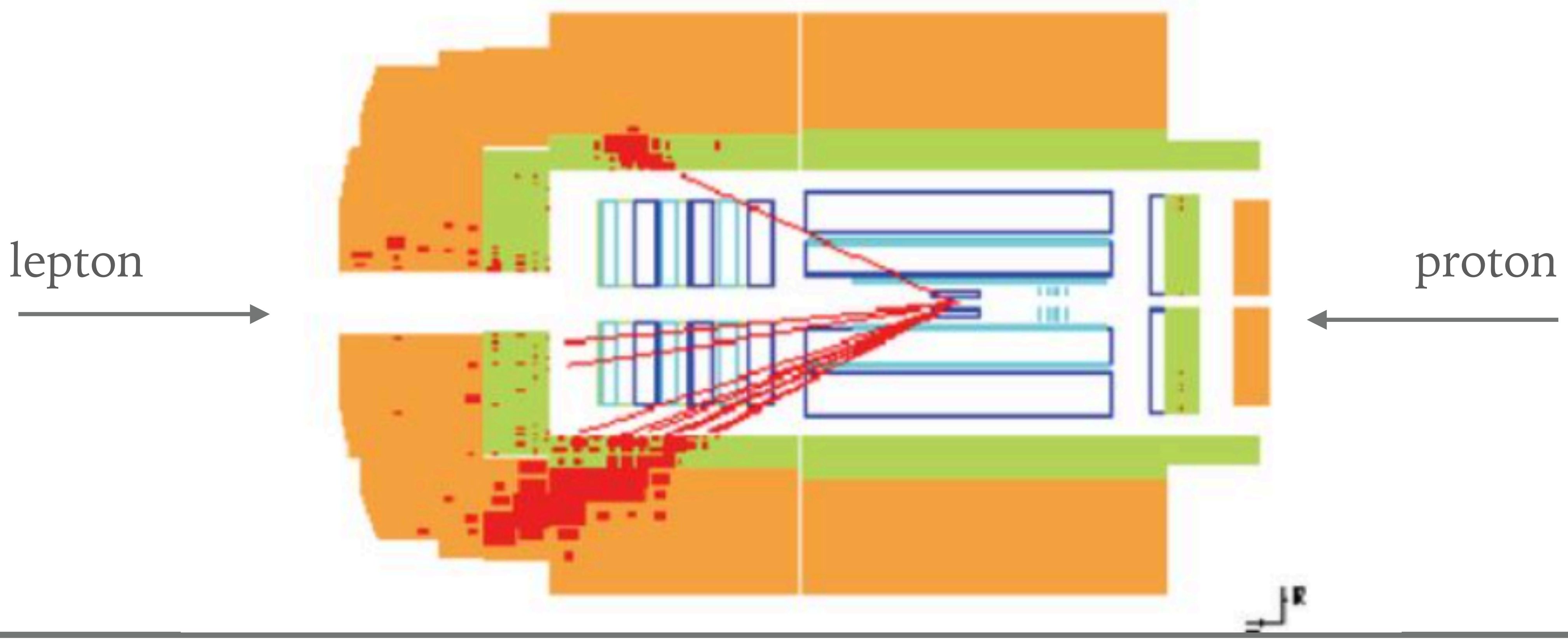
$$\sigma_{r,NC} = \frac{d^2\sigma_{NC}}{dx dQ^2}\frac{Q^4 x}{2\pi\alpha_{\rm em} Y_+} = F_2 - \frac{y^2}{Y_+} F_L$$

Dominated by the F_2 structure function except for large y

Deep Inelastic Scattering at large Q^2

- Lepton undergoes wide angle scattering at high Q^2
- Over short distance scale the struck parton interaction with the rest of target can be neglected
- Incoming parton can be approximately treated as free particle
- Single struck quark dominates since other partons are separated from it by hadronic scale $\sim 1 \text{ fm} \gg \frac{1}{Q}$

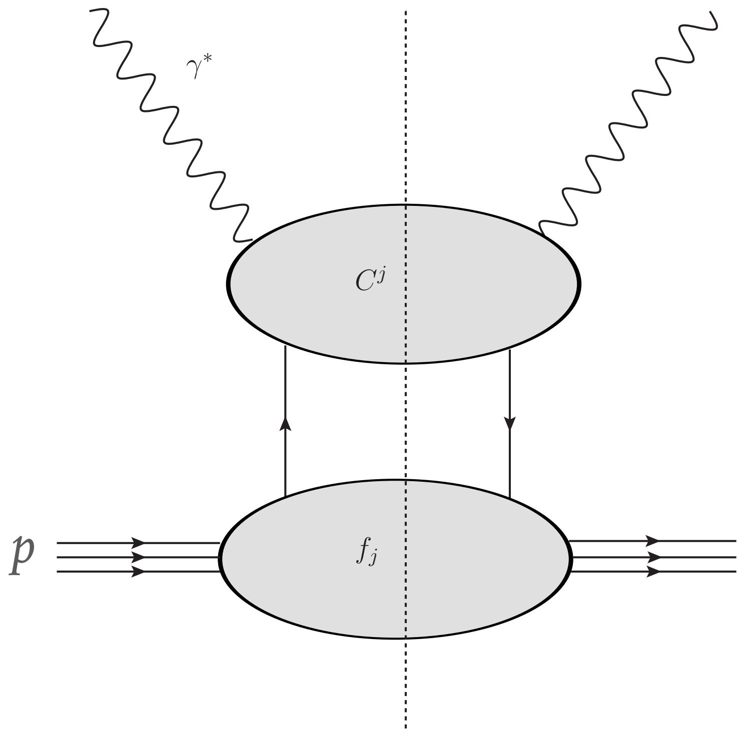




Collinear factorization

Schematic picture of

collinear factorization in DIS



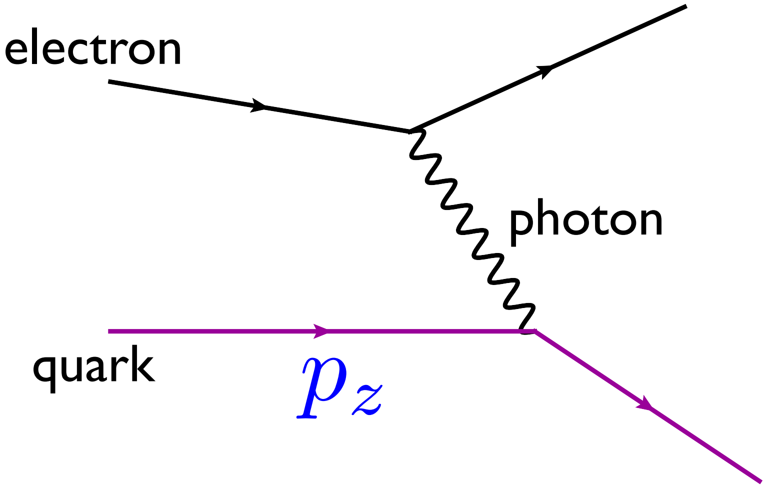
$$F_{2,L}(x,Q^2) = x\sum_q e_q^2 \sum_j \int_x^1 \frac{dz}{z}\, C^j_{2,L}(x/z,Q^2/\mu^2,\alpha_s)\, f_j(z,\mu^2)$$

$C^j_{2,L}(x/z,Q^2/\mu^2,\alpha_s)$ **Coefficient functions:** calculable order by order in perturbation theory

$f_j(z,\mu^2)$ **Parton densities:** non-perturbative distributions in longitudinal momentum fractions z at a given scale μ^2

Radiation in QCD

Parton model



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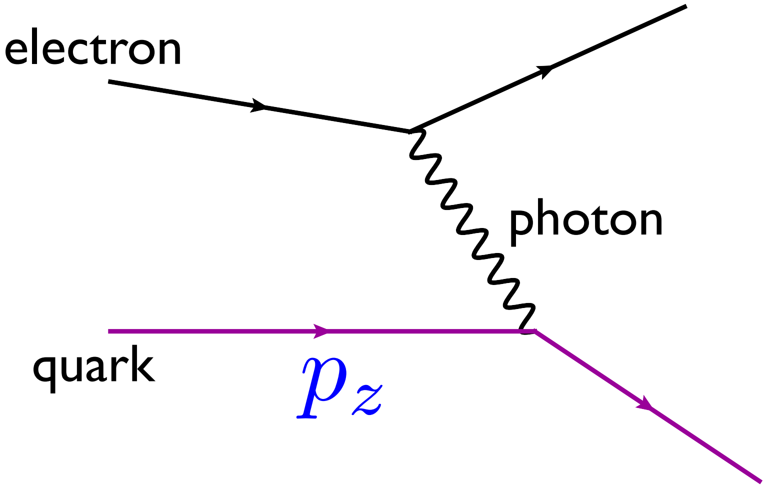
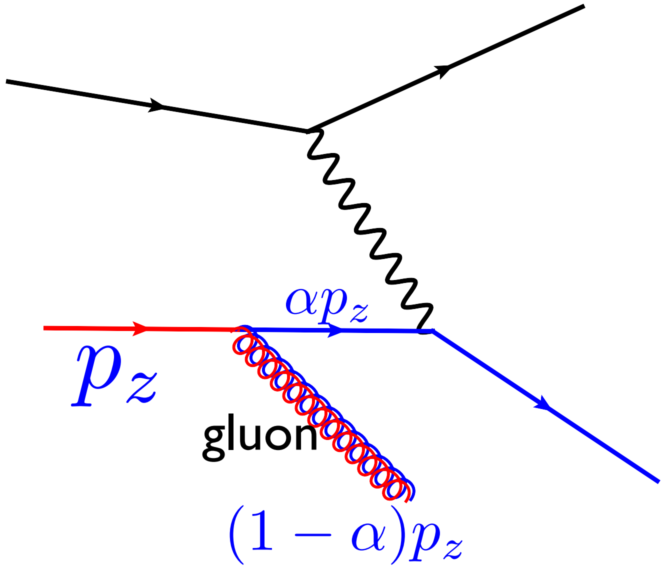


QCD radiation



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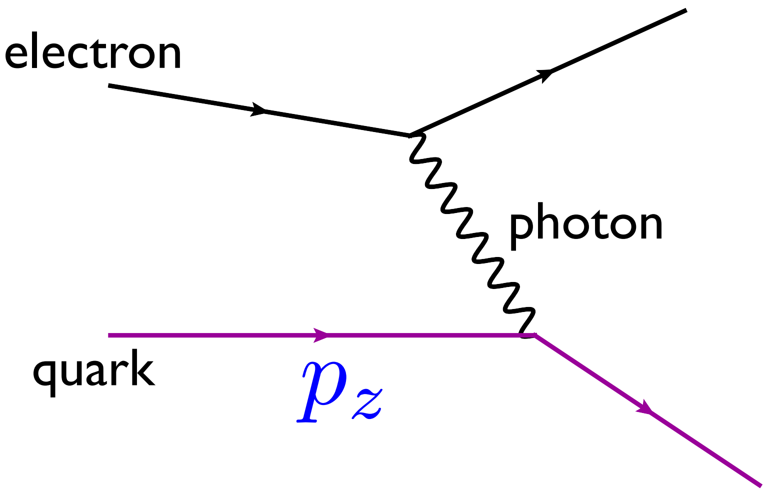

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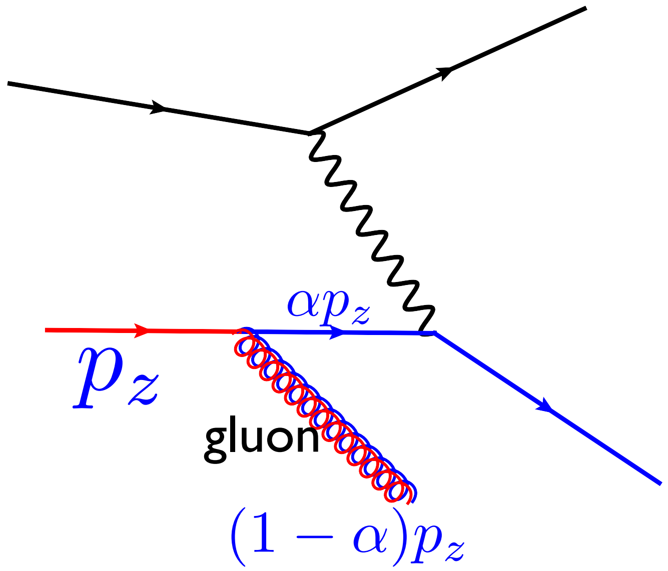

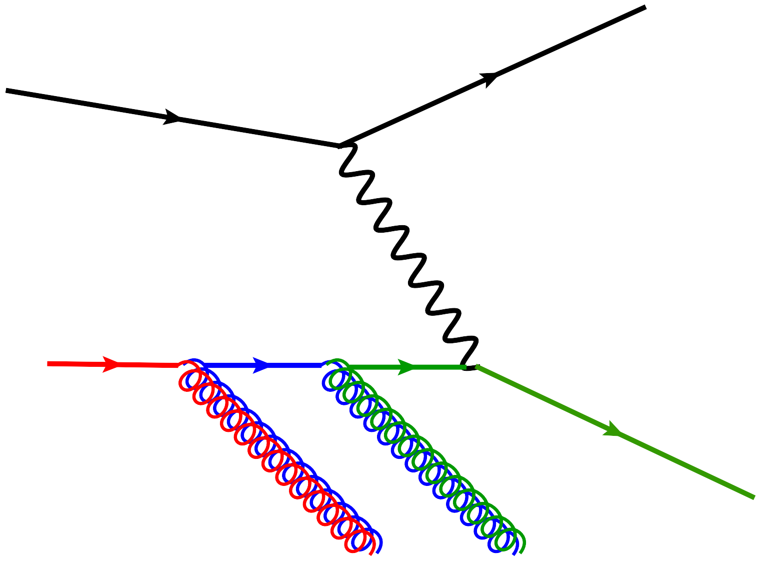

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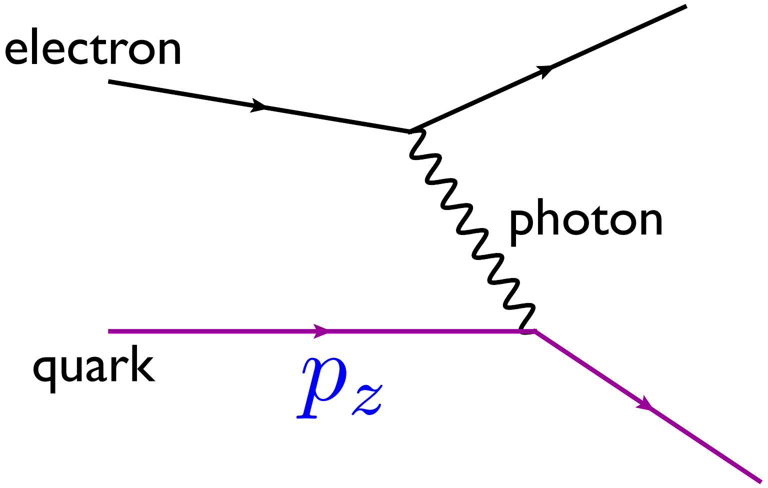

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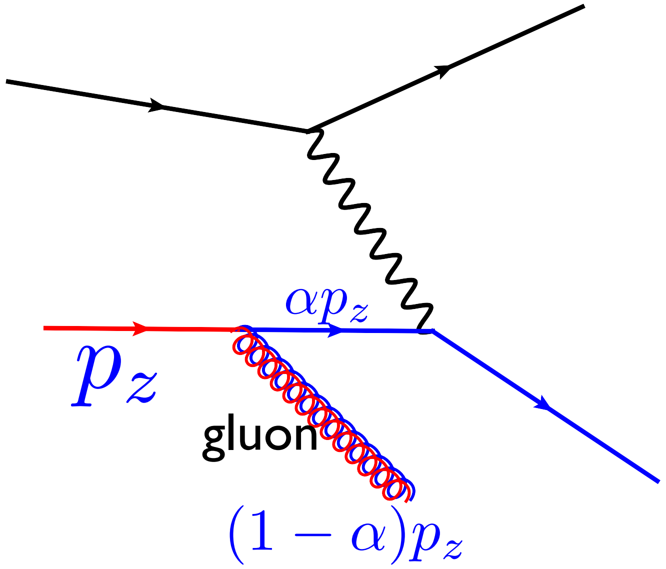
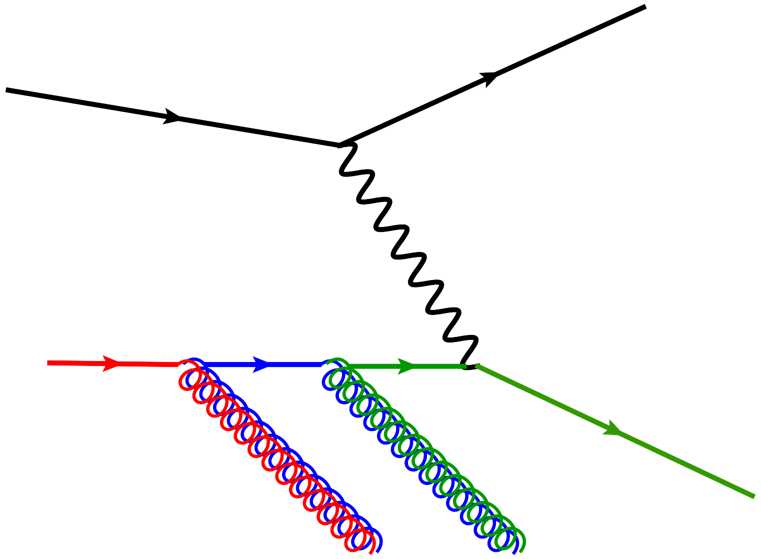
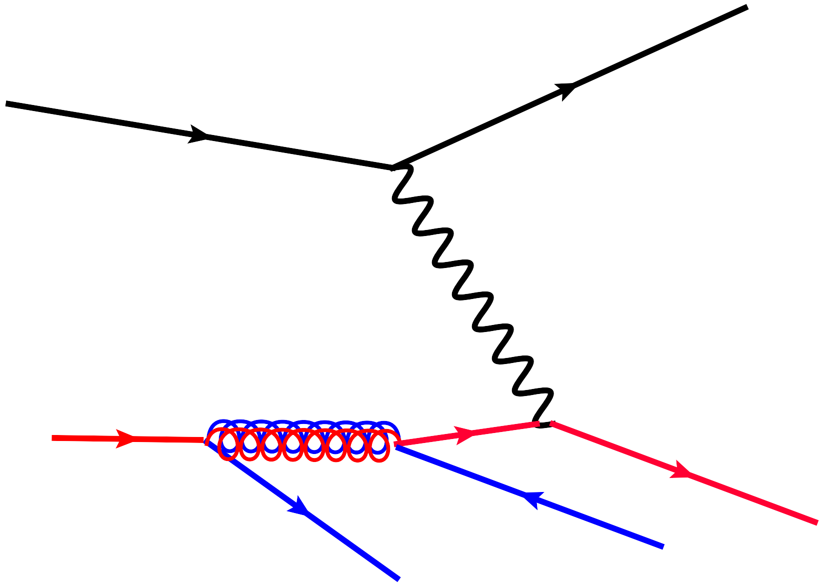




Pair production of sea quarks



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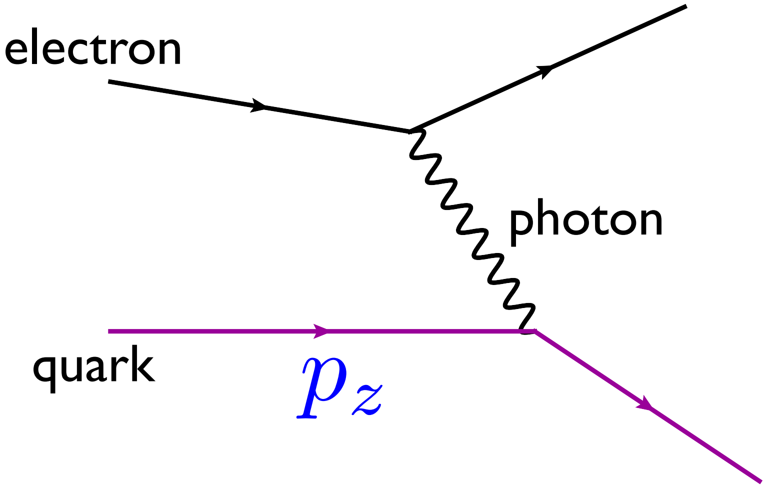

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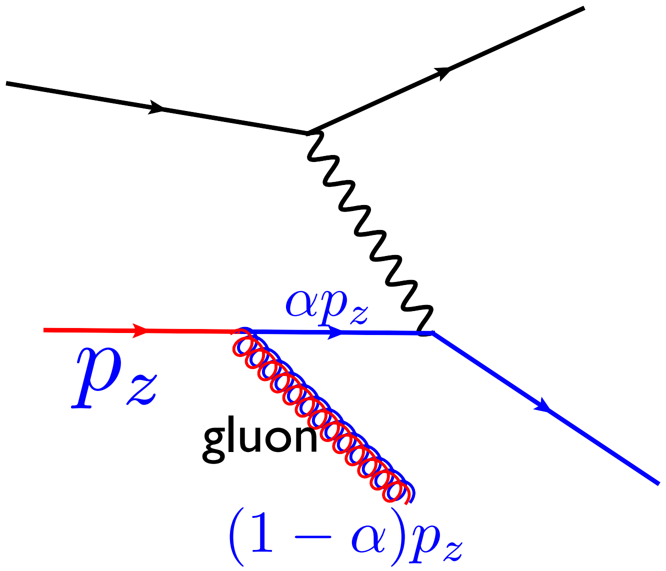

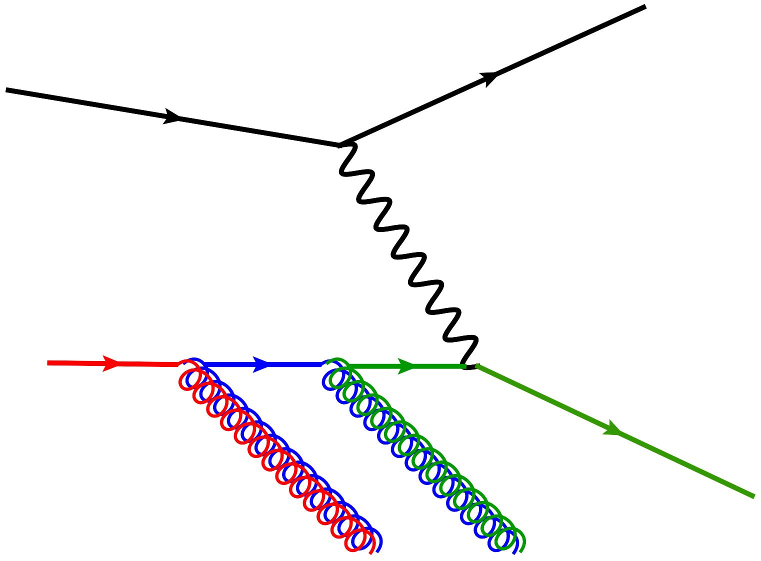

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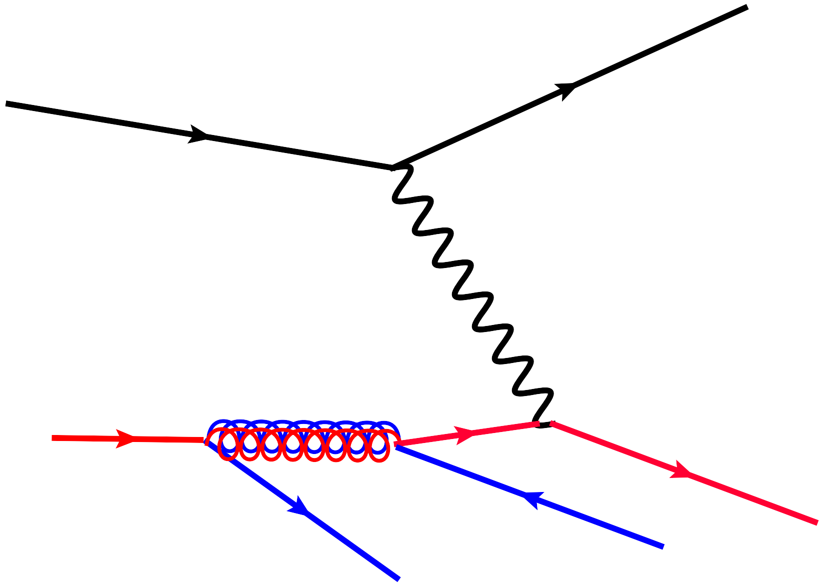
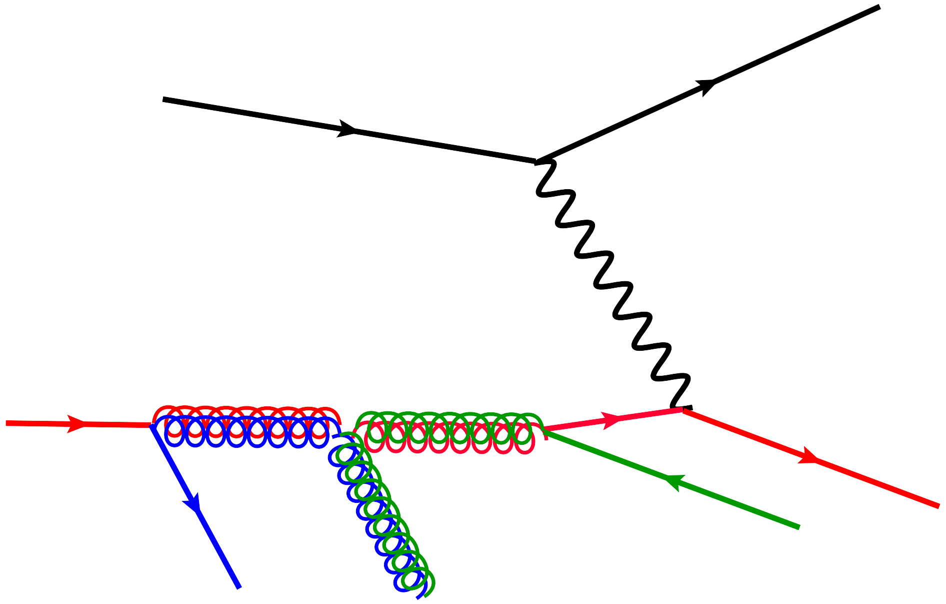


Gluon splitting



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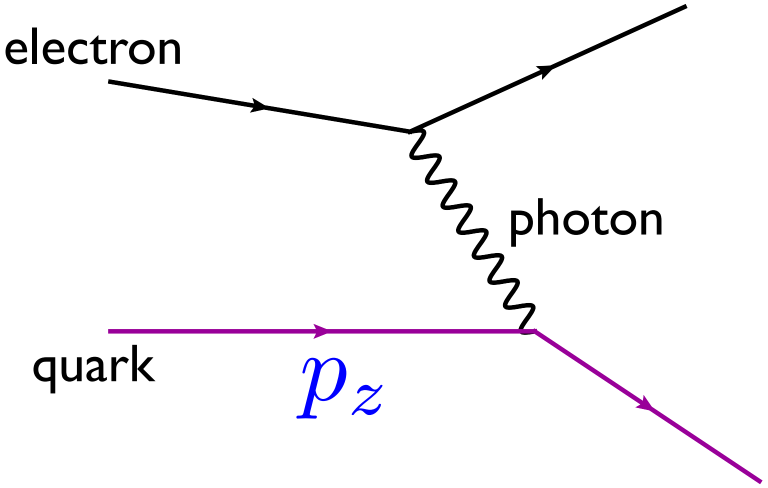

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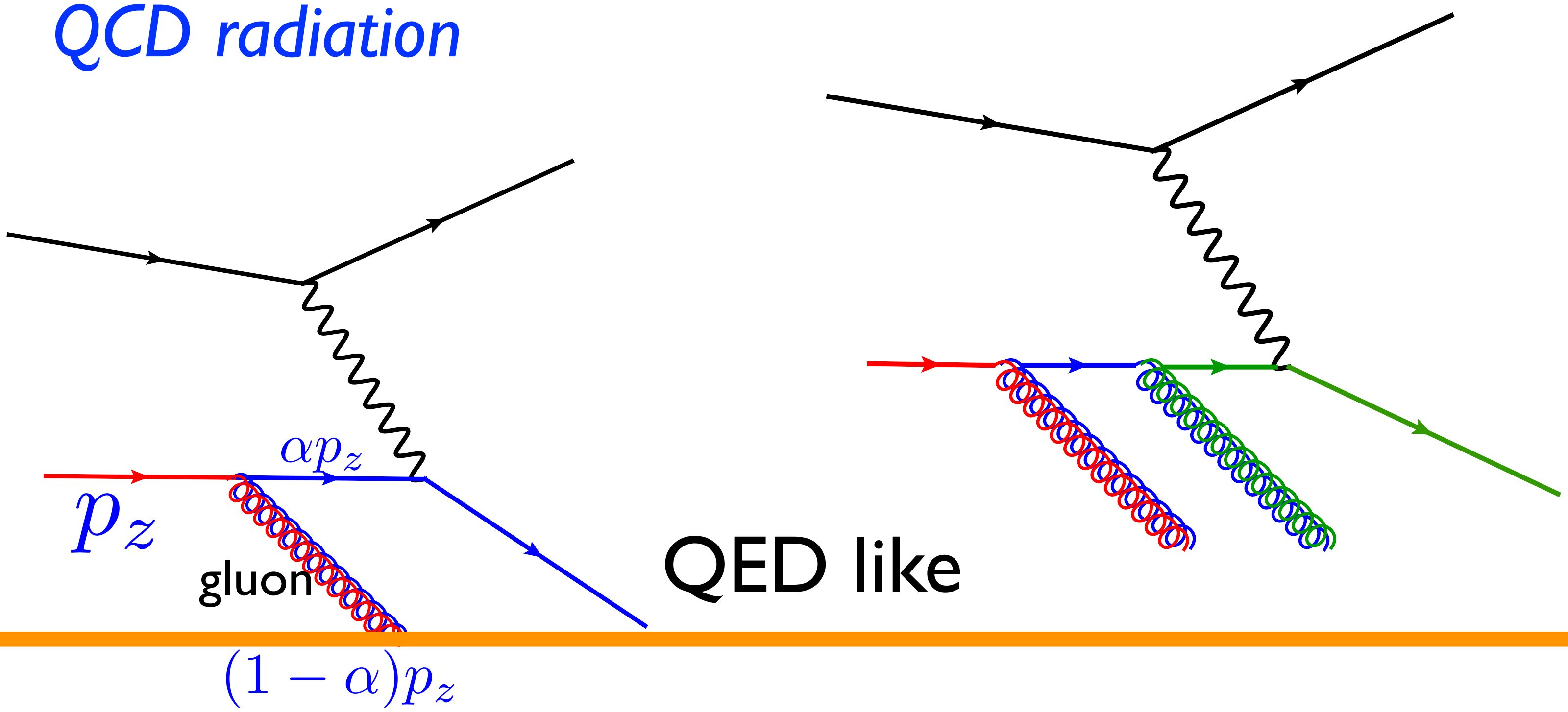
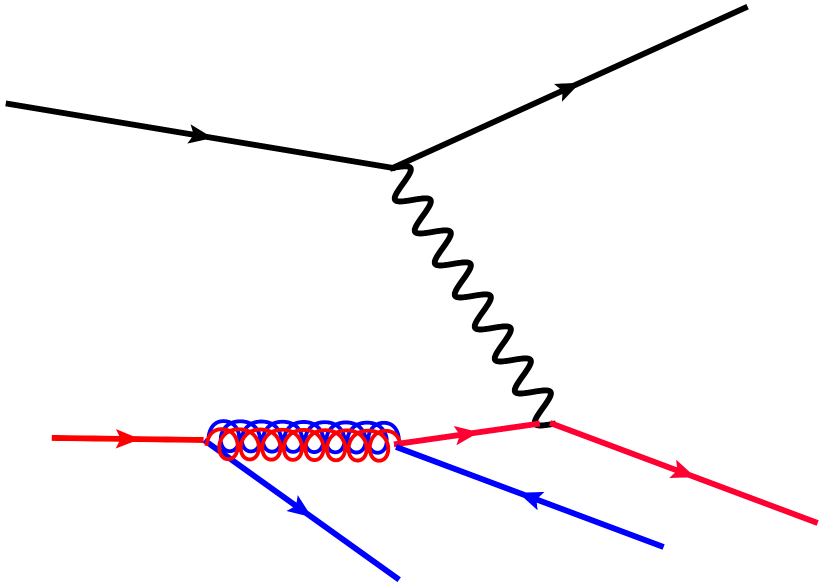


QED like

Pair production of sea quarks



QED like

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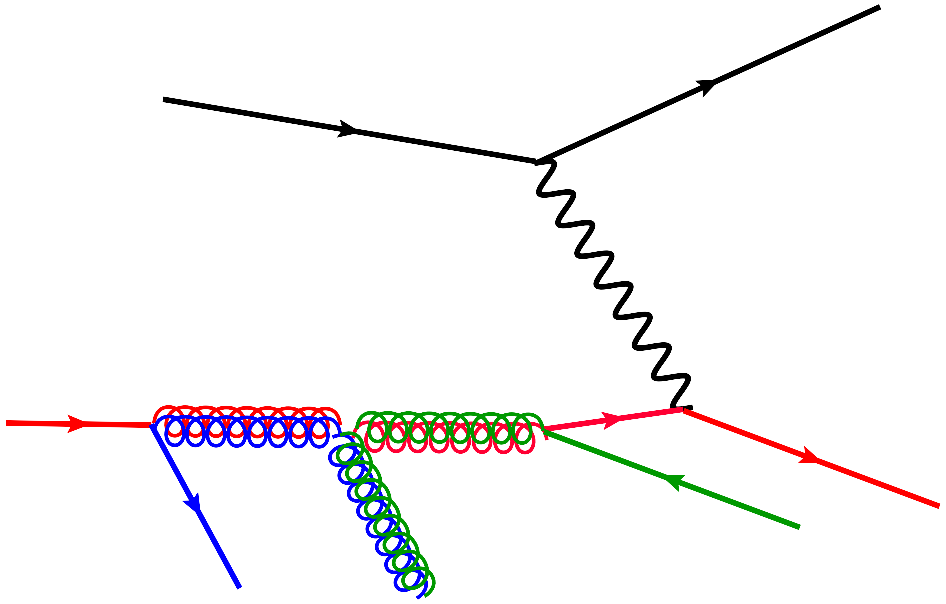
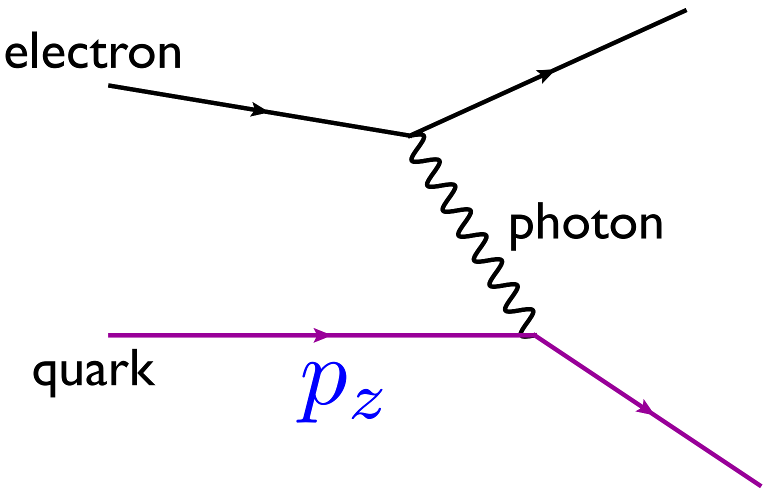


Radiation in QCD

Parton model



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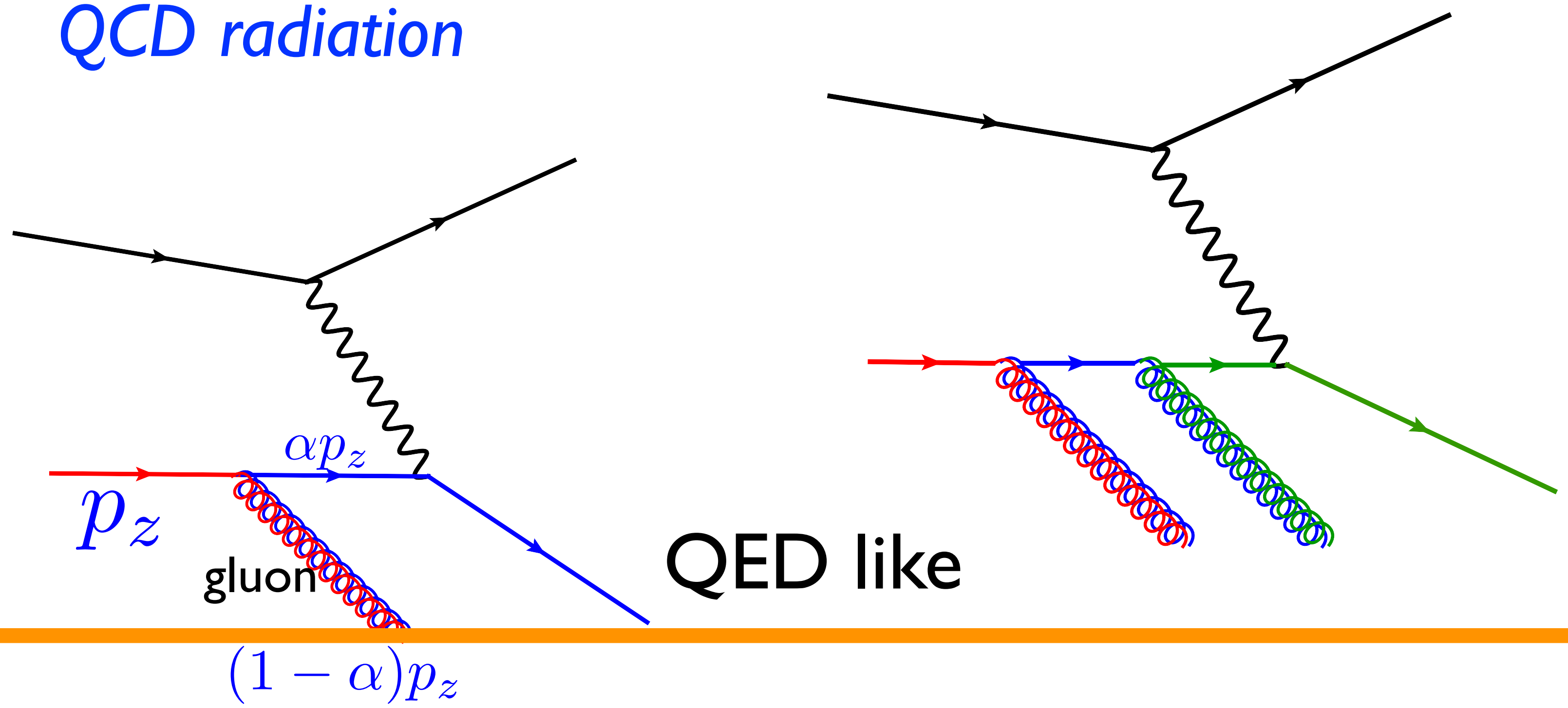

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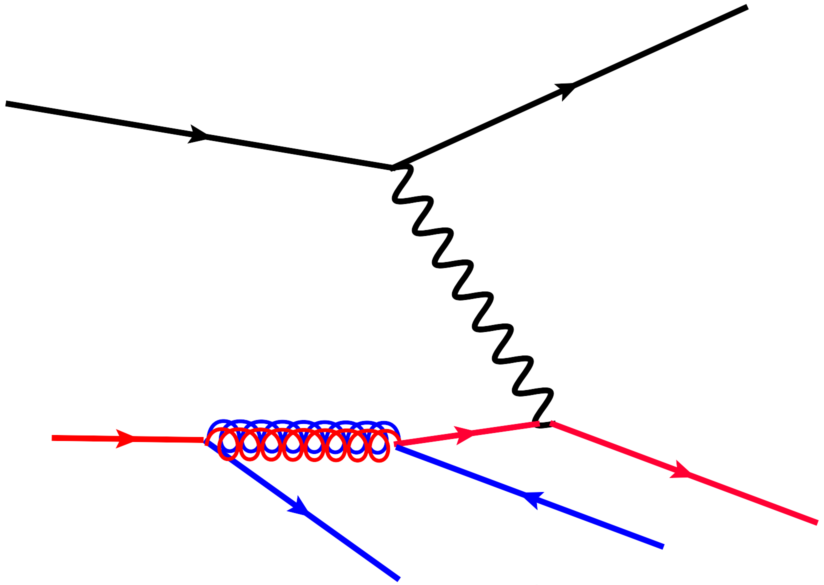

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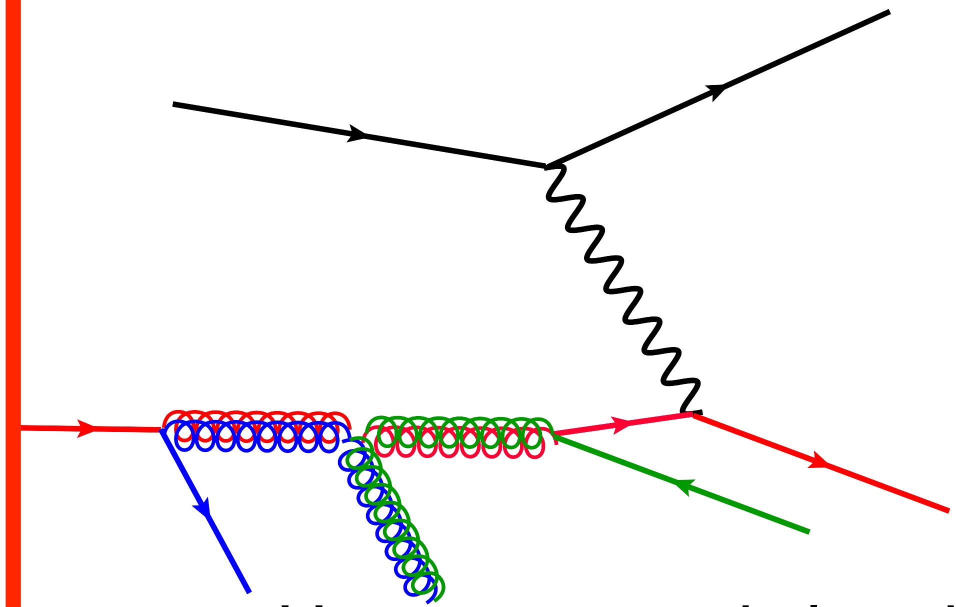
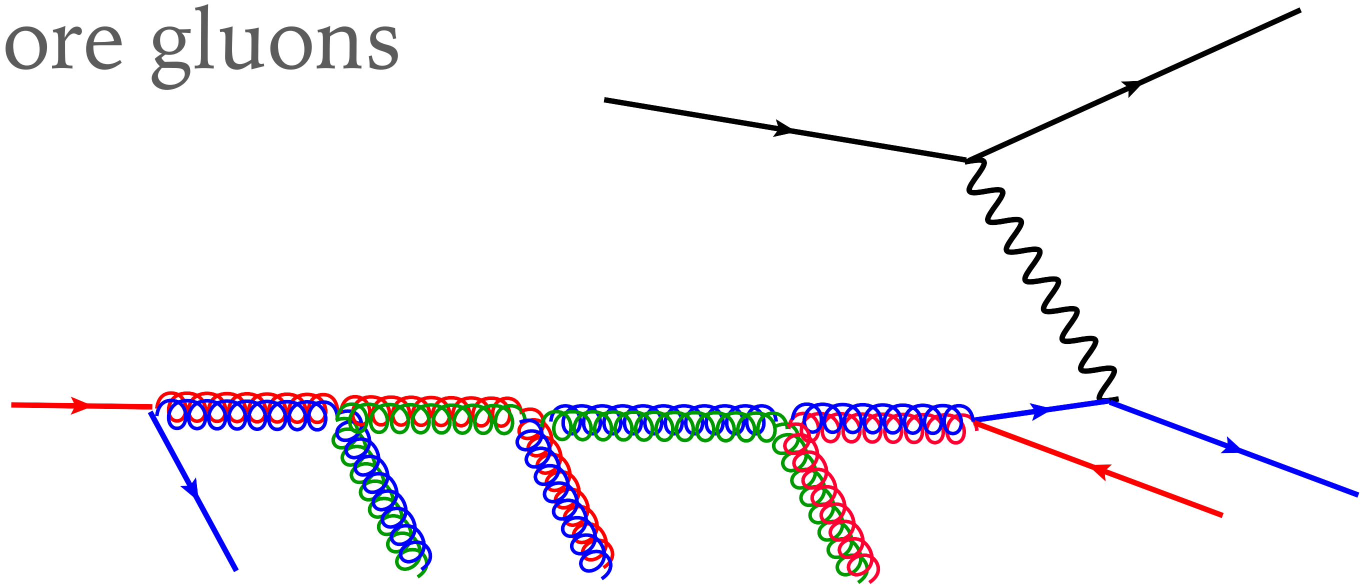


Unique to non-abelian theory

Radiation in QCD

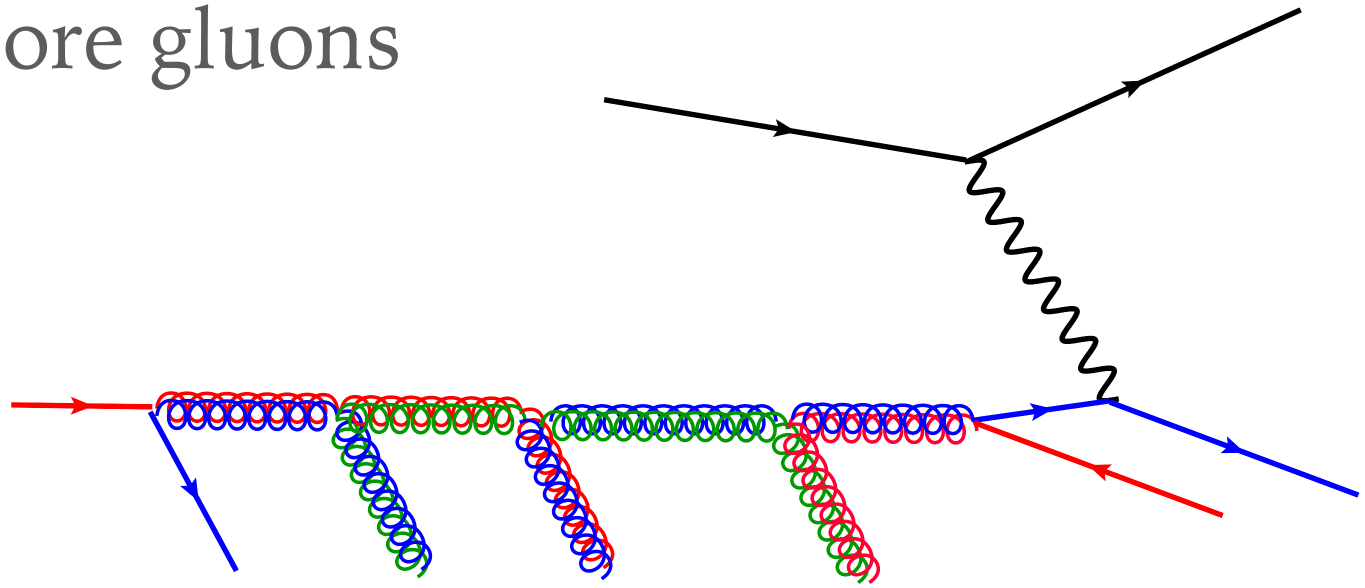
More gluons



...and even more...

Radiation in QCD

More gluons



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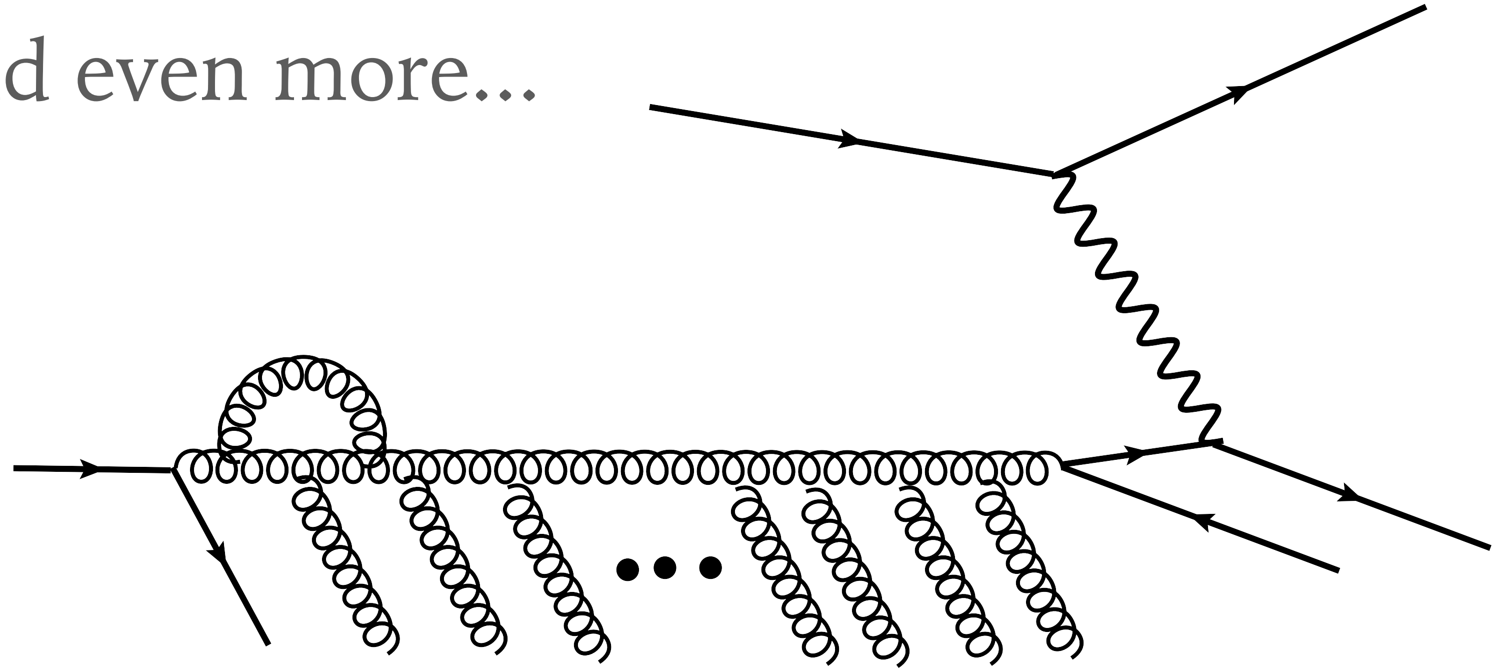
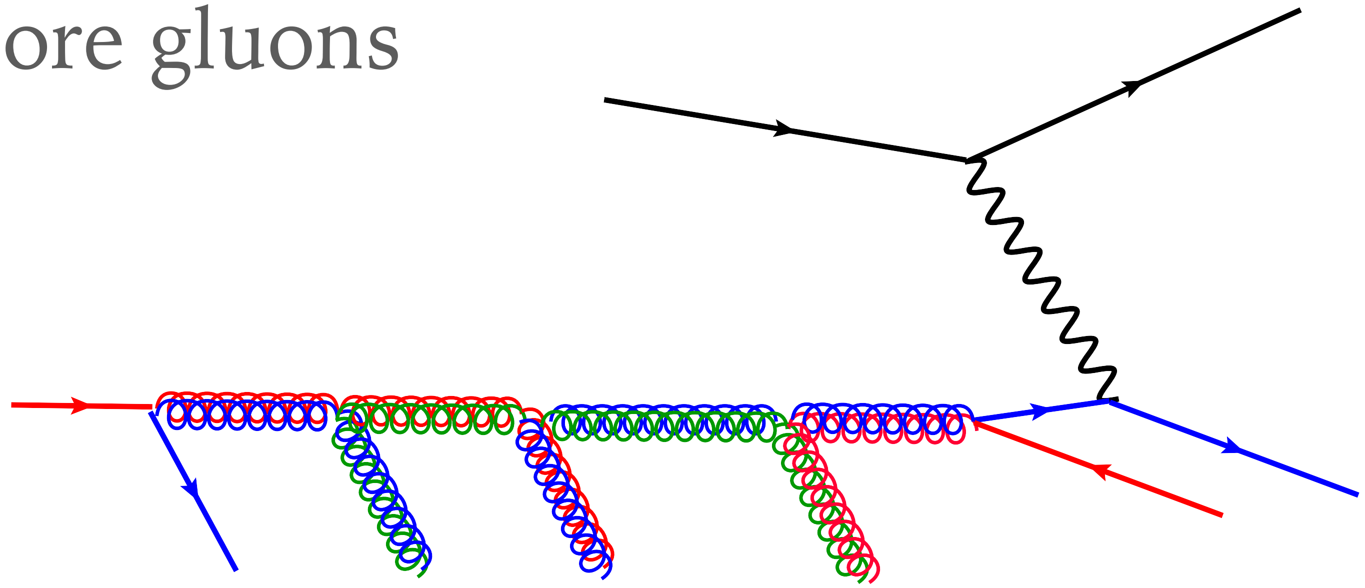


Radiation in QCD

More gluons



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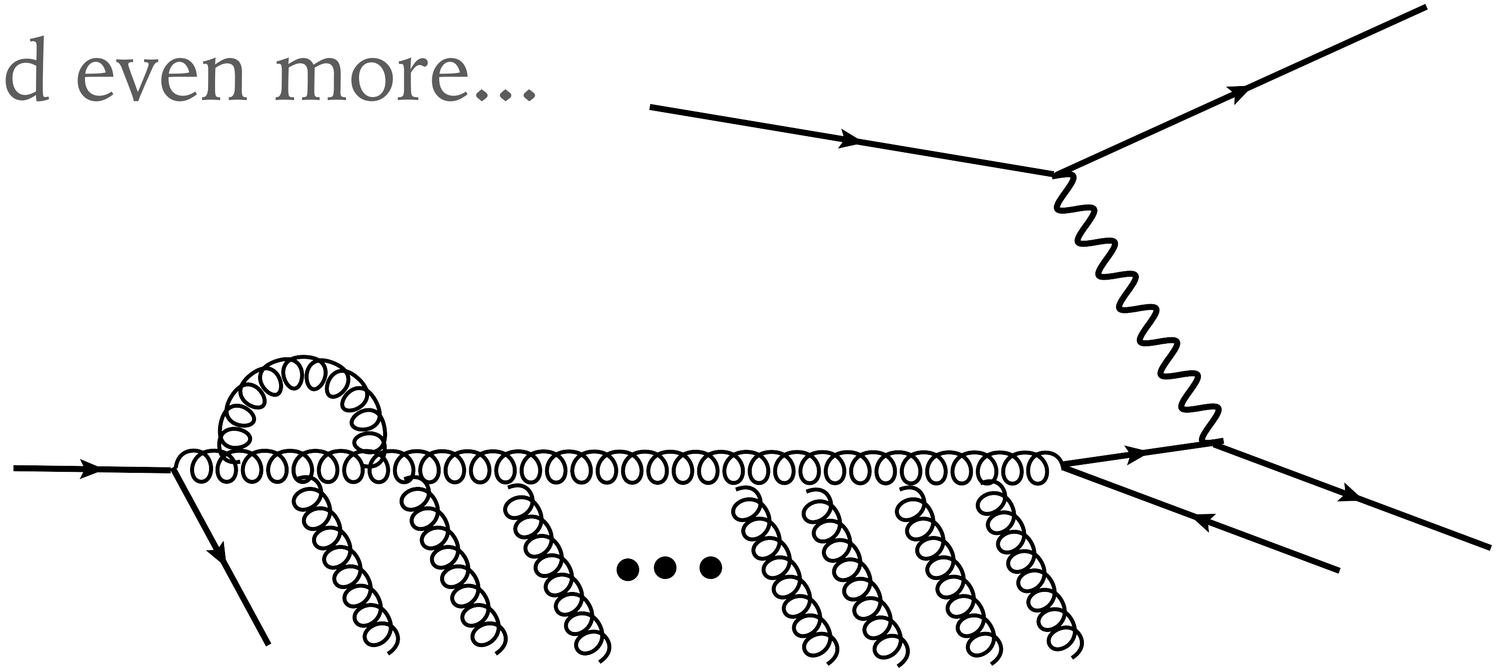
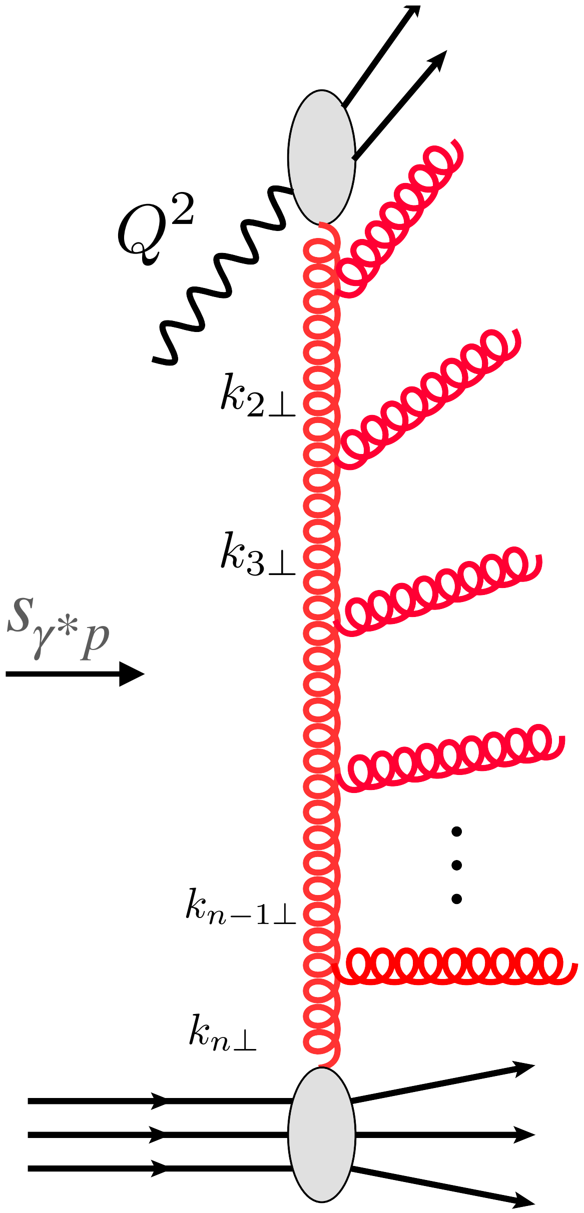


These emissions suppressed by powers of coupling constant but enhanced by large (kinematical) logarithms

Arbitrarily many gluon emissions

Collinear approach

$\gamma^* N$ as a template



Focusing on gluon emissions

Large parameter

$$Q^2 \rightarrow \infty$$

x is fixed

Probing small distances

Strong ordering in transverse momenta

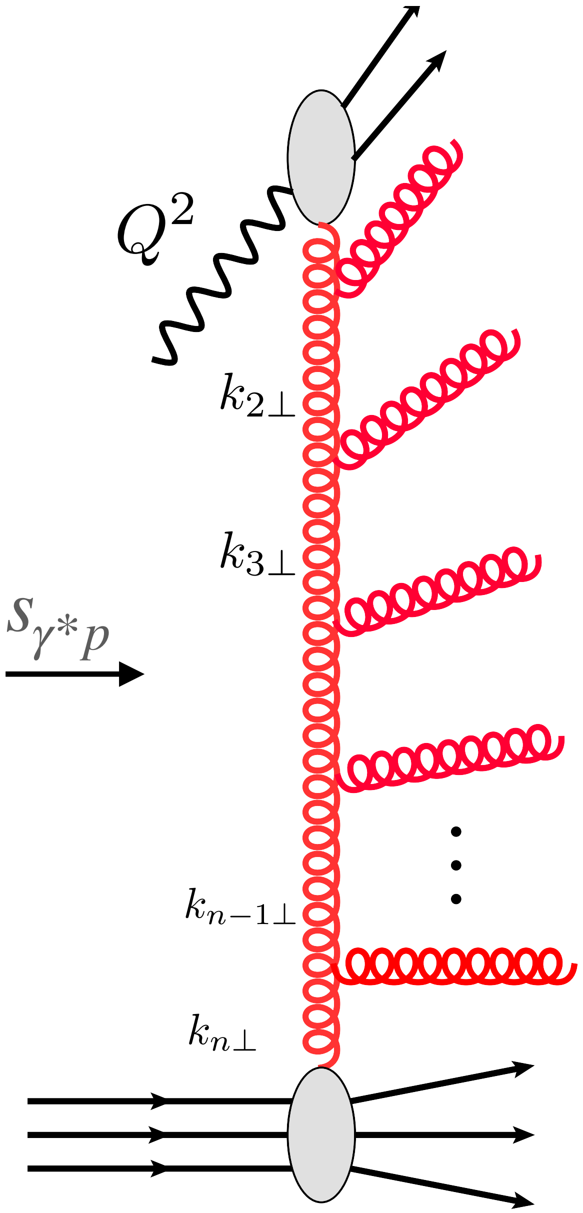
$$Q^2 \gg k_{1\perp}^2 \gg k_{2\perp}^2 \gg k_{3\perp}^2 \cdots \gg k_{n\perp}^2$$

Resummation of large logarithms

$$\int_{\mu_0^2}^{Q^2} \frac{dk_{1\perp}^2}{k_{1\perp}^2} g^2 \int_{\mu_0^2}^{k_{1\perp}^2} \frac{dk_{2\perp}^2}{k_{2\perp}^2} g^2 \int_{\mu_0^2}^{k_{2\perp}^2} \frac{dk_{3\perp}^2}{k_{3\perp}^2} g^2 \cdots \int_{\mu_0^2}^{k_{n-1\perp}^2} \frac{dk_{n\perp}^2}{k_{n\perp}^2} g^2 \simeq \left(g^2 \log \frac{Q^2}{\mu_0^2} \right)^n$$

DGLAP evolution

Dokshitzer-Gribov-Lipatov-Altarelli-Parisi



DGLAP evolution equations for parton densities

$$\mu^2 \frac{\partial}{\partial \mu^2} \begin{pmatrix} q_i(x,\mu^2) \\ g(x,\mu^2) \end{pmatrix} = \sum_j \int_x^1 \frac{dz}{z} \begin{pmatrix} P_{q_i q_j}(z,\alpha_s) & P_{q_i g}(z,\alpha_s) \\ P_{g q_j}(z,\alpha_s) & P_{gg}(z,\alpha_s) \end{pmatrix} \begin{pmatrix} q_j(\frac{x}{z},\mu^2) \\ g(\frac{x}{z},\mu^2) \end{pmatrix}$$

q_j : quark density, g : gluon density

Splitting functions calculated perturbatively

$$P_{ab}(z,\alpha_s) \equiv P_{b\to a}(z,\alpha_s) = \underbrace{\frac{\alpha_s}{2\pi} P_{ab}^{(0)}(z)}_{\text{LO}} + \underbrace{\left(\frac{\alpha_s}{2\pi}\right)^2 P_{ab}^{(1)}(z)}_{\text{NLO}} + \underbrace{\left(\frac{\alpha_s}{2\pi}\right)^3 P_{ab}^{(2)}(z)}_{\text{NNLO}} + \dots$$

Leading order splitting functions

$$P_{qq}^{(0)}(z) = C_F\left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\right]$$

$$P_{qg}^{(0)}(z) = T_R\left[z^2 + (1-z)^2\right]$$

$$P_{gq}^{(0)}(z) = C_F\left[\frac{z^2+(1-z)^2}{z}\right]$$

$$P_{gg}^{(0)}(z) = 2C_A\left[\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z) + \delta(1-z)\frac{11C_A - 4n_f T_R}{6}\right]$$

Successful description of HERA data

Reduced cross section at HERA

Combined measurement H1 and ZEUS

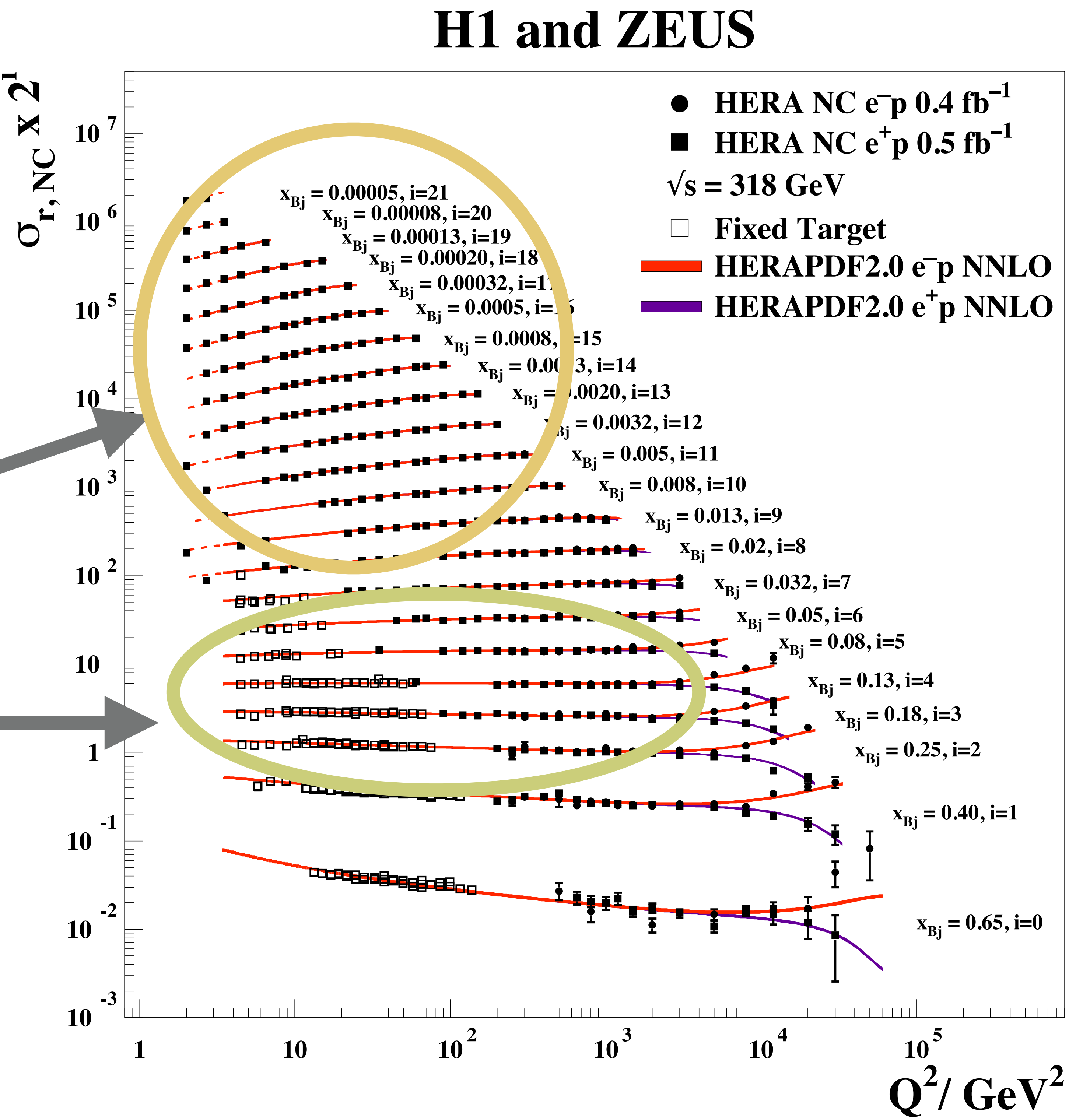
Function of Q^2 for fixed x

Scaling violations

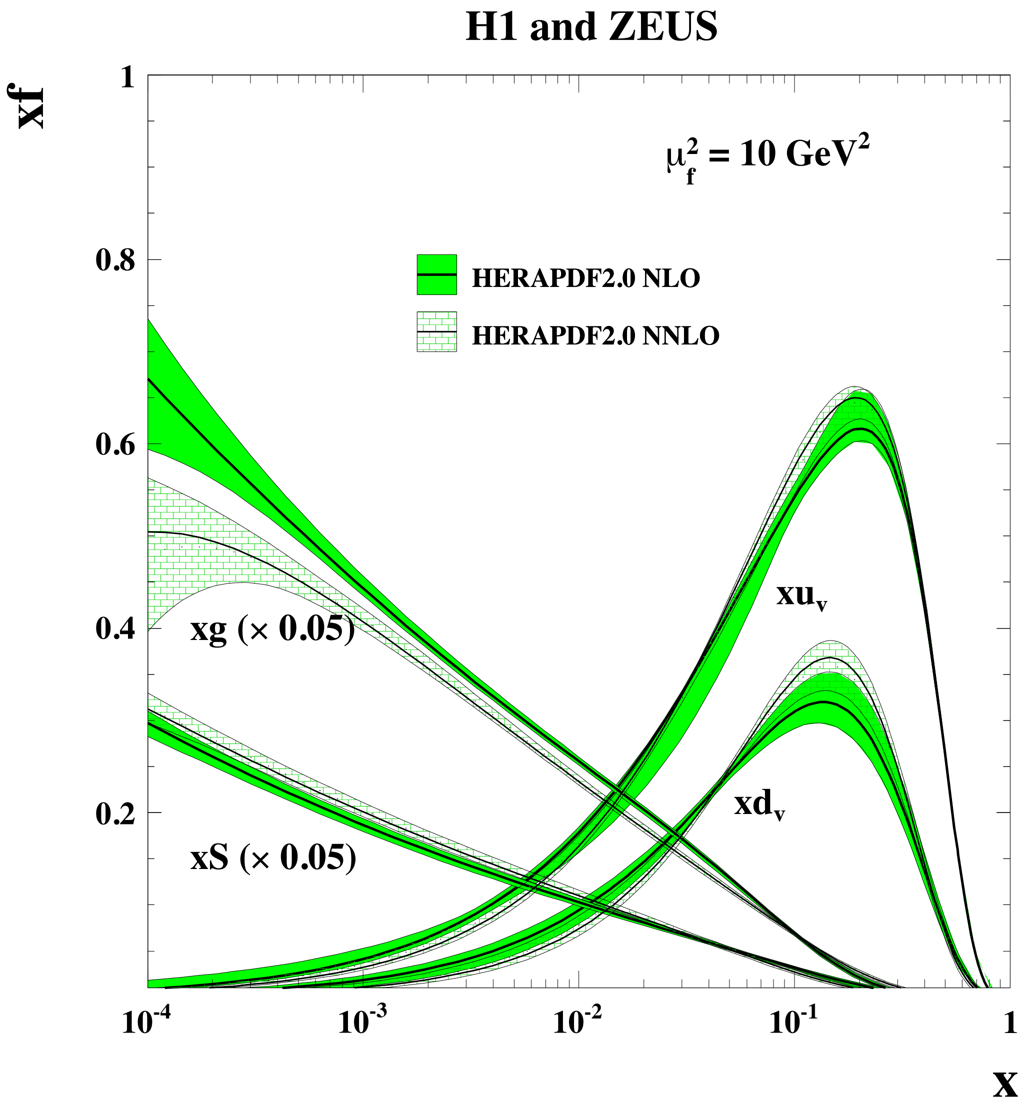
Scaling region : independent of Q^2

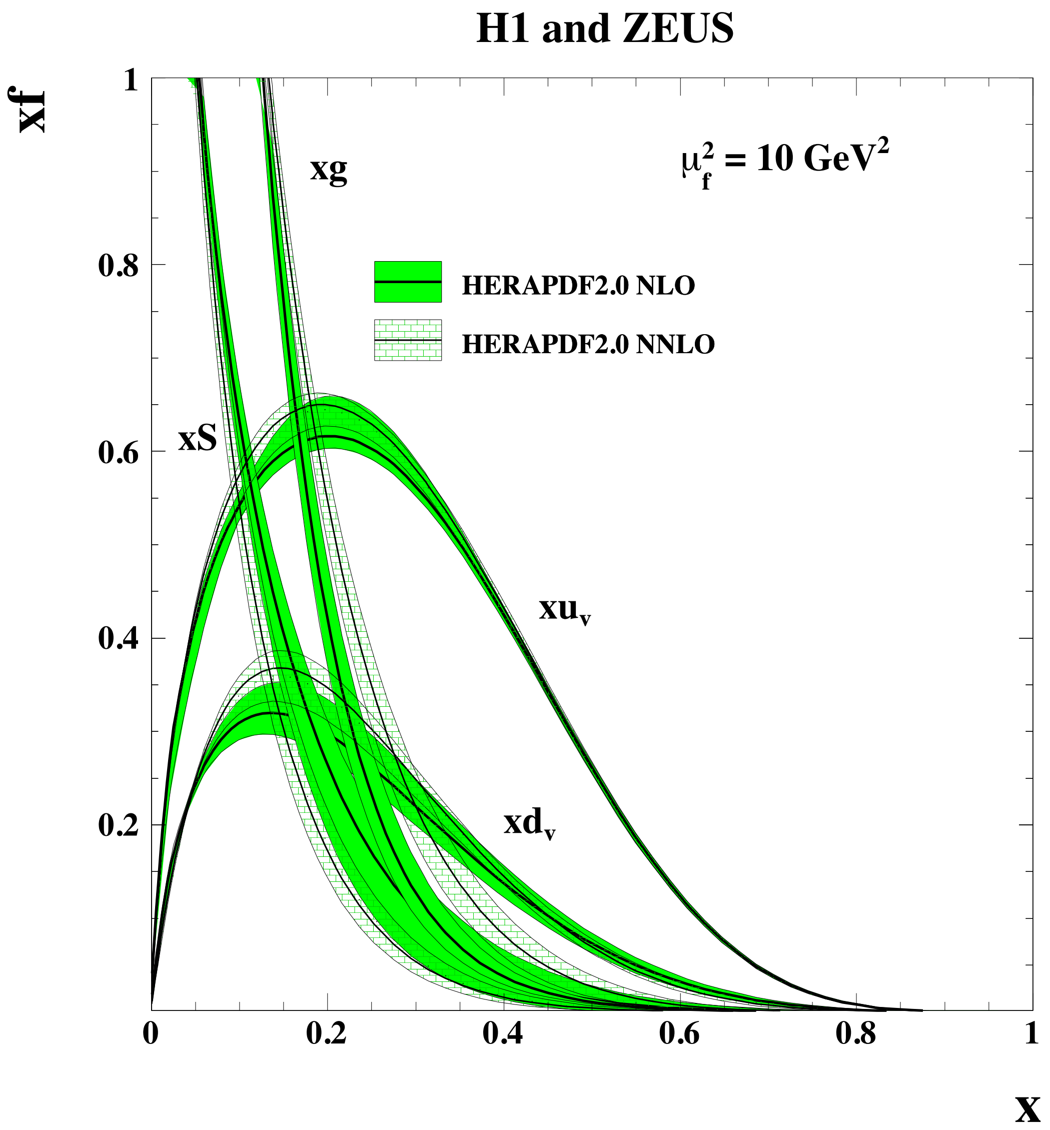
Excellent description using DGLAP

HERAPDF parton densities



DGLAP parton densities





Gluon density dominates at small x

NLO vs NNLO small x behavior

What happens at small x ?

Small x means large energy

$$x = \frac{Q^2}{Q^2 + W^2}$$

$$W^2 = s_{\gamma^* p}$$

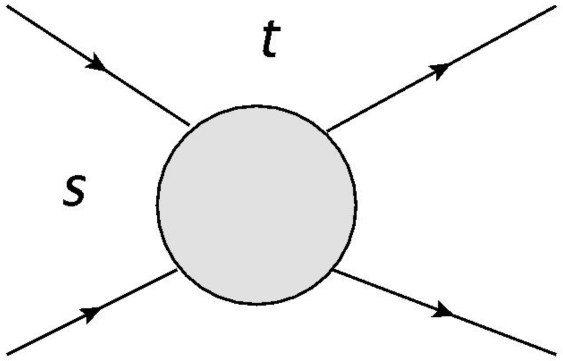
S-matrix and Regge limit

Properties of S matrix:

- Lorentz invariance
- crossing
- unitarity
- analyticity

$\mathcal{A}(s,t)$

ex. 2 to 2 scattering



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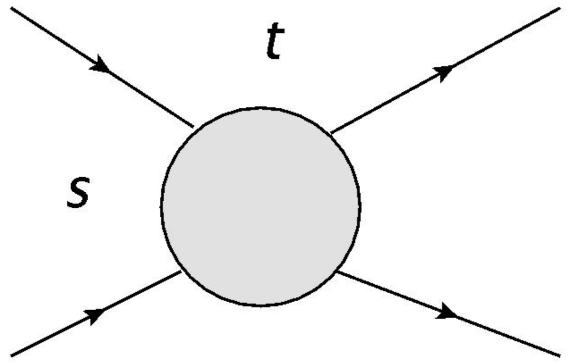
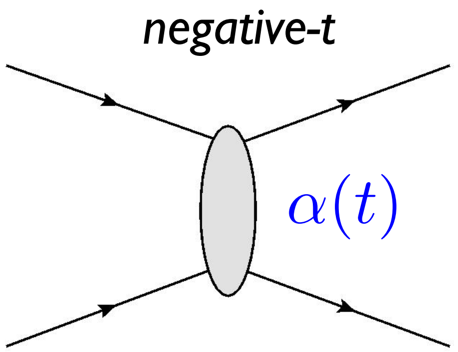


Regge limit: $s \to \infty$ $t = \text{const}$



$$\mathcal{A}(s,t) \sim \tilde{\beta}(t) s^{\alpha(t)}$$

Amplitude dominated by exchange of the Regge trajectory $\alpha(t) = \alpha(0) + \alpha' t$

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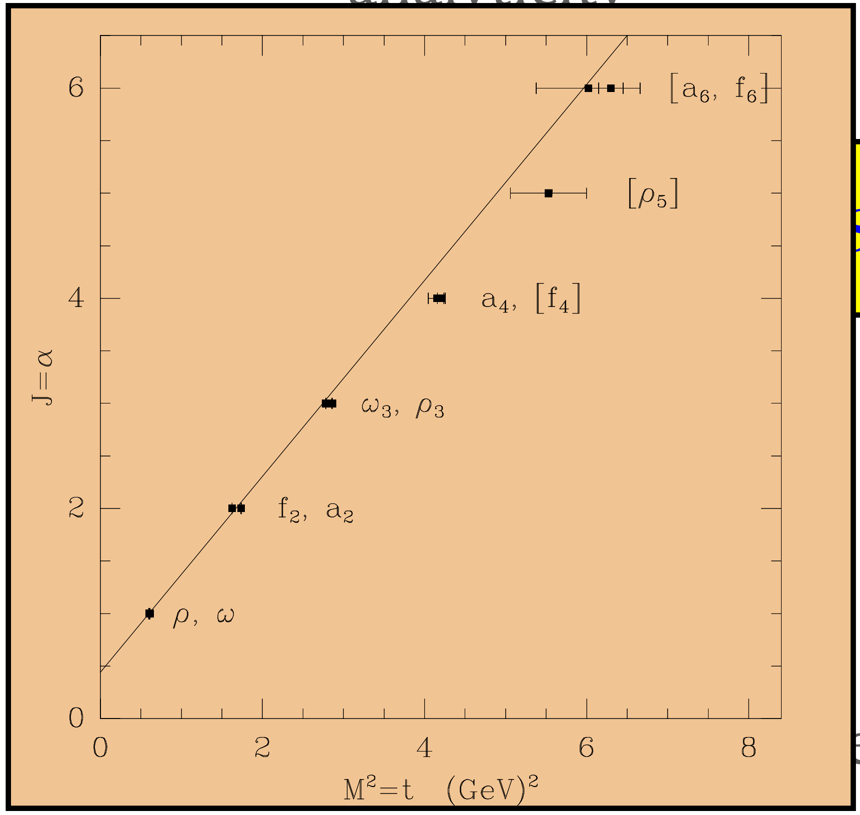
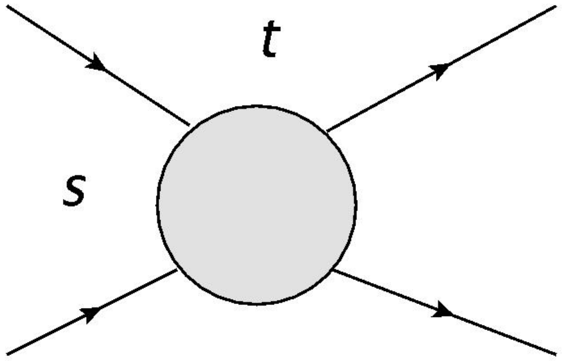
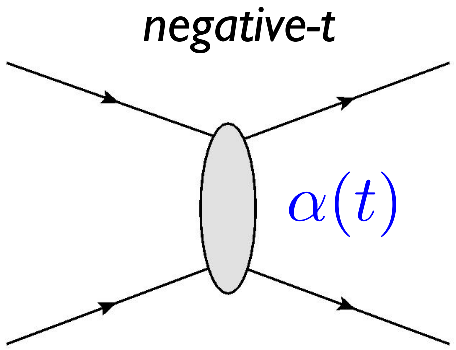


$$\mathcal{A}(s,t)$$

ex. 2 to 2 scattering



$$s \to \infty \qquad t = \text{const}$$



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exchange of the Regge trajectory $\alpha(t) = \alpha(0) + \alpha' t$

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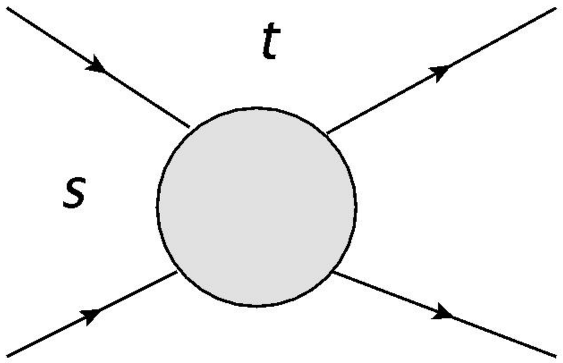

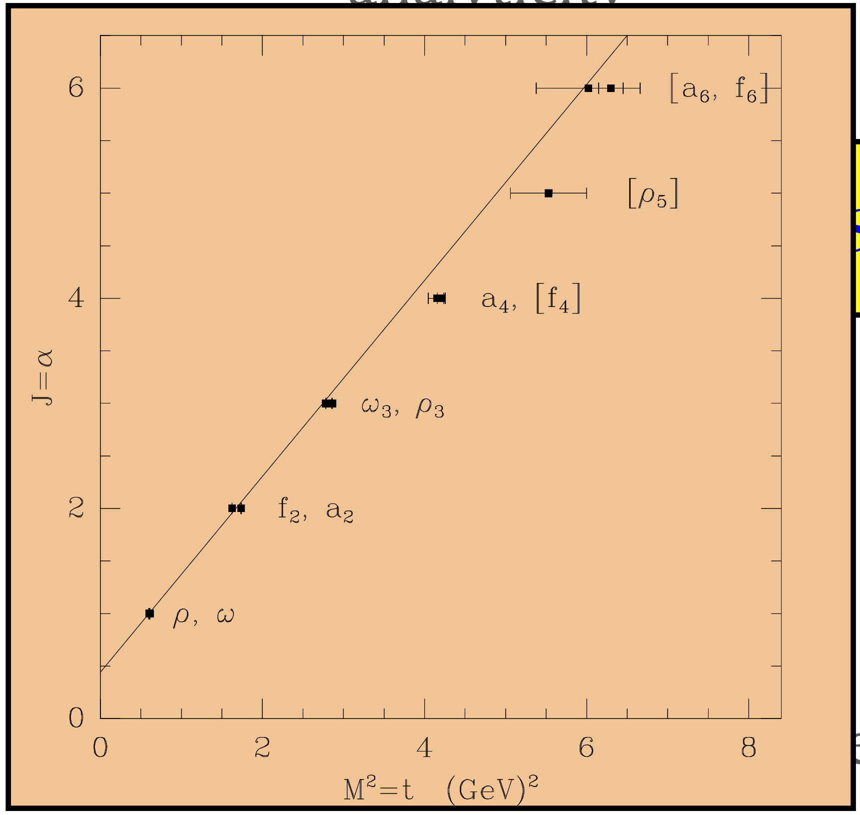

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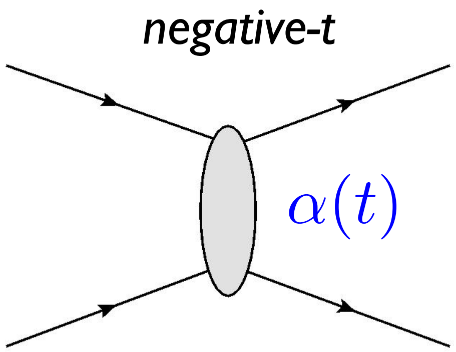


$$\mathcal{A}(s,t) \sim \tilde{\beta}(t) s^{\alpha(t)}$$

exchange of the Regge trajectory $\alpha(t) = \alpha(0) + \alpha' t$

From optical theorem $\sigma_{\text{tot}} = s^{-1} \text{Im} \mathcal{A}(s,0) \sim s^{\alpha(0)-1}$

Intercept $\alpha(0)$ of Regge trajectory determines the behavior of the cross section

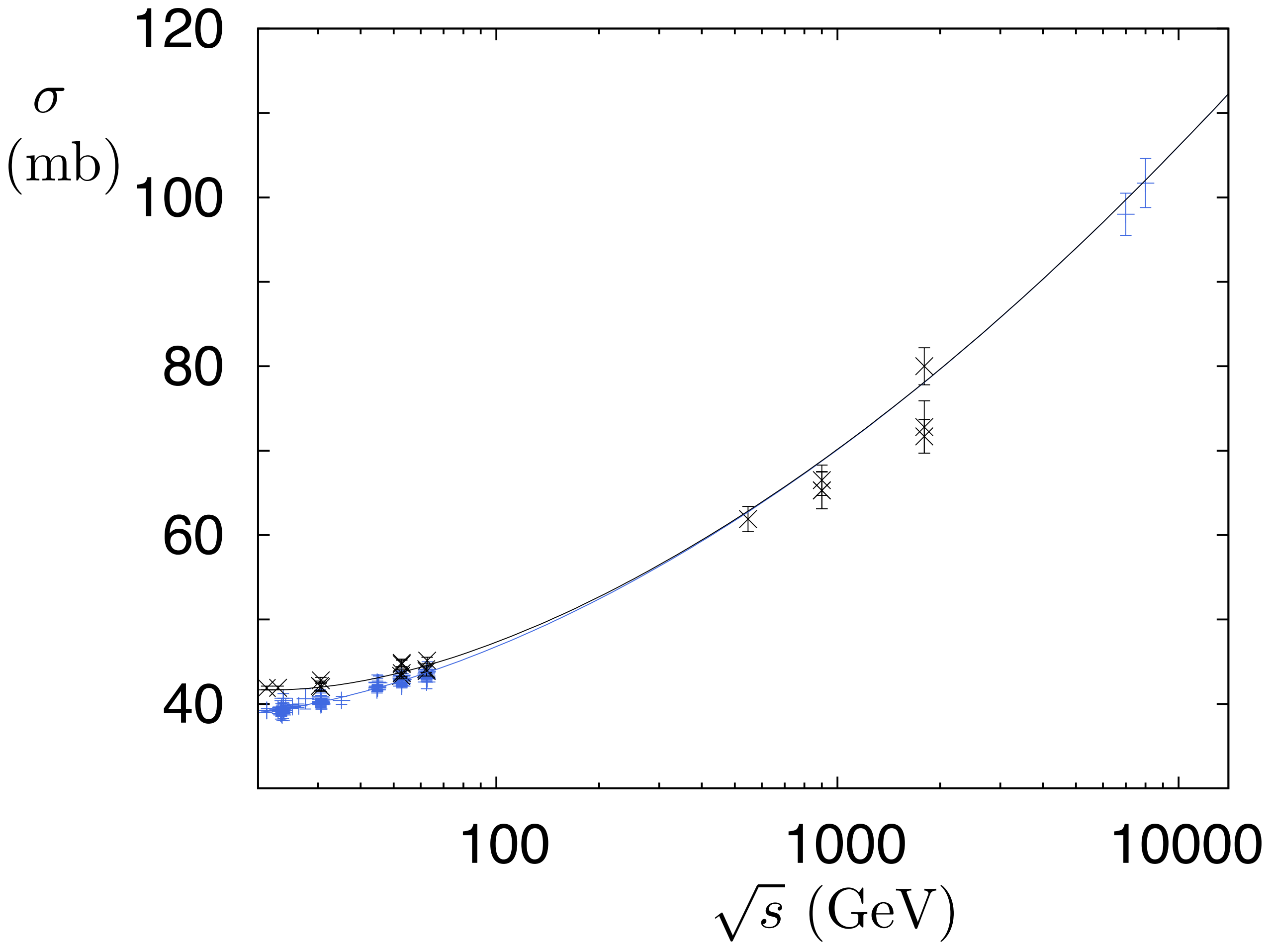
Pomeron

Okun,Pomeranchuk;
Foldy,Peierls

Pomeron:

- Reggeon with intercept greater than unity.
- Corresponds to the exchange of the vacuum quantum numbers.
- Dominates the cross section at asymptotically high energies

Donnachie, Landshoff



Soft Pomeron

$$\alpha_P(t) = 1.11 + 0.165\text{GeV}^{-2} t$$

However, such soft pomeron power behavior is potentially in conflict with Froissart bound which stems from unitarity requirements:

$$\sigma^{\text{tot}}(s) \leq C \log^2(s/s_0)$$

Note: the exact value of the constant C is of crucial importance here.

Pomeron

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Okun,Pomeranchuk;
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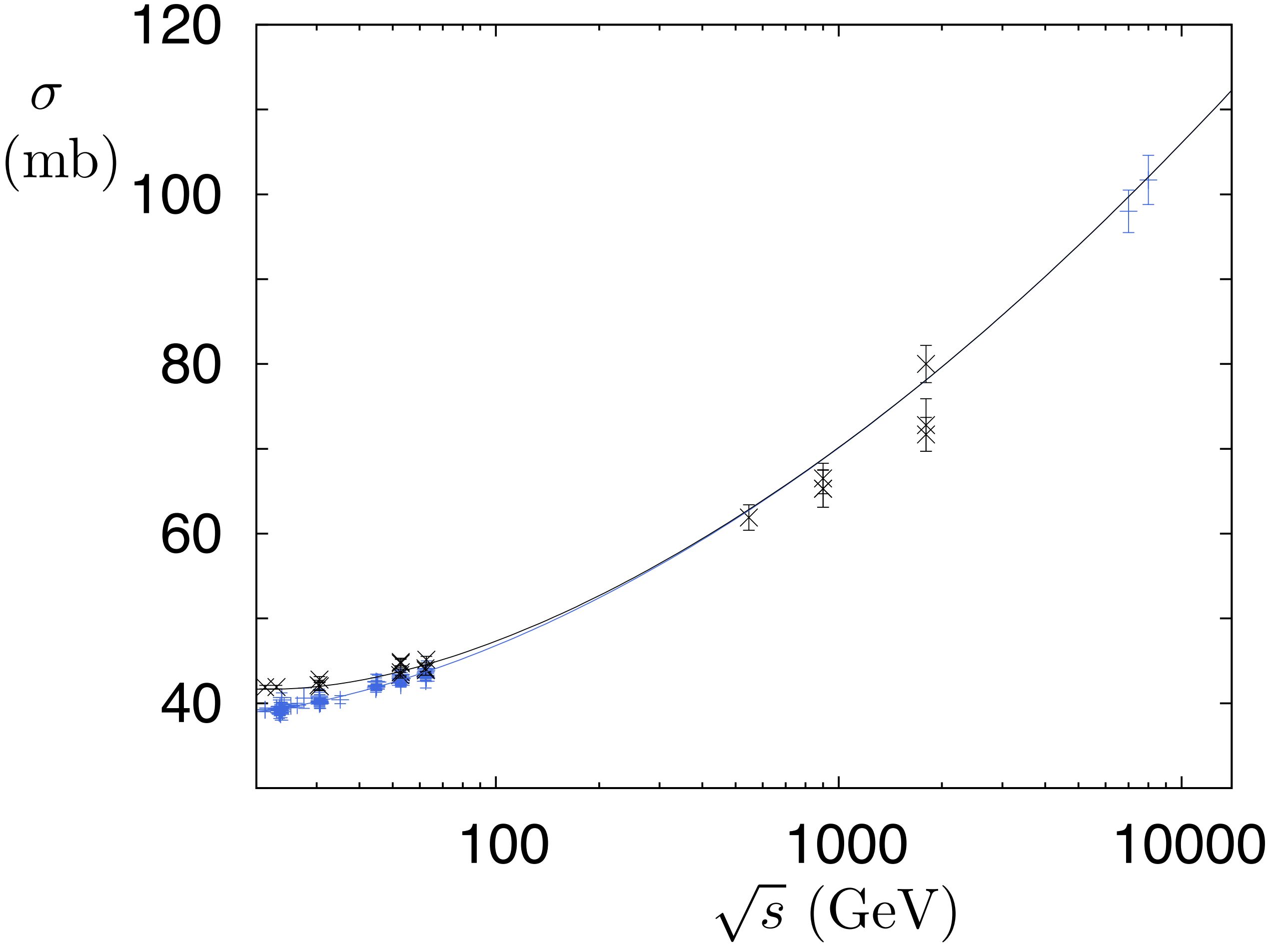

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However, such soft pomeron power behavior is potentially in conflict with Froissart bound which stems from unitarity requirements:

$$\sigma^{\text{tot}}(s) \leq C \log^2(s/s_0)$$

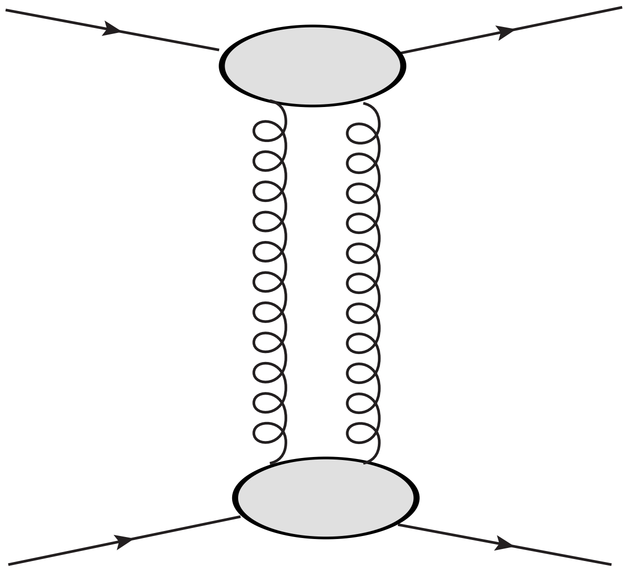
Note: the exact value of the constant C is of crucial importance here.

Pomeron in QCD

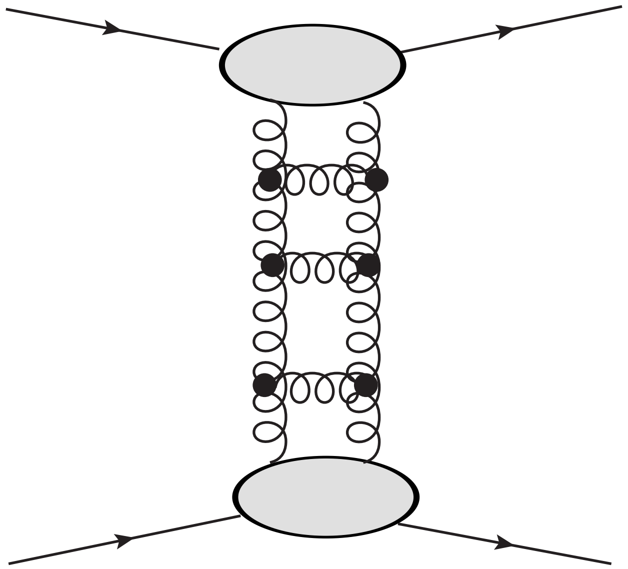
What is a Pomeron in QCD?

High energy limit
in perturbative QCD:

$$s \gg |t| \qquad \alpha_s \ll 1 \qquad \alpha_s \log s/s_0 \sim 1$$



Low-Nussinov model:
2 gluon exchange



BFKL Pomeron
gluon ladder in the multi-Regge kinematics

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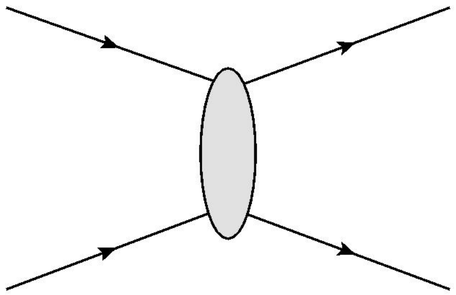

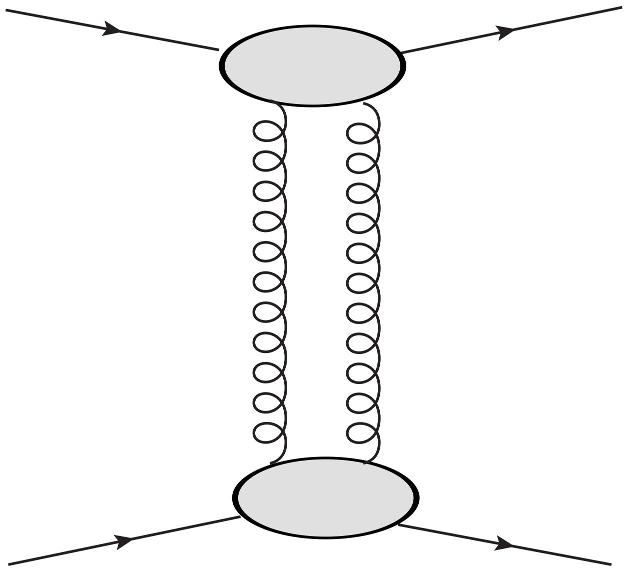

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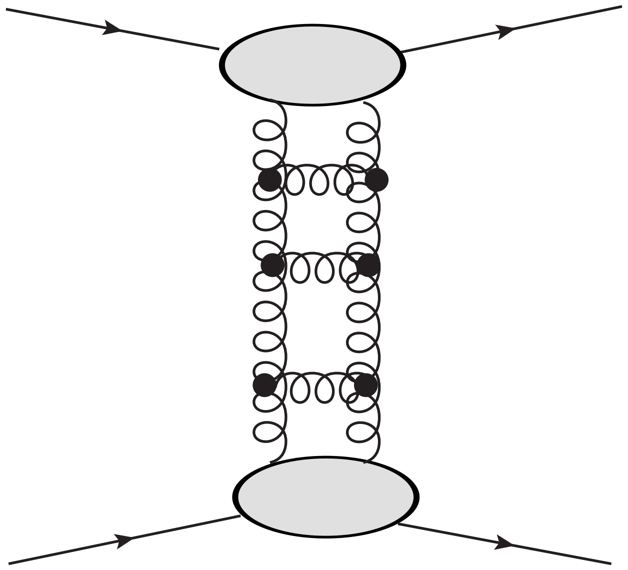
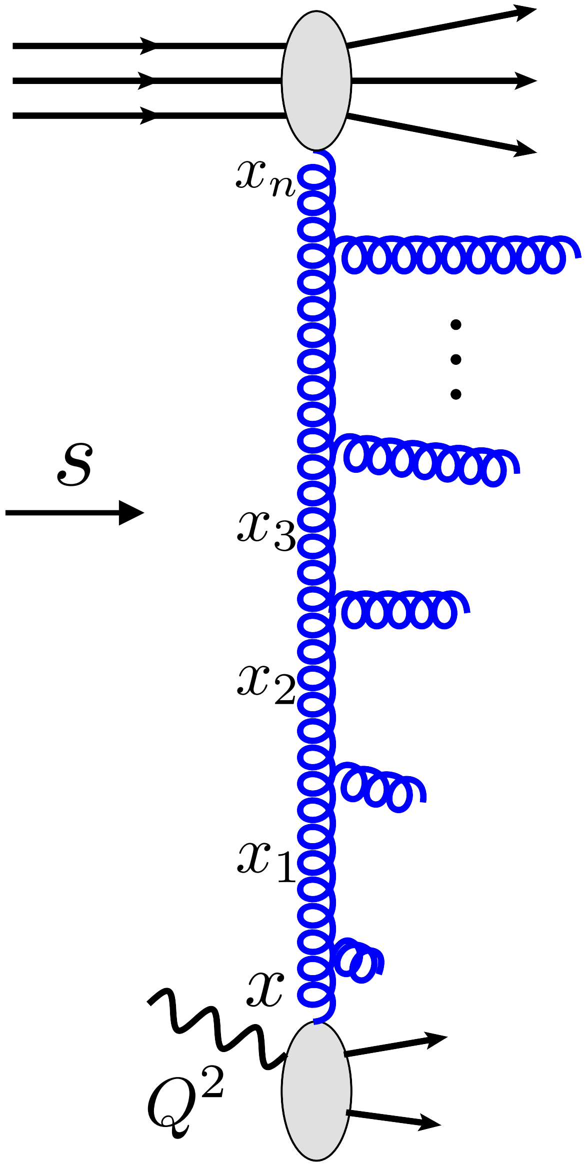


BFKL Pomeron
gluon ladder in the multi-Regge
kinematics

Gluon emissions at small x



Large parameter

$$s \to \infty$$

Q^2 fixed, perturbative

High energy or Regge limit

$$s \gg Q^2 \gg \Lambda^2$$

Light cone proton momentum

$$p^+ = p^0 + p^z \qquad k_i^+ = x_i p^+$$

Strong ordering in longitudinal momenta

$$x \ll x_1 \ll x_2 \ll \cdots \ll x_n$$

Perturbative coupling but large logarithm

$$\bar{\alpha}_s \ll 1 \qquad \ln\frac{1}{x} \simeq \ln\frac{s}{Q^2} \gg 1$$

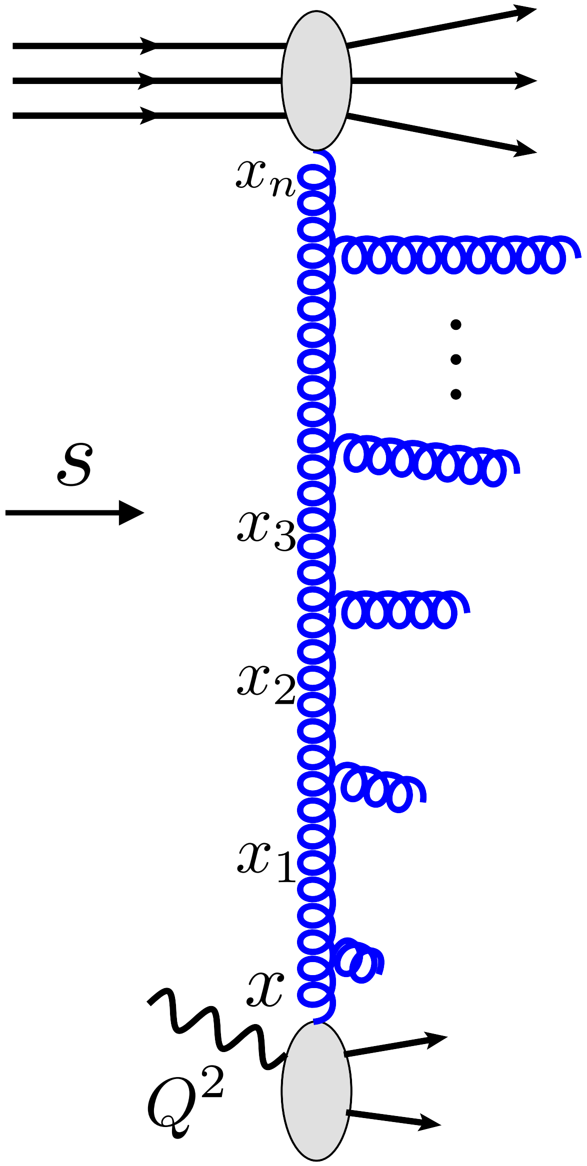
Large logarithms

$$\frac{\alpha_s N_c}{\pi}\int_x^1 \frac{dz}{z} = \frac{\alpha_s N_c}{\pi}\ln\frac{1}{x} = \bar{\alpha}_s \ln\frac{1}{x}$$

Leading logarithmic resummation

$$\left(\bar{\alpha}_s \ln\frac{1}{x}\right)^n \qquad \left(\bar{\alpha}_s \ln\frac{s}{s_0}\right)^n$$

BFKL evolution



Resummation performed by BFKL evolution equation

Balitsky-Fadin-Kuraev-Lipatov (BFKL)

$$\frac{\partial f_g(x,k_T)}{\partial \ln 1/x} = \int \frac{d^2k'_T}{\pi k'^2_T} \mathcal{K}(k_T,k'_T) f_g(x,k'_T)$$

Branching kernel (perturbative expansion)

$$\mathcal{K} = \bar{\alpha}_s \mathcal{K}^{LLx} + \bar{\alpha}_s^2 \mathcal{K}^{NLLx} + \bar{\alpha}_s^3 \mathcal{K}^{NNLLx} + \dots$$

QCD

N=4 SYM

Unintegrated, (transverse momentum dependent) gluon density

$$f_g(x,k_T)$$

compare with DGLAP-collinear approach

$$\frac{\partial f_i(x,Q^2)}{\partial \log(Q^2)} = \sum_j \int_x^1 \frac{dz}{z} P_{j\to i}(z) f_j(\frac{x}{z},Q^2)$$

Solution to BFKL

$$\frac{\partial f_g(x,k_T)}{\partial \ln 1/x} = \int \frac{d^2k'_T}{\pi k'^2_T} \mathcal{K}(k_T,k'_T) f_g(x,k'_T)$$

Mellin space: $\tilde{g}(\omega) = \int_0^1 \frac{dx}{x} x^\omega g(x)$ $\tilde{h}(\gamma) = \int_0^\infty \frac{dk_T^2}{k_T^2} (k_T^2)^{-\gamma} h(k_T^2)$

Mellin variables: $\gamma \leftrightarrow \ln k_T^2$ $\omega \leftrightarrow \ln 1/x$

$$\tilde{f}(\omega,\gamma) = \int_0^1 \frac{dx}{x} x^\omega \int_0^\infty \frac{dk_T^2}{k_T^2} (k_T^2)^{-\gamma} f(x,k_T^2) \qquad \bar{\alpha}_s \chi(\gamma) = \int \frac{dk'^2_T}{k'^2_T} \mathcal{K}(k_T,k'_T) \left(\frac{k'^2_T}{k_T^2}\right)^\gamma$$

$$\tilde{f}(\omega,\gamma) = \frac{\tilde{f}^{(0)}(\omega,\gamma)}{\boxed{\omega - \bar{\alpha}_s \chi(\gamma)}}$$

← Inhomogenous term

Singularity determining the energy behavior

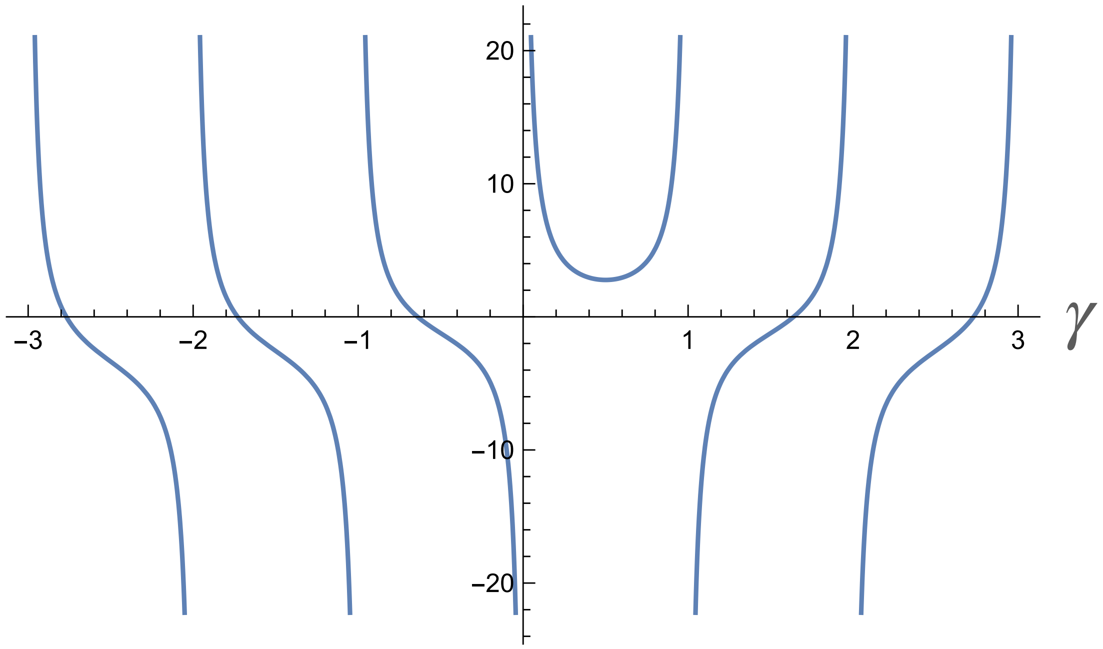
Solution to BFKL

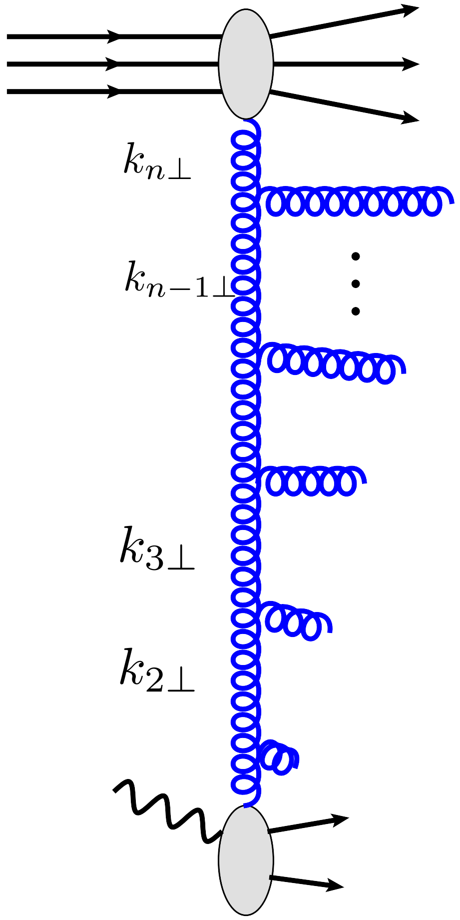
LL kernel in Mellin space $\gamma \leftrightarrow \ln \boldsymbol{k}^2$

$\psi(z) = \Gamma'(z)/\Gamma(z)$

$$\chi_0(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma) \simeq \frac{1}{\gamma} + \frac{1}{1-\gamma}$$

collinear & anti-collinear poles





$$\frac{1}{\gamma} \longleftrightarrow k_{2T}^2 \gg k_{3T}^2 \gg \cdots \gg k_{nT}^2$$

$$\frac{1}{1-\gamma} \longleftrightarrow k_{2T}^2 \ll k_{3T}^2 \ll \cdots \ll k_{nT}^2$$

Solution to BFKL

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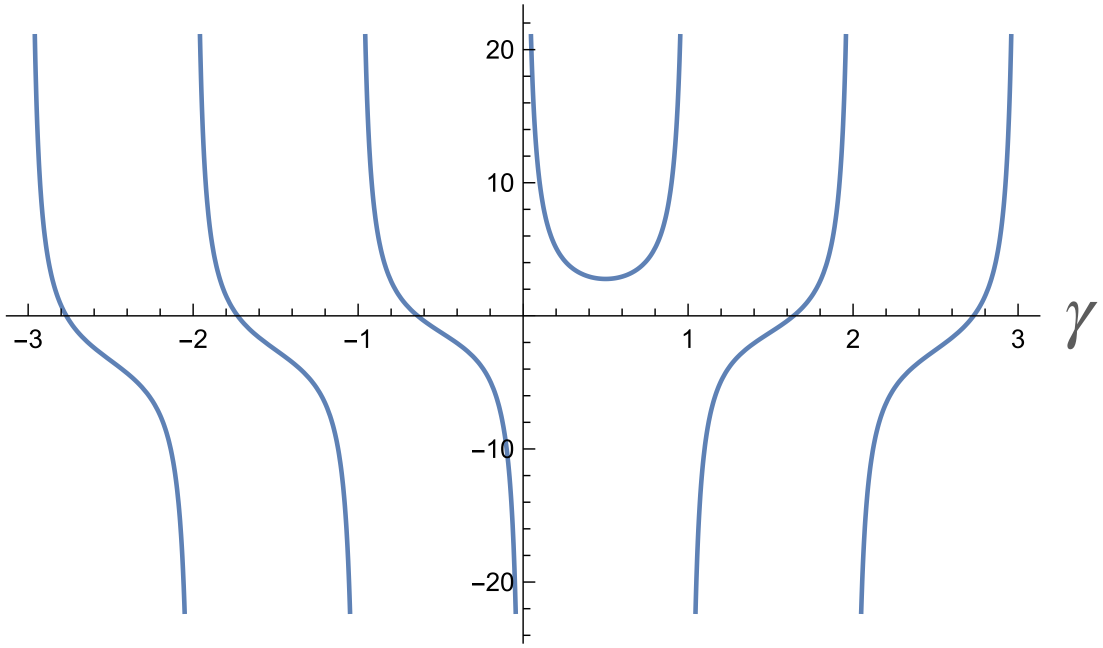
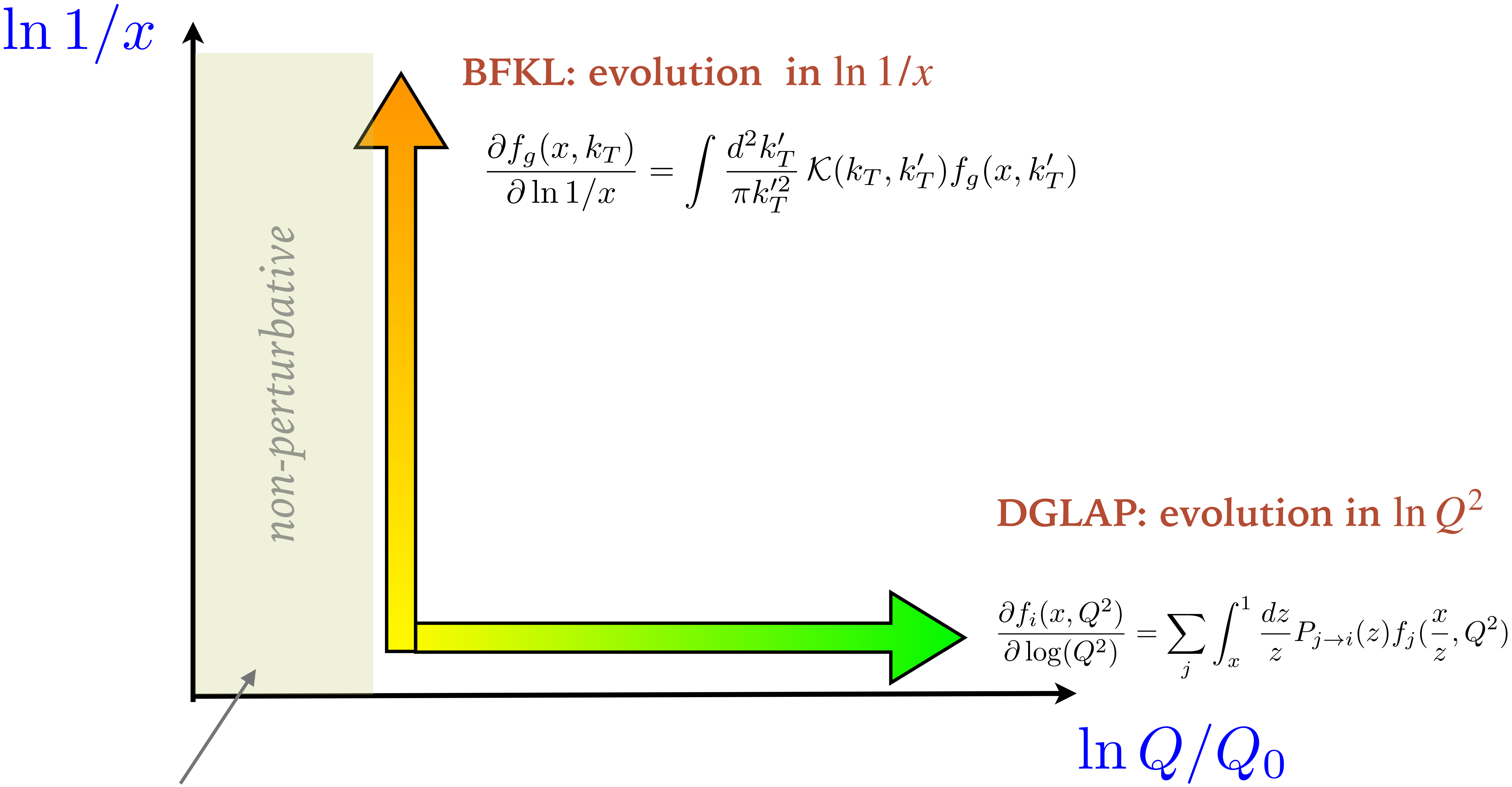


Solution to the gluon density $f_g(x, k_T) \sim x^{-\omega_P}$

$$\omega_P = \bar{\alpha}_s \chi(\gamma = 1/2) \to 4 \ln 2 \bar{\alpha}_s \simeq 2.77 \bar{\alpha}_s = 2.77 \frac{\alpha_s N_c}{\pi}$$

$$\sigma^{DIS}_{\gamma^* p} \sim s^{\omega_P}$$

BFKL vs DGLAP



BFKL: evolution in $\ln 1/x$

$$\frac{\partial f_g(x,k_T)}{\partial \ln 1/x} = \int \frac{d^2k'_T}{\pi k'^2_T}\mathcal{K}(k_T,k'_T)f_g(x,k'_T)$$

DGLAP: evolution in $\ln Q^2$

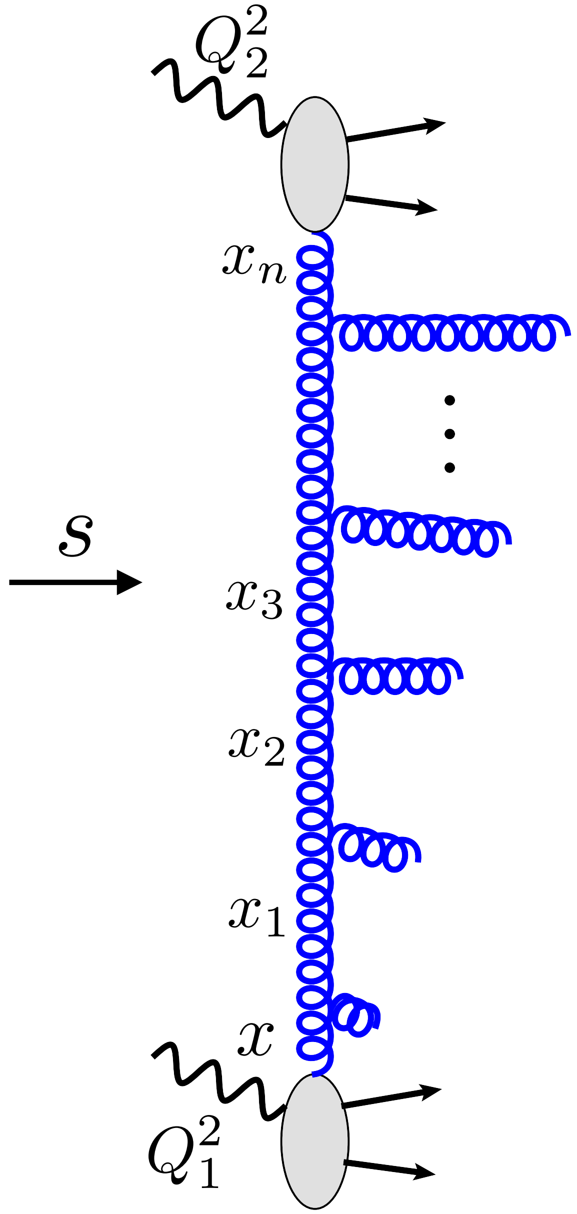
$$\frac{\partial f_i(x,Q^2)}{\partial \log(Q^2)} = \sum_j \int_x^1 \frac{dz}{z}P_{j\to i}(z)f_j(\frac{x}{z},Q^2)$$

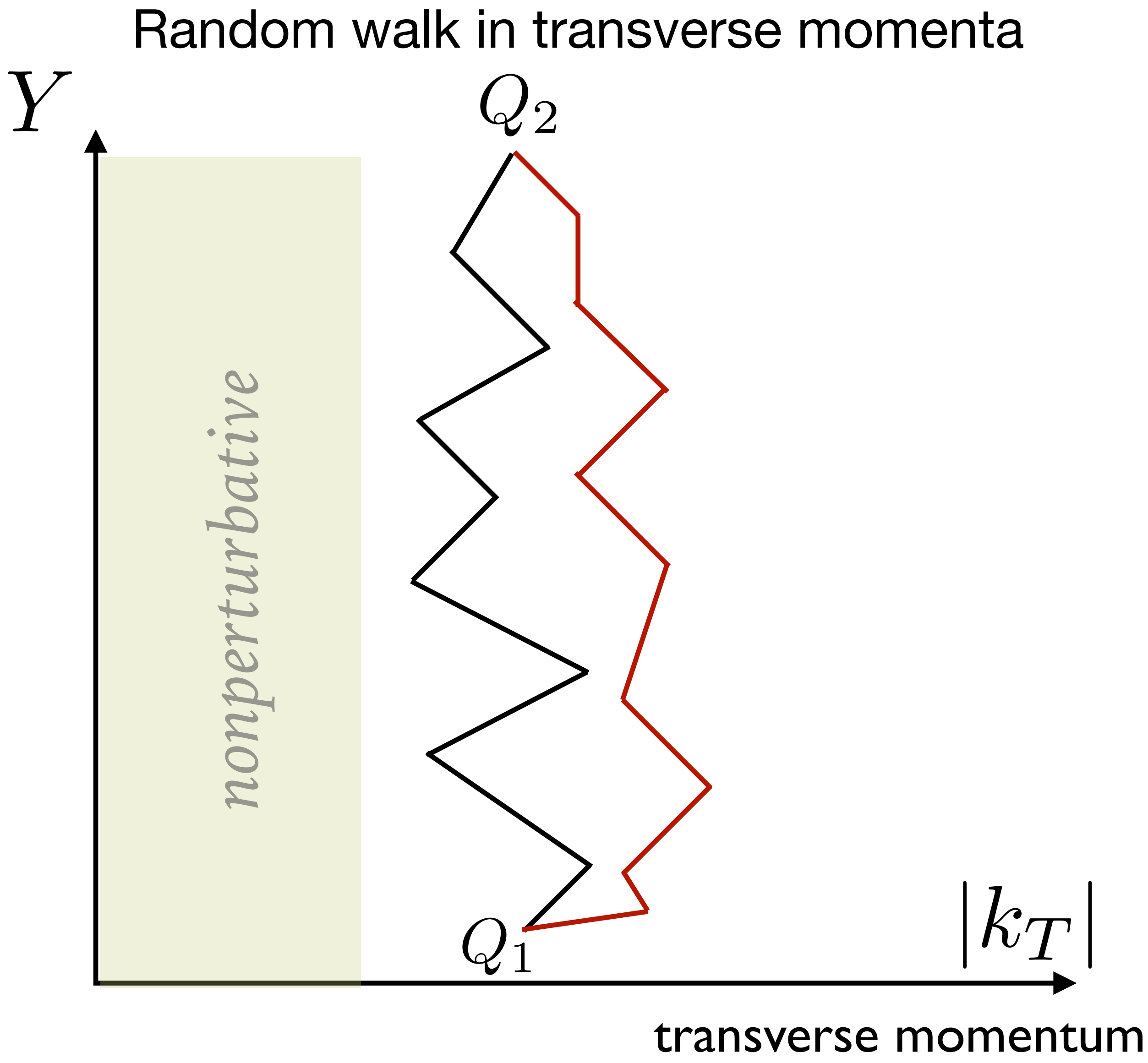
BFKL evolution is sensitive to the **non-perturbative** region

Diffusion into infrared

Consider a process with two large scales (ex. $\gamma^* \gamma^*$ scattering) with $Q_1^2 \sim Q_2^2 \gg \Lambda_{QCD}^2$

Large comparable scales to suppress DGLAP, large rapidity for BFKL evolution, keep perturbative

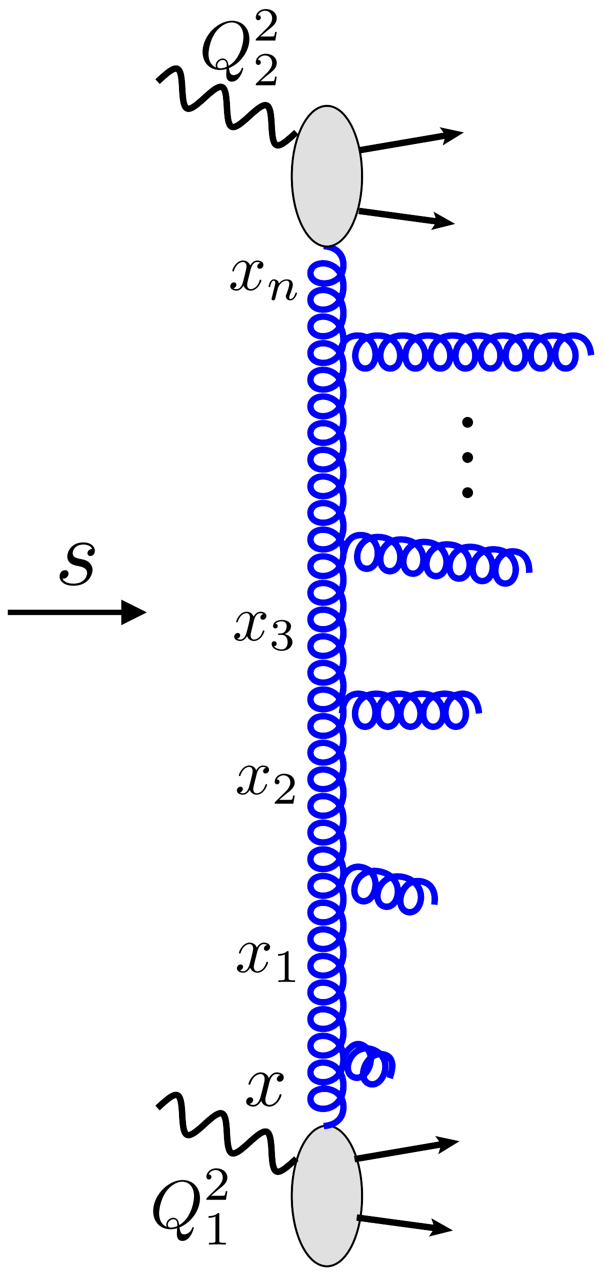


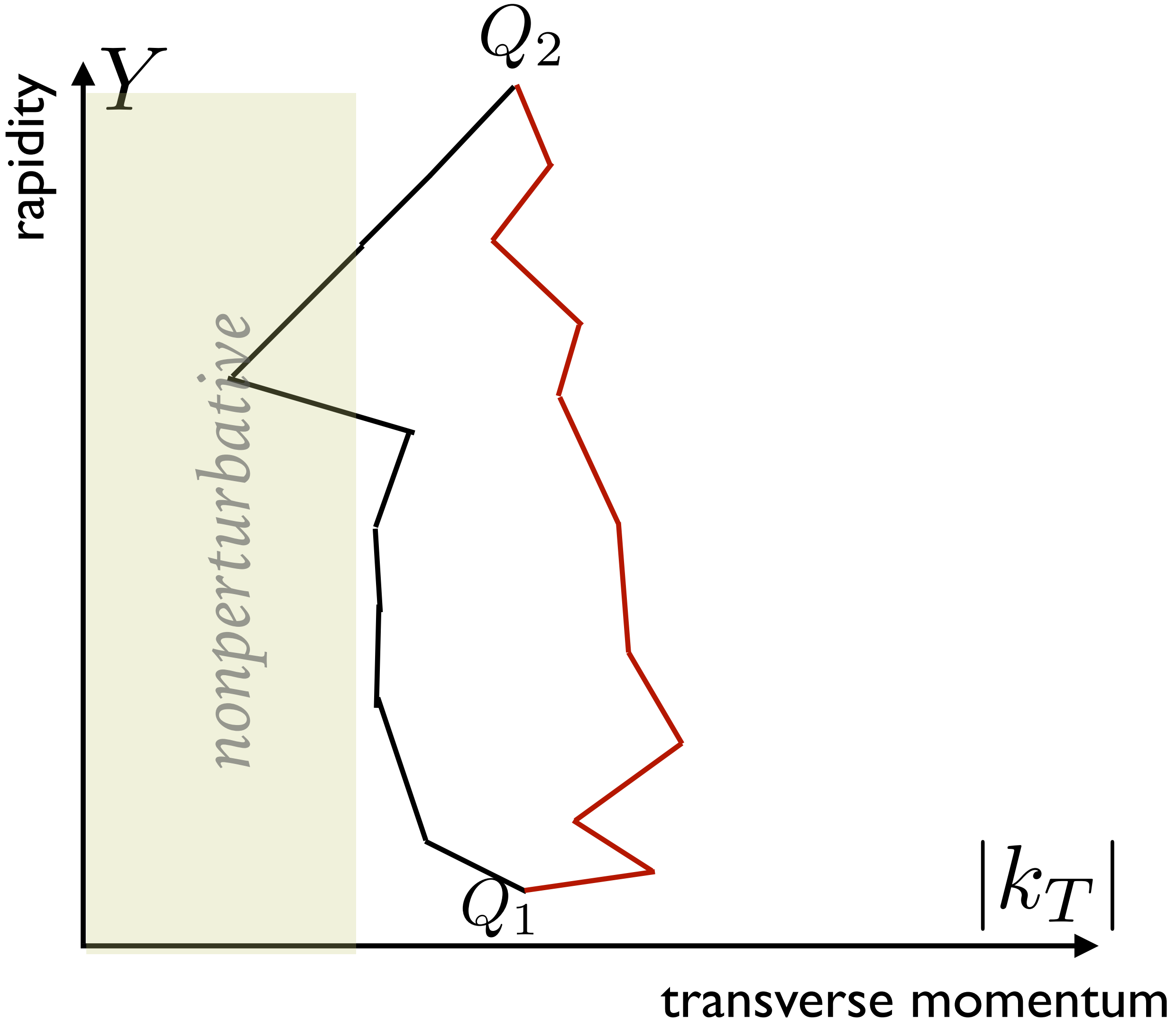


Diffusion of transverse momenta towards IR and UV.
For large energies momenta can diffuse to low scales even when starting from large scales.

Diffusion into infrared with running coupling in BFKL

Effects of running coupling:
‘pull’ towards the infrared region





Large non-perturbative effects for large energies.

NLL corrections to BFKL

NLL corrections to BFKL equation are **large** and **negative**

Main sources:

- running coupling (double poles)
- kinematical constraint (triple poles)
- DGLAP anomalous dimension (double poles)

LLx kernel in Mellin space

$$\chi_0(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma)$$

NLLx kernel in Mellin space

$$\chi_1(\gamma) = \boxed{-\frac{b}{2}[\chi_0^2(\gamma) + \chi_0'(\gamma)]} - \boxed{\frac{1}{4}\chi_0''(\gamma)} - \boxed{\frac{1}{4}\left(\frac{\pi}{\sin\pi\gamma}\right)^2} \frac{\cos\pi\gamma}{3(1-2\gamma)}\left(11 + \frac{\gamma(1-\gamma)}{(1+2\gamma)(3-2\gamma)}\right)$$
$$+ \left(\frac{67}{36} - \frac{\pi^2}{12}\right)\chi_0(\gamma) + \frac{3}{2}\zeta(3) + \frac{\pi^3}{4\sin\pi\gamma}$$
$$- \sum_{n=0}^{\infty}(-1)^n \left[\frac{\psi(n+1+\gamma)-\psi(1)}{(n+\gamma)^2} + \frac{\psi(n+2-\gamma)-\psi(1)}{(n+1-\gamma)^2}\right]$$

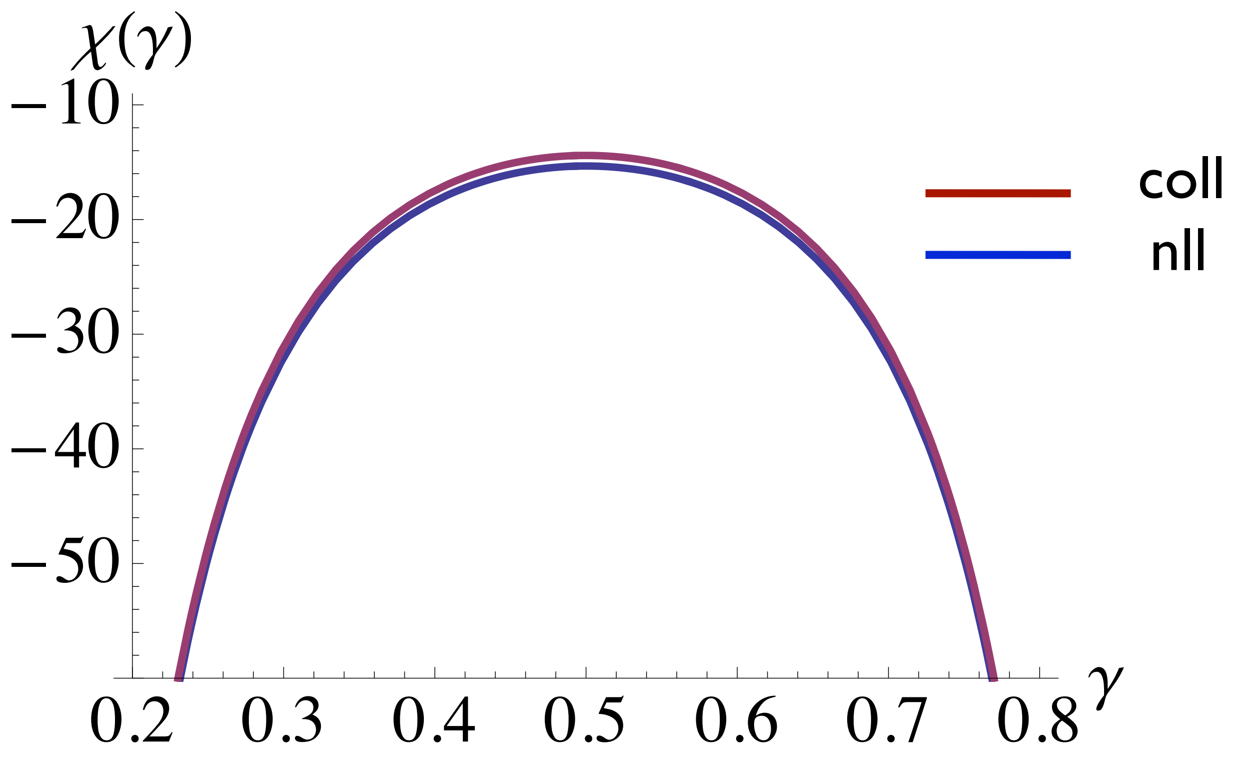
Collinear poles in NLL BFKL

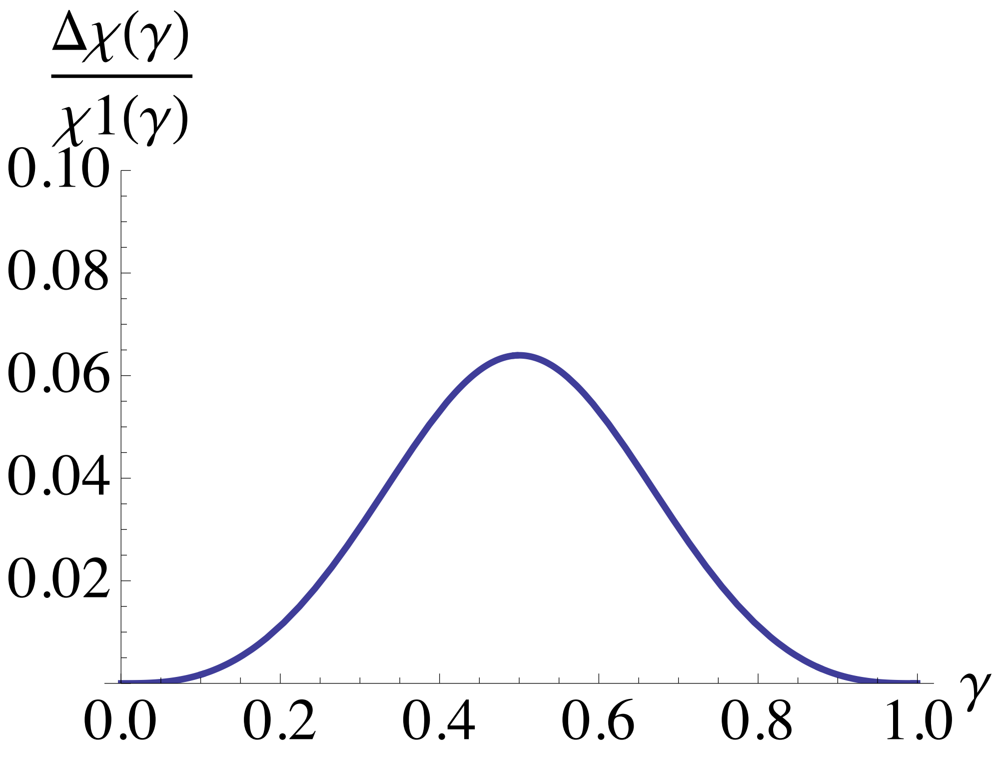
$$\chi_1^{\rm coll}(\gamma) = \boxed{-\frac{1}{2\gamma^3} - \frac{1}{2(1-\gamma)^3}} + \boxed{\frac{A_1(0)}{\gamma^2} + \frac{A_1(0)-b}{(1-\gamma)^2}}$$

double and triple poles
of the NLL part

LO DGLAP anomalous dimension $\gamma_{gg}^{(0)}(\omega) = \frac{\bar{\alpha}_s}{\omega} + \bar{\alpha}_s A_1(\omega)$ $A_1(\omega) = -\frac{11}{12} + \mathcal{O}(\omega)$

Difference of about 7% at most





Origin of NLL corrections in BFKL

NLLx kernel in Mellin space

$$\chi_1(\gamma) = \boxed{-\frac{b}{2}[\chi_0^2(\gamma)+\chi_0'(\gamma)]} - \boxed{\frac{1}{4}\chi_0''(\gamma)} - \boxed{\frac{1}{4}\left(\frac{\pi}{\sin\pi\gamma}\right)^2} \frac{\cos\pi\gamma}{3(1-2\gamma)}\left(11+\frac{\gamma(1-\gamma)}{(1+2\gamma)(3-2\gamma)}\right)$$
$$+\left(\frac{67}{36}-\frac{\pi^2}{12}\right)\chi_0(\gamma)+\frac{3}{2}\zeta(3)+\frac{\pi^3}{4\sin\pi\gamma}$$
$$-\sum_{n=0}^{\infty}(-1)^n\left[\frac{\psi(n+1+\gamma)-\psi(1)}{(n+\gamma)^2}+\frac{\psi(n+2-\gamma)-\psi(1)}{(n+1-\gamma)^2}\right]$$

Running coupling can be resummed into LL kernel

DGLAP anomalous dimension

$$\gamma_{gg}(\omega)=\int_0^1 dz P_{gg}(z) z^{-\omega}$$

$$P_{gg}(z)=\frac{\alpha_s}{2\pi}P_{gg}^{(0)}+\dots$$

$$P_{gg}^{(0)}(z)=2C_A\Big[\frac{z}{(1-z)_+}+\frac{1-z}{z}+z(1-z)+\delta(1-z)\frac{11C_A-4n_fT_R}{6}\Big]$$

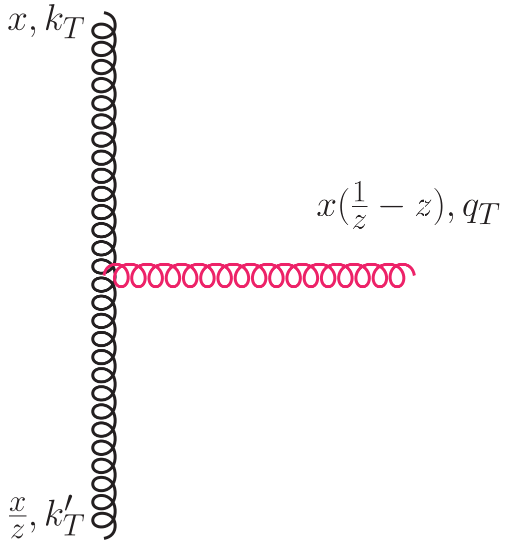
$$\gamma_{gg}^{(0)}(\omega)=\frac{\alpha_s C_A}{\pi}\Big(\frac{1}{\omega}+A_1(\omega)\Big) \qquad A_1(0)=-\frac{11}{12}$$

Triple poles: kinematical constraint and energy scales

$$f_g(x,k_T) = f_g^{(0)}(k_T) + \int_x^1 \frac{dz}{z} \int \frac{d^2k_T'^2}{\pi k_T'^2} \mathcal{K}(k_T,k_T') f_g(\frac{x}{z},k_T')$$

The integrals are unrestricted

However in Regge kinematics, virtualities of exchanged momenta dominated by transverse components



This leads to constraint:

$$k_T'^2 < \frac{k_T^2}{z}$$

(on the real emission kernel)

$$f_g(x,k_T) = f_g^{(0)}(k_T) + \int_x^1 \frac{dz}{z} \int \frac{d^2k_T'^2}{\pi k_T'^2} \Theta_R(k_T^2/z - k_T'^2) \mathcal{K}(k_T,k_T') f_g(\frac{x}{z},k_T')$$

In the Mellin space: shift of poles

Kernel with kinematical constraint has **shifted pole**

$$\chi(\gamma,\omega) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma+\omega)$$

Expanding to first order in ω

$$\chi(\gamma,\omega) \simeq \chi^{(0)}(\gamma) - \omega\psi'(1-\gamma) \simeq \chi^{(0)}(\gamma) - \omega\frac{1}{(1-\gamma)^2}$$

Using the solution at LL to eliminate ω

$$\omega = \bar{\alpha}_s\chi^{(0)}(\gamma) \simeq \bar{\alpha}_s(\frac{1}{\gamma} + \frac{1}{1-\gamma})$$

Generate triple poles:

$$-\bar{\alpha}_s\frac{1}{(1-\gamma)^3}$$

Scale choices

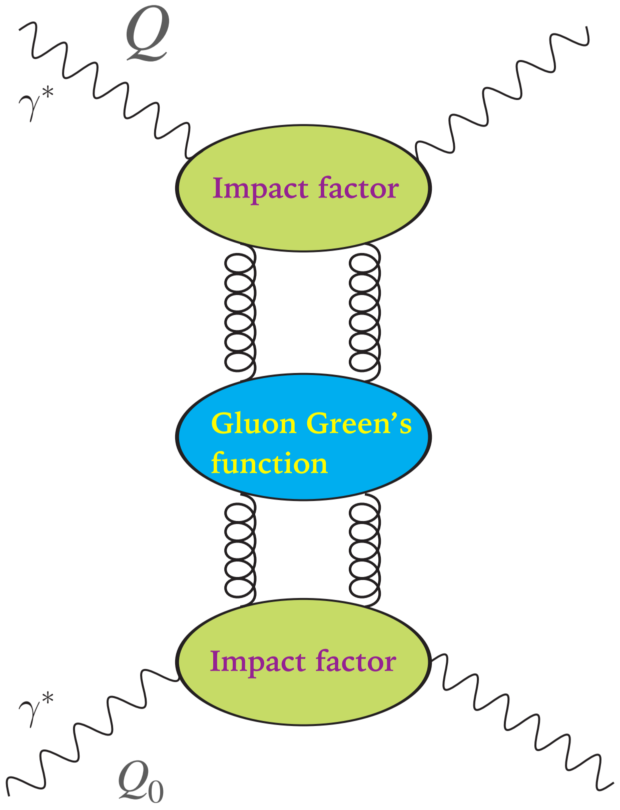
This is related to the scale choice in BFKL. Consider a high energy process

$$\boxed{\sigma_{AB}(s;Q,Q_0) = \int \frac{d\omega}{2\pi i} \frac{d^2\boldsymbol{k}}{\boldsymbol{k}^2} \frac{d^2\boldsymbol{k}_0}{\boldsymbol{k}_0^2} \left(\frac{s}{QQ_0}\right)^{\omega} h_\omega^A(Q,\boldsymbol{k})\, \mathcal{G}_\omega(\boldsymbol{k},\boldsymbol{k}_0)\, h_\omega^B(Q_0,\boldsymbol{k}_0)}$$

Impact factors

$$\boxed{\omega \mathcal{G}_\omega(\boldsymbol{k},\boldsymbol{k}_0) = \delta^2(\boldsymbol{k}-\boldsymbol{k}_0) + \int \frac{d^2\boldsymbol{k}'}{\pi} \mathcal{K}_\omega(\boldsymbol{k},\boldsymbol{k}')\, \mathcal{G}_\omega(\boldsymbol{k}',\boldsymbol{k}_0)}$$

Gluon Green's function



Different possible scale choices:

symmetric (ex. two photons) $s_0 = QQ_0$ $Q \sim Q_0$

DIS type configuration

$s_0 = Q^2$ $Q^2 \gg Q_0^2$

$s_0 = Q_0^2$ $Q^2 \ll Q_0^2$

Triple poles: scale choice

Different scale choices matter beyond LLx

Need to put different kinematical constraints

Kernel will be different

asymmetric scale choice

$$\chi^u(\gamma,\omega) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma+\omega)$$

$$\chi^l(\gamma,\omega) = 2\psi(1) - \psi(\gamma+\omega) - \psi(1-\gamma)$$

symmetric scale choice

$$\chi^s(\gamma,\omega) = 2\psi(1) - \psi(\gamma+\frac{\omega}{2}) - \psi(1-\gamma+\frac{\omega}{2})$$

The shift resums towers of subleading terms to **all orders**

Shift of poles: triple poles

Expansion reproduces higher order poles (NLL):

symmetric scale choice

$$\chi^s(\gamma,\omega) \simeq \chi^{(0)}(\gamma) - \frac{1}{2}\frac{\bar{\alpha}_s}{\gamma^3} - \frac{1}{2}\frac{\bar{\alpha}_s}{(1-\gamma)^3}$$

The same poles (with the exact same coefficients are in QCD and N=4 sYM)

In N=4 sYM NNLO result is available, can check if shifts reproduce the poles

$$\chi^{(2)}(\gamma) \sim +\frac{1}{2}\frac{\bar{\alpha}_s^2}{\gamma^5} + \frac{1}{2}\frac{\bar{\alpha}_s^2}{(1-\gamma)^5} + \dots$$

Coincides with the result obtained by *Gromov,Levkovich-Masyluk,Sizov;Velizhanin; Caron-Huot, Herranen*

Form of resummed kernel

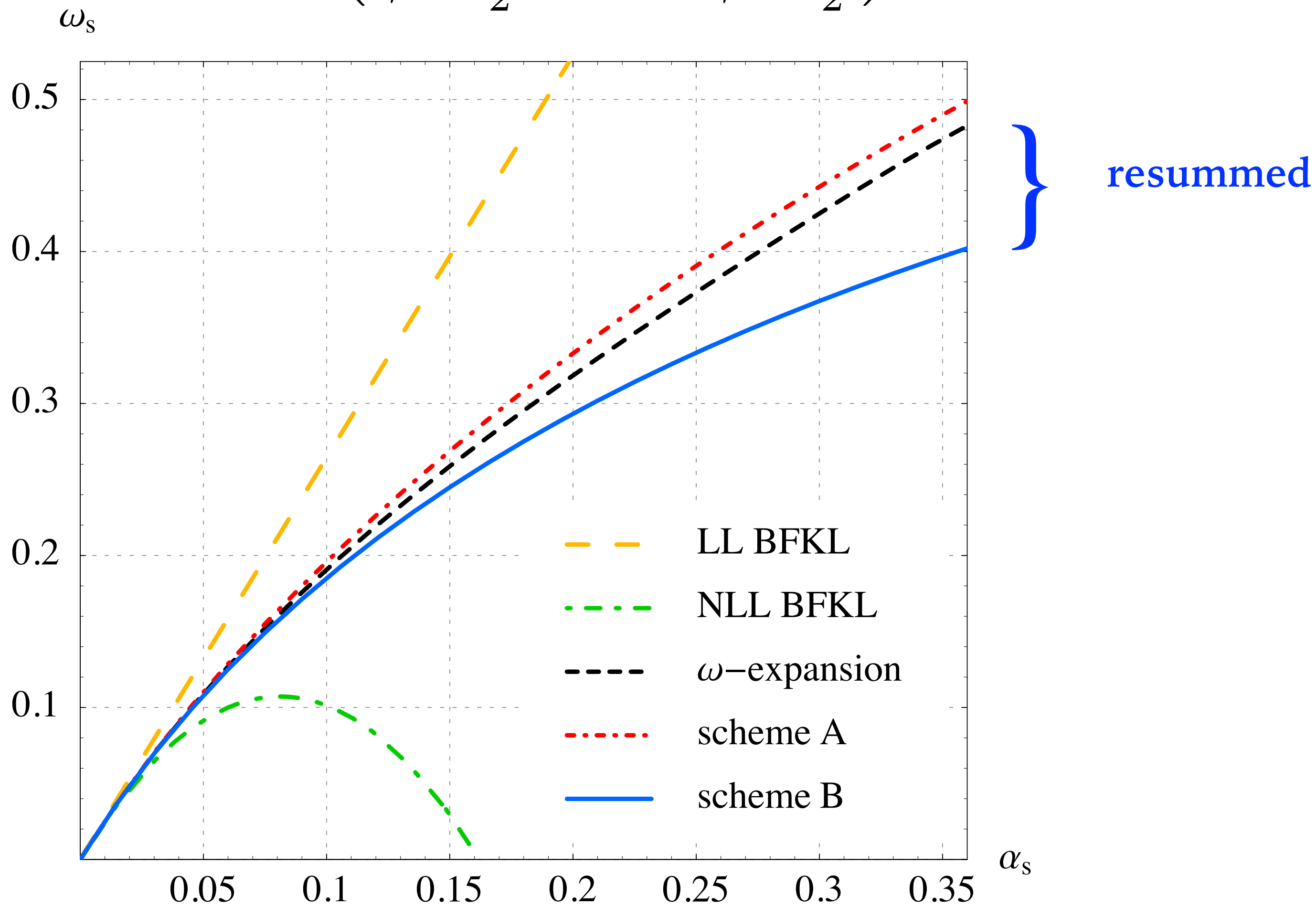
CCSS resummation (RGI renormalization group improved small x evolution):

- Include **kinematical constraint** : leads to **shifts of poles**
- Include DGLAP **splitting function** and **running coupling** in the leading part
- Suitable subtractions to avoid double counting, guarantee momentum sum rule
- Motivation in Mellin space, final equation in the **momentum** space

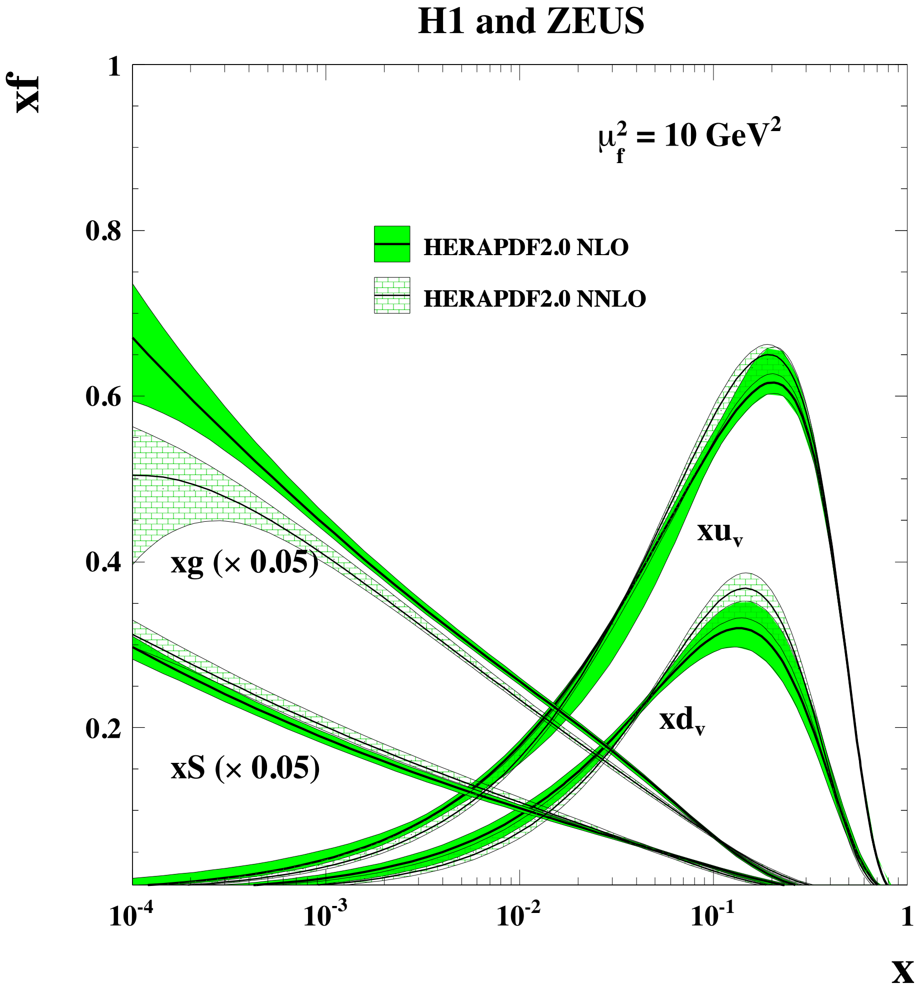
$$X(\gamma,\omega) = 2\psi(1) - \psi(\gamma+\frac{\omega}{2}) - \psi(1-\gamma+\frac{\omega}{2}) + \omega A_{gg}(\omega)\left(\frac{1}{\gamma+\frac{\omega}{2}} + \frac{1}{1-\gamma+\frac{\omega}{2}}\right) + \bar{\alpha}_s\tilde{\chi}_1(\gamma,\omega)$$

$A_{gg}(\omega)$	DGLAP anomalous dimension without the $1/\omega$ term
$\tilde{\chi}1$	NLL term with subtractions

Much more `phenomenology friendly' result



Can the gluon density rise forever ? How fast ?



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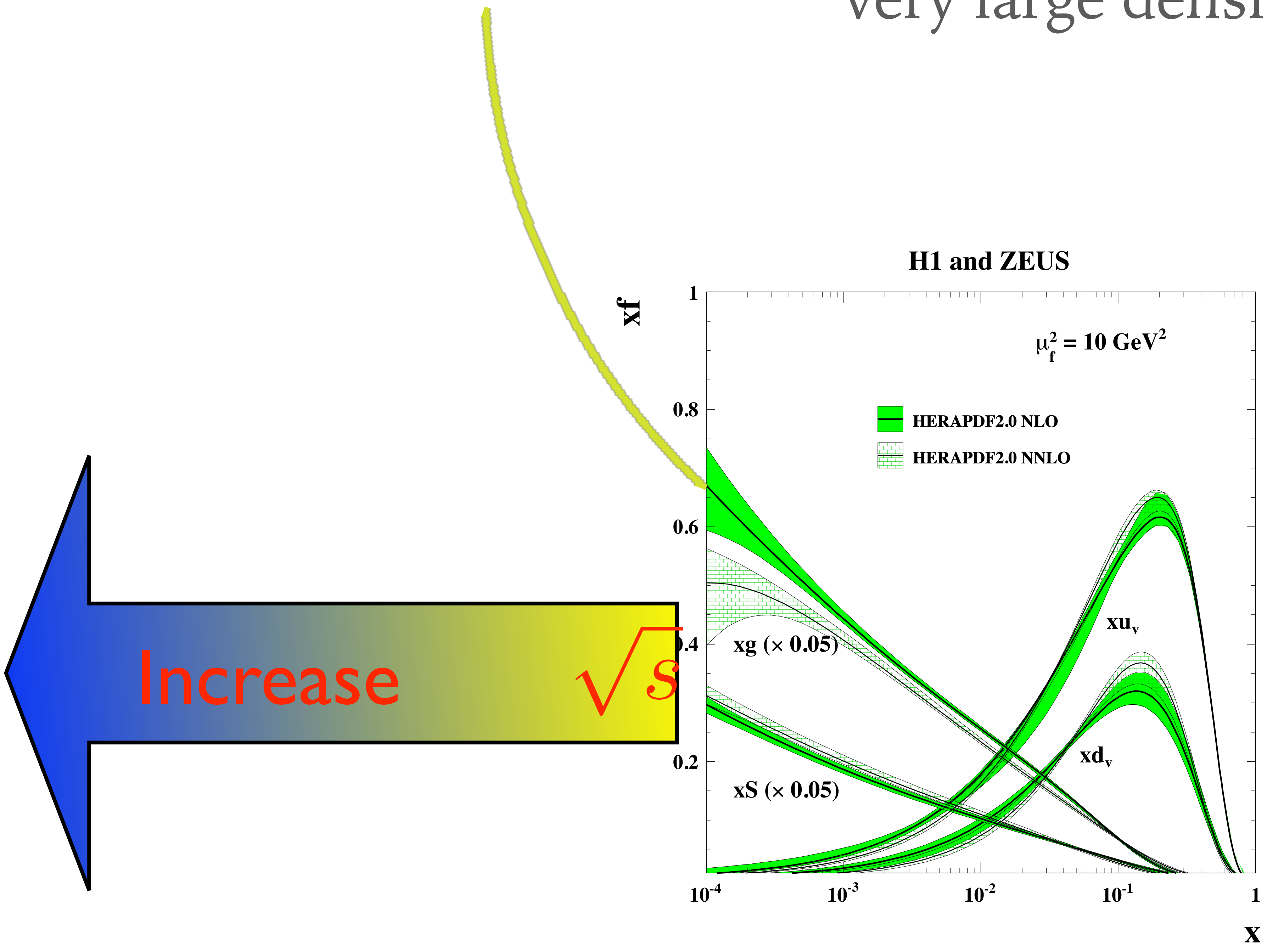
very large density of gluons



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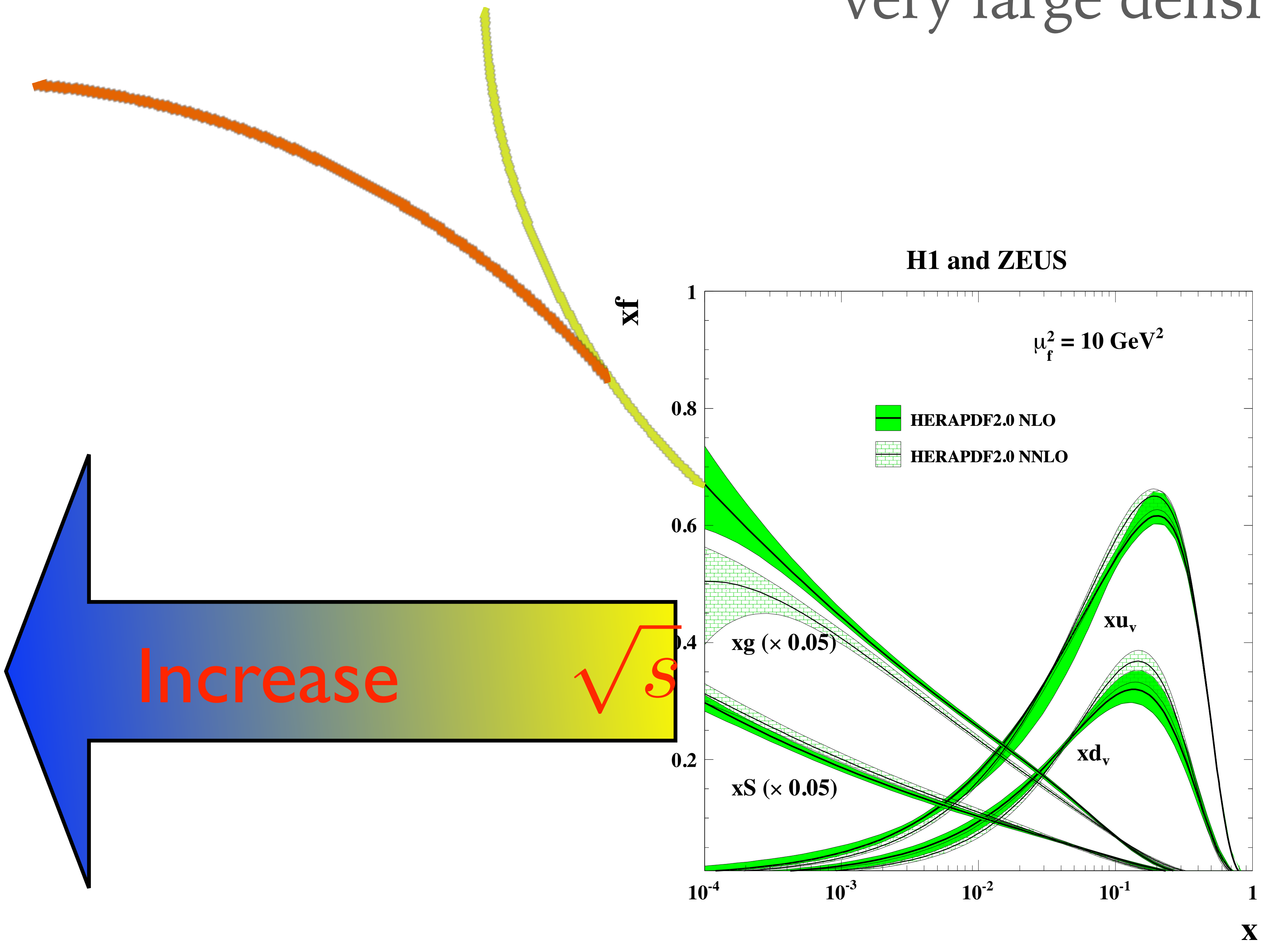

Gluon density at high energies

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- Gluon **recombination** need to be taken into account in addition to the radiation of gluons at small x.

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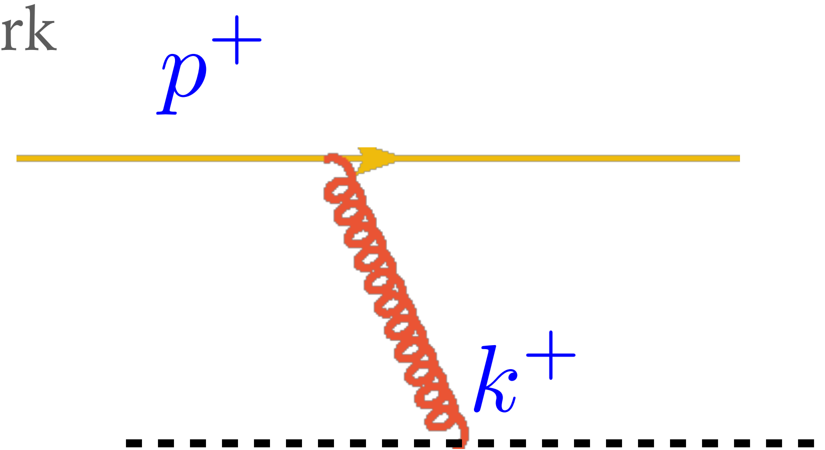
- Both **small x and large A** (nuclear effects) can be addressed in this formalism.

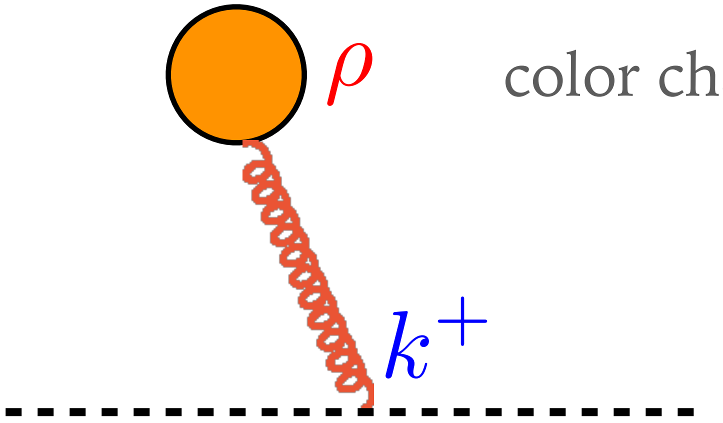
Towards the non-linear equation

fast quark



probe(for example a quark-antiquark pair from the virtual photon in DIS)

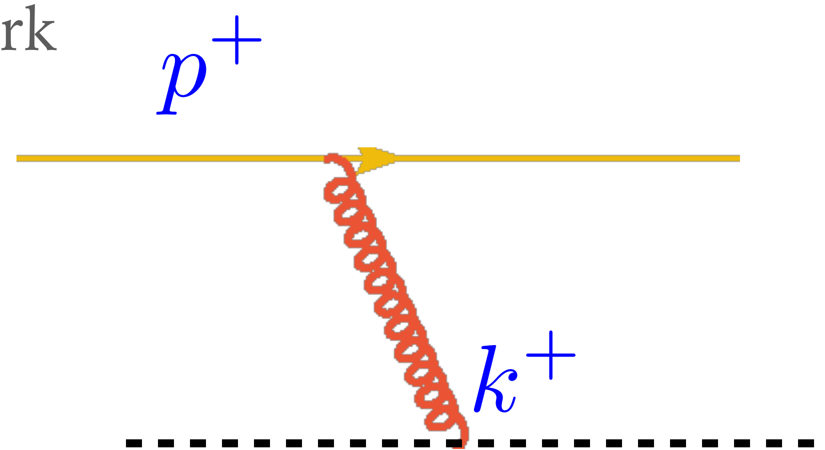
$$p^+ \gg k^+$$



color charge

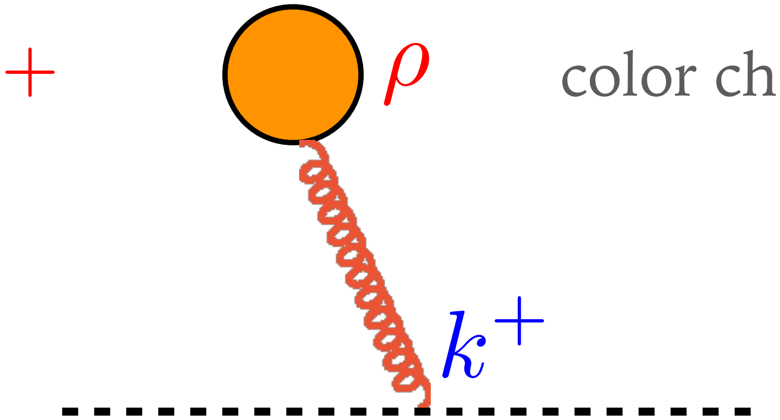
Towards the non-linear equation

fast quark

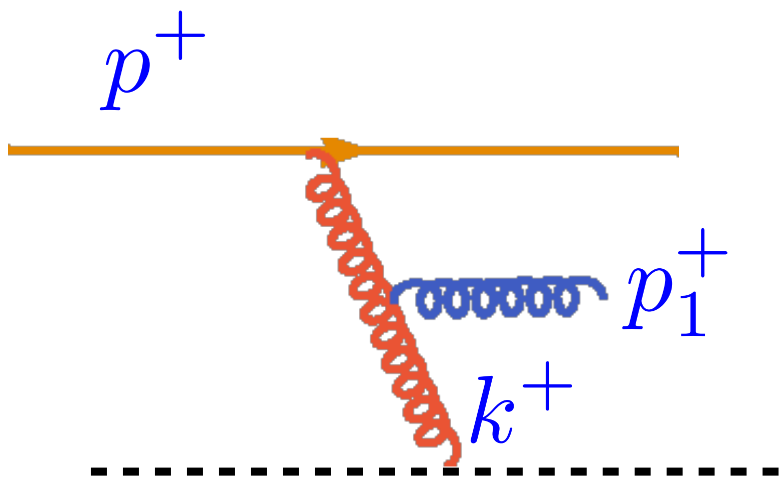


probe(for example a quark-antiquark pair from the virtual photon in DIS)

$$p^+ \gg k^+$$



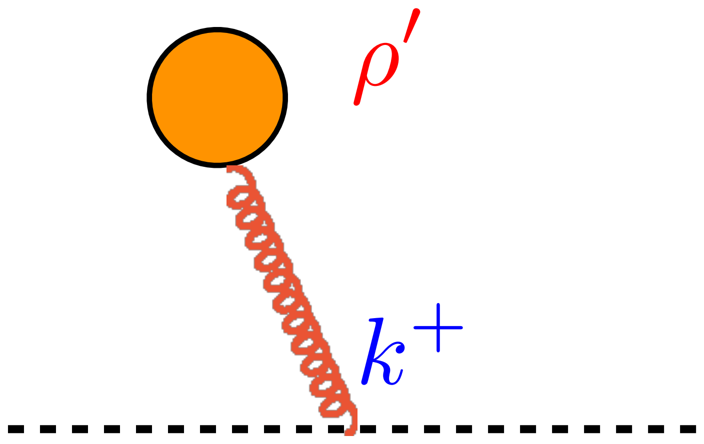
color charge



One gluon emission

$$p^+ \gg p_1^+ \gg k^+$$

Separation of scales (ordering in energies)

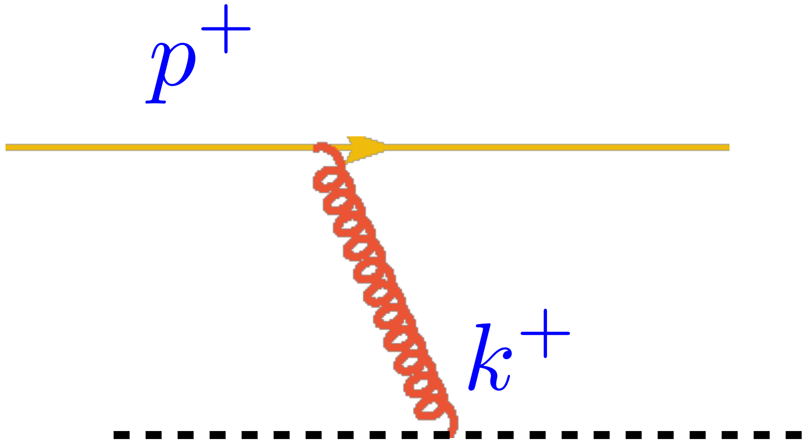


Radiation of gluons: Bremsstrahlung

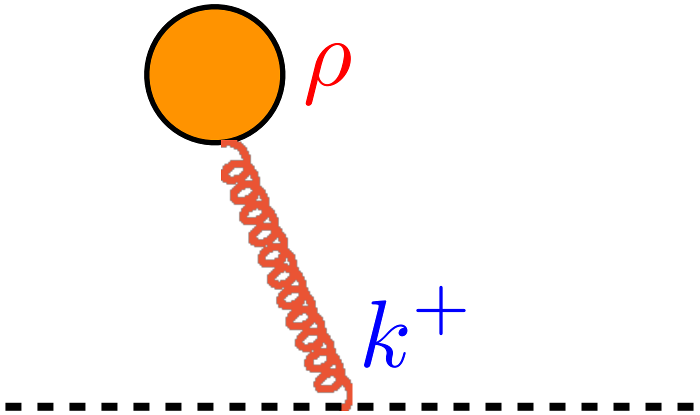
Renormalized charge

The effect of the additional gluon emission is to renormalize the effective color charge.

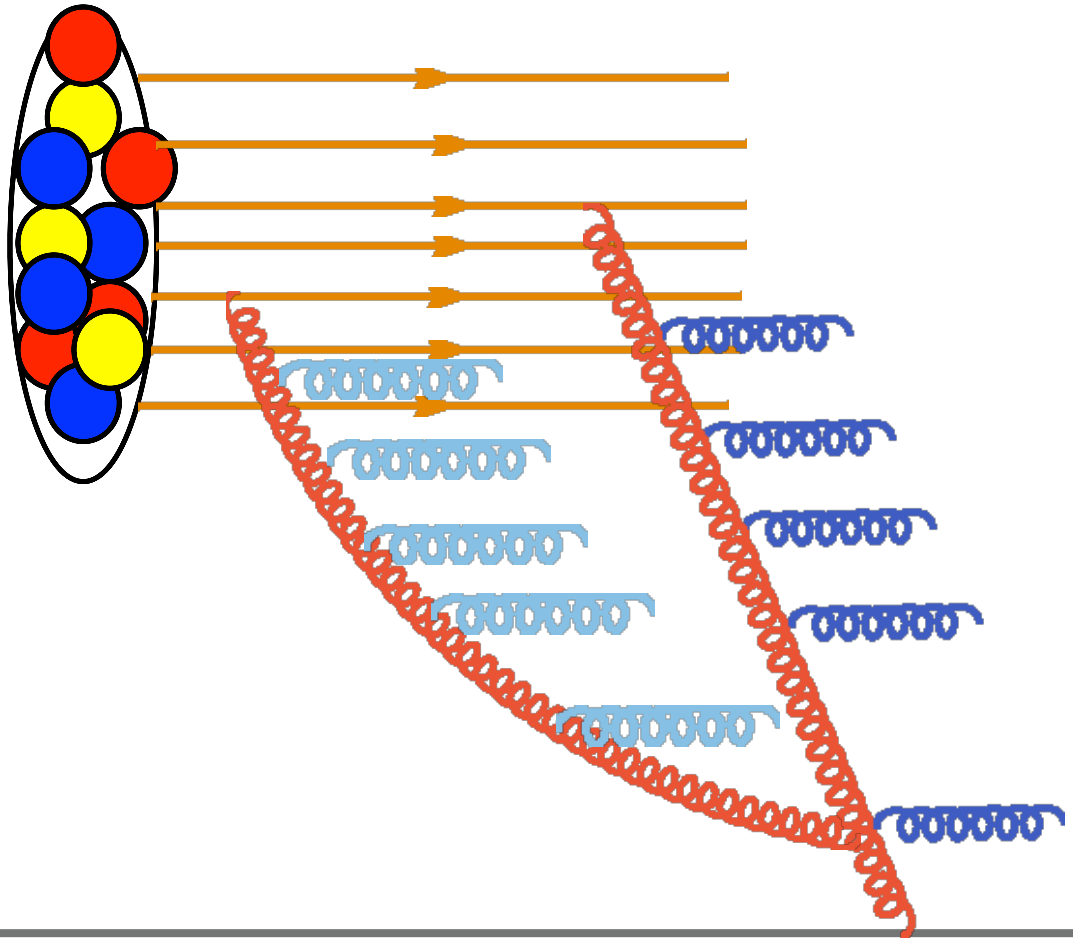
Towards the non-linear equation



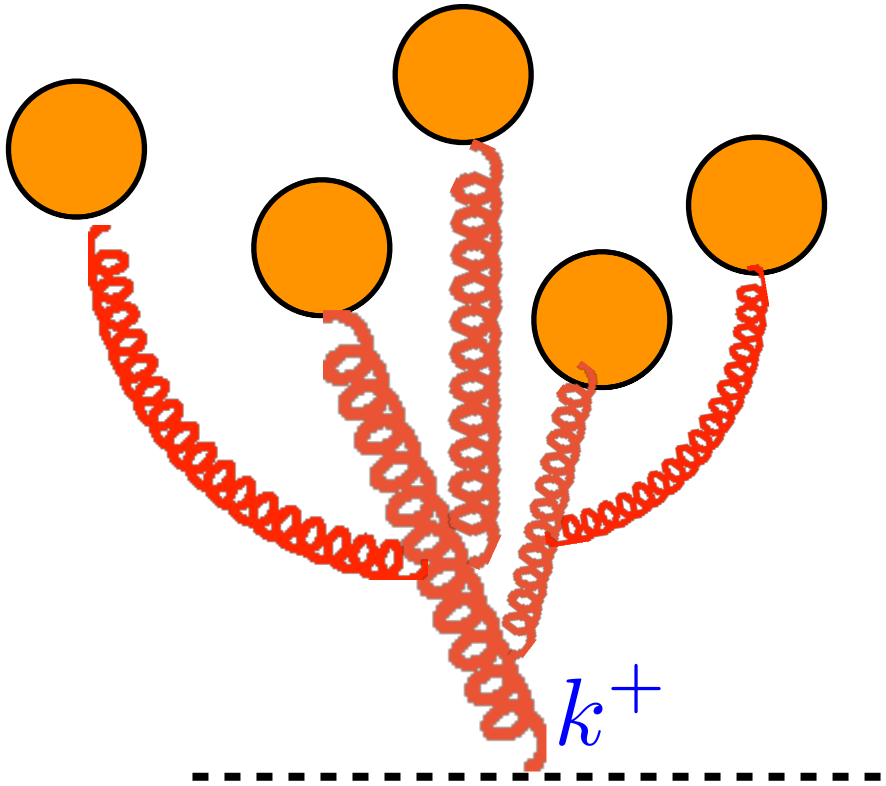
Single charge(source)



Fast nucleus

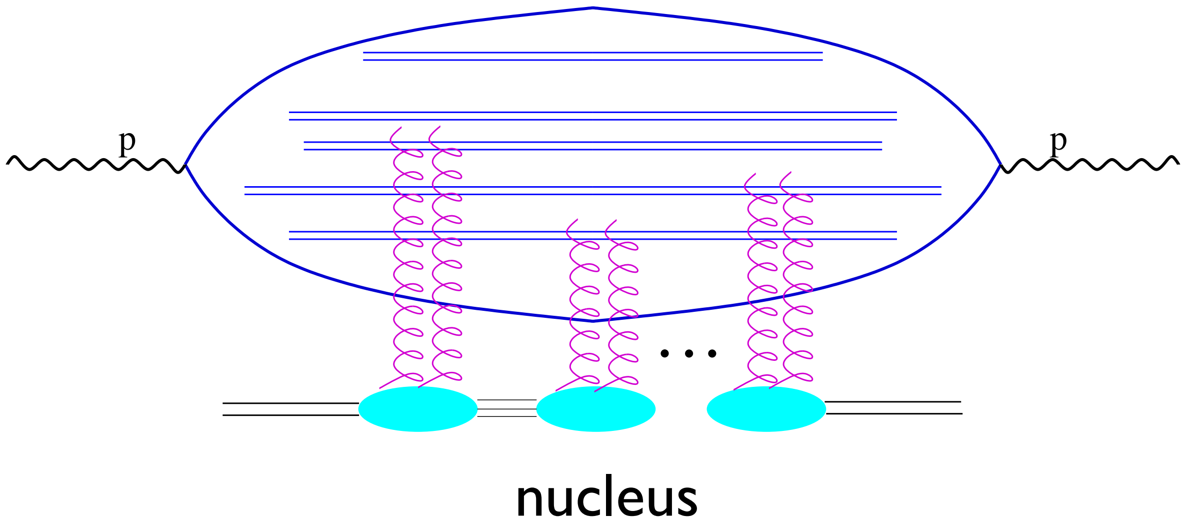


Many charges (sources)



Recombination vs multiple scattering

Now the nucleus is at rest. The photon develops a small x wave function in terms of many quark-antiquark dipoles



Multiple scatterings of different components of the small x photon wave function on the nucleus.

Multiple scattering in rest frame of the nucleus is viewed as **recombination** of gluons in the frame in which the nucleus moves very fast.

Gluon density at high energies

Evolution equation for the dipole-hadron(nucleus) scattering amplitude:

$$\boxed{\begin{aligned}\frac{dN(\mathbf{b}_{01},\mathbf{x}_{01},Y)}{dY} = \bar{\alpha}_s \int \frac{d^2\mathbf{x}_2\,\mathbf{x}_{01}^2}{\mathbf{x}_{20}^2\,\mathbf{x}_{12}^2}\Big[& N(\mathbf{b}_{01}+\frac{\mathbf{x}_{12}}{2},\mathbf{x}_{20},Y)+N(\mathbf{b}_{01}-\frac{\mathbf{x}_{20}}{2},\mathbf{x}_{12},Y) \\ & - N(\mathbf{b}_{01},\mathbf{x}_{01},Y) - N(\mathbf{b}_{01}+\frac{\mathbf{x}_{12}}{2},\mathbf{x}_{20},Y)\,N(\mathbf{b}_{01}-\frac{\mathbf{x}_{20}}{2},\mathbf{x}_{12},Y)\Big]\end{aligned}}$$

Dipole amplitude is related to the unintegrated gluon density (impact parameter neglected)

$$N(b,r,Y=\ln 1/x) \longleftrightarrow f(x,k_T) \qquad r \leftrightarrow \frac{1}{k_T}$$

$$\boxed{\frac{\partial f_g(x,k_T)}{\partial \ln 1/x} = \int \frac{d^2k'_T}{\pi k'^2_T}\mathcal{K}(k_T,k'_T)f_g(x,k'_T) - \frac{\alpha_s N_c}{\pi}(f_g(x,k_T))^2}$$

Linear term is the BFKL evolution

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I.Balitsky,Y.Kovchegov;J.Jalilian-Marian, E.Iancu,L.McLerran,H.Weigert, Leonidov

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Linear term: gluon splitting (governed by BFKL), increase of the density

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Nonlinear term: gluon merging, slow down the growth of the density with the energy

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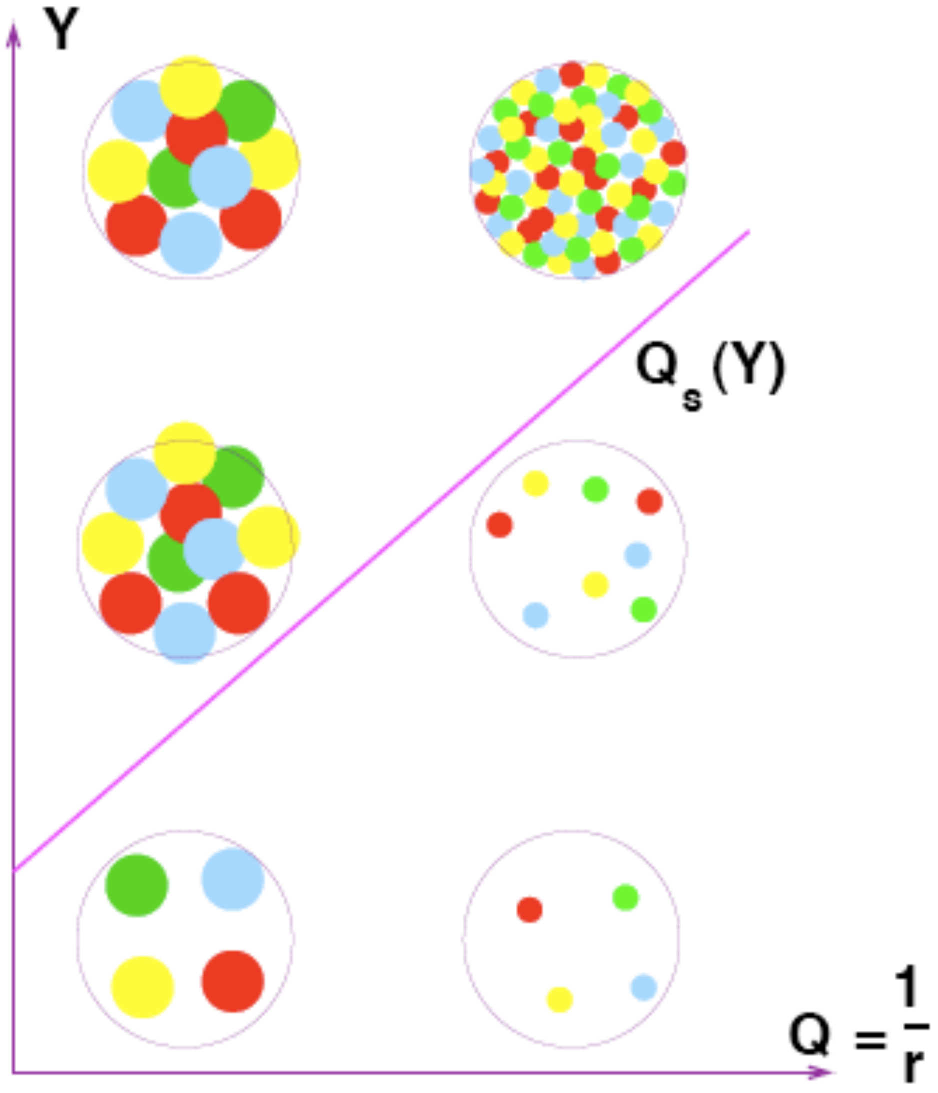
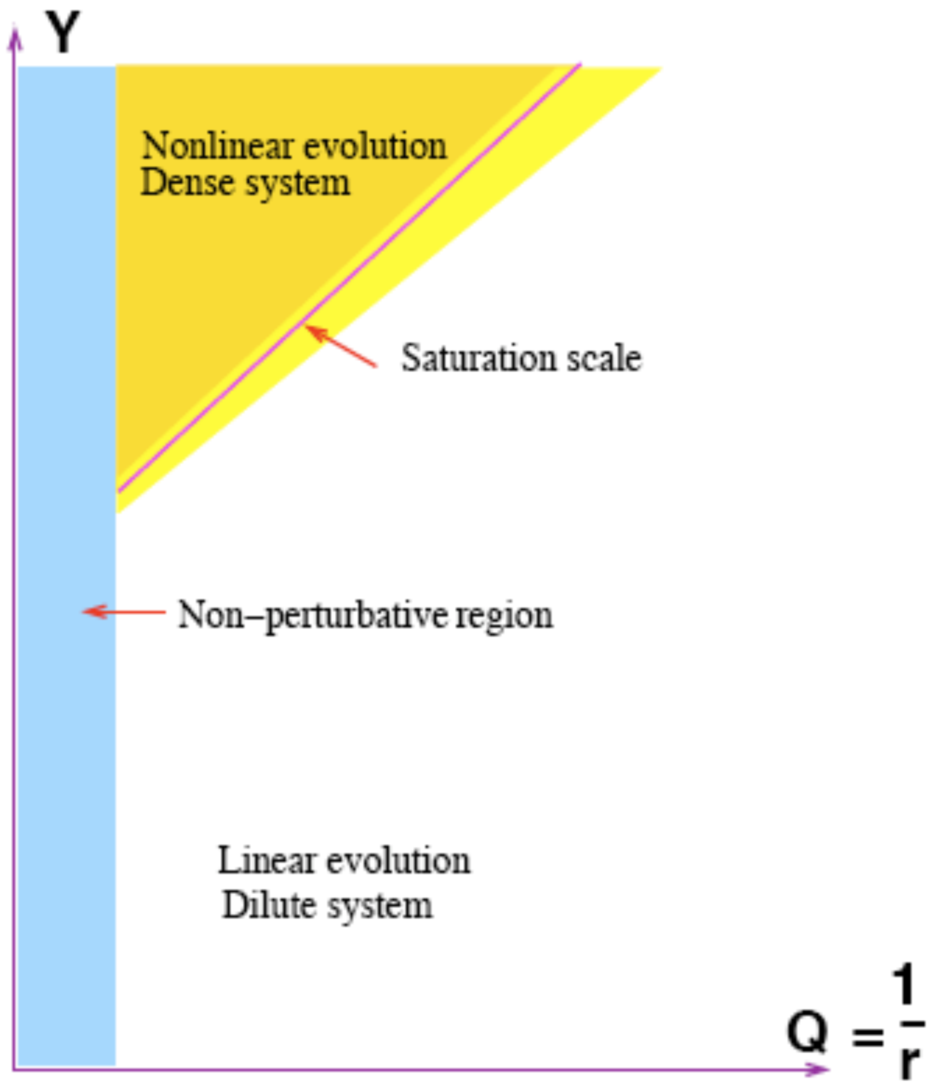
Nonlinear term: gluon merging, slow down the growth of the density with the energy

Equilibrium: nonlinear term compensates the linear term.

Gluon density saturates: parton saturation

Saturation scale

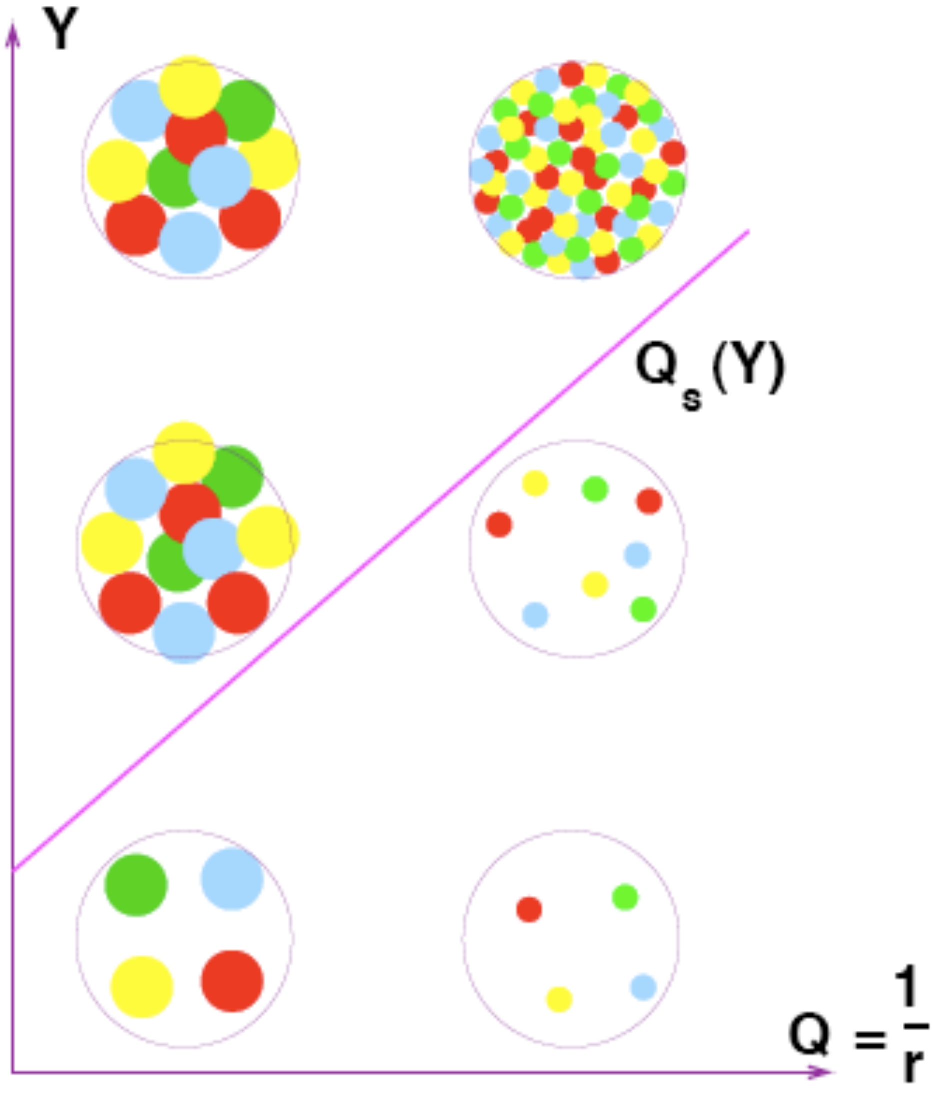


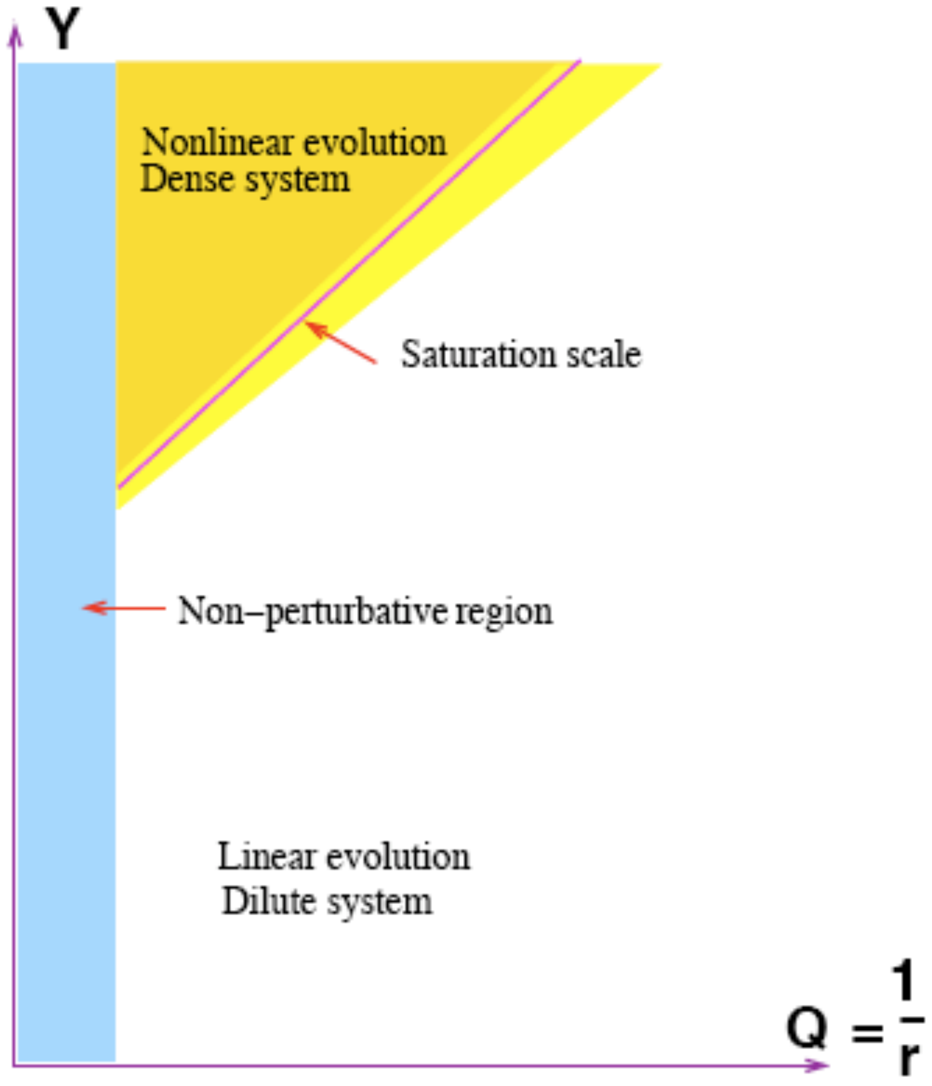


Dynamically generated saturation scale

Regulates the diffusion into infrared

Saturation scale





Dynamically generated saturation scale

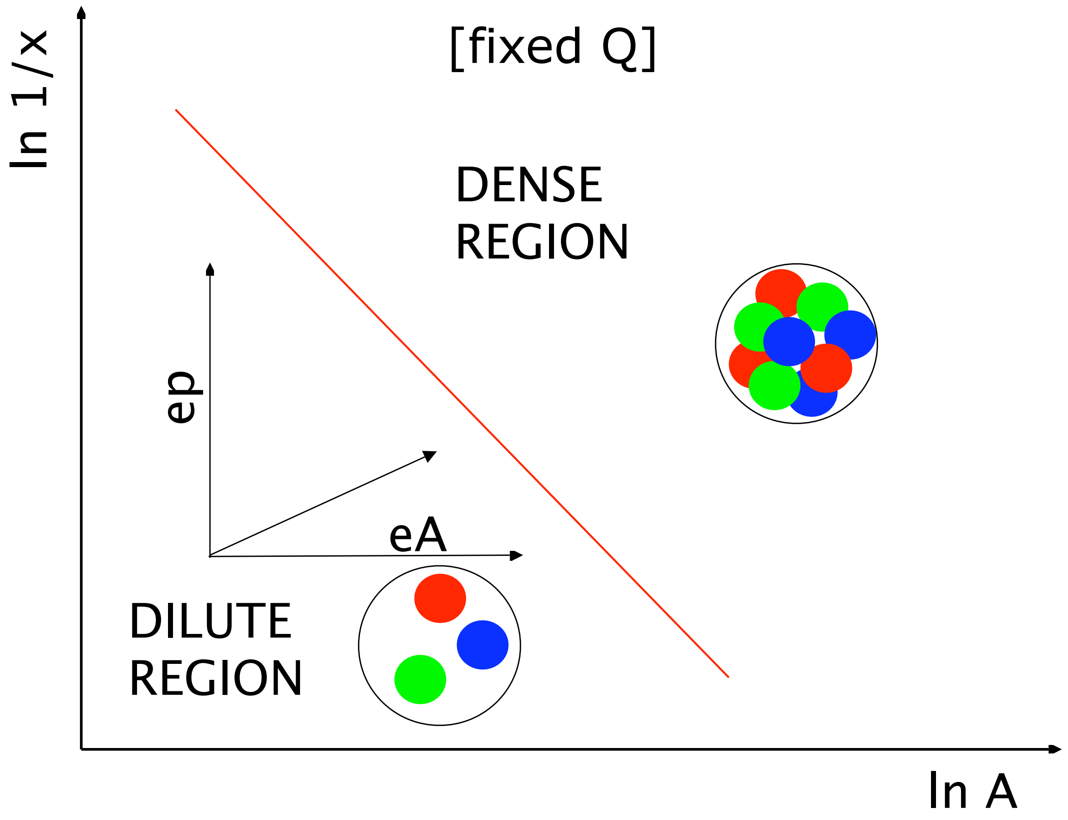
Regulates the diffusion into infrared

$$\frac{A \times xg(x, Q_s^2)}{\pi A^{2/3}} \times \frac{\alpha_s(Q_s^2)}{Q_s^2} \sim 1 \qquad Q_s^2 \sim A^{1/3} Q_0^2 \left(\frac{1}{x}\right)^{\lambda}$$

Saturation scale: nuclear enhancement

Dynamically generated
saturation scale

For a nucleus there is an enhancement factor related to the nuclear size. The dense region is approached either by selecting larger nucleus and probing smaller impact parameters or by decreasing value of x.



Can search for saturation either:

➡ decreasing x

➡ DIS on nuclei

➡ combination of both

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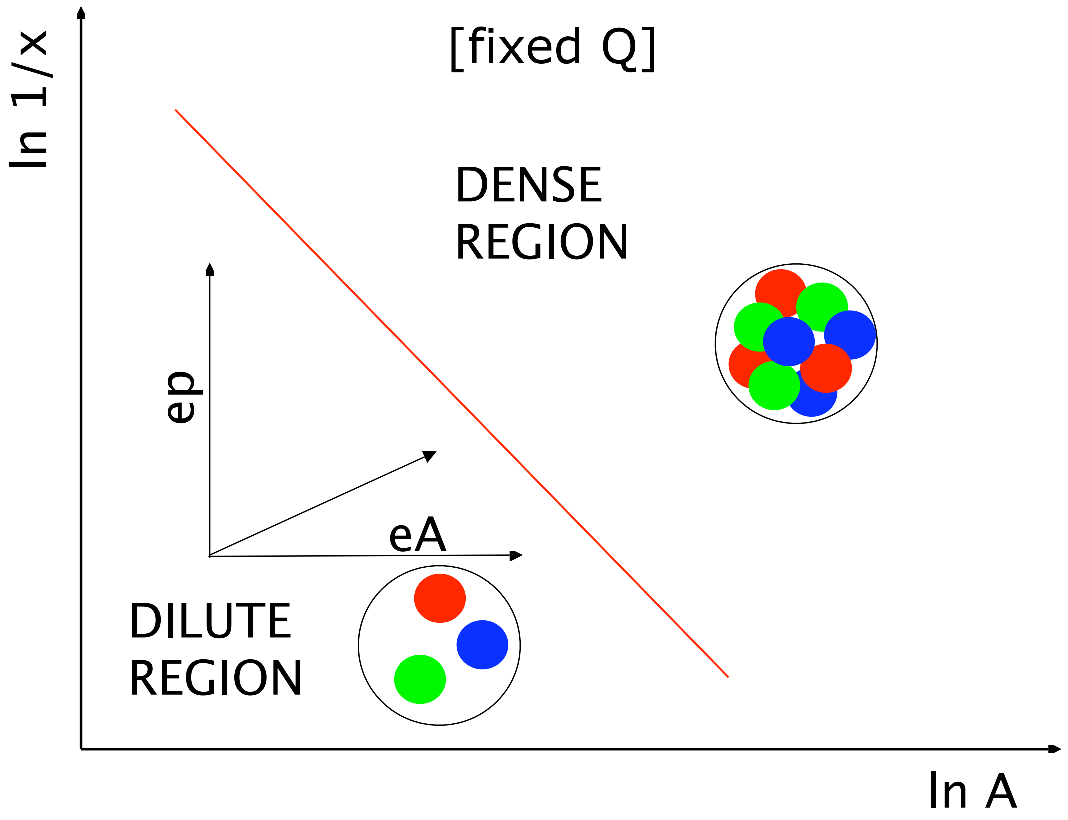

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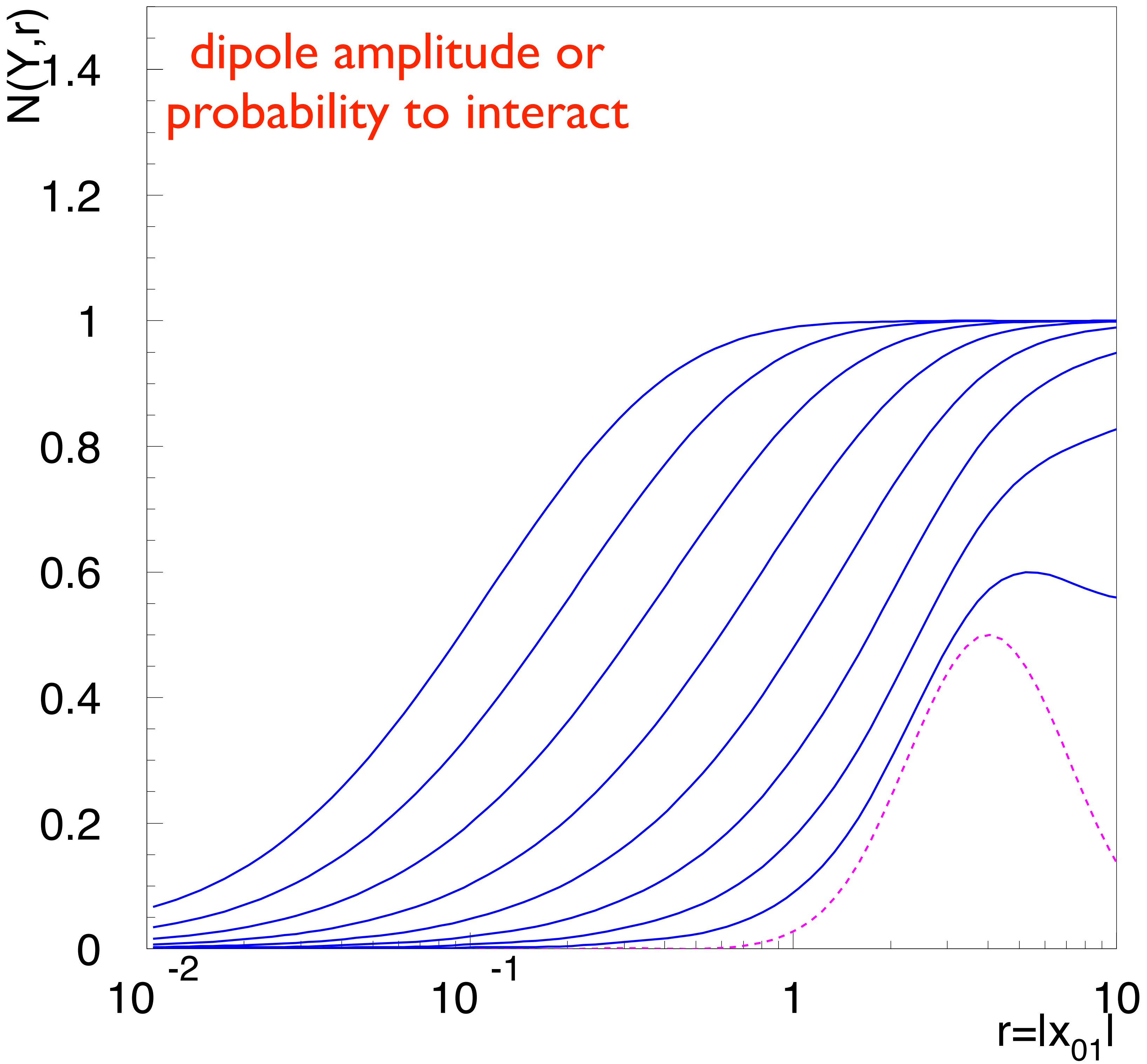
Solution to the BK equation (no impact parameter dependence)



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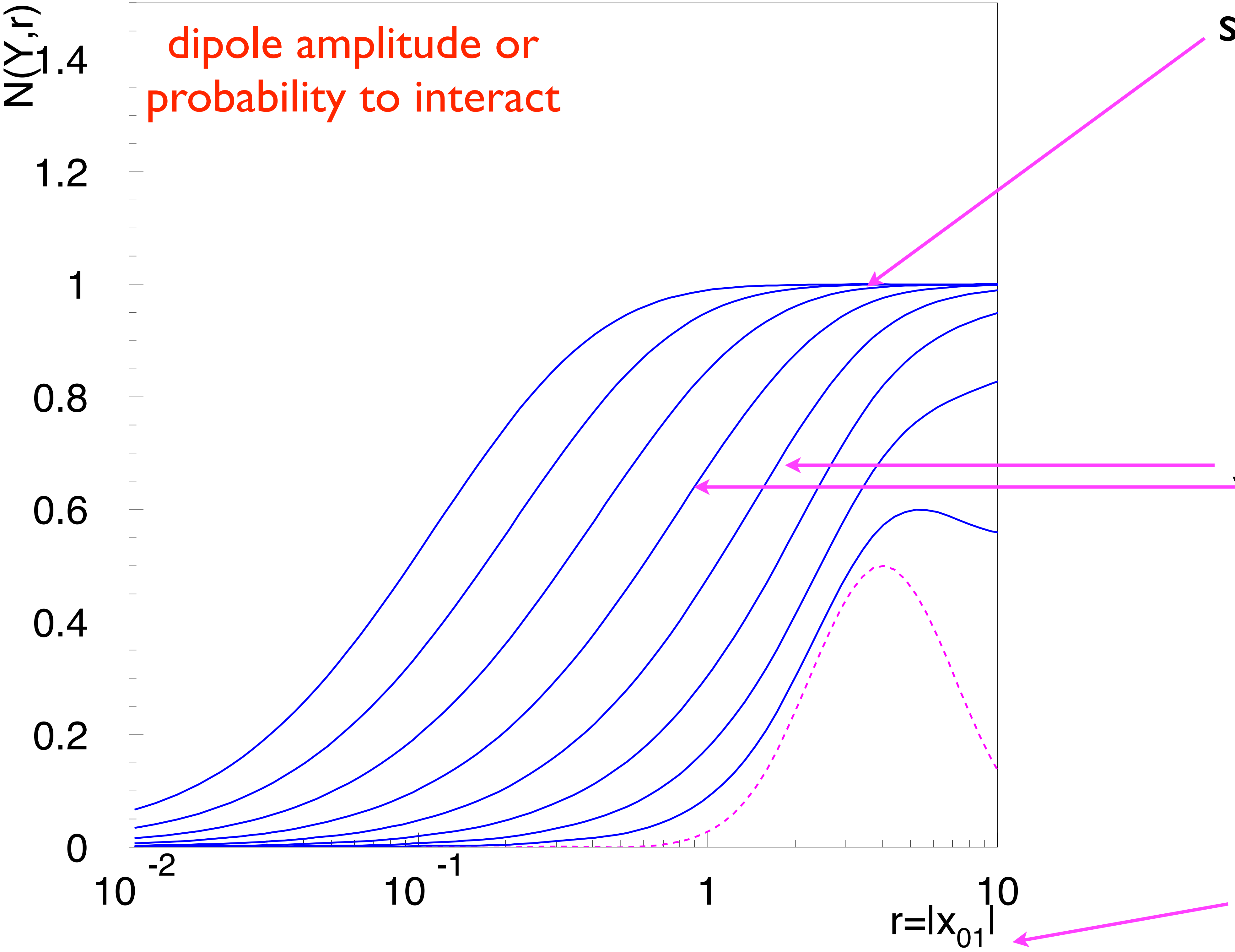

dipole amplitude or probability to interact

saturation

decreasing values of x (increasing energy)

size

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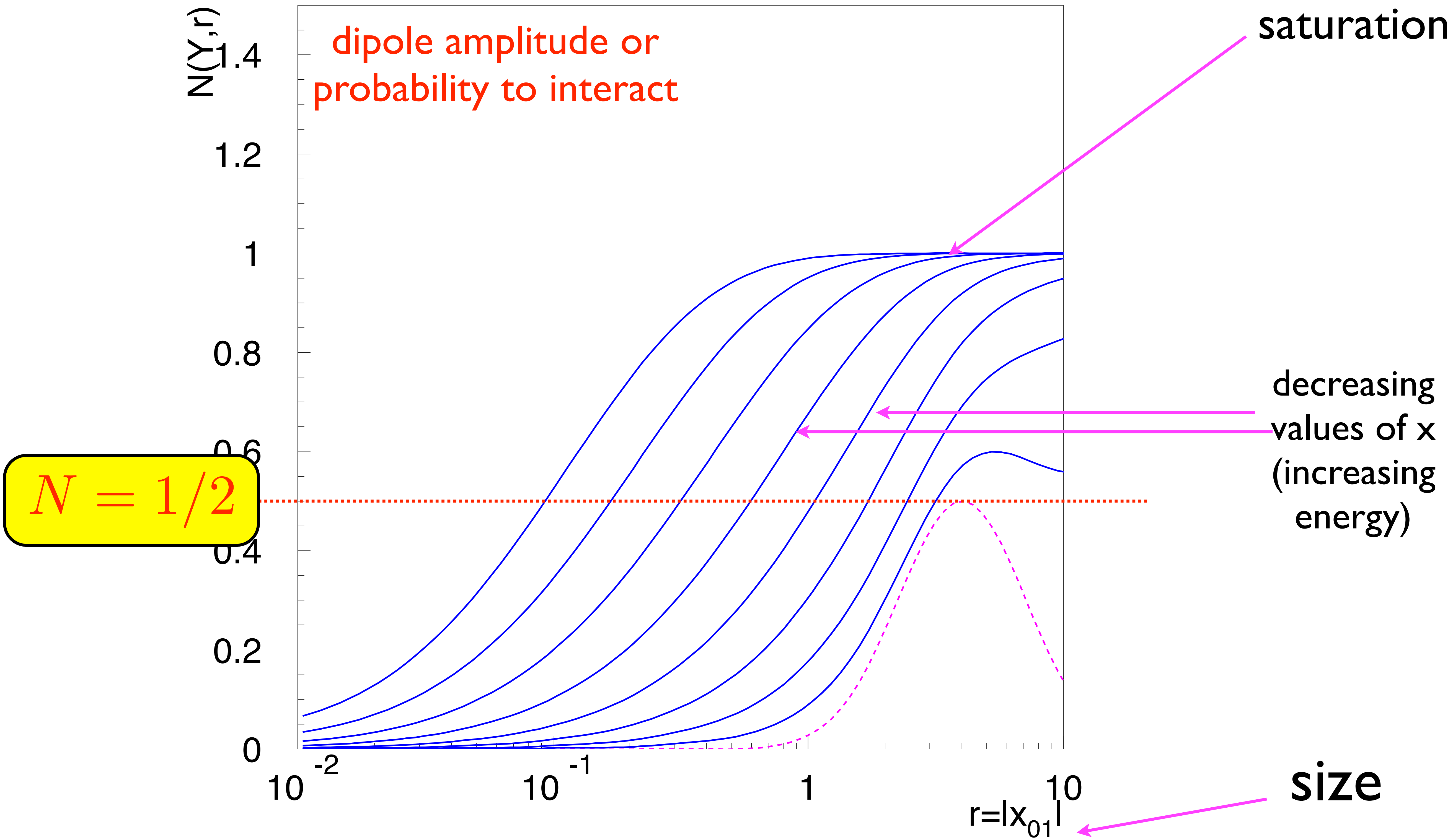

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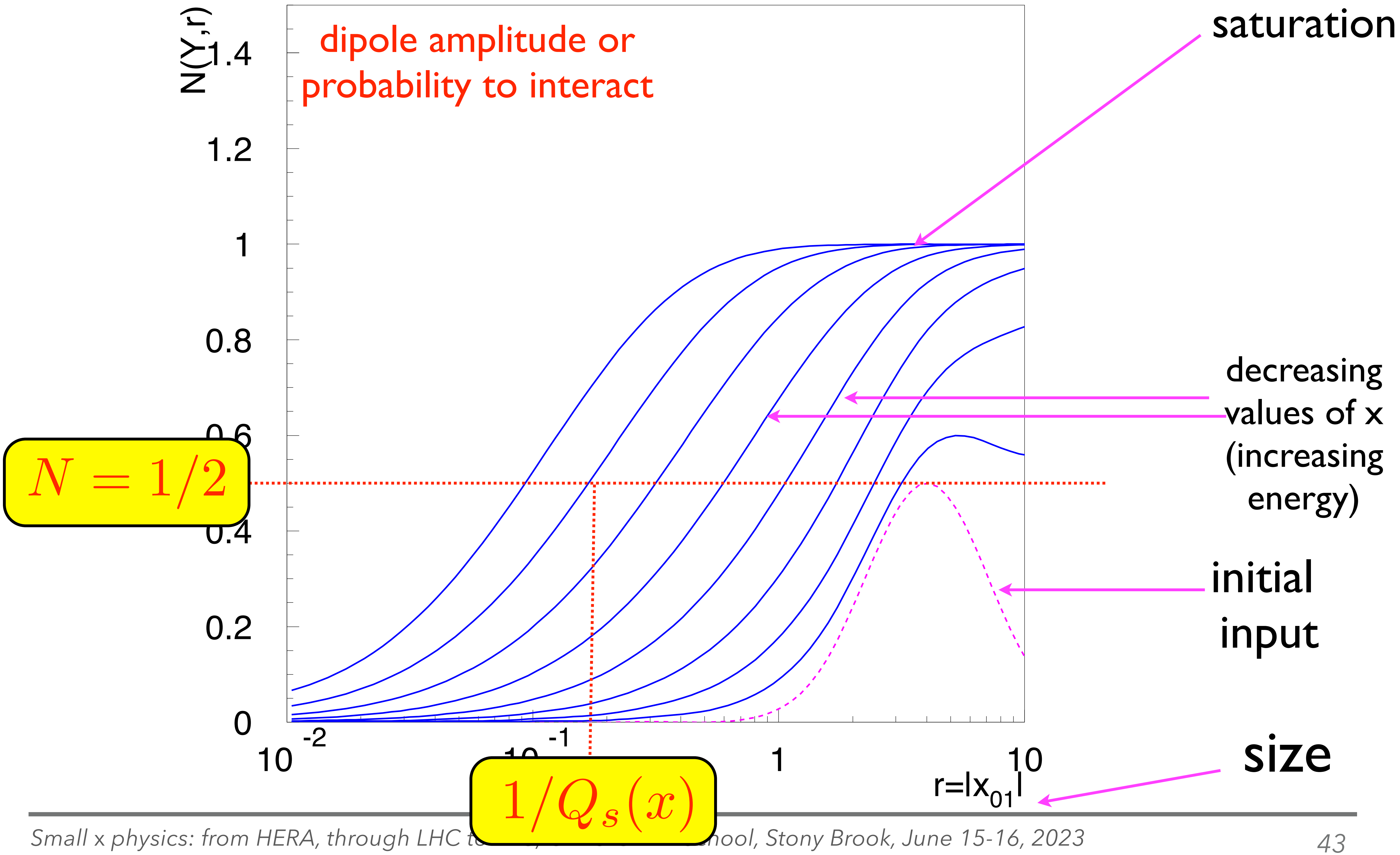


dipole amplitude or probability to interact

saturation

decreasing values of x (increasing energy)

initial input

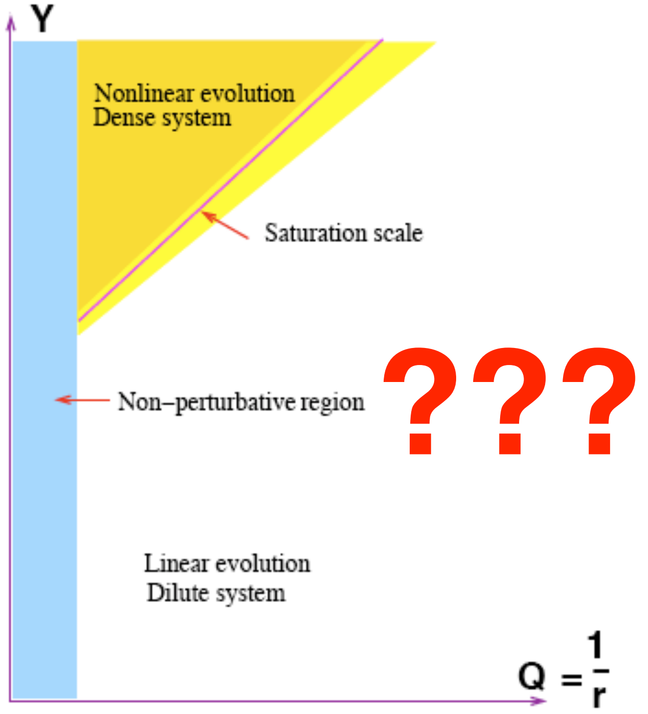
size

$N = 1/2$

$1/Q_s(x)$

Bonus: impact parameter dependence and small x

Impact parameter dependence

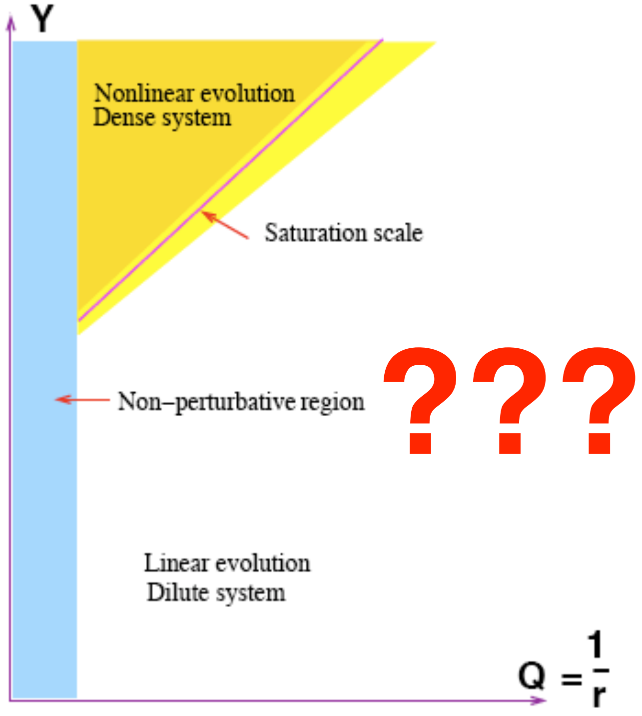


Usual approximation:

$$N(Y;r,b) \to N(Y;r)$$

- The target has infinite size.
- Local approximation suggests that the system becomes more perturbative as the energy grows.
- But this cannot be true everywhere (IR in QCD)

Impact parameter dependence



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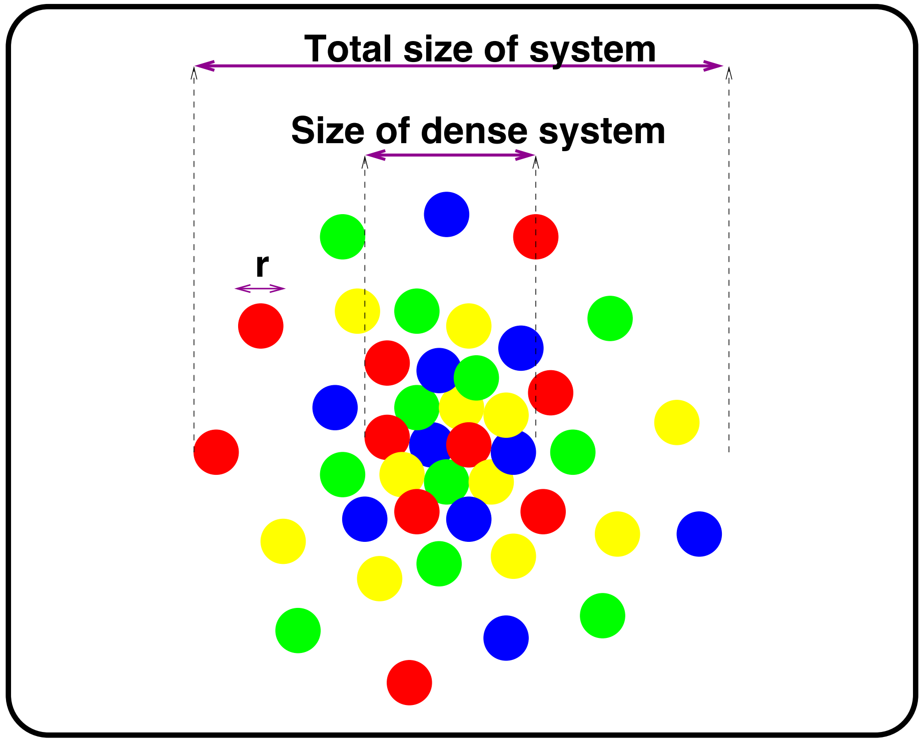
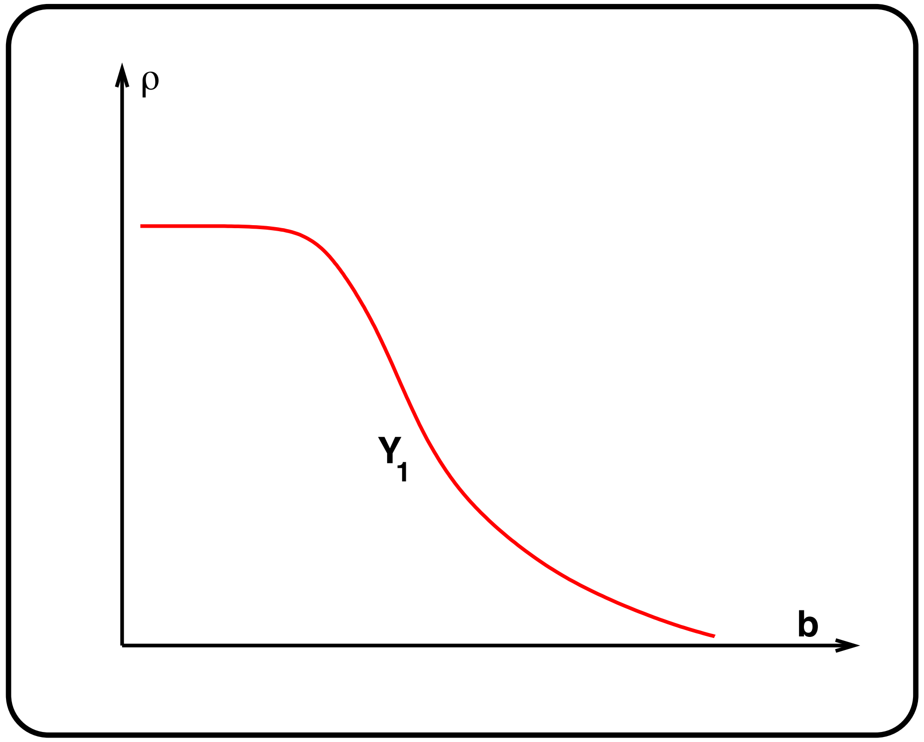
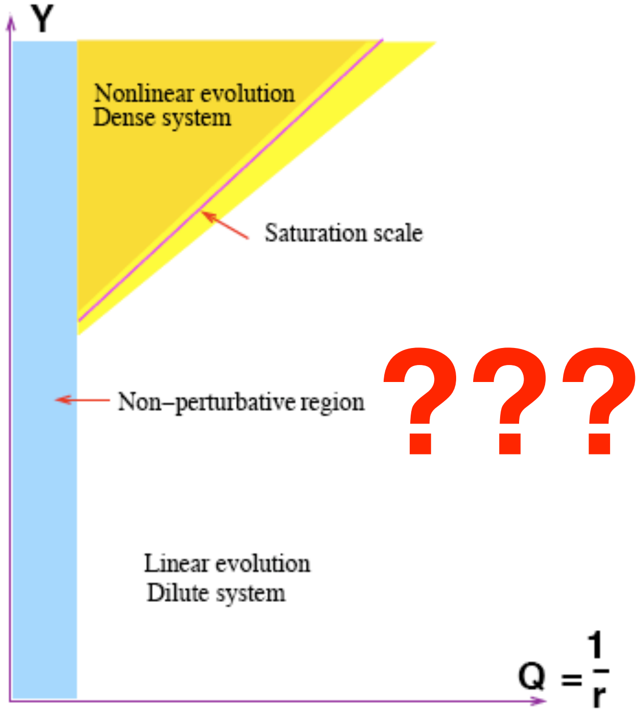




Impact parameter dependence



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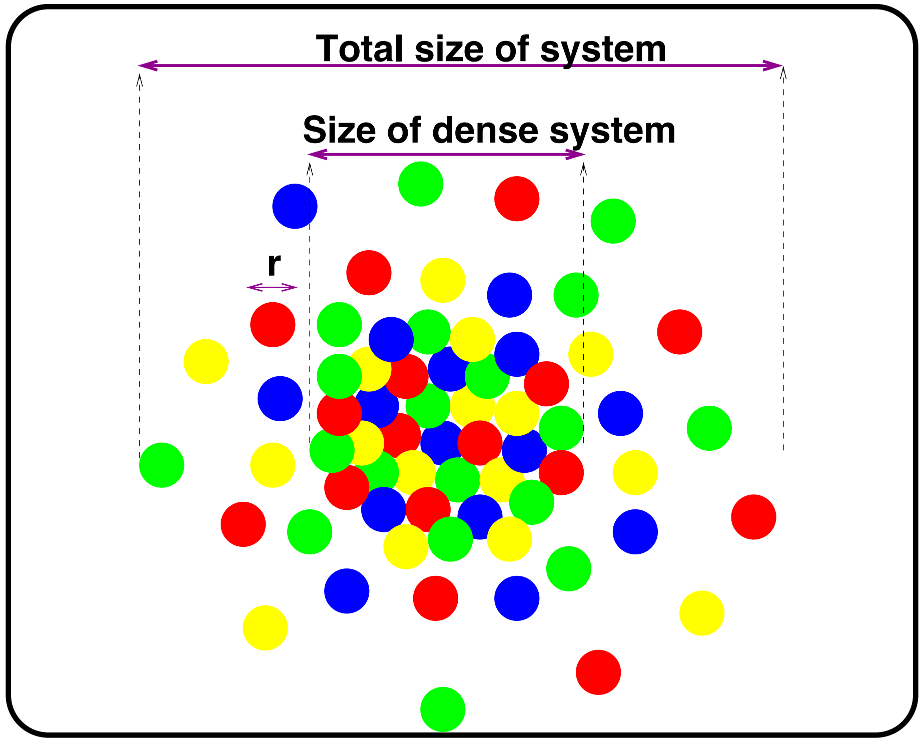
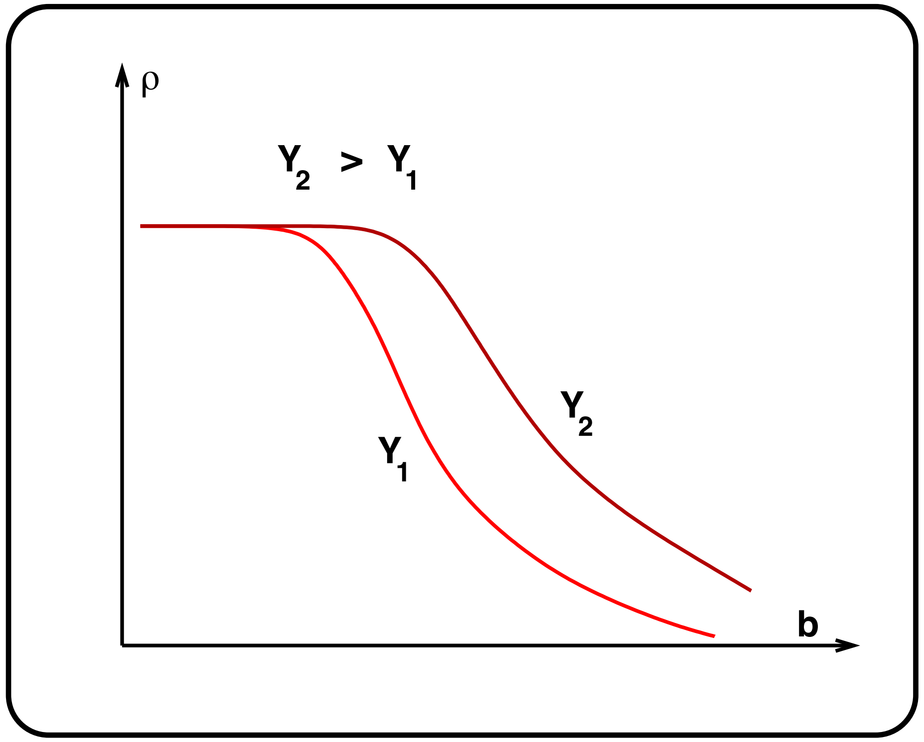




Impact parameter representation

Why do we care about impact parameter?

Impact parameter profile can provide the information how close the amplitudes are to the unitarity limit.
Important to address the issue of correlations and in the double parton scattering context.

Impact parameter representation for total, elastic and inelastic

$$\sigma_{tot}(s) = 2\int d^2\mathbf{b}\,\mathrm{Re}\,\Gamma(s,b),$$

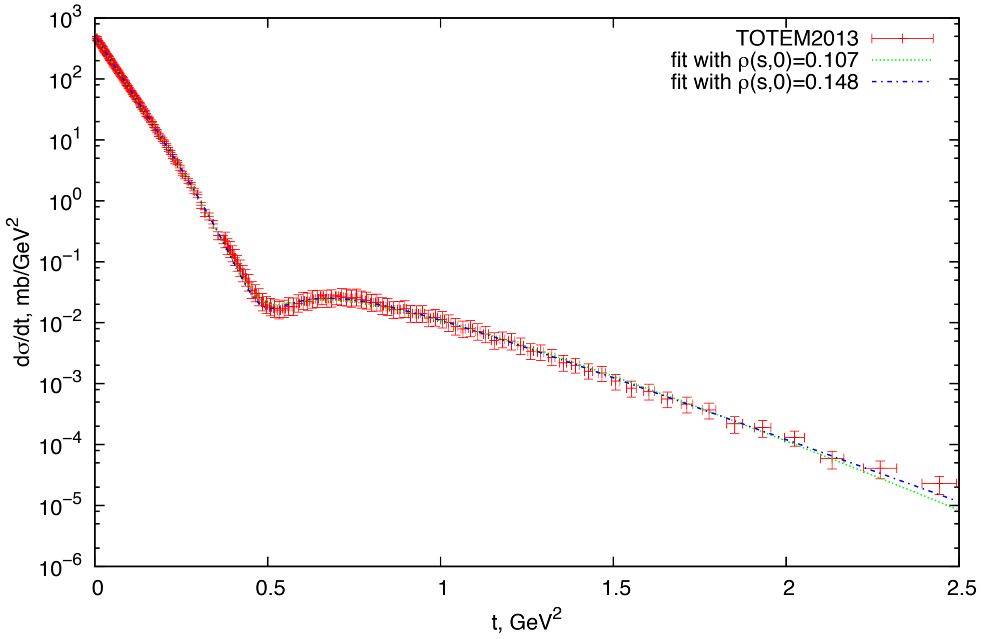
$$\sigma_{el}(s) = \int d^2\mathbf{b}\,|\Gamma(s,b)|^2,$$

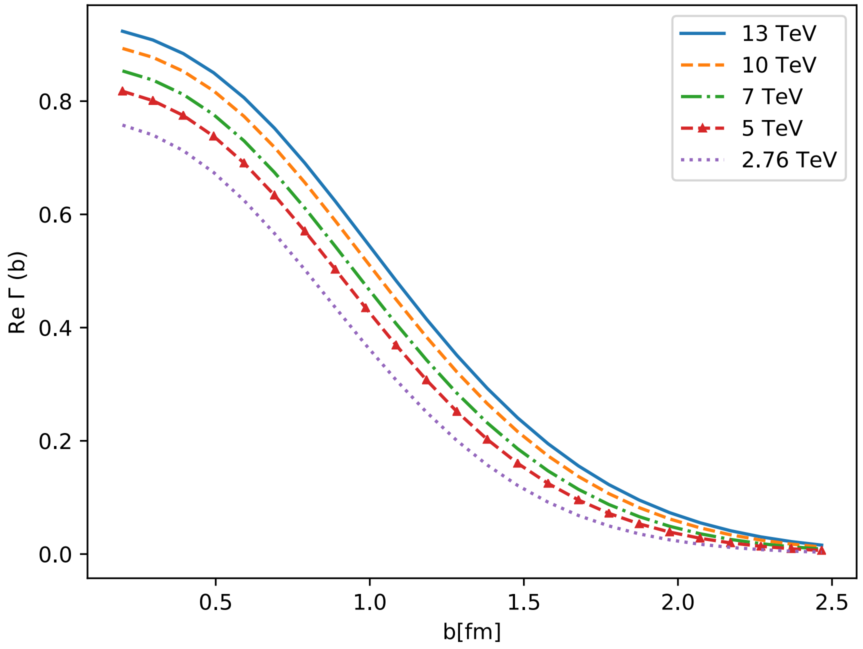
$$\sigma_{inel}(s) = \int d^2\mathbf{b}\left(2\,\mathrm{Re}\,\Gamma(s,b) - |\Gamma(s,b)|^2\right),$$

$$\Gamma(s,b) = \frac{1}{2is(2\pi)^2}\int d^2\mathbf{q}\,e^{i\mathbf{q}\cdot\mathbf{b}}A(s,t)$$

Unitarity limit:

$$\Gamma(s,b) \leq 1$$



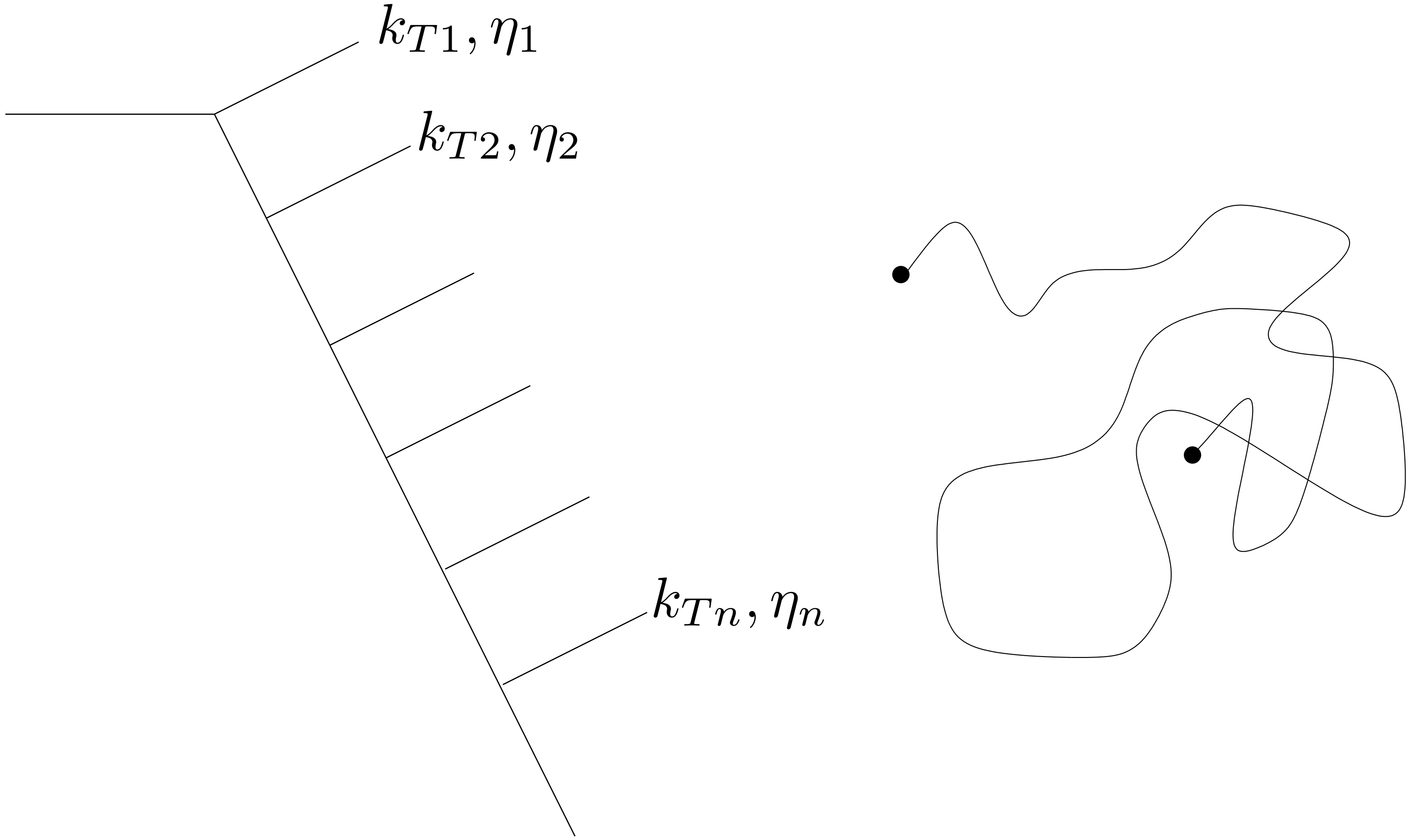


Impact parameter amplitude provides information about the unitarity limit.

Gribov diffusion in the parton model

Gribov

Emission of particles, with some transverse momenta leads to the diffusion in impact parameter space. Rapidity η plays a role of `time'



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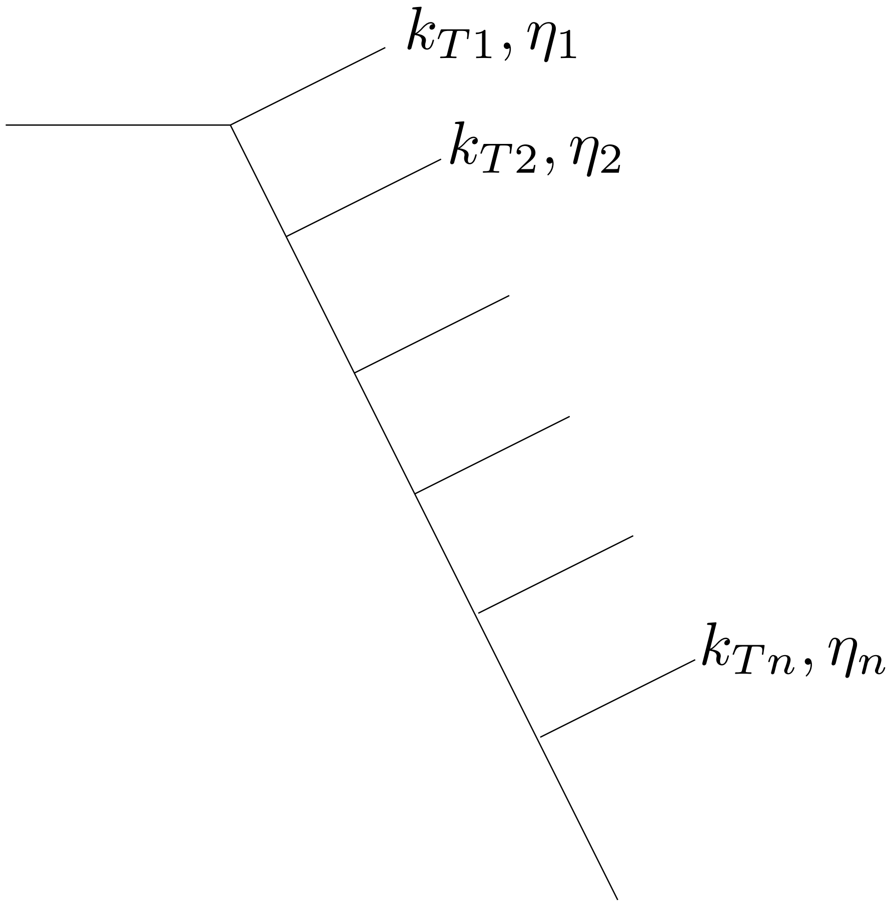

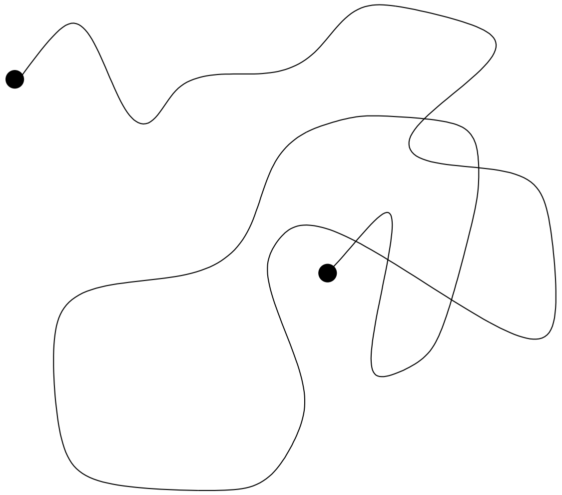

Assumption:
each emission leads to the change of impact parameter of the order of some scale

$$b \sim \frac{1}{\mu}$$

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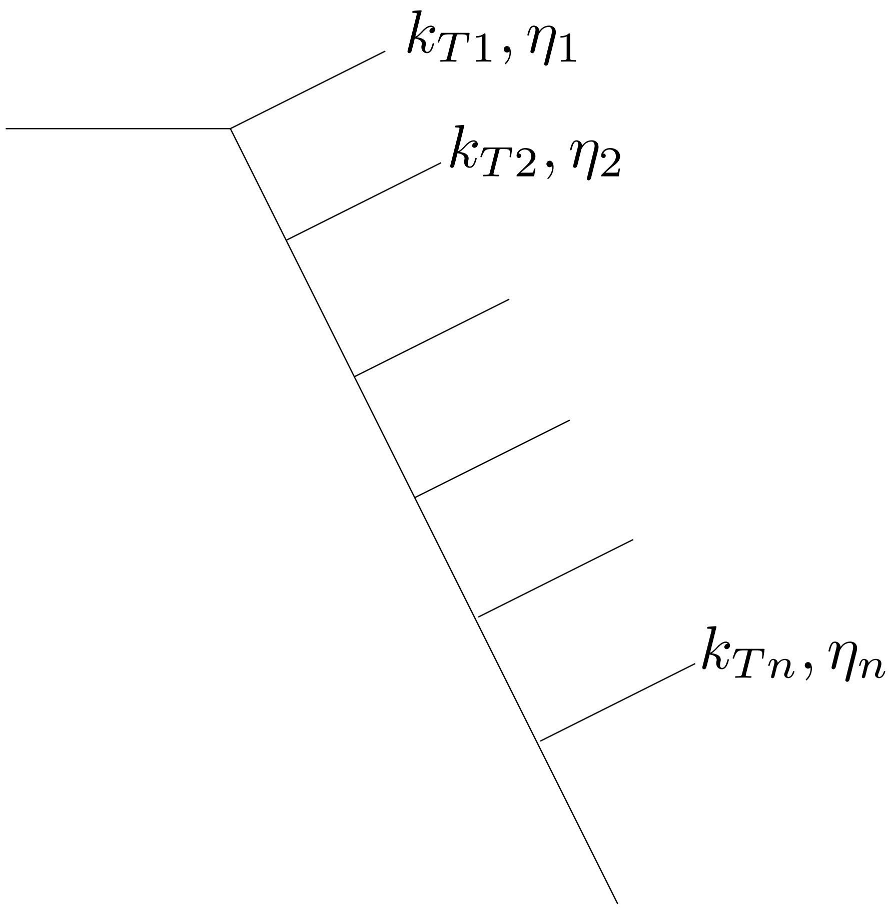

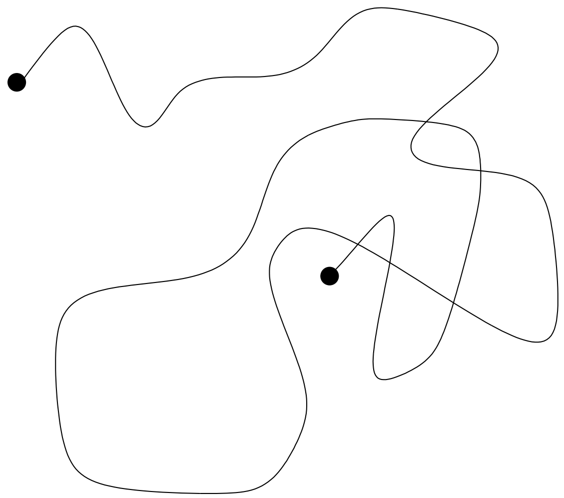

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$$\langle (\Delta b)^2 \rangle = c(\eta_1 - \eta_n)$$

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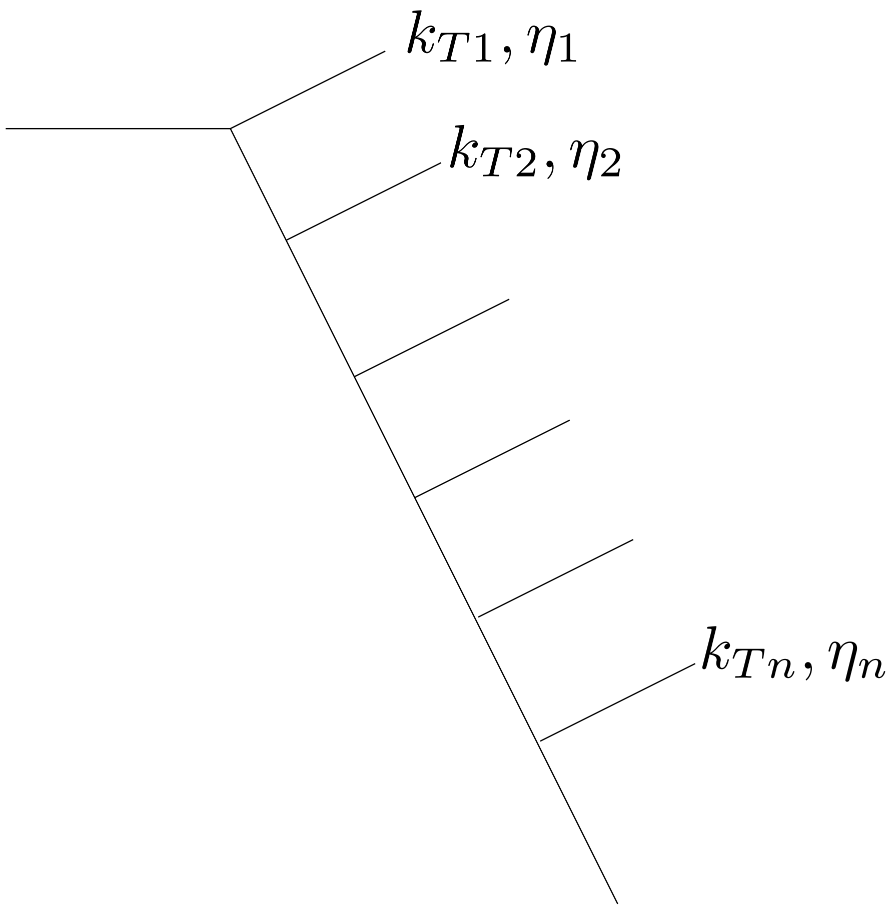

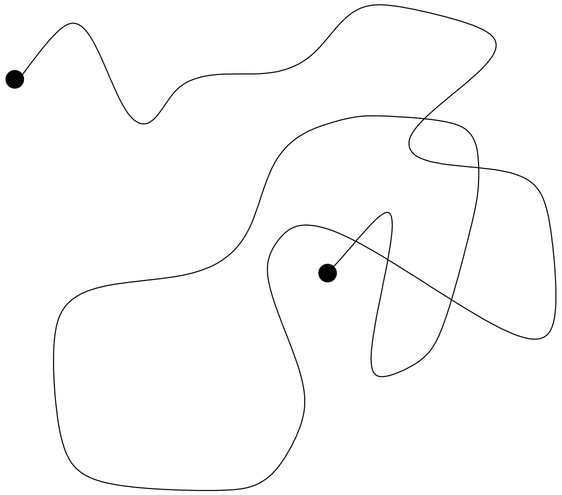

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Spatial distribution of partons

$$\phi(b,\eta) \sim \frac{1}{c(\eta_1 - \eta_n)} \exp(-\frac{b^2}{c(\eta_1 - \eta_n)})$$

Impact parameter dependence in BK equation

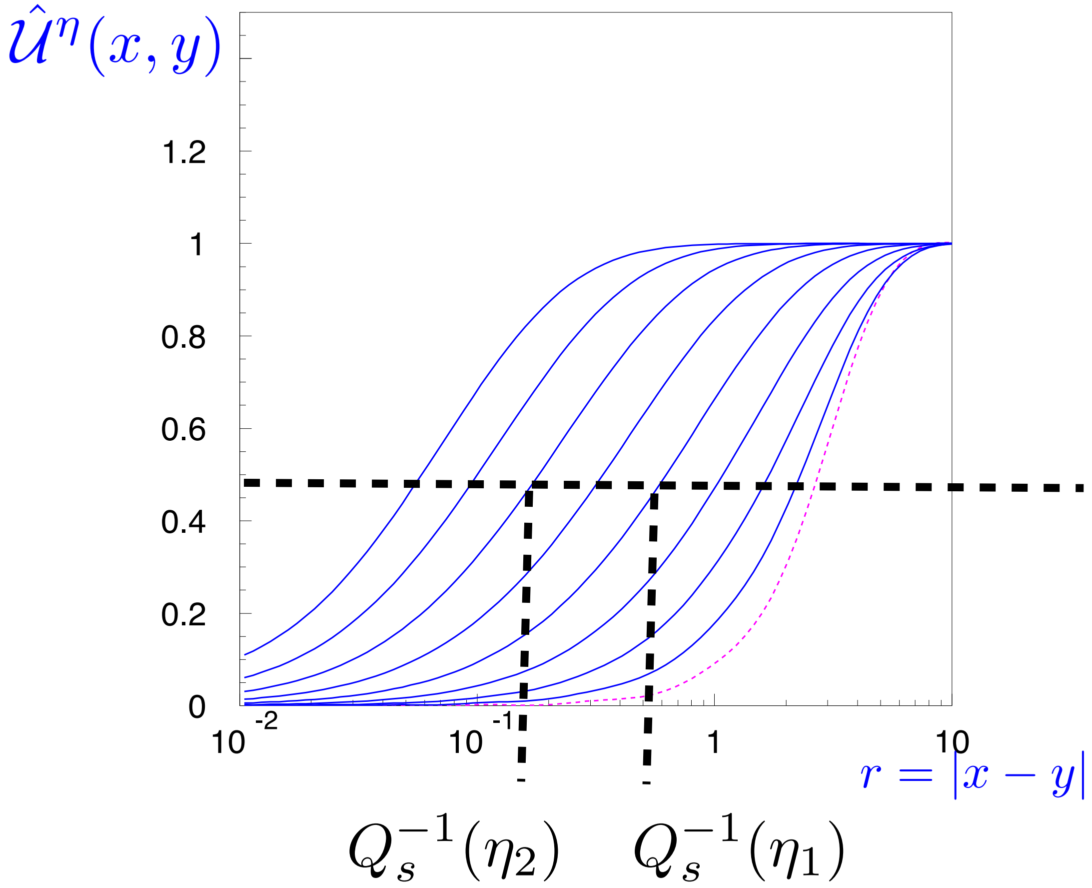
Solution to the BK equation with impact parameter dependence

Initial condition $N^{(0)} = 1 - \exp(-c_r r^2 \exp(-c_b b^2))$

initial profile in impact parameter

Golec-Biernat,AS

Without impact parameter



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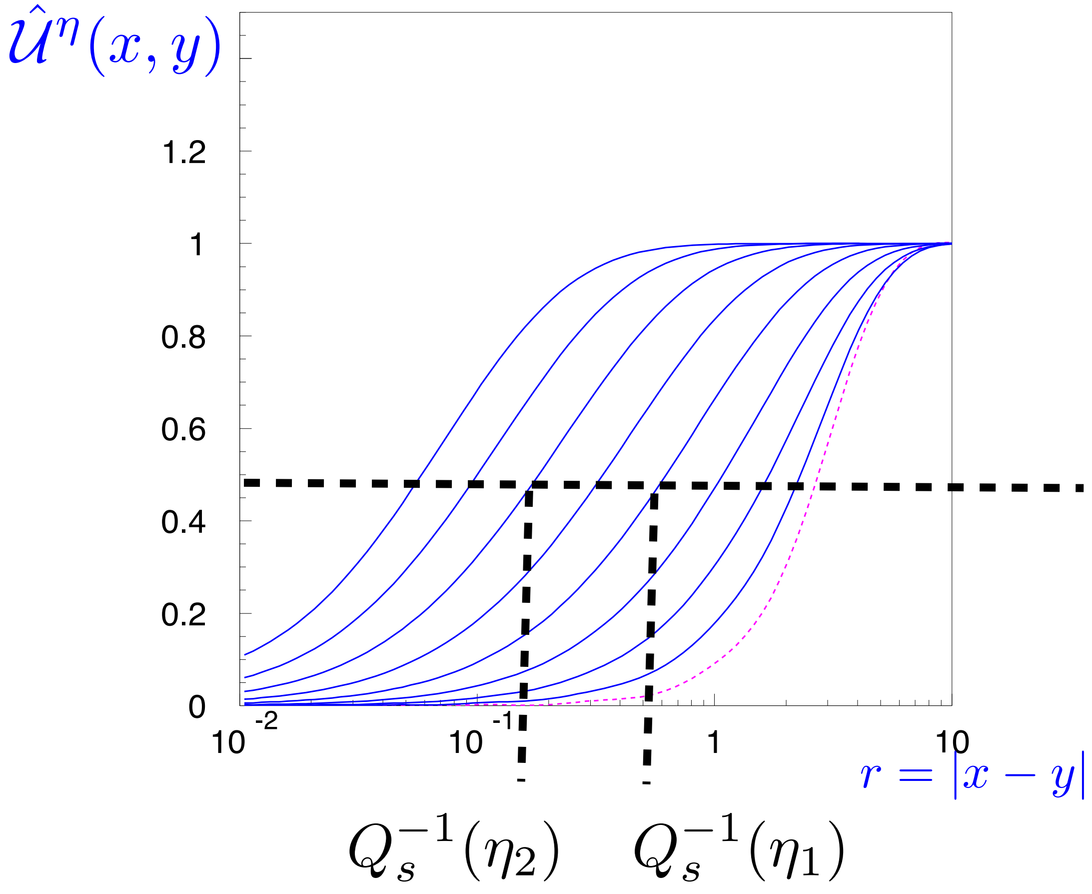

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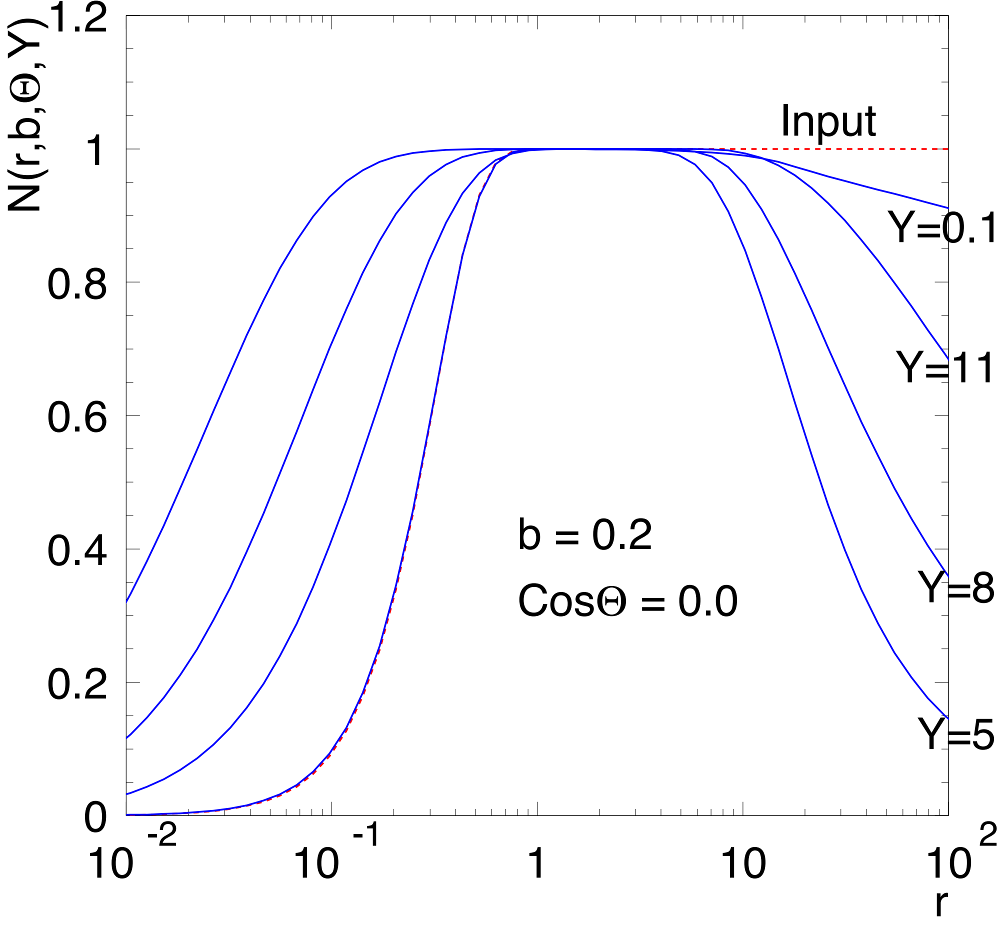

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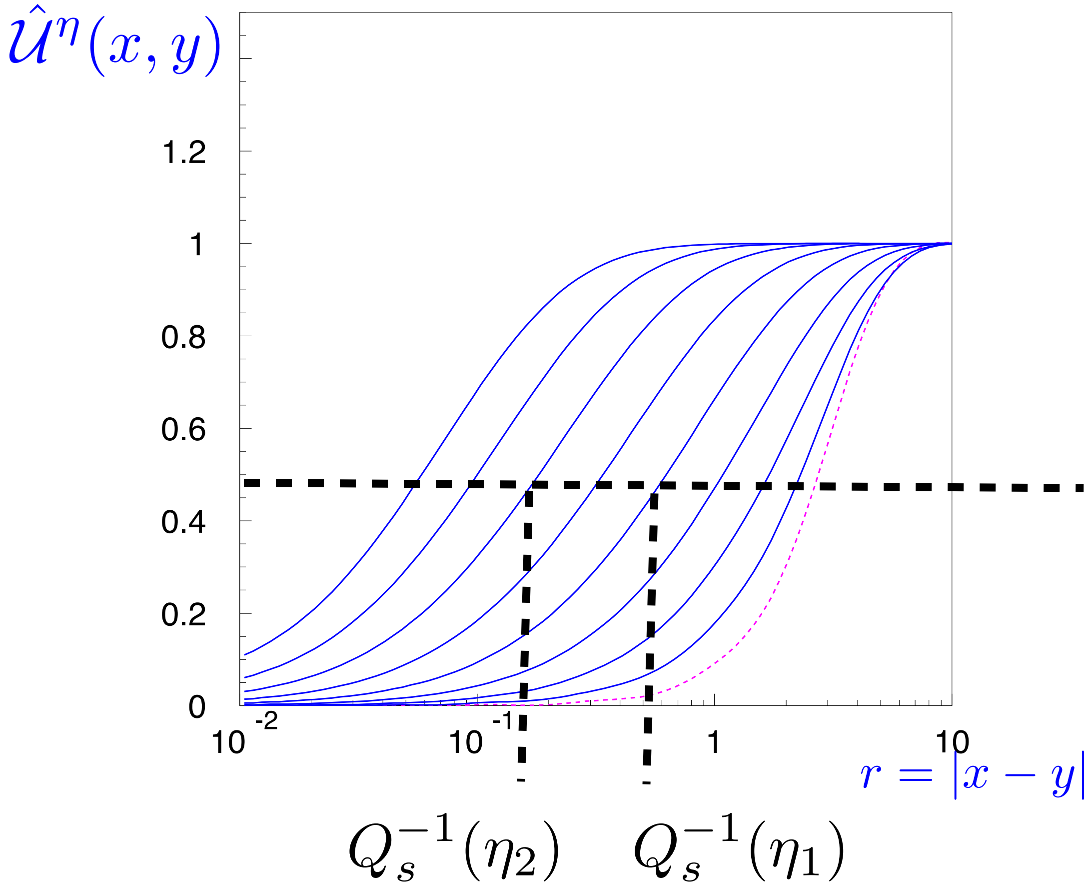
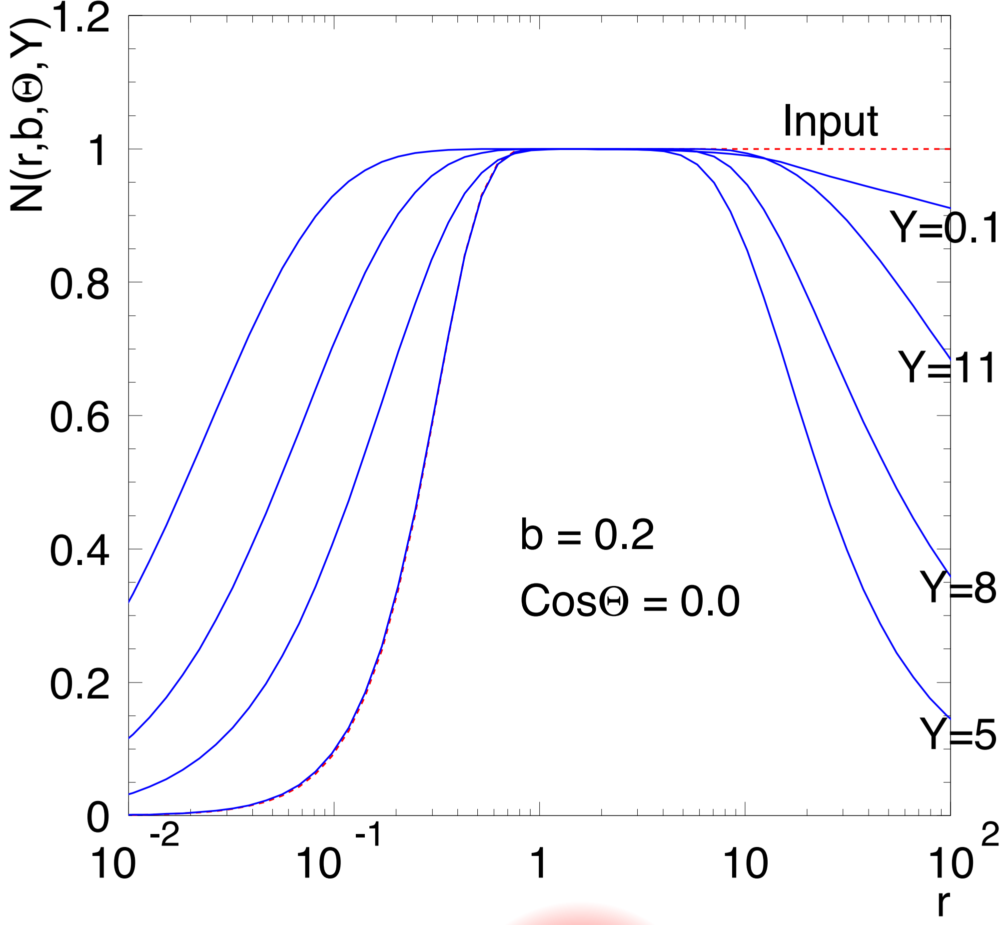
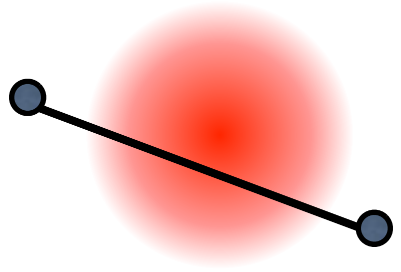


With impact parameter



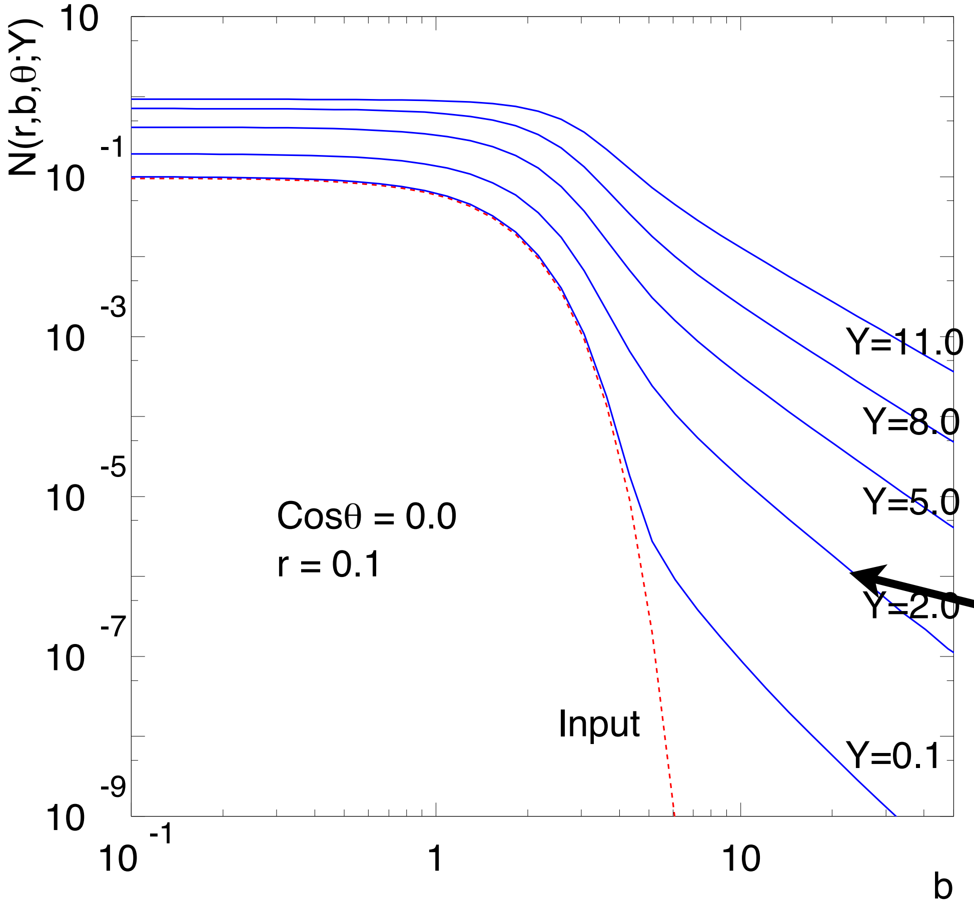
When dipole is larger than the target it can miss it.
At LL it is a manifestation of conformal invariance.



Impact parameter profile of scattering amplitude

Kovner, Wiedemann

It was argued that the nonlinear equation leads to saturation but there will be long Coulomb tails due to the massless gluons.



power tail

- Saturation for small impact parameters
- No saturation for large impact parameters (system is still dilute)
- Initial impact parameter profile is not preserved
- Power tail in impact parameter is generated

Perturbative LL QCD gives leads to the power tails: lack of confinement, conformal invariance