The massless single off-shell scalar box integral

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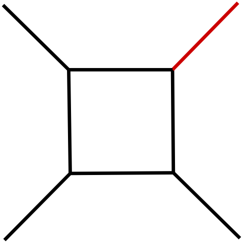


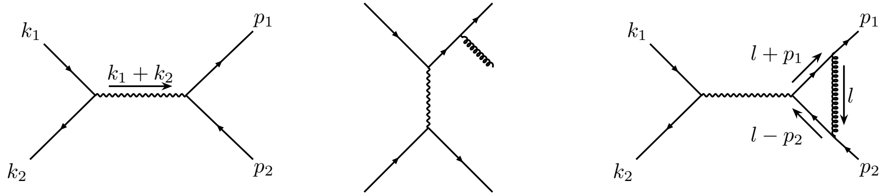
JH, Fabian Wunder, JHEP 02 (2023) 177, arXiv:2211.14110

Outline

1. Loop integrals and dimensional regularization
2. The massless single off-shell scalar box integral
3. Conclusion and Outlook

1. Loop integrals and dimensional regularization

Loop Integrals



↓

Feynman rules

↓

Mathematical expression for scattering amplitude

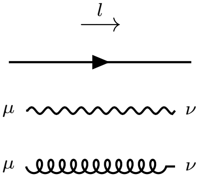
- Sum over all possible loop momenta with $\int d^4 l$

(One) Loop Integrals

Typical integrand:

- Denominator of the form

$$\prod_n \left[(l-p_n)^2 - m_n^2 + \mathrm{i}0\right]$$



$$= \frac{i(\not{l}-m)}{l^2-m^2+i0}$$

$$= \frac{-i\,g_{\mu\nu}}{l^2+i0}$$

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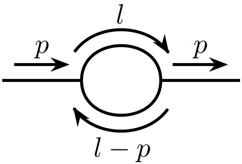
- Passarino-Veltman reduction of tensor one-loop integrals
  $\rightarrow$ Only need to evaluate scalar integrals of the form

$$\int \mathrm{d}^4 l \frac{1}{[l^2-m_1^2+\mathrm{i}0]\,[(l-p_2)^2-m_2^2+\mathrm{i}0]\ldots}$$

Dimensional Regularization



$$= \int d^4 l \, \frac{1}{[l^2 + i0]\,[(l-p)^2 + i0]}$$

- For large l, this integral behaves like

$$\int \frac{d^4 l}{l^4} \sim \ln(l) \to \text{diverges logarithmically for large } l$$

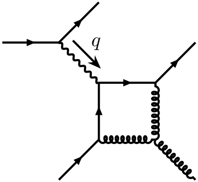
- Must parametrize these divergences (*regularization*) before adding all contributing diagrams
- If we set the number of space-time dimensions to $d = 4 - 2\varepsilon$, then

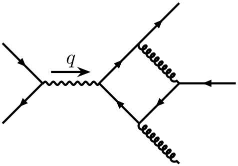
$$\int \frac{d^4 l}{l^4} \to \int \frac{d^{4-2\varepsilon} l}{l^4} \sim \frac{1}{l^{2\varepsilon}} \to \text{finite for large } l \ \ (\text{and } \varepsilon > 0)$$

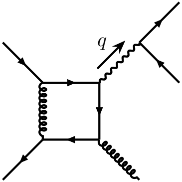
- divergences manifest themselves as poles for $\varepsilon \to 0$

2. The massless single off-shell scalar box integral

Motivation

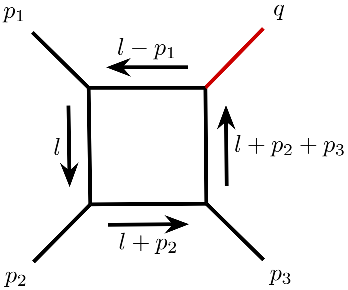






- NNLO processes DIS, SIA, DY feature single off-shell boson
- Light quark masses negligible in high energy limit
- Passarino-Veltman reduction of tensor one loop integrals to scalar integrals
- All-order ε-expansion valid in all kinematic regions (DIS $q^2 < 0$, SIA & DY $q^2 > 0$; DIS & DY $s > 0$ & $t, u < 0$, SIA $s, t, u > 0$)
- Explicitly give real and imaginary parts

The scalar box integral in dimensional regularization



- massless propagators
- $q^2 \neq 0,\ p_i^2 = 0$
- Dimensional regularization with $d = 4 - 2\varepsilon$
- Keep causal $+i0$ throughout

$$\mathrm{D}_0 \equiv \frac{\mu^{4-d}}{i\pi^{d/2}} \int d^d l \, \frac{1}{[l^2 + i0]\,[(l+p_2)^2 + i0]\,[(l+p_2+p_3)^2 + i0]\,[(l-p_1)^2 + i0]}$$

Feynman parametrization

$$\mathrm{D}_0 \equiv \frac{\mu^{4-d}}{\mathrm{i}\pi^{d/2}} \int \mathrm{d}^d l \, \frac{1}{[l^2 + \mathrm{i}0]\,[(l+p_2)^2 + \mathrm{i}0]\,[(l+p_2+p_3)^2 + \mathrm{i}0]\,[(l-p_1)^2 + \mathrm{i}0]}$$

- Apply *Feynman parametrization*

$$\frac{1}{abcd} = 6 \int_0^1 \mathrm{d}\alpha_1\, \mathrm{d}\alpha_2\, \mathrm{d}\alpha_3\, \mathrm{d}\alpha_4 \, \frac{\delta(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)}{[a\alpha_1 + b\alpha_2 + c\alpha_3 + d\alpha_4]^4}$$

- Shift loop momentum $\rightarrow$

$$\mathrm{D}_0 = \mu^{4-d} \int_0^1 \mathrm{d}\alpha_1\, \mathrm{d}\alpha_2\, \mathrm{d}\alpha_3\, \mathrm{d}\alpha_4 \, \delta(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)$$
$$\times \frac{1}{\mathrm{i}\pi^{d/2}} \int \mathrm{d}^d l' \, \frac{6}{[l'^2 + s_1\alpha_1(\alpha_2+\alpha_3) + s_2\alpha_3(\alpha_1+\alpha_4) + s_3\alpha_1\alpha_3 + \mathrm{i}0]^4}$$

with Mandelstam variables

$$s_1 \equiv (p_1+p_2)^2, \quad s_2 \equiv (p_2+p_3)^2, \quad s_3 \equiv (p_1+p_3)^2$$

Feynman parameter integral

$$D_0 = \mu^{4-d} \int_0^1 \mathrm{d}\alpha_1\, \mathrm{d}\alpha_2\, \mathrm{d}\alpha_3\, \mathrm{d}\alpha_4\, \delta(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4) \times \frac{1}{\mathrm{i}\pi^{d/2}} \int \mathrm{d}^d l' \frac{6}{[l'^2 + s_1\alpha_1(\alpha_2+\alpha_3) + s_2\alpha_3(\alpha_1+\alpha_4) + s_3\alpha_1\alpha_3 + \mathrm{i}0]^4}$$

- Evaluate l'-integral using

$$\frac{1}{\mathrm{i}\pi^{d/2}} \int \mathrm{d}^d l \frac{1}{[l^2 - \Delta + \mathrm{i}0]^\beta} = (-1)^\beta \frac{\Gamma(\beta - d/2)}{\Gamma(\beta)} (\Delta - \mathrm{i}0)^{d/2-\beta}$$

- Leads to

$$D_0 = \mu^{2\varepsilon}\, \Gamma(2+\varepsilon) \int_0^1 \frac{\mathrm{d}\alpha_1\, \mathrm{d}\alpha_2\, \mathrm{d}\alpha_3\, \mathrm{d}\alpha_4\, \delta(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)}{[-s_1\alpha_1(\alpha_2+\alpha_3) - s_2\alpha_3(\alpha_1+\alpha_4) - s_3\alpha_1\alpha_3 - \mathrm{i}0]^{2+\varepsilon}}$$

Decoupling of Feynman parameter integrals

$$D_0 \;=\; \mu^{2\varepsilon}\,\Gamma(2+\varepsilon)\int_0^1 \frac{\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3\,\mathrm{d}\alpha_4\,\delta(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)}{\left[-s_1\alpha_1(\alpha_2+\alpha_3)-s_2\alpha_3(\alpha_1+\alpha_4)-s_3\alpha_1\alpha_3-\mathrm{i}0\right]^{2+\varepsilon}}$$

Decouple Feynman parameter integrals through [Smirnov, 2012]

$$\alpha_1 \;\to\; \eta_1\xi_1\,,\quad \alpha_4 \;\to\; \eta_1(1-\xi_1)\,,\quad \alpha_3 \;\to\; \eta_2\xi_2\,,\quad \alpha_2 \;\to\; \eta_2(1-\xi_2)$$

Evaluate η-integrals in terms of Gamma functions, factor out $s_1 s_2/s_3 \to$

$$D_0\left(s_1,s_2,q^2\right) \;=\; \mu^{2\varepsilon}\,\frac{\Gamma(2+\varepsilon)\Gamma^2(-\varepsilon)}{\Gamma(-2\varepsilon)}\left(\frac{s_1 s_2}{s_3}-\mathrm{i}0\right)^{-\varepsilon-2}$$
$$\times\int_0^1 \mathrm{d}\xi_1\int_0^1 \mathrm{d}\xi_2\,\left[x_2\xi_1+x_1\xi_2-x_1x_2\xi_1\xi_2-\mathrm{i}0\,\mathrm{sgn}\!\left(\frac{s_3}{s_1 s_2}\right)\right]^{-\varepsilon-2}$$

Depends on only 2 dimensionless variables

$$x_1 \;\equiv\; -\frac{s_3}{s_1}\,,\quad x_2 \;\equiv\; -\frac{s_3}{s_2}$$

Epsilon expansion of the scalar box integral

$$D_0\left(s_1, s_2, q^2\right) = \frac{1}{\varepsilon^2} \frac{\Gamma(1+\varepsilon)\,\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \frac{2}{s_1 s_2} \left|\frac{s_3 \mu^2}{s_1 s_2}\right|^{\varepsilon}$$

$$\times \Big[\left(\Theta(-s_2) + \Theta(s_2) e^{i\pi\varepsilon}\right) \mathfrak{F}(\varepsilon; x_1) + \left(\Theta(-s_1) + \Theta(s_1) e^{i\pi\varepsilon}\right) \mathfrak{F}(\varepsilon; x_2)$$

$$-\left(\Theta(-q^2) + \Theta(q^2) e^{i\pi\varepsilon}\right) \mathfrak{F}(\varepsilon; x_1 + x_2 - x_1 x_2)\Big]$$

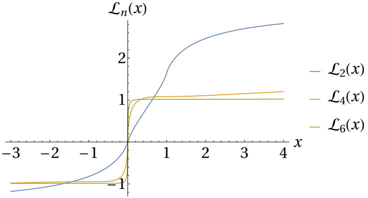
where

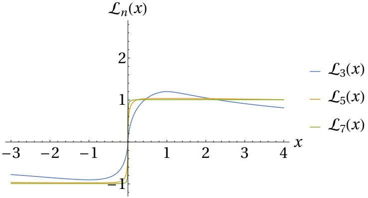
$$\mathfrak{F}(\varepsilon; x) \equiv 1 + \ln\left|\frac{x}{1-x}\right| \sum_{n=1}^{\infty} \frac{(-\varepsilon)^n}{n!} \ln^{n-1}|x| - \sum_{n=2}^{\infty} \varepsilon^n \mathcal{L}_n(x)$$

- Diverges like $1/\varepsilon^2$ for $\varepsilon \to 0$ (box integral itself is no measurable quantity)
- Real and imaginary parts can be easily read off
- All-order ε-expansion
- Valid in all kinematic regions ($s_1, s_2, q^2 \in \mathbb{R}$)

Single-valued polylogarithms

$$\mathcal{L}_n(x) \equiv \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \ln^k |x| \, \mathrm{Li}_{n-k}(x) + \frac{(-1)^{n-1}}{n!} \ln^{n-1} |x| \ln |1-x|$$





- $\mathcal{L}_n(x)$ is single-valued, in contrast to $\mathrm{Li}_n(x)$
- $\mathcal{L}_n(x)$ is continuous for all $x \in \mathbb{R}$
- $\mathcal{L}_n(x)$ is bounded on $\mathbb{R}$, in contrast to $\mathrm{Li}_n(x)$
- $\mathcal{L}_n(x)$ satisfies *clean* versions of the functional equations of $\mathrm{Li}_n(x)$, i.e. without product terms

3. Conclusion and Outlook

Conclusion and Outlook

- All-order ε-expansion of massless single off-shell scalar box integral in terms of newly defined single-valued polylogarithms
- Real and imaginary parts to all orders in ε

- Result can be generalized to two non-adjacent endpoints off-shell [J. Haug and F. Wunder, 2023]
- Additional methods needed for further generalization of results
- Starting point: one-fold integral representation of general box integral with massless propagators [Tarasov, 2019]

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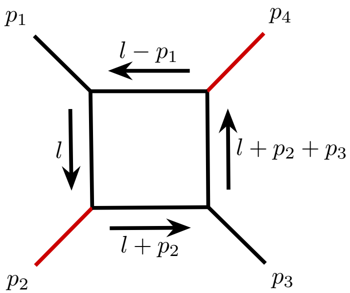
Backup slides

4. Generalization to two non-adjacent off-shell external particles

5. The massless scalar box integral in the literature

6. Detailed calculation of the scalar box integral

4. Generalization to two non-adjacent off-shell external particles



JH, Fabian Wunder, JHEP 05 (2023) 059, arXiv:2302.01956

Generalization to two non-adjacent off-shell external particles

General box integral with massless propagators after Feynman parametrization:

$$D_0 = \mu^{2\varepsilon}\,\Gamma(2+\varepsilon) \times \int_0^1 \frac{\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}x_4\,\delta(1-x_1-x_2-x_3-x_4)}{\left[-x_1x_2s_1 - x_1x_3p_4^2 - x_1x_4p_1^2 - x_2x_3p_3^2 - x_2x_4p_2^2 - x_3x_4s_2 - \mathrm{i}0\right]^{2+\varepsilon}}$$

With the same substitution as before,

$$x_1 = \eta_1\xi_1\,,\quad x_2 = \eta_2(1-\xi_2)\,,\quad x_3 = \eta_2\xi_2\,,\quad x_4 = \eta_1(1-\xi_1)$$

term in denominator becomes

$$-\eta_1\eta_2(1-\xi_1)(1-\xi_2)\,s_1 - \eta_1\eta_2\xi_1\xi_2\,s_2 - \eta_1\eta_2\xi_1(1-\xi_2)\,p_2^2 - \eta_1\eta_2\xi_2(1-\xi_1)\,p_4^2 - \eta_1^2\xi_1(1-\xi_1)\,p_1^2 - \eta_2^2\xi_2(1-\xi_2)\,p_3^2 - \mathrm{i}0$$

- Factorization of η- and ξ-integrals if $p_1^2 = p_3^2 = 0$
- Proceed analogously to single off-shell case for two non-adjacent external particles off light cone

Result in terms of 4 hypergeometric functions

$$
\begin{aligned}
D_0\left(s_1, s_2, 0, p_2^2, 0, p_4^2\right) = {} & \frac{1}{\varepsilon^2} \frac{\Gamma(1+\varepsilon)\,\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \frac{2}{s_1 s_2 - p_2^2 p_4^2} \\
& \times \left\{ \left[\frac{\mu^2}{-s_1 - i0}\right]^{\varepsilon} {}_2F_1\left(1, -\varepsilon, 1-\varepsilon; -\frac{(p_2^2 + p_4^2 - s_1 - s_2)s_1}{s_1 s_2 - p_2^2 p_4^2} + i\tilde{0}\,\mathrm{sgn}(s_1 - s_2)\right)\right. \\
& + \left[\frac{\mu^2}{-s_2 - i0}\right]^{\varepsilon} {}_2F_1\left(1, -\varepsilon, 1-\varepsilon; -\frac{(p_2^2 + p_4^2 - s_1 - s_2)s_2}{s_1 s_2 - p_2^2 p_4^2} + i\tilde{0}\,\mathrm{sgn}(s_2 - s_1)\right) \\
& - \left[\frac{\mu^2}{-p_2^2 - i0}\right]^{\varepsilon} {}_2F_1\left(1, -\varepsilon, 1-\varepsilon; -\frac{(p_2^2 + p_4^2 - s_1 - s_2)p_2^2}{s_1 s_2 - p_2^2 p_4^2} + i\tilde{0}\,\mathrm{sgn}\left(p_4^2 - p_2^2\right)\right) \\
& \left. - \left[\frac{\mu^2}{-p_4^2 - i0}\right]^{\varepsilon} {}_2F_1\left(1, -\varepsilon, 1-\varepsilon; -\frac{(p_2^2 + p_4^2 - s_1 - s_2)p_4^2}{s_1 s_2 - p_2^2 p_4^2} + i\tilde{0}\,\mathrm{sgn}\left(p_2^2 - p_4^2\right)\right)\right\}
\end{aligned}
$$

- Reduces to single off-shell integral in limits $p_2^2, p_4^2 \to 0$
- Similar result found via functional equations in [Tarasov, 2019]

Epsilon expansion in the non-adjacent double off-shell case

$$D_0\left(s_1, s_2, p_2^2, p_4^2\right) = \frac{1}{\varepsilon^2}\frac{\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)}\frac{2}{s_1 s_2 - p_2^2 p_4^2}\left|\frac{(p_2^2+p_4^2-s_1-s_2)\mu^2}{s_1 s_2 - p_2^2 p_4^2}\right|^{\varepsilon}$$
$$\times\Bigg\{\left(\Theta(-s_1)+\Theta(s_1)e^{i\pi\varepsilon}\right)\mathfrak{F}\left(\varepsilon; -\frac{(p_2^2+p_4^2-s_1-s_2)s_1}{s_1 s_2 - p_2^2 p_4^2}\right)$$
$$+\left(\Theta(-s_2)+\Theta(s_2)e^{i\pi\varepsilon}\right)\mathfrak{F}\left(\varepsilon; -\frac{(p_2^2+p_4^2-s_1-s_2)s_2}{s_1 s_2 - p_2^2 p_4^2}\right)$$
$$-\left(\Theta(-p_2^2)+\Theta(p_2^2)e^{i\pi\varepsilon}\right)\mathfrak{F}\left(\varepsilon; -\frac{(p_2^2+p_4^2-s_1-s_2)p_2^2}{s_1 s_2 - p_2^2 p_4^2}\right)$$
$$-\left(\Theta(-p_4^2)+\Theta(p_4^2)e^{i\pi\varepsilon}\right)\mathfrak{F}\left(\varepsilon; -\frac{(p_2^2+p_4^2-s_1-s_2)p_4^2}{s_1 s_2 - p_2^2 p_4^2}\right)\Bigg\}$$

- All-order ε-expansion (not previously known)
- Real and imaginary parts can be easily read off
- Valid in all kinematic regions $(s_1, s_2, p_2^2, p_4^2 \in \mathbb{R})$
- No spurious branch cuts

5. The massless scalar box integral in the literature

The massless scalar box integral in the literature

Massless single-off shell case:

- K. Fabricius and I. Schmitt [1979]: Box integral with explicit imaginary part up to $\mathcal{O}(\varepsilon^0)$
- Matsuura et al. [1989]: Result in terms of 3 Gauss hypergeometric functions for general d
- Bern et al. [1994]: Rule for analytic continuation of result up to $\mathcal{O}(\varepsilon^0)$
- G. Duplančić and B. Nižić [2001]: Systematically keep causal $+i0$, result up to $\mathcal{O}(\varepsilon^0)$
- Lyubovitskij et al. [2021]: All-order ε-expansion, branch cuts not discussed

Non-adjacent double off-shell case:

- Tarasov [2019]: Functional equation approach yields result in terms of 4 Gauss hypergeometric functions, expansion up to $\mathcal{O}(\varepsilon^0)$

Comparison to literature

Combine prefactors $(\ldots)^{\varepsilon}$ and $[\ldots]^{-\varepsilon}$ $\rightarrow$

$$
\begin{aligned}
D_0\left(s_1, s_2, q^2\right) = \; & \frac{1}{\varepsilon^2} \frac{\Gamma(1+\varepsilon)\,\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \frac{2}{s_1 s_2} \\
& \times \left\{ \left[\frac{\mu^2}{-s_2 - \mathrm{i}0}\right]^{\varepsilon} {}_2\mathrm{F}_1\left(1, -\varepsilon, 1-\varepsilon; -\frac{s_3}{s_1} + \mathrm{i}\tilde{0}\,\mathrm{sgn}\left(\frac{s_3}{s_2} - \frac{s_3}{s_1}\right)\right) \right. \\
& \quad + \left[\frac{\mu^2}{-s_1 - \mathrm{i}0}\right]^{\varepsilon} {}_2\mathrm{F}_1\left(1, -\varepsilon, 1-\varepsilon; -\frac{s_3}{s_2} + \mathrm{i}\tilde{0}\,\mathrm{sgn}\left(\frac{s_3}{s_1} - \frac{s_3}{s_2}\right)\right) \\
& \quad \left. - \left[\frac{\mu^2}{-q^2 - \mathrm{i}0}\right]^{\varepsilon} {}_2\mathrm{F}_1\left(1, -\varepsilon, 1-\varepsilon; -\frac{s_3 q^2}{s_1 s_2} + \mathrm{i}\tilde{0}\right) \right\}
\end{aligned}
$$

- Agrees with [Matsuura et al., 1989] and [Lyubovitskij et al., 2021] if $s_1, s_2, q^2 < 0$ and all three hypergeometric functions are away from their branch cut
- Agrees with [Bern et al., 1994] if all three hypergeometric functions are away from their branch cut

6. Detailed calculation of the scalar box integral

Factoring out $s_1 s_2 / s_3$

Use

$$(a - \mathrm{i}0)^{\alpha} = (b - \mathrm{i}0)^{\alpha} \left(\frac{a}{b} - \mathrm{i}0 \,\mathrm{sgn}(b) \right)^{\alpha}, \quad \text{where } a \in \mathbb{R}\,,\ b \in \mathbb{R} \backslash \{0\}\,,\ \alpha \in \mathbb{C}\,,$$

to factor out $s_1 s_2 / s_3 \to$

$$D_0\left(s_1, s_2, q^2\right) = \mu^{2\varepsilon} \frac{\Gamma(2+\varepsilon)\Gamma^2(-\varepsilon)}{\Gamma(-2\varepsilon)} \left(\frac{s_1 s_2}{s_3} - \mathrm{i}0 \right)^{-\varepsilon-2}$$
$$\times \int_0^1 \mathrm{d}\xi_1 \int_0^1 \mathrm{d}\xi_2 \left[x_2 \xi_1 + x_1 \xi_2 - x_1 x_2 \xi_1 \xi_2 - \mathrm{i}0 \,\mathrm{sgn}\left(\frac{s_3}{s_1 s_2} \right) \right]^{-\varepsilon-2},$$

depends on only 2 dimensionless variables

$$x_1 \equiv -\frac{s_3}{s_1}\,, \quad x_2 \equiv -\frac{s_3}{s_2}$$

Several substitutions & evaluating 1 integral & splitting of integrals $\rightarrow$

$$D_0\left(s_1, s_2, q^2\right) = -\frac{1}{\varepsilon}\frac{\Gamma(1+\varepsilon)\,\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)}\,\frac{2}{s_1 s_2}\left(\frac{s_3\mu^2}{s_1 s_2} + \mathrm{i}0\right)^{\varepsilon} \times \left\{I(x_1) + I(x_2) - I(1-(1-x_1)(1-x_2))\right\}, \qquad (1)$$

where

$$I(\chi) \equiv \int_0^{\chi} \frac{\mathrm{d}\zeta}{1-\zeta}\left(\left[\zeta - \mathrm{i}0\,\mathrm{sgn}_{123}\right]^{-\varepsilon-1} - 1\right),$$

with abbreviation

$$\mathrm{sgn}_{123} \equiv \mathrm{sgn}\left(\frac{s_3}{s_1 s_2}\right)$$

Note

$$1-(1-x_1)(1-x_2) = -\frac{s_3 q^2}{s_1 s_2} \xrightarrow{q^2\to 0} 0$$

$I(\chi)$ vanishes for $\chi = 0 \rightarrow$ set third integral to 0 in case $q^2 = 0$

Calculating $I(\chi)$

In case $\chi \neq 0$, substitute $\zeta \to \chi^{-1}\zeta \to$

$$I(\chi) = \int_0^1 \frac{\mathrm{d}\zeta}{1-\chi\zeta} \left([\chi - \mathrm{i}0\,\mathrm{sgn}_{123}]^{-\varepsilon}\, \zeta^{-\varepsilon-1} - \chi \right)$$

Denominator $1-\chi\zeta$ diverges for $\chi > 1$

$\to$ Introduce regulator $\chi \to \chi + \mathrm{i}\tilde{0}$ to split integral in two

$$\begin{aligned} I(\chi) &= [\chi - \mathrm{i}0\,\mathrm{sgn}_{123}]^{-\varepsilon} \int_0^1 \mathrm{d}\zeta\, \zeta^{-\varepsilon-1} \left(1 - (\chi + \mathrm{i}\tilde{0})\zeta\right)^{-1} - \int_0^1 \mathrm{d}\zeta\, \frac{\chi}{1-(\chi+\mathrm{i}\tilde{0})\zeta} \\ &= -\frac{1}{\varepsilon} [\chi - \mathrm{i}0\,\mathrm{sgn}_{123}]^{-\varepsilon}\, {}_2F_1\left(1, -\varepsilon, 1-\varepsilon; \chi + \mathrm{i}\tilde{0}\right) + \ln(1 - \chi - \mathrm{i}\tilde{0}) \end{aligned}$$

- Both ${}_2F_1$ and ln evaluated on branch cut for $\chi > 1$
- Branch cuts are spurious and must cancel

Cancellation of spurious branch cuts

Add integrals in eq. (1),

$$D_0\left(s_1, s_2, q^2\right) = -\frac{1}{\varepsilon}\frac{\Gamma(1+\varepsilon)\,\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)}\,\frac{2}{s_1 s_2}\left(\frac{s_3\mu^2}{s_1 s_2}+\mathrm{i}0\right)^{\varepsilon} \times \{I(x_1) + I(x_2) - I(1-(1-x_1)(1-x_2))\}\,,$$

use different regulator $\mathrm{i}\tilde{0}_i$ for each integral

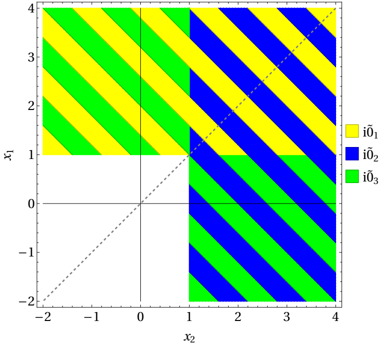
Sum $I(x_1) + I(x_2) - I(1-(1-_1)(1-x_2))$ contains following logarithms

$$\ln(1-x_1-\mathrm{i}\tilde{0}_1) + \ln(1-x_2-\mathrm{i}\tilde{0}_2) - \ln\big((1-x_1)(1-x_2)-\mathrm{i}\tilde{0}_3\big)$$

- Real parts cancel
- Choose signs of $\mathrm{i}\tilde{0}_i$ such that imaginary parts cancel as well

Cancellation of imaginary parts

$$i\pi \left[-\,\mathrm{sgn}\left(\tilde{0}_1\right) \Theta(x_1-1) - \mathrm{sgn}\left(\tilde{0}_2\right) \Theta(x_2-1) + \mathrm{sgn}\left(\tilde{0}_3\right) \left\{\Theta(x_1-1)\,\Theta(1-x_2) + \Theta(1-x_1)\,\Theta(x_2-1)\right\}\right] \stackrel{!}{=} 0 \qquad (2)$$



- Conditions for eq. (2) to hold:

$$\mathrm{sgn}\left(\tilde{0}_1\right) \stackrel{!}{=} \mathrm{sgn}\left(\tilde{0}_3\right) \quad \text{(yellow-green region)}$$

$$\mathrm{sgn}\left(\tilde{0}_1\right) \stackrel{!}{=} -\,\mathrm{sgn}\left(\tilde{0}_2\right) \quad \text{(yellow-blue region)}$$

$$\mathrm{sgn}\left(\tilde{0}_2\right) \stackrel{!}{=} \mathrm{sgn}\left(\tilde{0}_3\right) \quad \text{(blue-green region)}$$

- Choose (only relative signs matter)

$$i\tilde{0}_1 \equiv i\tilde{0}\,\mathrm{sgn}(x_1-x_2)$$

$$i\tilde{0}_2 \equiv i\tilde{0}\,\mathrm{sgn}(x_2-x_1)$$

$$i\tilde{0}_3 \equiv i\tilde{0}$$

Result in terms of hypergeometric functions

$$\begin{aligned}
D_0\left(s_1, s_2, q^2\right) = \; & \frac{1}{\varepsilon^2} \frac{\Gamma(1+\varepsilon)\,\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \frac{2}{s_1 s_2} \left(\frac{s_3 \mu^2}{s_1 s_2} + \mathrm{i}0\right)^{\varepsilon} \\
& \times \left\{ \left[-\frac{s_3}{s_1} - \mathrm{i}0\,\mathrm{sgn}\left(\frac{s_3}{s_1 s_2}\right)\right]^{-\varepsilon} {}_2F_1\left(1, -\varepsilon, 1-\varepsilon;\; -\frac{s_3}{s_1} + \mathrm{i}\tilde{0}\,\mathrm{sgn}\left(\frac{s_3}{s_2} - \frac{s_3}{s_1}\right)\right)\right. \\
& + \left[-\frac{s_3}{s_2} - \mathrm{i}0\,\mathrm{sgn}\left(\frac{s_3}{s_1 s_2}\right)\right]^{-\varepsilon} {}_2F_1\left(1, -\varepsilon, 1-\varepsilon;\; -\frac{s_3}{s_2} + \mathrm{i}\tilde{0}\,\mathrm{sgn}\left(\frac{s_3}{s_1} - \frac{s_3}{s_2}\right)\right) \\
& \left. - \left[-\frac{s_3 q^2}{s_1 s_2} - \mathrm{i}0\,\mathrm{sgn}\left(\frac{s_3}{s_1 s_2}\right)\right]^{-\varepsilon} {}_2F_1\left(1, -\varepsilon, 1-\varepsilon;\; -\frac{s_3 q^2}{s_1 s_2} + \mathrm{i}\tilde{0}\right)\right\} \qquad (3)
\end{aligned}$$

- Imaginary parts of hypergeometric functions will cancel by construction
- Last term vanishes for $q^2 = 0$