Can spin chains describe colored d.o.f. in DIS? (II)

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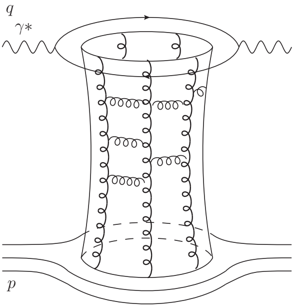
CFNS-CTEQ Summer School

June 16, 2023

Lecture plan

The scattering of a lepton on a hadron is a sum of Feynman diagrams. In leading logarithmic approximation ladder diagrams dominate. Quarks exchange gluons. The Hamiltonian of LNS describes interactions of the gluons.

BFKL tried to sum these diagrams.
They derived a linear integral
equation in terms of z [complex variable describing plane transverse (perpendicular) to p, where TMD lives]. The Fourier transform of the kernel of this equation leads to a spin Hamiltonian [lattice nonlinear Schrödinger]



BCFW recursion relation

We shall start with tree approximation: R. Britto, F. Cachazo, B. Feng, and E. Witten, Direct Proof of the Tree-Level Scattering Amplitude Recursion Relation in Yang-Mills Theory, Phys. Rev. Lett. 94:181602, (2005)

The tree-level amplitudes are studied in Yang-Mills theory. A set of recursion relations for tree-level amplitudes of gluons derived. These relations express any tree-level amplitude of gluons as a sum over terms constructed from the product of two subamplitudes with fewer gluons times a Feynman propagator. The subamplitudes are physical, on-shell amplitudes with shifted momenta.

The recursion relations are:

$$A_n = \sum_r A^h_{r+1} \frac{1}{P_r^2} A^{-h}_{n-r+1}.$$

Here, for any positive integer s, A_s denotes the tree-level scattering amplitudes for s cyclically ordered gluons. In writing a recursion relation for A_n, one "marks" two of the gluons and sums over products of subamplitudes, with r external gluons on one side, $n - r$ external gluons on the other side, and one internal gluon connecting them, and with the two marked or reference gluons being on opposite sides. (The sum in above equation is really a sum over decompositions with one marked gluon on each side, not just a sum over r.) P is the momentum and h the helicity of the internal gluon. Momenta are shifted so that this gluon as well as the external ones are on-shell.

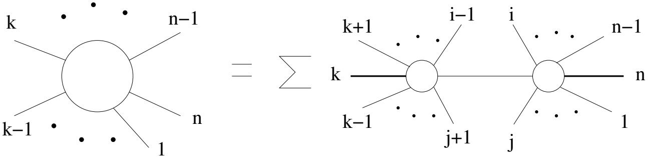
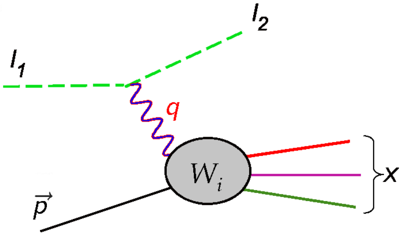


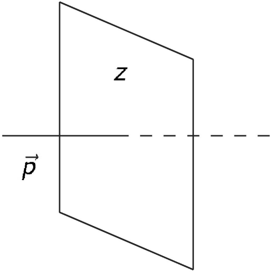
Figure 1: Pictorial representation of the recursion relation. Thick lines represent the two marked gluons. The sum is over all cyclically ordered distributions of gluons with at least two gluons on each subamplitude and over the two choices of helicity for the internal gluon.

Color factorization: The QCD tree scattering amplitudes can be formulated in a way separating the gauge group (color) factors from the momentum factors. Related to this the interaction of reggeized gluons is not affected by the color structure.

Bjorken x and coordinate z

In 1968, Bjorken proposed that the structure functions measured in electron-nucleon deep inelastic scattering, may exhibit scaling behavior in the asymptotic limit,





The dimensionless variable $x = Q^2/2M\nu$ is the Bjorken x scaling variable, here $Q^2 = -q^2$.
p is perpendicular to the z coordinate plane.

Radiative corrections:
A one-particle exchanges in the t channel leads to the disappearance of the non-analytic terms [in z] of tree approximation [the Reggeization].
The local Hamiltonians are given by the equivalent representations

$$\begin{aligned} H_{j,k} &= P_j^{-1} \ln(z_{jk}) P_j + P_k^{-1} \ln(z_{jk}) P_k + \ln(P_j P_k) + 2\gamma_E \\ &= 2\ln(z_{jk}) + (z_{jk}) \ln(P_j P_k)(z_{jk})^{-1} + 2\gamma_E, \end{aligned}$$

where $P_j = i\partial/\partial z_j = i\partial_j$, $z_{jk} = z_j - z_k$, and γ_E is the Euler constant.

$$\mathcal{H} = \sum_{k=1}^{L} H_{k,k+1},$$

Gell-Mann formulated necessary conditions for Reggeization.

BFKL Hamiltonian and Yangian symmetry

R. Kirschner derived the BFKL Hamiltonian from Yangian symmetry.
Yangian symmetry applied to Quantum chromodynamics, arXiv:2302.00449
https://arxiv.org/abs/2302.00449
https://www.worldscientific.com/doi/abs/10.1142/S0217751X23300065

Kirschner started from BCFW recursion relation, considered Reggeized gluons, then obtained BFKL Hamiltonian.

For Reggeization of elementary particles, see Chapter 4: BFKL – Past and Future (by Victor S. Fadin) of the book "From the Past to the Future The Legacy of Lev Lipatov"
https://www.worldscientific.com/worldscibooks/10.1142/12127

Algebraic Bethe Ansatz [equivalent to MPS].

Consider the L operator in the case $n = 2$ after projection to the weight 2ℓ in normal coordinates $x = \frac{x_1}{x_2}$. The matrix $L(v^{(1)}, v^{(2)})$ factorizes as

$$L(v^{(1)}, v^{(2)}) = \begin{pmatrix} v^{(2)} + 1 + x\partial & \partial \\ x(-x\partial + 2\ell) & v^{(1)} - x\partial \end{pmatrix} = \quad v^{(1)} \hat{V}^{-1}(v^{(1)}) \hat{D} \hat{V}(v^{(2)}),$$

$$\hat{V}(v) = \begin{pmatrix} v & 0 \\ -x & -1 \end{pmatrix}, \quad \hat{D} = \begin{pmatrix} 1 & -\partial \\ 0 & 1 \end{pmatrix}.$$

The convenient form with the same eigenvalues is obtained as

$$\mathbb{R}(v_1 - v_2; \ell_1, \ell_2) = \frac{\Gamma(-\hat{m} + v_1 - v_2)}{\Gamma(-\hat{m} + v_2 - v_1)}$$

in terms of the operator $\hat{m}$ related to the tensor Casimir as $\hat{m}(1 - \hat{m}) = x_{12}^2 \partial_1 \partial_2$.
The BFKL Hamiltonian is obtained up to normalization as the first non-trivial term,

$$H = \psi(\hat{m}) + \psi(1 - \hat{m}) - 2\psi(1)$$

The ψ is the logarithmic derivative of the Gamma funciton. This Hamiltonian describes the nearest-neighbour interaction in the multiple exchange of gluon reggeons. It is also the Hamiltonian of lattice nonlinear Schrödinger.
The decomposition of the R matrix results in the set of commuting local observables of the corresponding spin chain.

The holomorphic multicolor QCD Hamiltonian describes the nearest neighbor interactions of L particles (reggeized gluons):

$$\mathcal{H} = \sum_{k=1}^{L} H_{k,k+1},$$

with periodic boundary conditions $H_{L,L+1} = H_{L,1}$. The local Hamiltonians are given by the equivalent representations

$$\begin{aligned} H_{j,k} &= P_j^{-1} \ln(z_{jk}) P_j + P_k^{-1} \ln(z_{jk}) P_k + \ln(P_j P_k) + 2\gamma_E \\ &= 2\ln(z_{jk}) + (z_{jk}) \ln(P_j P_k)(z_{jk})^{-1} + 2\gamma_E, \end{aligned}$$

where $P_j = i\partial/\partial z_j = i\partial_j$, $z_{jk} = z_j - z_k$, and γ_E is the Euler constant.

The $su(2)$ representation

BFKL used holomorphic representation of $su(2)$

$$S_k^+ = z_k^2 \partial_k - 2 s z_k, \qquad S_k^- = -\partial_k, \qquad S_k^z = z_k \partial_k - s$$

Later we shall change the space of representation to a set of interacting harmonic oscillators [lattice Bose filed]. The definition of the chain is based on the fundamental matrix $R_{jk}^{(s,s)}(\lambda)$ which obeys the Yang-Baxter equation

$$R_{jk}^{(s,s)}(\lambda) = f(s,\lambda) \frac{\Gamma(i\lambda - 2s)\Gamma(i\lambda + 2s + 1)}{\Gamma(i\lambda - J_{jk})\Gamma(i\lambda + J_{jk} + 1)}.$$

Here $f(s,\lambda)$ is a complex valued function. The operator J_{jk} is defined in the space $V \otimes V$ as a solution of the operator equation,

$$J_{jk}(J_{jk} + 1) = 2 \boldsymbol{S}_j \otimes \boldsymbol{S}_k + 2s(s+1).$$

This is also the lattice nonlinear Schrödinger.

The Hamiltonian of the XXX model with spin $s = -1$ describes interaction of nearest neighbors

$$H_{jk} = \frac{-1}{i}\frac{d}{d\lambda}\ln R_{jk}(\lambda)\bigg|_{\lambda=0}, \quad H_{jk} = \psi(-J_{jk}) + \psi(J_{jk}+1) - 2\psi(1).$$

Here $\psi(x) = d\ln\Gamma(x)/dx$, and $\psi(1) = -\gamma_E$.
The operator J_{jk} is a solution of the operator equation when $s = -1$,

$$J_{jk}(J_{jk}+1) = -(z_j - z_k)^2\partial_j\partial_k,$$

Transfer matrix is a part of Algebraic Bethe Ansatz

We define the fundamental monodromy matrix by taking the ordered product of the fundamental Lax operators $L^{(s,s)}_{f,k}(\lambda) = R^{(s,s)}_{f,k}(\lambda)$ along the lattice (with both its auxiliary space and quantum space being spin s)

$$T_f(\lambda) = L^{(s,s)}_{f,L}(\lambda) L^{(s,s)}_{f,L-1}(\lambda) \dots L^{(s,s)}_{f,1}(\lambda).$$

The fundamental transfer matrix,

$$\tau(\lambda) = \mathrm{tr}_f\, T_f(\lambda), \quad [\tau(\lambda), \tau(\mu)] = 0$$

The matrices commute with each other for different values of the spectral parameter.

The Hamiltonian of spin $s = -1$ model can be obtained from the first order derivative of the transfer matrix τ,

$$\mathcal{H}^{(s=-1)} = \frac{-1}{i} \frac{d}{d\lambda} \ln \tau^{(s=-1)}(\lambda) \Big|_{\lambda=0}.$$

Two transfer matrices

On the other hand, if we choose the L operator:

$$L_{a,k}^{(\frac{1}{2},s)}(\lambda) = \begin{pmatrix} \lambda \mathbb{1}_k + iS_k^z & iS_k^- \\ iS_k^+ & \lambda \mathbb{1}_k - iS_k^z \end{pmatrix}.$$

The auxiliary monodromy matrix:

$$T_a(\lambda) = [L_{a,L}^{(\frac{1}{2},s)}(\lambda) \cdots L_{a,1}^{(\frac{1}{2},s)}(\lambda)] = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix},$$

The transfer matrix:

$$t(\lambda) = \mathrm{tr}_a T_a(\lambda) = A(\lambda) + D(\lambda).$$

We have

$$[t(\lambda), t(\mu)] = 0, \quad [t(\lambda), \tau(\mu)] = 0.$$

Both of the two transfer matrices $\tau(\lambda)$ and $t(\lambda)$ act on the full quantum space and commute with each other.

Bethe state

By using the explicit form of the spin operators, one can find that for $s = -1$ the equations

$$S_k^+|\omega_k\rangle = 0, \quad S_k^z|\omega_k\rangle = -|\omega_k\rangle$$

have the solution $|\omega_k\rangle = 1/z_k^2$. This allows us to construct the pseudovacuum state as

$$|\Omega\rangle = (z_1^2 z_2^2 \ldots z_L^2)^{-1}.$$

Then the Bethe states for spin $s = -1$ are given in terms of operator B

$$|\hat{\varphi}_N(\{\lambda\})\rangle = B(\lambda_1)B(\lambda_2)\ldots B(\lambda_N)(z_1^2 z_2^2 \ldots z_L^2)^{-1}.$$

These are the eigenvectors of Lipatov's spin chain.

The corresponding Bethe equations for $s = -1$ determine the parameters $(\lambda_1, \dots, \lambda_N)$:

$$\left(\frac{\lambda_k + is}{\lambda_k - is}\right)^L = \prod_{\substack{j=1 \\ j\neq k}}^{N} \frac{\lambda_k - \lambda_j + i}{\lambda_k - \lambda_j - i}$$

$$\xrightarrow{s=-1} \left(\frac{\lambda_k - i}{\lambda_k + i}\right)^L = \prod_{\substack{j=1 \\ j\neq k}}^{N} \frac{\lambda_k - \lambda_j + i}{\lambda_k - \lambda_j - i}, \quad k = 1, \cdots, N.$$

For $s = -1$, the eigenvalues of the Hamiltonian

$$E \equiv \sum_{j=1}^{N} \frac{-1}{i} \frac{d}{d\lambda_j} \ln \frac{\lambda_j + i}{\lambda_j - i} = \sum_{j=1}^{N} \frac{2}{\lambda_j^2 + 1}.$$

The Bethe equations for lattice NLS model,

$$\left(\frac{1+i\lambda_k\Delta/2}{1-i\lambda_k\Delta/2}\right)^L=\prod_{j\neq k}^{N}\frac{\lambda_k-\lambda_j+i\kappa}{\lambda_k-\lambda_j-i\kappa}\iff\left(\frac{i(\frac{-2}{\kappa\Delta})\kappa+\lambda_k}{i(\frac{-2}{\kappa\Delta})\kappa-\lambda_k}\right)^L=\prod_{j\neq k}^{N}\frac{\lambda_k-\lambda_j+i\kappa}{\lambda_k-\lambda_j-i\kappa}.$$

When we take coupling constant $\kappa=1$, and $\Delta=2$, the Bethe equations become

$$\xRightarrow{\kappa=1,\Delta=2}(-1)^L\left(\frac{\lambda_k-i}{\lambda_k+i}\right)^L=\prod_{j\neq k}^{N}\frac{\lambda_k-\lambda_j+i}{\lambda_k-\lambda_j-i}.$$

One can calculate the ground state energy in thermodynamic limit $N\to\infty$:

$$E=L\varepsilon-\frac{\pi v}{6L}+\text{higher orgers in}\frac{1}{L}$$

Let us compare a classical random variable with quantum state.

1) Classical X outcomes are x_j with probabilities p_j $j = 1...N$. The entropy [information] $S = -\sum_{j=1}^{N} p_j \log p_j$. If the total entropy is $S = 0$ then there is no entropy in any subsystem.
2) Quantum: no so. The total entropy can be zero, but positive entropy in a subsystem: quantum fluctuation. Entanglement. Example is singlet [spin zero state] of two interacting spins $1/2$.
   Einstein, A; B Podolsky; N Rosen (1935)

Let us study entanglement entropy production in spin chains.
At positive temperature the thermo-fluctuation dominate quantum fluctuation. Dynamics of classical entropy was studied in 19 century by Rudolf Clausius.
https://en.wikipedia.org/wiki/Entropy_production]. A theory of classical shock waves shows that after local quench [explosion] the entropy is a linear function of time:

$$\mathcal{S}(t) = \frac{2\pi c}{3} T t$$

Here T is the temperature. The shock wave changes the density of the entropy.

At zero temperature the entropy is some function f of time,

$$\mathcal{S}(t) = f(t).$$

Conformal mapping leads us to the following expression for positive $T > 0$

$$\mathcal{S}_T(t) = f\left(\frac{v}{\pi T}\sinh\frac{\pi T}{v}(x + vt)\right),$$

where v is velocity.

Put $x = 0$ and consider the limit of large time. The result for the entanglement entropy at time t is

$$\mathcal{S}_T(t) = f\left(\exp[\pi T(t - t_0)]\right),$$

where t_0 is an inessential constant. Now we have two expressions for the entropy for positive temperature

$$\mathcal{S}_T(t) = \frac{2\pi c}{3} Tt = f\left(\exp[\pi T(t - t_0)]\right).$$

This means that we have found the function, and it is given by:

$$f(t) = \frac{2c}{3} \ln t + \text{constant.}$$

Each end of the block contributes equally, so for the local quench we get

$$\mathcal{S}(t) = \frac{c}{3} \ln t + \text{constant.}$$

It agrees with the entanglement entropy evolution after local quench described by Essler, Calabrese and Cardy.

The central charge can be extracted from finite size corrections to the ground state energy:

The entanglement entropy behaves logarithmically:

$$\mathcal{S}_{BFKL}(t) = \frac{c}{3} \ln \frac{t}{\tau} = \frac{1}{3} \ln(mt)$$

Here the m is the mass of the hadron. The central charge c of lattice nonlinear Schrödinger is the same as its continuous version [Lieb-Liniger] $c = 1$. It can extracted from the ground state enegry

$$E = L\varepsilon - \frac{c\pi v}{6L}$$

The $\mathcal{S}_{BFKL}(t)$ is actually a change of entanglement entropy caused by BFKL interaction. The total entropy is positive. Duration of BFKL phase is about $mt \sim 1/x$, after that the hadronization starts.

Physical Review D 104, L031503 (2021).
https://journals.aps.org/prd/abstract/10.1103/PhysRevD.104.L031503

Parton distributions can be defined in terms of the entropy of entanglement between the spatial region probed by deep inelastic scattering (DIS) and the rest of the proton. For very small x, the proton becomes a maximally entangled state [maximally mixed state]. This approach leads to a simple relation $S = \ln \mathcal{N}$ between the average number $\mathcal{N}$ of color-singlet dipoles in the proton wave function and the entropy of the produced hadronic state S. At small x, the multiplicity of dipoles is given by the gluon structure function, $\mathcal{N} = xG(x, Q^2)$. Recently, the H1 Collaboration analyzed the entropy of the produced hadronic state in DIS, and studied its relation to the gluon structure function; poor agreement with the predicted relation was found. In this letter we argue that a more accurate account of the number of color-singlet dipoles in the kinematics of H1 experiment (where hadrons are detected in the current fragmentation region) is given not by $xG(x, Q^2)$ but by the sea quark structure function $x\Sigma(x, Q^2)$. Sea quarks originate from the splitting of gluons, so at small x $x\Sigma(x, Q^2) \sim xG(x, Q^2)$, but in the current fragmentation region this proportionality is distorted by the contribution of the quark-antiquark pair produced by the virtual photon splitting. In addition, the multiplicity of color-singlet dipoles in the current fragmentation region is quite small, and one needs to include $\sim 1/N$ corrections to $S = \ln \mathcal{N}$ asymptotic formula. Taking both of these modifications into account, we find that the data from the H1 Collaboration in fact agree well with the prediction based on entanglement.

Gluon structure function

The average number of color-singlet dipoles:

$$\mathcal{N} = e^{S_{BFKL}} = xG(x)$$

Bjorken x represents the fraction of the nucleon's momentum carried by struck quark. It is the longitudinal momentum fraction of the nucleon carried by the struck quark. The hadron will come to maximally entangled state [maximally mixed stsate] at $mt \sim 1/x$ [then phase transition to hadronization starts]. [1] The gluon structure function growing at small Bjorken x as

$$xG(x) \sim \frac{1}{x^{1/3}}$$

One can try to fit HERA data to power of x. Our $1/3$ seems to fit into the range, but it is close to the boundary. We looking forward for EIC results.

[1] During BFKL phase the temperature rases from zero [when photon splits into color-singlet dipole] to Hagedorn temperature [the hadronization starts].

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