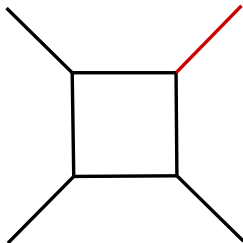


The massless single off-shell scalar box integral

Juliane Haug

Universität Tübingen

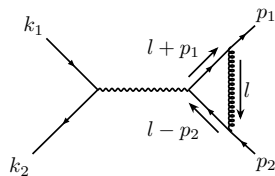
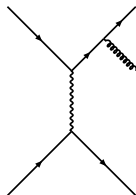
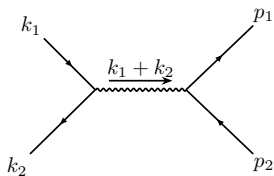
CFNS-CTEQ Summer School, June 2023



1. Loop integrals and dimensional regularization
2. The massless single off-shell scalar box integral
3. Conclusion and Outlook

1. Loop integrals and dimensional regularization

Loop Integrals



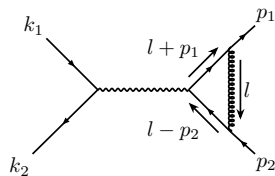
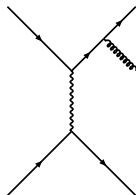
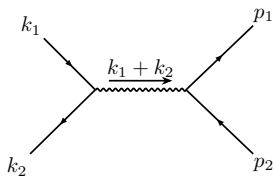
Feynman rules



Mathematical expression for scattering amplitude

- ▶ Sum over all possible loop momenta with $\int d^4l$

Loop Integrals



Feynman rules



Mathematical expression for scattering amplitude

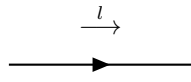
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(One) Loop Integrals

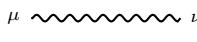
Typical integrand:

- Denominator of the form

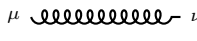
$$\prod_n \left[(l - p_n)^2 - m_n^2 + i0 \right]$$



$$= \frac{i(l - m)}{l^2 - m^2 + i0}$$



$$= \frac{-i g_{\mu\nu}}{l^2 + i0}$$



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- Passarino-Veltman reduction of tensor one-loop integrals
→ Only need to evaluate scalar integrals of the form

$$\int d^4 l \frac{1}{[l^2 - m_1^2 + i0] [(l - p_2)^2 - m_2^2 + i0] \dots}$$

Dimensional Regularization

The diagram shows a bubble loop with two external lines. The left external line has momentum p pointing right. The right external line has momentum p pointing right. The top arc of the loop has momentum l pointing clockwise. The bottom arc has momentum $l - p$ pointing clockwise.

$$= \int d^4 l \frac{1}{[l^2 + i0] [(l - p)^2 + i0]}$$

- ▶ For large l , this integral behaves like

$$\int \frac{d^4 l}{l^4} \sim \ln(l) \rightarrow \text{diverges logarithmically for large } l$$

- ▶ Must parametrize these divergences (*regularization*) before adding all contributing diagrams
- ▶ If we set the number of space-time dimensions to $d = 4 - 2\varepsilon$, then

$$\int \frac{d^4 l}{l^4} \rightarrow \int \frac{d^{4-2\varepsilon} l}{l^4} \sim \frac{1}{l^{2\varepsilon}} \rightarrow \text{finite for large } l \text{ (and } \varepsilon > 0)$$

- ▶ divergences manifest themselves as poles for $\varepsilon \rightarrow 0$

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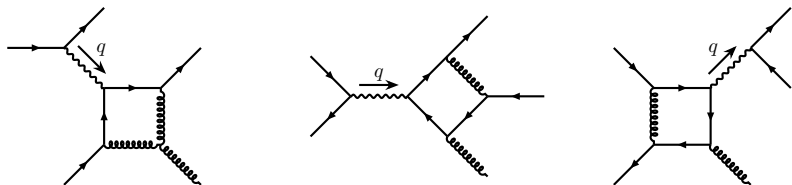
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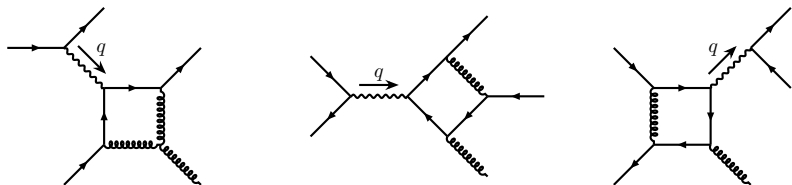
2. The massless single off-shell scalar box integral

Motivation



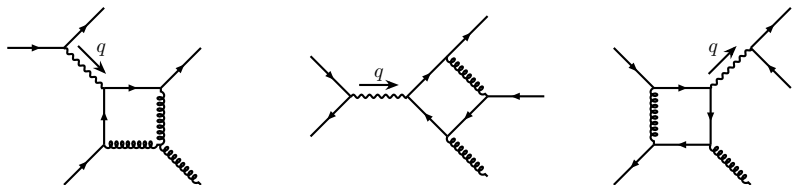
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- ▶ Light quark masses negligible in high energy limit
- ▶ Passarino-Veltman reduction of tensor one loop integrals to scalar integrals
- ▶ All-order ε -expansion valid in all kinematic regions (DIS $q^2 < 0$, SIA & DY $q^2 > 0$; DIS & DY $s > 0$ & $t, u < 0$, SIA $s, t, u > 0$)
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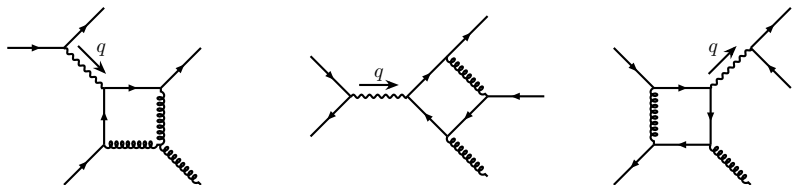
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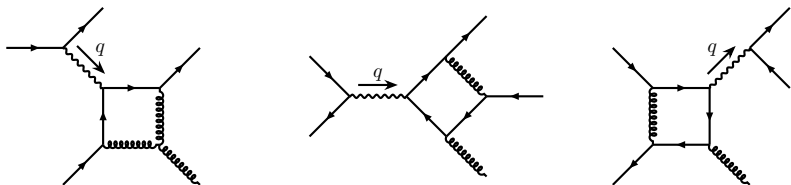
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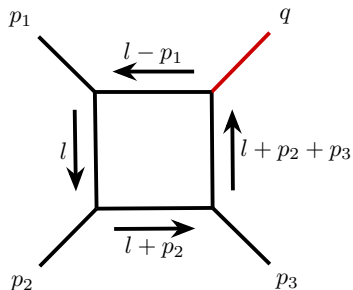
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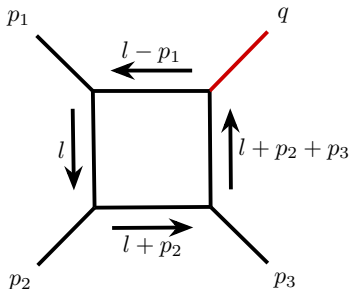
The scalar box integral in dimensional regularization



- ▶ massless propagators
- ▶ $q^2 \neq 0, p_i^2 = 0$
- ▶ Dimensional regularization with $d = 4 - 2\epsilon$
- ▶ Keep causal $+i0$ throughout

$$D_0 \equiv \frac{\mu^{4-d}}{i\pi^{d/2}} \int d^d l \frac{1}{[l^2 + i0] [(l + p_2)^2 + i0] [(l + p_2 + p_3)^2 + i0] [(l - p_1)^2 + i0]}$$

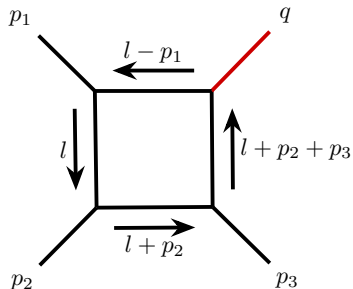
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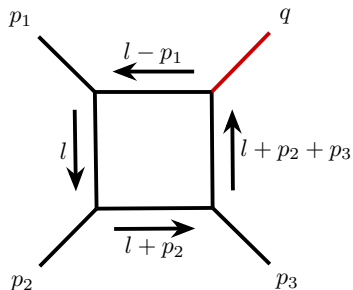
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Feynman parametrization

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- Apply *Feynman parametrization*

$$\frac{1}{abcd} = 6 \int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4 \frac{\delta(1 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4)}{[a\alpha_1 + b\alpha_2 + c\alpha_3 + d\alpha_4]^4}$$

- Shift loop momentum \rightarrow

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with Mandelstam variables

$$s_1 \equiv (p_1 + p_2)^2, \quad s_2 \equiv (p_2 + p_3)^2, \quad s_3 \equiv (p_1 + p_3)^2$$

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- Evaluate l' -integral using

$$\frac{1}{i\pi^{d/2}} \int d^d l' \frac{1}{[l'^2 - \Delta + i0]^\beta} = (-1)^\beta \frac{\Gamma(\beta - d/2)}{\Gamma(\beta)} (\Delta - i0)^{d/2 - \beta}$$

- Leads to

$$D_0 = \mu^{2\epsilon} \Gamma(2 + \epsilon) \int_0^1 \frac{d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4 \delta(1 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4)}{[-s_1\alpha_1(\alpha_2 + \alpha_3) - s_2\alpha_3(\alpha_1 + \alpha_4) - s_3\alpha_1\alpha_3 - i0]^{2+\epsilon}}$$

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Decoupling of Feynman parameter integrals

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Decouple Feynman parameter integrals through [Smirnov, 2012]

$$\alpha_1 \rightarrow \eta_1 \xi_1, \quad \alpha_4 \rightarrow \eta_1(1 - \xi_1), \quad \alpha_3 \rightarrow \eta_2 \xi_2, \quad \alpha_2 \rightarrow \eta_2(1 - \xi_2)$$

Evaluate η -integrals in terms of Gamma functions, factor out $s_1 s_2 / s_3 \rightarrow$

$$D_0(s_1, s_2, q^2) = \mu^{2\epsilon} \frac{\Gamma(2 + \epsilon) \Gamma^2(-\epsilon)}{\Gamma(-2\epsilon)} \left(\frac{s_1 s_2}{s_3} - i0 \right)^{-\epsilon-2} \\ \times \int_0^1 d\xi_1 \int_0^1 d\xi_2 \left[x_2 \xi_1 + x_1 \xi_2 - x_1 x_2 \xi_1 \xi_2 - i0 \operatorname{sgn} \left(\frac{s_3}{s_1 s_2} \right) \right]^{-\epsilon-2}$$

Depends on only 2 dimensionless variables

$$x_1 \equiv -\frac{s_3}{s_1}, \quad x_2 \equiv -\frac{s_3}{s_2}$$

Decoupling of Feynman parameter integrals

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Epsilon expansion of the scalar box integral

$$\begin{aligned}
 D_0(s_1, s_2, q^2) &= \frac{1}{\varepsilon^2} \frac{\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \frac{2}{s_1 s_2} \left| \frac{s_3 \mu^2}{s_1 s_2} \right|^\varepsilon \\
 &\times \left[(\Theta(-s_2) + \Theta(s_2)e^{i\pi\varepsilon}) \mathfrak{F}(\varepsilon; x_1) + (\Theta(-s_1) + \Theta(s_1)e^{i\pi\varepsilon}) \mathfrak{F}(\varepsilon; x_2) \right. \\
 &\quad \left. - (\Theta(-q^2) + \Theta(q^2)e^{i\pi\varepsilon}) \mathfrak{F}(\varepsilon; x_1 + x_2 - x_1 x_2) \right]
 \end{aligned}$$

where

$$\mathfrak{F}(\varepsilon; x) \equiv 1 + \ln \left| \frac{x}{1-x} \right| \sum_{n=1}^{\infty} \frac{(-\varepsilon)^n}{n!} \ln^{n-1} |x| - \sum_{n=2}^{\infty} \varepsilon^n \mathcal{L}_n(x)$$

- ▶ Diverges like $1/\varepsilon^2$ for $\varepsilon \rightarrow 0$ (box integral itself is no measurable quantity)
- ▶ Real and imaginary parts can be easily read off
- ▶ All-order ε -expansion
- ▶ Valid in all kinematic regions ($s_1, s_2, q^2 \in \mathbb{R}$)

Epsilon expansion of the scalar box integral

$$D_0(s_1, s_2, q^2) = \frac{1}{\varepsilon^2} \frac{\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \frac{2}{s_1 s_2} \left| \frac{s_3 \mu^2}{s_1 s_2} \right|^\varepsilon \\ \times \left[(\Theta(-s_2) + \Theta(s_2)e^{i\pi\varepsilon}) \mathfrak{F}(\varepsilon; x_1) + (\Theta(-s_1) + \Theta(s_1)e^{i\pi\varepsilon}) \mathfrak{F}(\varepsilon; x_2) \right. \\ \left. - (\Theta(-q^2) + \Theta(q^2)e^{i\pi\varepsilon}) \mathfrak{F}(\varepsilon; x_1 + x_2 - x_1 x_2) \right]$$

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Epsilon expansion of the scalar box integral

$$D_0(s_1, s_2, q^2) = \frac{1}{\varepsilon^2} \frac{\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \frac{2}{s_1 s_2} \left| \frac{s_3 \mu^2}{s_1 s_2} \right|^\varepsilon \\ \times \left[(\Theta(-s_2) + \Theta(s_2)e^{i\pi\varepsilon}) \mathfrak{F}(\varepsilon; x_1) + (\Theta(-s_1) + \Theta(s_1)e^{i\pi\varepsilon}) \mathfrak{F}(\varepsilon; x_2) \right. \\ \left. - (\Theta(-q^2) + \Theta(q^2)e^{i\pi\varepsilon}) \mathfrak{F}(\varepsilon; x_1 + x_2 - x_1 x_2) \right]$$

where

$$\mathfrak{F}(\varepsilon; x) \equiv 1 + \ln \left| \frac{x}{1-x} \right| \sum_{n=1}^{\infty} \frac{(-\varepsilon)^n}{n!} \ln^{n-1} |x| - \sum_{n=2}^{\infty} \varepsilon^n \mathcal{L}_n(x)$$

- ▶ Diverges like $1/\varepsilon^2$ for $\varepsilon \rightarrow 0$ (box integral itself is no measurable quantity)
- ▶ Real and imaginary parts can be easily read off
- ▶ All-order ε -expansion
- ▶ Valid in all kinematic regions ($s_1, s_2, q^2 \in \mathbb{R}$)

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 \end{aligned}$$

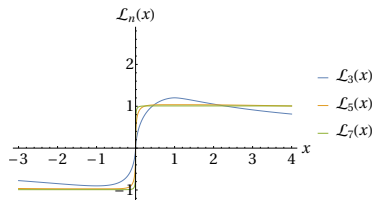
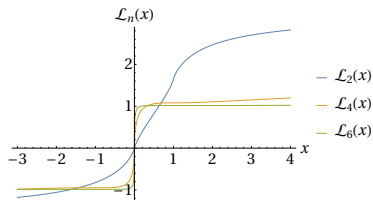
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Single-valued polylogarithms

$$\mathcal{L}_n(x) \equiv \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \ln^k |x| \operatorname{Li}_{n-k}(x) + \frac{(-1)^{n-1}}{n!} \ln^{n-1} |x| \ln |1-x|$$



- ▶ $\mathcal{L}_n(x)$ is single-valued, in contrast to $\operatorname{Li}_n(x)$
- ▶ $\mathcal{L}_n(x)$ is continuous for all $x \in \mathbb{R}$
- ▶ $\mathcal{L}_n(x)$ is bounded on \mathbb{R} , in contrast to $\operatorname{Li}_n(x)$
- ▶ $\mathcal{L}_n(x)$ satisfies *clean* versions of the functional equations of $\operatorname{Li}_n(x)$, i.e. without product terms

3. Conclusion and Outlook

Conclusion and Outlook

- ▶ All-order ε -expansion of massless single off-shell scalar box integral in terms of newly defined single-valued polylogarithms
- ▶ Real and imaginary parts to all orders in ε
- ▶ Result can be generalized to two non-adjacent endpoints off-shell [J. Haug and F. Wunder, 2023]
- ▶ Additional methods needed for further generalization of results
- ▶ Starting point: one-fold integral representation of general box integral with massless propagators [Tarasov, 2019]

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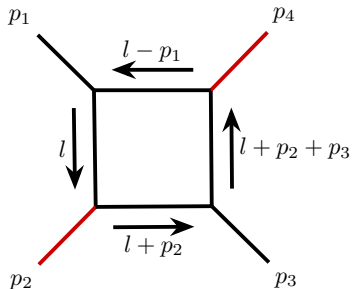
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4. Generalization to two non-adjacent off-shell external particles
5. The massless scalar box integral in the literature
6. Detailed calculation of the scalar box integral

4. Generalization to two non-adjacent off-shell external particles



JH, Fabian Wunder, JHEP 05 (2023) 059, arXiv:2302.01956

Generalization to two non-adjacent off-shell external particles

General box integral with massless propagators after Feynman parametrization:

$$D_0 = \mu^{2\varepsilon} \Gamma(2 + \varepsilon) \times \int_0^1 \frac{dx_1 dx_2 dx_3 dx_4 \delta(1 - x_1 - x_2 - x_3 - x_4)}{[-x_1 x_2 s_1 - x_1 x_3 p_4^2 - x_1 x_4 p_1^2 - x_2 x_3 p_3^2 - x_2 x_4 p_2^2 - x_3 x_4 s_2 - i0]^{2+\varepsilon}}$$

With the same substitution as before,

$$x_1 = \eta_1 \xi_1, \quad x_2 = \eta_2 (1 - \xi_2), \quad x_3 = \eta_2 \xi_2, \quad x_4 = \eta_1 (1 - \xi_1)$$

term in denominator becomes

$$\begin{aligned} & -\eta_1 \eta_2 (1 - \xi_1)(1 - \xi_2) s_1 - \eta_1 \eta_2 \xi_1 \xi_2 s_2 - \eta_1 \eta_2 \xi_1 (1 - \xi_2) p_2^2 - \eta_1 \eta_2 \xi_2 (1 - \xi_1) p_4^2 \\ & - \eta_1^2 \xi_1 (1 - \xi_1) p_1^2 - \eta_2^2 \xi_2 (1 - \xi_2) p_3^2 - i0 \end{aligned}$$

- ▶ Factorization of η - and ξ -integrals if $p_1^2 = p_3^2 = 0$
- ▶ Proceed analogously to single off-shell case for two non-adjacent external particles off light cone

Result in terms of 4 hypergeometric functions

$$\begin{aligned}
D_0(s_1, s_2, 0, p_2^2, 0, p_4^2) &= \frac{1}{\varepsilon^2} \frac{\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \frac{2}{s_1 s_2 - p_2^2 p_4^2} \\
&\times \left\{ \left[\frac{\mu^2}{-s_1 - i0} \right]^\varepsilon {}_2F_1\left(1, -\varepsilon, 1-\varepsilon; -\frac{(p_2^2 + p_4^2 - s_1 - s_2)s_1}{s_1 s_2 - p_2^2 p_4^2} + i\tilde{0} \operatorname{sgn}(s_1 - s_2)\right) \right. \\
&+ \left[\frac{\mu^2}{-s_2 - i0} \right]^\varepsilon {}_2F_1\left(1, -\varepsilon, 1-\varepsilon; -\frac{(p_2^2 + p_4^2 - s_1 - s_2)s_2}{s_1 s_2 - p_2^2 p_4^2} + i\tilde{0} \operatorname{sgn}(s_2 - s_1)\right) \\
&- \left[\frac{\mu^2}{-p_2^2 - i0} \right]^\varepsilon {}_2F_1\left(1, -\varepsilon, 1-\varepsilon; -\frac{(p_2^2 + p_4^2 - s_1 - s_2)p_2^2}{s_1 s_2 - p_2^2 p_4^2} + i\tilde{0} \operatorname{sgn}(p_4^2 - p_2^2)\right) \\
&\left. - \left[\frac{\mu^2}{-p_4^2 - i0} \right]^\varepsilon {}_2F_1\left(1, -\varepsilon, 1-\varepsilon; -\frac{(p_2^2 + p_4^2 - s_1 - s_2)p_4^2}{s_1 s_2 - p_2^2 p_4^2} + i\tilde{0} \operatorname{sgn}(p_2^2 - p_4^2)\right) \right\}
\end{aligned}$$

- ▶ Reduces to single off-shell integral in limits $p_2^2, p_4^2 \rightarrow 0$
- ▶ Similar result found via functional equations in [Tarasov, 2019]

Epsilon expansion in the non-adjacent double off-shell case

$$\begin{aligned}
D_0(s_1, s_2, p_2^2, p_4^2) &= \frac{1}{\varepsilon^2} \frac{\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \frac{2}{s_1 s_2 - p_2^2 p_4^2} \left| \frac{(p_2^2 + p_4^2 - s_1 - s_2)\mu^2}{s_1 s_2 - p_2^2 p_4^2} \right|^\varepsilon \\
&\times \left\{ \left(\Theta(-s_1) + \Theta(s_1)e^{i\pi\varepsilon} \right) \mathfrak{F}\left(\varepsilon; -\frac{(p_2^2 + p_4^2 - s_1 - s_2)s_1}{s_1 s_2 - p_2^2 p_4^2}\right) \right. \\
&+ \left(\Theta(-s_2) + \Theta(s_2)e^{i\pi\varepsilon} \right) \mathfrak{F}\left(\varepsilon; -\frac{(p_2^2 + p_4^2 - s_1 - s_2)s_2}{s_1 s_2 - p_2^2 p_4^2}\right) \\
&- \left(\Theta(-p_2^2) + \Theta(p_2^2)e^{i\pi\varepsilon} \right) \mathfrak{F}\left(\varepsilon; -\frac{(p_2^2 + p_4^2 - s_1 - s_2)p_2^2}{s_1 s_2 - p_2^2 p_4^2}\right) \\
&\left. - \left(\Theta(-p_4^2) + \Theta(p_4^2)e^{i\pi\varepsilon} \right) \mathfrak{F}\left(\varepsilon; -\frac{(p_2^2 + p_4^2 - s_1 - s_2)p_4^2}{s_1 s_2 - p_2^2 p_4^2}\right) \right\}
\end{aligned}$$

- ▶ All-order ε -expansion (not previously known)
- ▶ Real and imaginary parts can be easily read off
- ▶ Valid in all kinematic regions ($s_1, s_2, p_2^2, p_4^2 \in \mathbb{R}$)
- ▶ No spurious branch cuts

5. The massless scalar box integral in the literature

The massless scalar box integral in the literature

Massless single-off shell case:

- ▶ K. Fabricius and I. Schmitt [1979]: Box integral with explicit imaginary part up to $\mathcal{O}(\varepsilon^0)$
- ▶ Matsuura et al. [1989]: Result in terms of 3 Gauss hypergeometric functions for general d
- ▶ Bern et al. [1994]: Rule for analytic continuation of result up to $\mathcal{O}(\varepsilon^0)$
- ▶ G. Duplanić and B. Nižić [2001]: Systematically keep causal $+i0$, result up to $\mathcal{O}(\varepsilon^0)$
- ▶ Lyubovitskij et al. [2021]: All-order ε -expansion, branch cuts not discussed

Non-adjacent double off-shell case:

- ▶ Tarasov [2019]: Functional equation approach yields result in terms of 4 Gauss hypergeometric functions, expansion up to $\mathcal{O}(\varepsilon^0)$

Comparison to literature

Combine prefactors $(\dots)^\varepsilon$ and $[\dots]^{-\varepsilon} \rightarrow$

$$\begin{aligned}
 D_0(s_1, s_2, q^2) &= \frac{1}{\varepsilon^2} \frac{\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \frac{2}{s_1 s_2} \\
 &\times \left\{ \left[\frac{\mu^2}{-s_2 - i0} \right]^\varepsilon {}_2F_1\left(1, -\varepsilon, 1-\varepsilon; -\frac{s_3}{s_1} + i\tilde{0} \operatorname{sgn}\left(\frac{s_3}{s_2} - \frac{s_3}{s_1}\right)\right) \right. \\
 &\quad + \left[\frac{\mu^2}{-s_1 - i0} \right]^\varepsilon {}_2F_1\left(1, -\varepsilon, 1-\varepsilon; -\frac{s_3}{s_2} + i\tilde{0} \operatorname{sgn}\left(\frac{s_3}{s_1} - \frac{s_3}{s_2}\right)\right) \\
 &\quad \left. - \left[\frac{\mu^2}{-q^2 - i0} \right]^\varepsilon {}_2F_1\left(1, -\varepsilon, 1-\varepsilon; -\frac{s_3 q^2}{s_1 s_2} + i\tilde{0}\right) \right\}
 \end{aligned}$$

- ▶ Agrees with [Matsuura et al., 1989] and [Lyubovitskij et al., 2021] if $s_1, s_2, q^2 < 0$ and all three hypergeometric functions are away from their branch cut
- ▶ Agrees with [Bern et al., 1994] if all three hypergeometric functions are away from their branch cut

6. Detailed calculation of the scalar box integral

Factoring out $s_1 s_2 / s_3$

Use

$$(a - i0)^\alpha = (b - i0)^\alpha \left(\frac{a}{b} - i0 \operatorname{sgn}(b) \right)^\alpha, \quad \text{where } a \in \mathbb{R}, b \in \mathbb{R} \setminus \{0\}, \alpha \in \mathbb{C},$$

to factor out $s_1 s_2 / s_3 \rightarrow$

$$D_0(s_1, s_2, q^2) = \mu^{2\varepsilon} \frac{\Gamma(2 + \varepsilon) \Gamma^2(-\varepsilon)}{\Gamma(-2\varepsilon)} \left(\frac{s_1 s_2}{s_3} - i0 \right)^{-\varepsilon - 2} \\ \times \int_0^1 d\xi_1 \int_0^1 d\xi_2 \left[x_2 \xi_1 + x_1 \xi_2 - x_1 x_2 \xi_1 \xi_2 - i0 \operatorname{sgn} \left(\frac{s_3}{s_1 s_2} \right) \right]^{-\varepsilon - 2},$$

depends on only 2 dimensionless variables

$$x_1 \equiv -\frac{s_3}{s_1}, \quad x_2 \equiv -\frac{s_3}{s_2}$$

Several substitutions & evaluating 1 integral & splitting of integrals \rightarrow

$$D_0(s_1, s_2, q^2) = -\frac{1}{\varepsilon} \frac{\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \frac{2}{s_1 s_2} \left(\frac{s_3 \mu^2}{s_1 s_2} + i0 \right)^\varepsilon \times \{I(x_1) + I(x_2) - I(1 - (1-x_1)(1-x_2))\}, \quad (1)$$

where

$$I(\chi) \equiv \int_0^\chi \frac{d\zeta}{1-\zeta} \left([\zeta - i0 \operatorname{sgn}_{123}]^{-\varepsilon-1} - 1 \right),$$

with abbreviation

$$\operatorname{sgn}_{123} \equiv \operatorname{sgn} \left(\frac{s_3}{s_1 s_2} \right)$$

Note

$$1 - (1-x_1)(1-x_2) = -\frac{s_3 q^2}{s_1 s_2} \xrightarrow{q^2 \rightarrow 0} 0$$

$I(\chi)$ vanishes for $\chi = 0 \rightarrow$ set third integral to 0 in case $q^2 = 0$

Calculating $I(\chi)$

In case $\chi \neq 0$, substitute $\zeta \rightarrow \chi^{-1}\zeta \rightarrow$

$$I(\chi) = \int_0^1 \frac{d\zeta}{1 - \chi\zeta} \left([\chi - i0 \operatorname{sgn}_{123}]^{-\varepsilon} \zeta^{-\varepsilon-1} - \chi \right)$$

Denominator $1 - \chi\zeta$ diverges for $\chi > 1$

\rightarrow Introduce regulator $\chi \rightarrow \chi + i\tilde{0}$ to split integral in two

$$\begin{aligned} I(\chi) &= [\chi - i0 \operatorname{sgn}_{123}]^{-\varepsilon} \int_0^1 d\zeta \zeta^{-\varepsilon-1} (1 - (\chi + i\tilde{0})\zeta)^{-1} - \int_0^1 d\zeta \frac{\chi}{1 - (\chi + i\tilde{0})\zeta} \\ &= -\frac{1}{\varepsilon} [\chi - i0 \operatorname{sgn}_{123}]^{-\varepsilon} {}_2F_1(1, -\varepsilon, 1 - \varepsilon; \chi + i\tilde{0}) + \ln(1 - \chi - i\tilde{0}) \end{aligned}$$

- ▶ Both ${}_2F_1$ and \ln evaluated on branch cut for $\chi > 1$
- ▶ Branch cuts are spurious and must cancel

Cancellation of spurious branch cuts

Add integrals in eq. (1),

$$D_0(s_1, s_2, q^2) = -\frac{1}{\varepsilon} \frac{\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \frac{2}{s_1 s_2} \left(\frac{s_3 \mu^2}{s_1 s_2} + i0 \right)^\varepsilon \\ \times \{I(x_1) + I(x_2) - I(1 - (1-x_1)(1-x_2))\},$$

use different regulator $i\tilde{0}_i$ for each integral

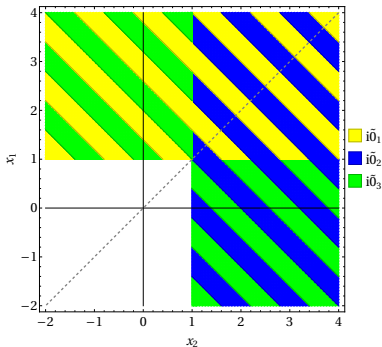
Sum $I(x_1) + I(x_2) - I(1 - (1-x_1)(1-x_2))$ contains following logarithms

$$\ln(1 - x_1 - i\tilde{0}_1) + \ln(1 - x_2 - i\tilde{0}_2) - \ln((1 - x_1)(1 - x_2) - i\tilde{0}_3)$$

- ▶ Real parts cancel
- ▶ Choose signs of $i\tilde{0}_i$ such that imaginary parts cancel as well

Cancellation of imaginary parts

$$i\pi \left[-\operatorname{sgn}(\tilde{\theta}_1) \Theta(x_1 - 1) - \operatorname{sgn}(\tilde{\theta}_2) \Theta(x_2 - 1) + \operatorname{sgn}(\tilde{\theta}_3) \{ \Theta(x_1 - 1) \Theta(1 - x_2) + \Theta(1 - x_1) \Theta(x_2 - 1) \} \right] \stackrel{!}{=} 0 \quad (2)$$



- Conditions for eq. (2) to hold:

$$\operatorname{sgn}(\tilde{\theta}_1) \stackrel{!}{=} \operatorname{sgn}(\tilde{\theta}_3) \quad (\text{yellow-green region})$$

$$\operatorname{sgn}(\tilde{\theta}_1) \stackrel{!}{=} -\operatorname{sgn}(\tilde{\theta}_2) \quad (\text{yellow-blue region})$$

$$\operatorname{sgn}(\tilde{\theta}_2) \stackrel{!}{=} \operatorname{sgn}(\tilde{\theta}_3) \quad (\text{blue-green region})$$

- Choose (only relative signs matter)

$$i\tilde{\theta}_1 \equiv i\tilde{\theta} \operatorname{sgn}(x_1 - x_2)$$

$$i\tilde{\theta}_2 \equiv i\tilde{\theta} \operatorname{sgn}(x_2 - x_1)$$

$$i\tilde{\theta}_3 \equiv i\tilde{\theta}$$

Result in terms of hypergeometric functions

$$\begin{aligned}
D_0(s_1, s_2, q^2) &= \frac{1}{\varepsilon^2} \frac{\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \frac{2}{s_1 s_2} \left(\frac{s_3 \mu^2}{s_1 s_2} + i0 \right)^\varepsilon \\
&\times \left\{ \left[-\frac{s_3}{s_1} - i0 \operatorname{sgn}\left(\frac{s_3}{s_1 s_2}\right) \right]^{-\varepsilon} {}_2F_1\left(1, -\varepsilon, 1-\varepsilon; -\frac{s_3}{s_1} + i\tilde{0} \operatorname{sgn}\left(\frac{s_3}{s_2} - \frac{s_3}{s_1}\right)\right) \right. \\
&\quad + \left[-\frac{s_3}{s_2} - i0 \operatorname{sgn}\left(\frac{s_3}{s_1 s_2}\right) \right]^{-\varepsilon} {}_2F_1\left(1, -\varepsilon, 1-\varepsilon; -\frac{s_3}{s_2} + i\tilde{0} \operatorname{sgn}\left(\frac{s_3}{s_1} - \frac{s_3}{s_2}\right)\right) \\
&\quad \left. - \left[-\frac{s_3 q^2}{s_1 s_2} - i0 \operatorname{sgn}\left(\frac{s_3}{s_1 s_2}\right) \right]^{-\varepsilon} {}_2F_1\left(1, -\varepsilon, 1-\varepsilon; -\frac{s_3 q^2}{s_1 s_2} + i\tilde{0}\right) \right\} \quad (3)
\end{aligned}$$

- ▶ Imaginary parts of hypergeometric functions will cancel by construction
- ▶ Last term vanishes for $q^2 = 0$