QCD Global Analysis of Single-Hadron Fragmentation TSSAs: the Role of Lattice QCD in Phenomenology & New Theory Developments for Dihadron Fragmentation

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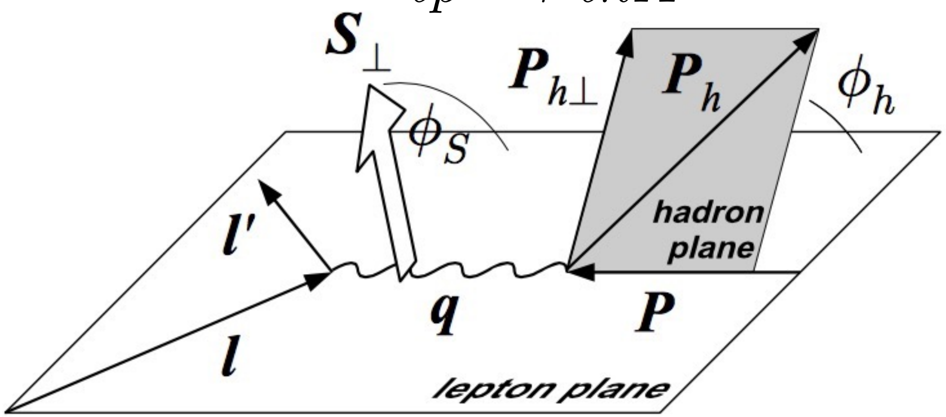
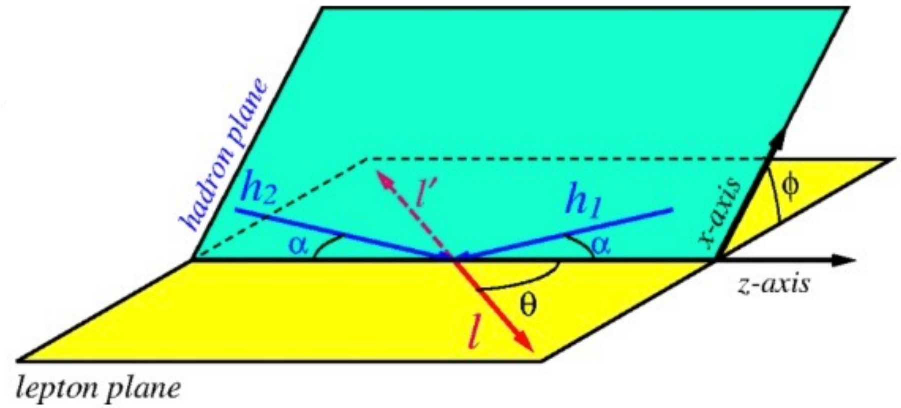
Background:
TSSAs for Single-Hadron Fragmentation

$\ell p^{\uparrow} \to \ell h X$



$\{\pi, p\} p^{\uparrow} \to \{\ell^+\ell^-, W^{\pm}, Z\} X$



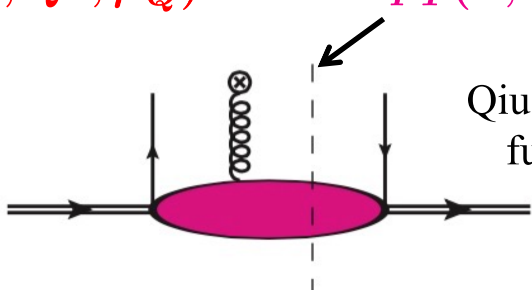
$$F_{UT}^{\sin(\phi_h - \phi_S)} = \mathcal{C}\left[-\frac{\hat{h}\cdot\vec{k}_T}{M} \boldsymbol{f_{1T}^{\perp}} D_1\right] \qquad F_{TU}^{\sin\phi} = \mathcal{C}\left[-\frac{\hat{h}\cdot\vec{k}_{aT}}{M_a} \boldsymbol{f_{1T}^{\perp}}\, \bar{f}_1\right]$$

TMD/Collins-Soper-Sterman (CSS) Evolution

OPE

Sudakov exponentials (gluon radiation)

$$\tilde{f}_{1T}^{\perp(1)}(x, b_T; Q^2, \mu_Q) \quad \sim \quad F_{FT}(x, x; \mu_{b_*}) \exp\left[-S_{pert}(b_*(b_T); \mu_{b_*}, Q, \mu_Q) - S_{NP}^{f_{1T}^{\perp}}(b_T, Q)\right]$$



Qiu-Sterman function

$$g_{f_{1T}^{\perp}}(x, b_T) + g_K(b_T) \ln(Q/Q_0)$$

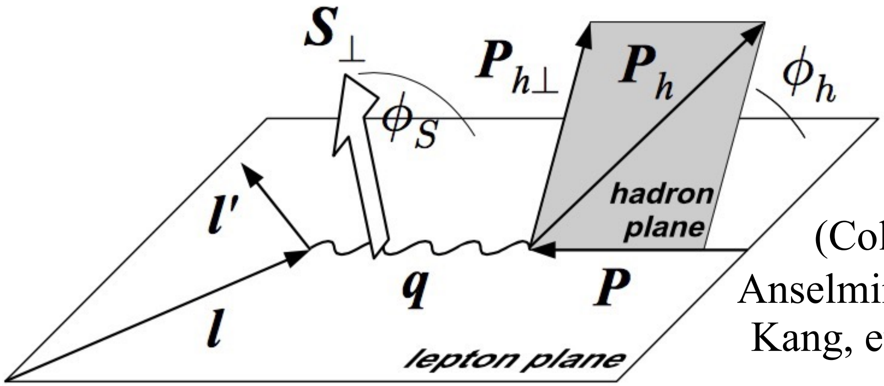
(Aybat, et al. (2012); Bury, et al. (2021); Echevarria, et al. (2014, 2021))

Parton model

$$\pi F_{FT}(x, x) = \int d^2\vec{k}_T \, \frac{k_T^2}{2M^2} f_{1T}^{\perp}(x, k_T^2) \equiv f_{1T}^{\perp(1)}(x)$$

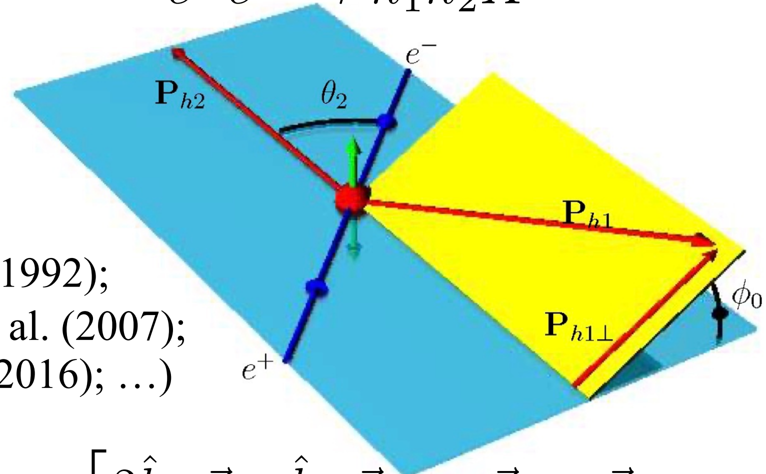
(Boer, Mulders, Pijlman (2003))

$$\ell\, N^{\uparrow} \to \ell\, h\, X$$



(Collins (1992); Anselmino, et al. (2007); Kang, et al. (2016); …)

$$e^{+}e^{-} \to h_1 h_2 X$$



$$F_{UT}^{\sin(\phi_h+\phi_S)} = \mathcal{C}\left[-\frac{\hat{h}\cdot\vec{p}_\perp}{M_h}\boldsymbol{h_1}\boldsymbol{H_1^\perp}\right]$$

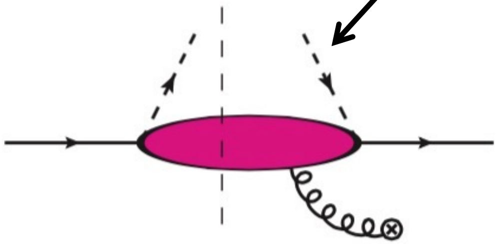
$$F_{UU}^{\cos(2\phi_0)} = \mathcal{C}\left[\frac{2\hat{h}\cdot\vec{p}_{a\perp}\,\hat{h}\cdot\vec{p}_{b\perp} - \vec{p}_{a\perp}\cdot\vec{p}_{b\perp}}{M_a M_b}\,\boldsymbol{H_1^\perp}\,\boldsymbol{\bar{H}_1^\perp}\right]$$

TMD/Collins-Soper-Sterman (CSS) Evolution

OPE — Sudakov exponentials (gluon radiation)

$$\boldsymbol{\tilde{h}_1(x, b_T; Q^2, \mu_Q)} \quad \sim \quad \boldsymbol{h_1(x; \mu_{b_*})}\exp\left[-S_{pert}(b_*(b_T); \mu_{b_*}, Q, \mu_Q) - S_{NP}^{h_1}(b_T, Q)\right]$$

$$\boldsymbol{\tilde{H}_1^{\perp(1)}(z, b_T; Q^2, \mu_Q)} \quad \sim \quad \boldsymbol{H_1^{\perp(1)}(z; \mu_{b_*})}\exp\left[-S_{pert}(b_*(b_T); \mu_{b_*}, Q, \mu_Q) - S_{NP}^{H_1^\perp}(b_T, Q)\right]$$



qgq fragmentation function

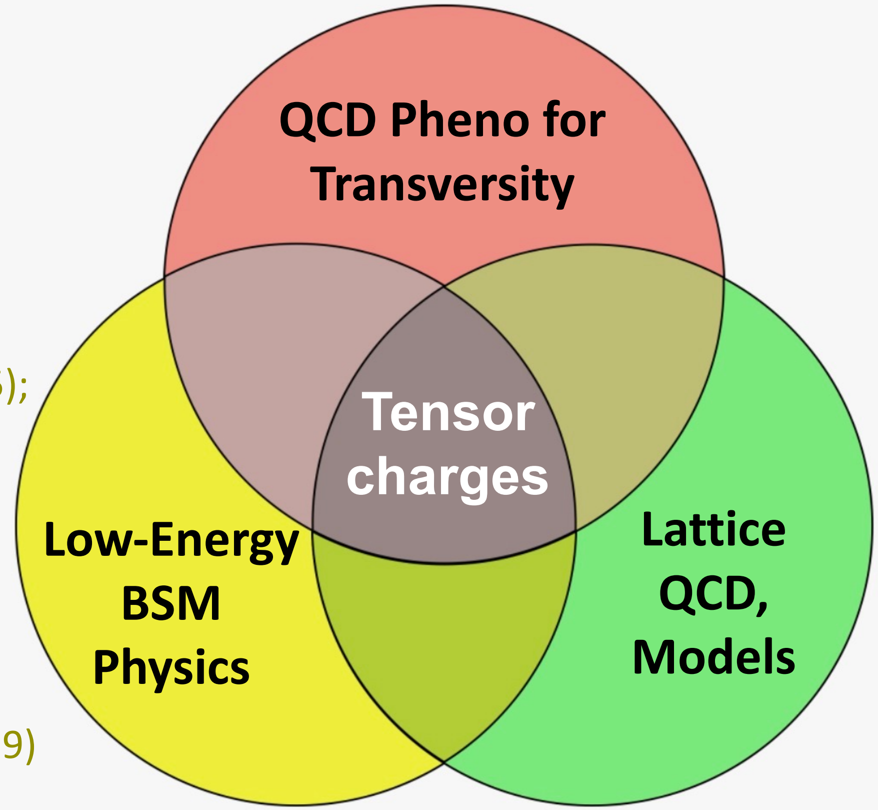
Parton model

$$h_1(x) = \int d^2\vec{k}_T\, h_1(x, \vec{k}_T^2)$$

$$H_1^{\perp(1)}(z) = z^2\int d^2\vec{p}_\perp \frac{p_\perp^2}{2M_h^2} H_1^\perp(z, z^2 p_\perp^2)$$

$$\delta q \equiv \int_0^1 dx\, [h_1^q(x) - h_1^{\bar{q}}(x)] \qquad g_T \equiv \delta u - \delta d$$

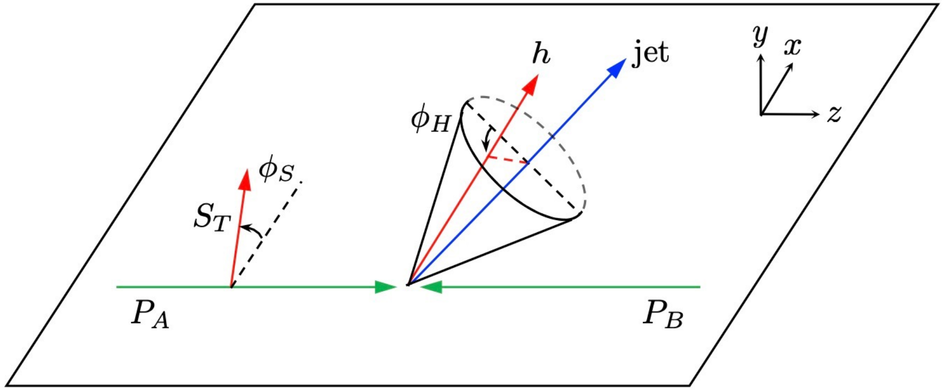
Anselmino, et al. (2007, 2009, 2013, 2015);
Goldstein, et al. (2014); Kang, et al. (2016); Radici, et al. (2013, 2015, 2018);
Benel, et al. (2020); D'Alesio, et al. (2020); Cammarota, et al. (2020);
Gamberg, et al. (2022); Cocuzza, et al. (2023)



Herczeg (2001);
Erler, Ramsey-Musolf (2005);
Pospelov, Ritz (2005);
Severijns, et al. (2006);
Cirigliano, et al. (2013);
Courtoy, et al. (2015);
Yamanaka, et al. (2017);
Liu, et al. (2018);
Gonzalez-Alonso, et al. (2019)

He, Ji (1995);
Barone, et al. (1997);
Schweitzer, et al. (2001);
Gamberg, Goldstein (2001);
Pasquini, et al. (2005);
Wakamatsu (2007);
Lorce (2009);
Gupta, et al. (2018);
Yamanaka, et al. (2018);
Hasan, et al. (2019);
Alexandrou, et al. (2019, 2023);
Yamanaka, et al. (2013);
Pitschmann, et al. (2015);
Xu, et al. (2015);
Wang, et al. (2018);
Liu, et al. (2019)

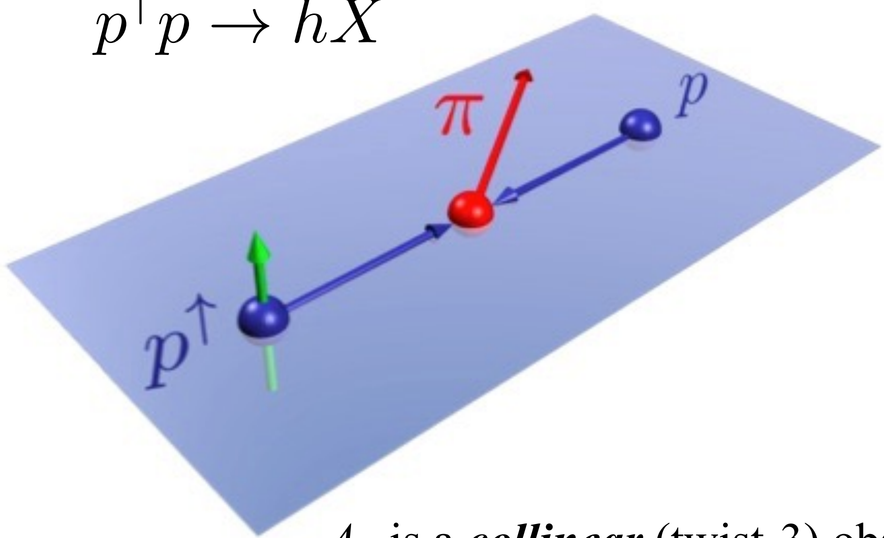
$$p^{\uparrow}p \to (h\, jet)\, X$$



(Yuan (2008); D'Alesio, Murgia, Pisano (2017); Kang, Prokudin, Ringer, Yuan (2017), …)

$$F_{UT}^{\sin(\phi_S-\phi_H)} \sim H_{ab\to c}^{\text{Collins}}(\hat{s},\hat{t},\hat{u}) \otimes \boldsymbol{h_1^a(x_1)} \otimes f_1^b(x_2) \otimes (j_\perp/(z_h M_h))\boldsymbol{H_1^{\perp h/c}(z_h, j_\perp^2)}$$

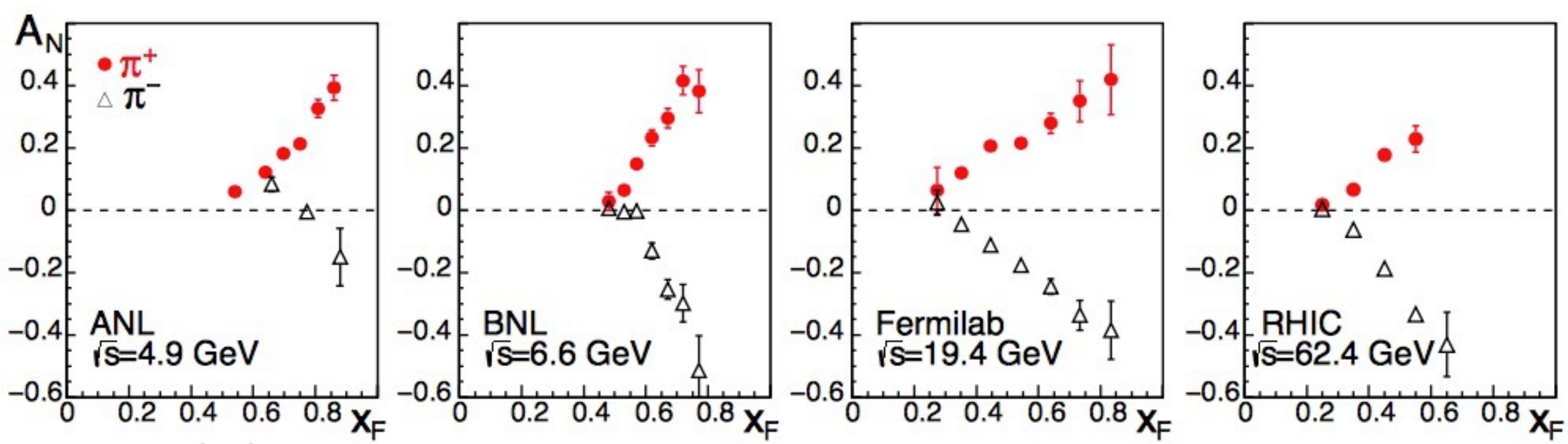
$$p^{\uparrow}p \to hX$$



$$d\Delta\sigma(S_T) \sim \underbrace{H_{QS} \otimes f_1 \otimes \boldsymbol{F_{FT}} \otimes D_1}_{\text{Qiu-Sterman term}}$$

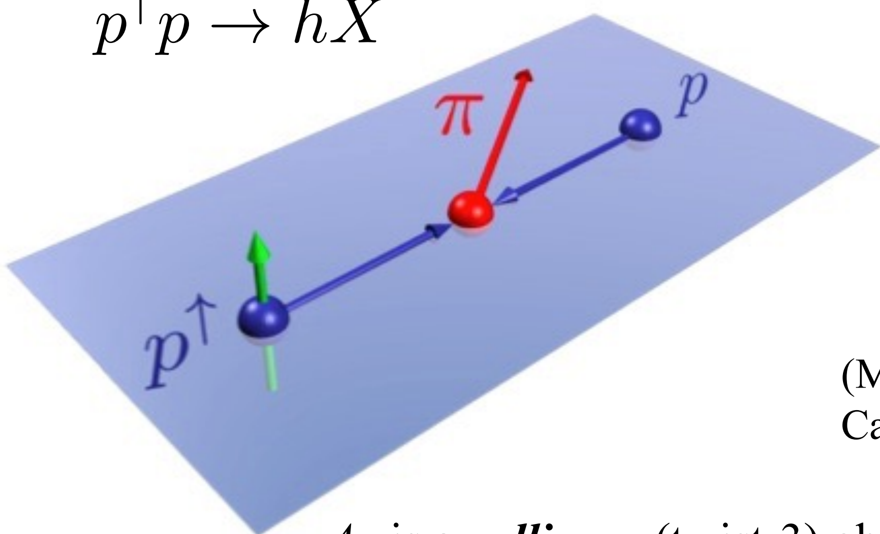
$$+ \underbrace{H_F \otimes f_1 \otimes \boldsymbol{h_1} \otimes \left(\boldsymbol{H_1^{\perp(1)}}, \boldsymbol{\tilde{H}}\right)}_{\text{Fragmentation term}}$$

A_N is a ***collinear*** (twist-3) observable



1976 →

$$p^{\uparrow}p \to hX$$

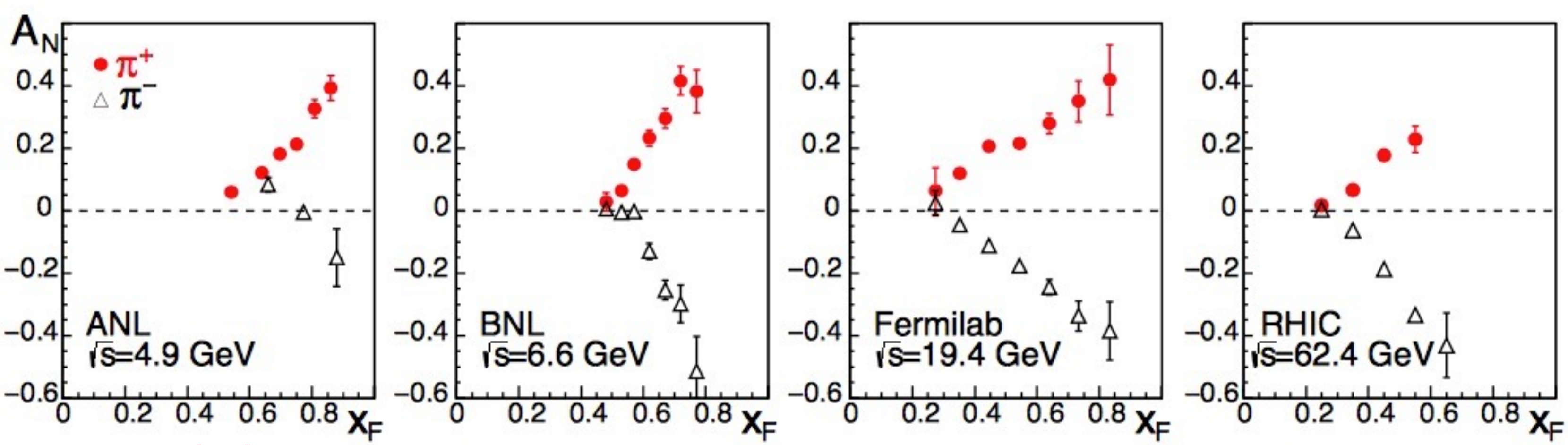


$$d\Delta\sigma(S_T) \sim \underbrace{H_{QS} \otimes f_1 \otimes \boldsymbol{F_{FT}} \otimes D_1}_{\text{Qiu-Sterman term}}$$

$$+ \underbrace{H_F \otimes f_1 \otimes \boldsymbol{h_1} \otimes \left(\boldsymbol{H_1^{\perp(1)}}, \boldsymbol{\tilde{H}}\right)}_{\text{Fragmentation term}}$$

(Metz, DP (2012); Kanazawa, et al. (2014); Cammarota, et al. (2020); Gamberg, et al. (2017, 2022))

A_N is a ***collinear*** (twist-3) observable



1976 →

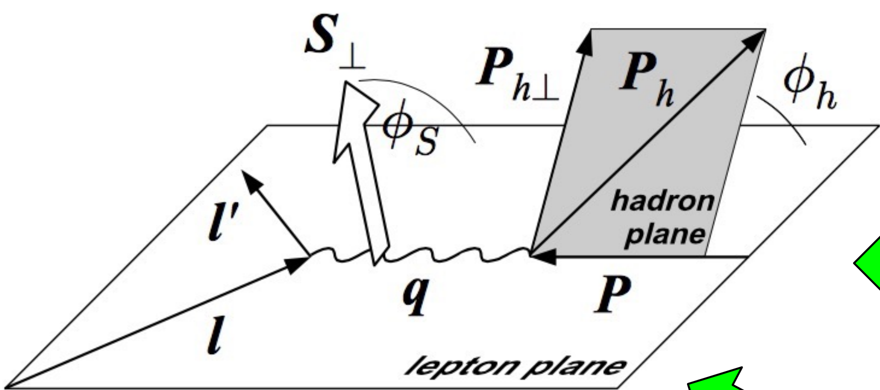
Updated QCD Global Analysis of TSSAs for Single-Hadron Fragmentation

Gamberg, Malda, Miller, DP, Prokudin, Sato, PRD **106**, 034014 (2022)

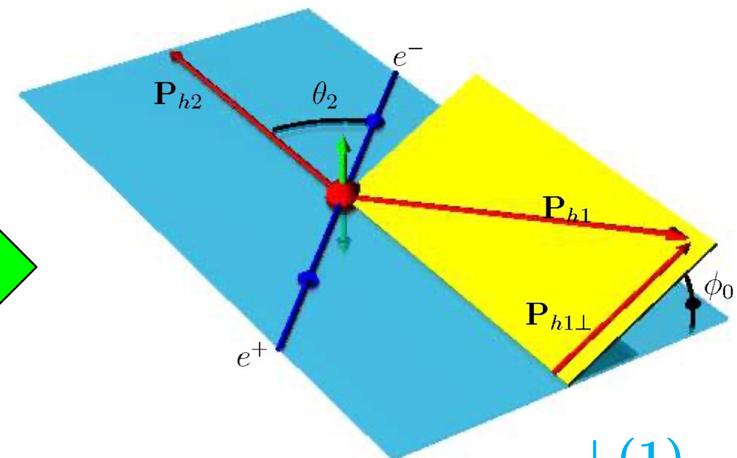
User-friendly jupyter notebook to calculate functions and asymmetries:
https://colab.research.google.com/github/pitonyak25/jam3d_dev_lib/blob/main/JAM3D_Library.ipynb

LHAPDF tables available (thanks to C. Cocuzza):
https://github.com/pitonyak25/jam3d_dev_lib/tree/main/LHAPDF_tables

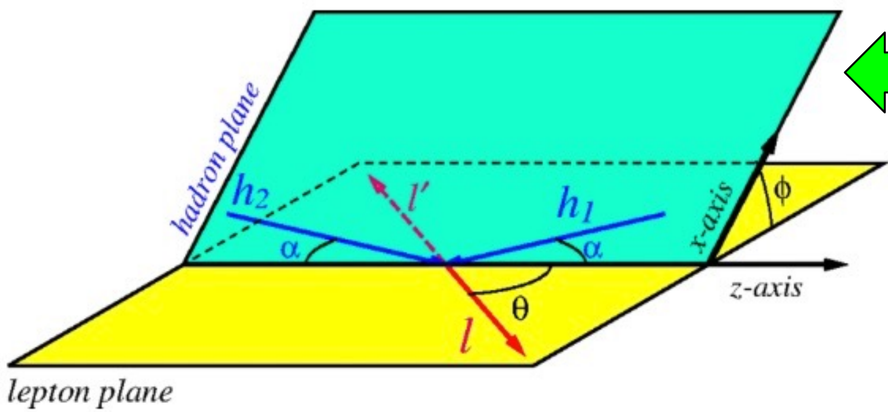
JAM Jefferson Lab Angular Momentum Collaboration



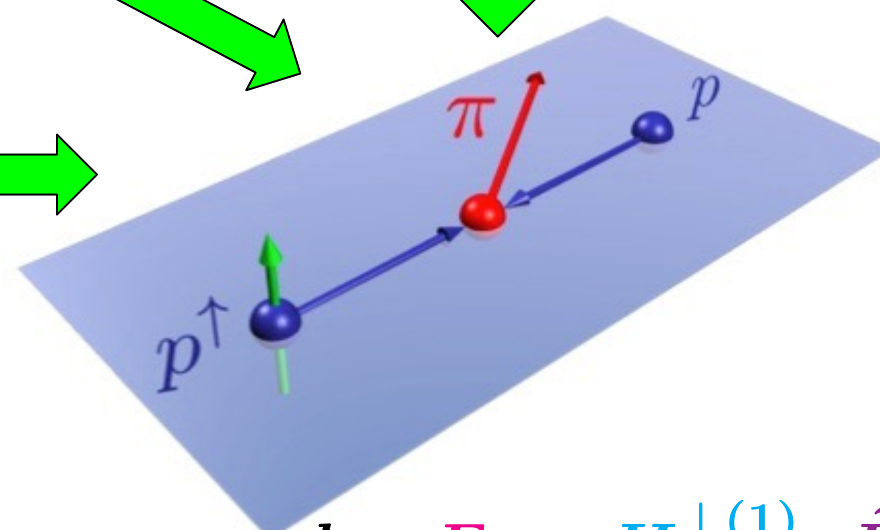
h_1, F_{FT}, $H_1^{\perp(1)}$, $\tilde{H}$



$H_1^{\perp(1)}$



F_{FT}



h_1, F_{FT}, $H_1^{\perp(1)}$, $\tilde{H}$

- Analyze TSSAs in SIDIS, Drell-Yan, e^+e^- annihilation, and proton-proton collisions and extract

$$h_1(x),\ F_{FT}(x,x),\ H_1^{\perp(1)}(z),\ \tilde{H}(z)$$

along with the relevant transverse momentum widths for the Sivers, transversity, and Collins functions: $\langle k_T^2\rangle_{f_{1T}^\perp},\ \langle k_T^2\rangle_{h_1},\ \langle p_\perp^2\rangle_{H_1^\perp}^{fav},\ \langle p_\perp^2\rangle_{H_1^\perp}^{unf}$

- We use a Gaussian ansatz: $F^q(x,k_T^2) \sim F^q(x)e^{-k_T^2/\langle k_T^2\rangle}$ where

$$F^q(x) = \frac{N_q\, x^{a_q}(1-x)^{b_q}(1+\gamma_q\, x^{\alpha_q}(1-x)^{\beta_q})}{\mathrm{B}[a_q+2,b_q+1]+\gamma_q\mathrm{B}[a_q+\alpha_q+2,b_q+\beta_q+1]}$$

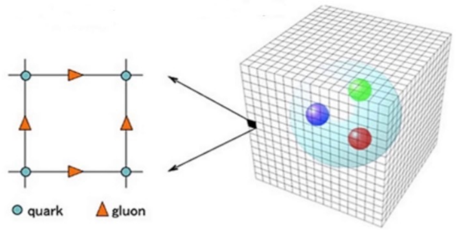
NB. $\{\gamma,\alpha,\beta\}$ only used for Collins function

- DGLAP-type evolution for the collinear functions analogous to Duke & Owens (1984): double-log Q^2-dependent term explicitly added to the parameters

- Additional data/constraints included in the fit compared to 2020:
  - Collins and Sivers effects (3D-binned) SIDIS data from HERMES (2020)
  - $A_{UT}^{\sin\phi_S}$ data (*x* and *z* projections only) from HERMES (2020)

$$\int d^2\vec{P}_{hT}\, F_{UT}^{\sin\phi_S} = -\frac{x}{z}\sum_q e_q^2\,\frac{2M_h}{Q}\,h_1^{q/N}(x)\,\tilde{H}^{h/q}(z)$$

  - Lattice data on g_T at the physical pion mass from ETMC (Alexandrou, et al. (2019))

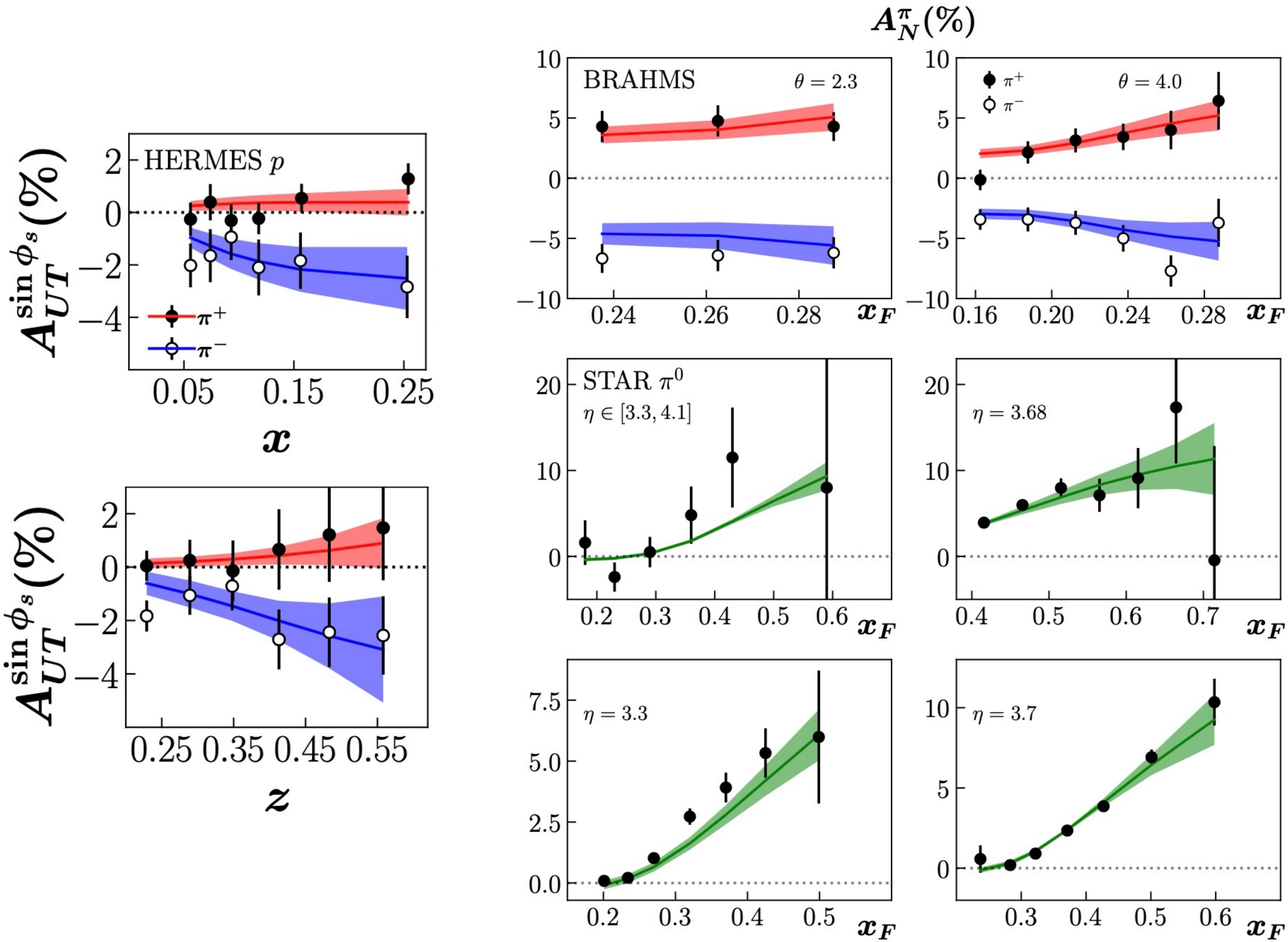


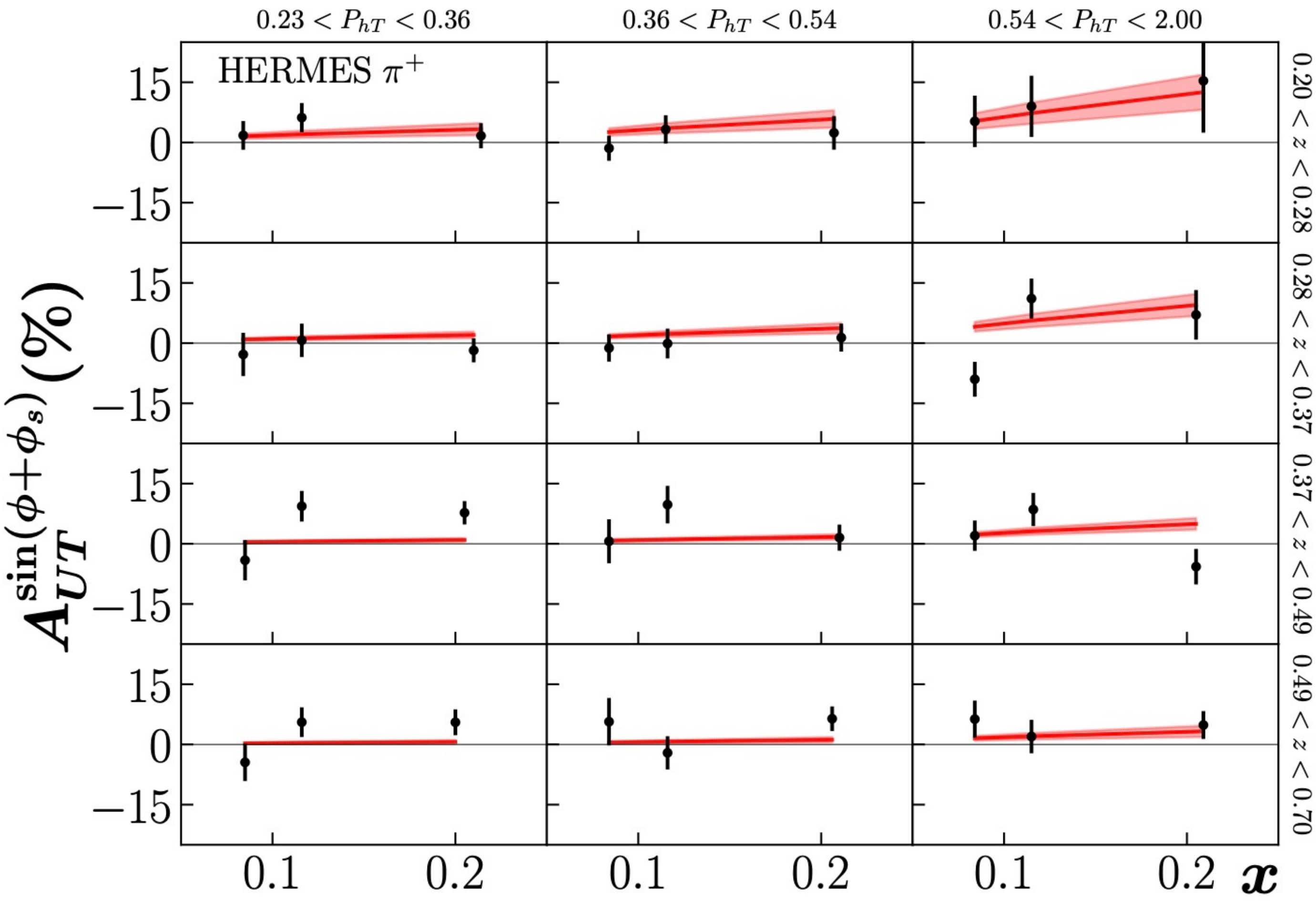
  - Imposing the Soffer bound on transversity: $|h_1^q(x)| \leq \frac{1}{2}(f_1^q(x) + g_1^q(x))$

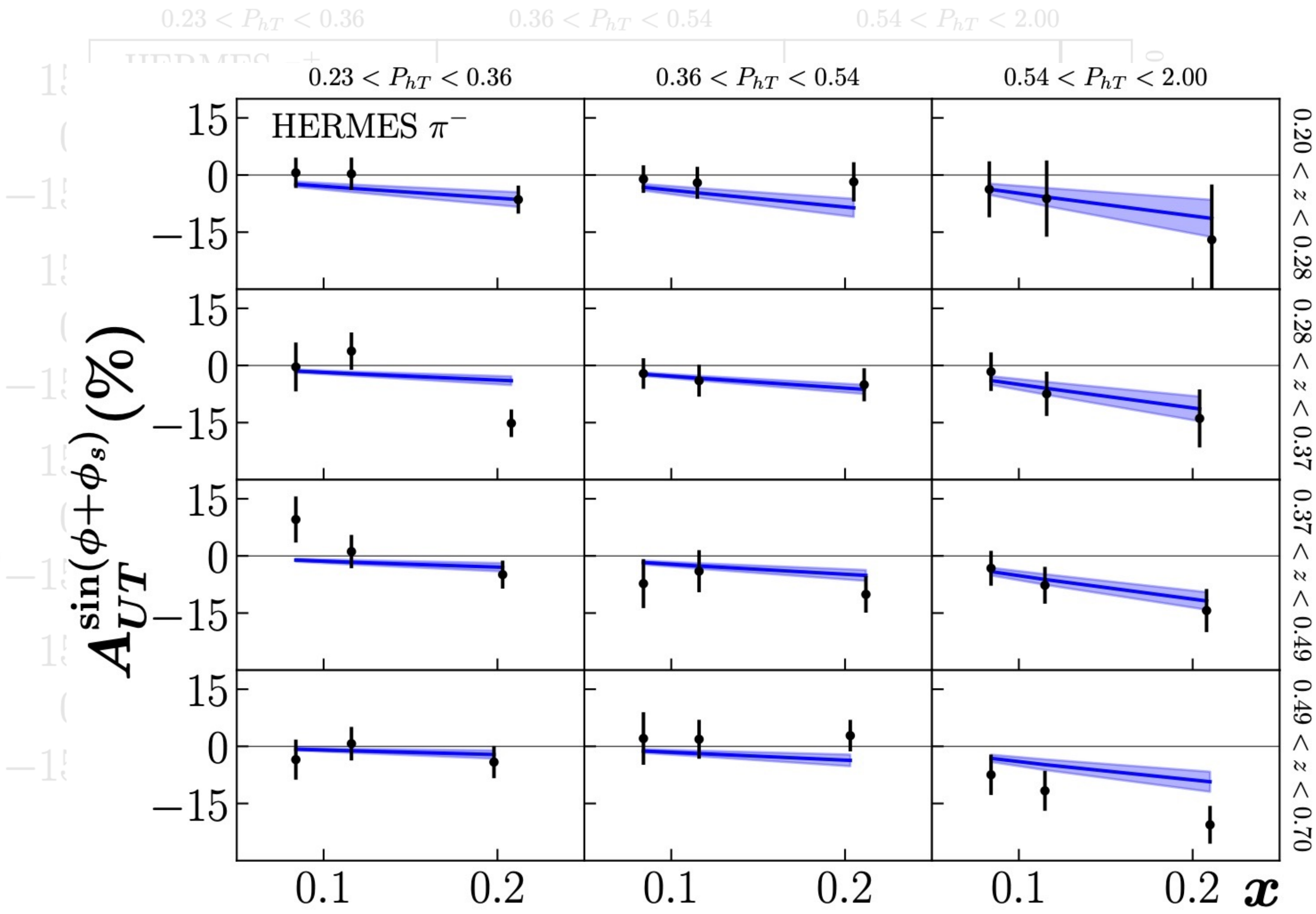
Generate "data" (central value and 1-σ uncertainty) using recent simultaneous fit of f_1 and g_1 from Cocuzza, et al. (2022) and add to the χ^2 if SB is violated by more than the uncertainty in the data

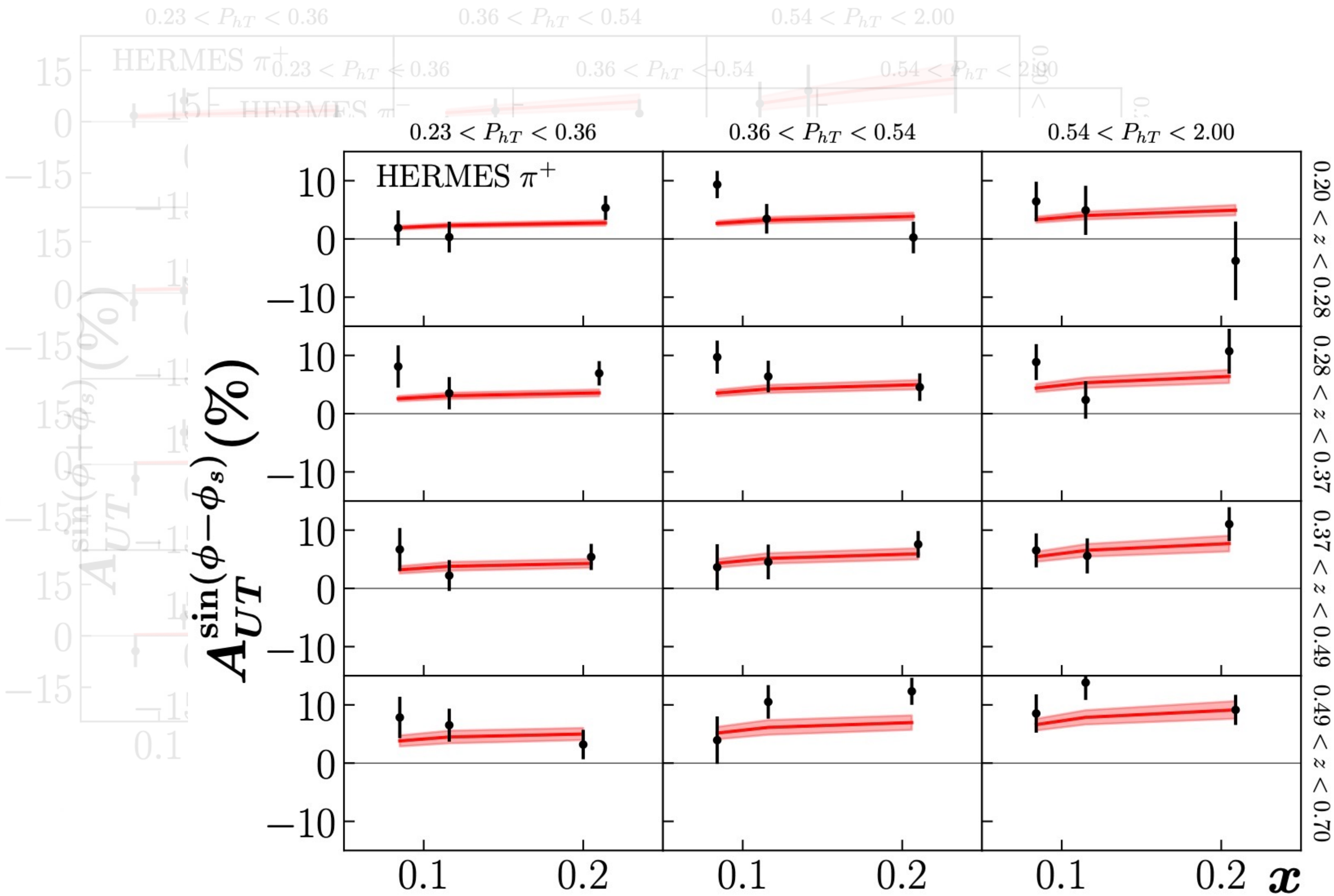
$$\chi^2/N_{\text{pts.}} = 647/634 = 1.02$$

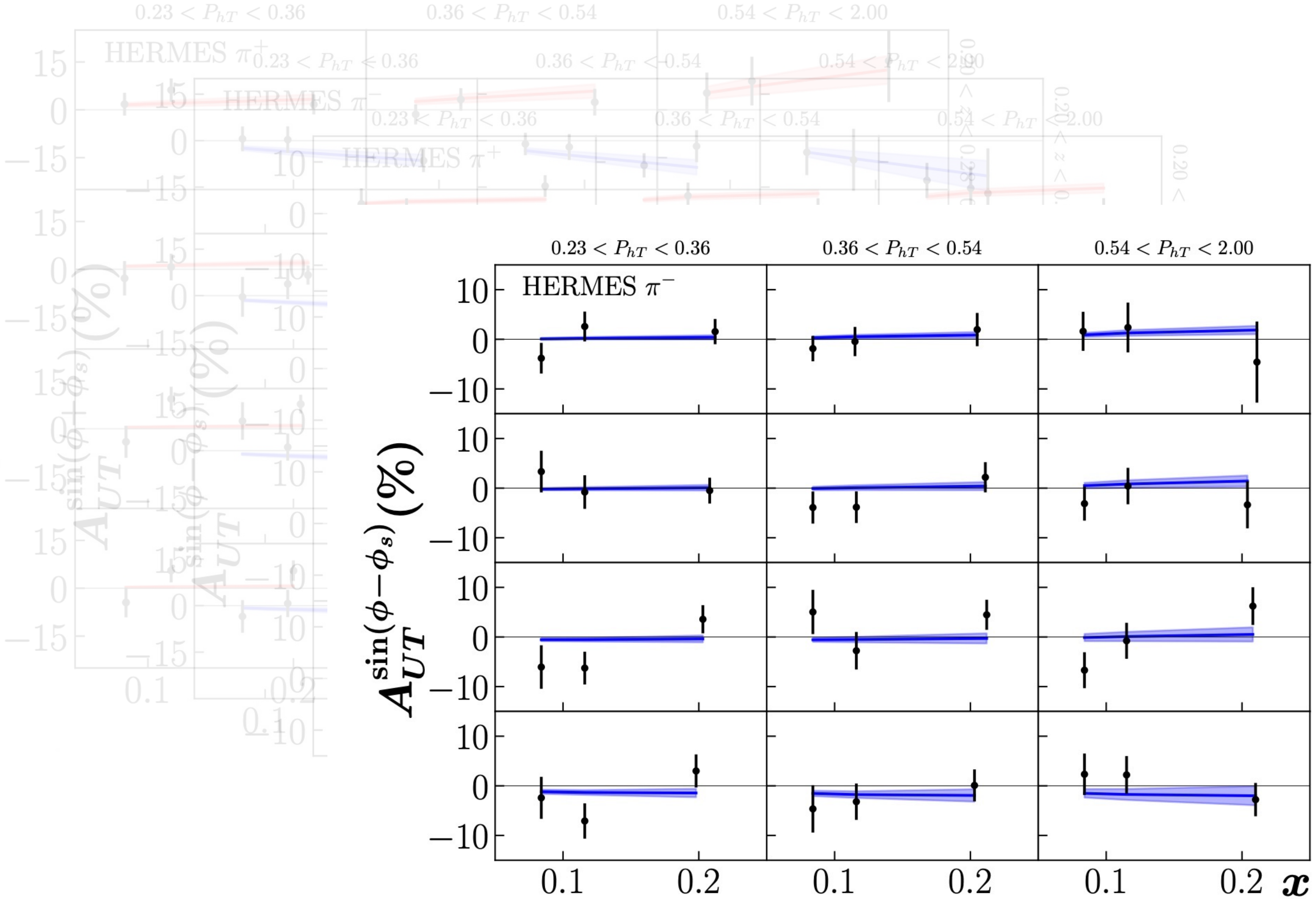
Observable	Reactions	Non-Perturbative Function(s)	χ^2/npts
$A_{UT}^{\sin(\phi_h-\phi_S)}$	$e+(p,d)^\uparrow \to e+(\pi^+,\pi^-,\pi^0)+X$	$f_{1T}^\perp(x,\vec{k}_T^2)$	$182.9/166 = 1.10$
$A_{UT}^{\sin(\phi_h+\phi_S)}$	$e+(p,d)^\uparrow \to e+(\pi^+,\pi^-,\pi^0)+X$	$h_1(x,\vec{k}_T^2), H_1^\perp(z,z^2\vec{p}_T^2)$	$181.0/166 = 1.09$
$^*A_{UT}^{\sin\phi_S}$	$e+p^\uparrow \to e+(\pi^+,\pi^-,\pi^0)+X$	$h_1(x), \tilde{H}(z)$	$18.6/36 = 0.52$
$A_{UC/UL}$	$e^+ + e^- \to \pi^+\pi^-(UC,UL)+X$	$H_1^\perp(z,z^2\vec{p}_T^2)$	$154.9/176 = 0.88$
$A_{T,\mu^+\mu^-}^{\sin\phi_S}$	$\pi^- + p^\uparrow \to \mu^+\mu^- + X$	$f_{1T}^\perp(x,\vec{k}_T^2)$	$6.92/12 = 0.58$
$A_N^{W/Z}$	$p^\uparrow + p \to (W^+,W^-,Z)+X$	$f_{1T}^\perp(x,\vec{k}_T^2)$	$30.8/17 = 1.81$
A_N^π	$p^\uparrow + p \to (\pi^+,\pi^-,\pi^0)+X$	$h_1(x), F_{FT}(x,x) = \frac{1}{\pi} f_{1T}^{\perp(1)}(x), H_1^{\perp(1)}(z), \tilde{H}(z)$	$70.4/60 = 1.17$
Lattice g_T	——	$h_1(x)$	$1.82/1 = 1.82$

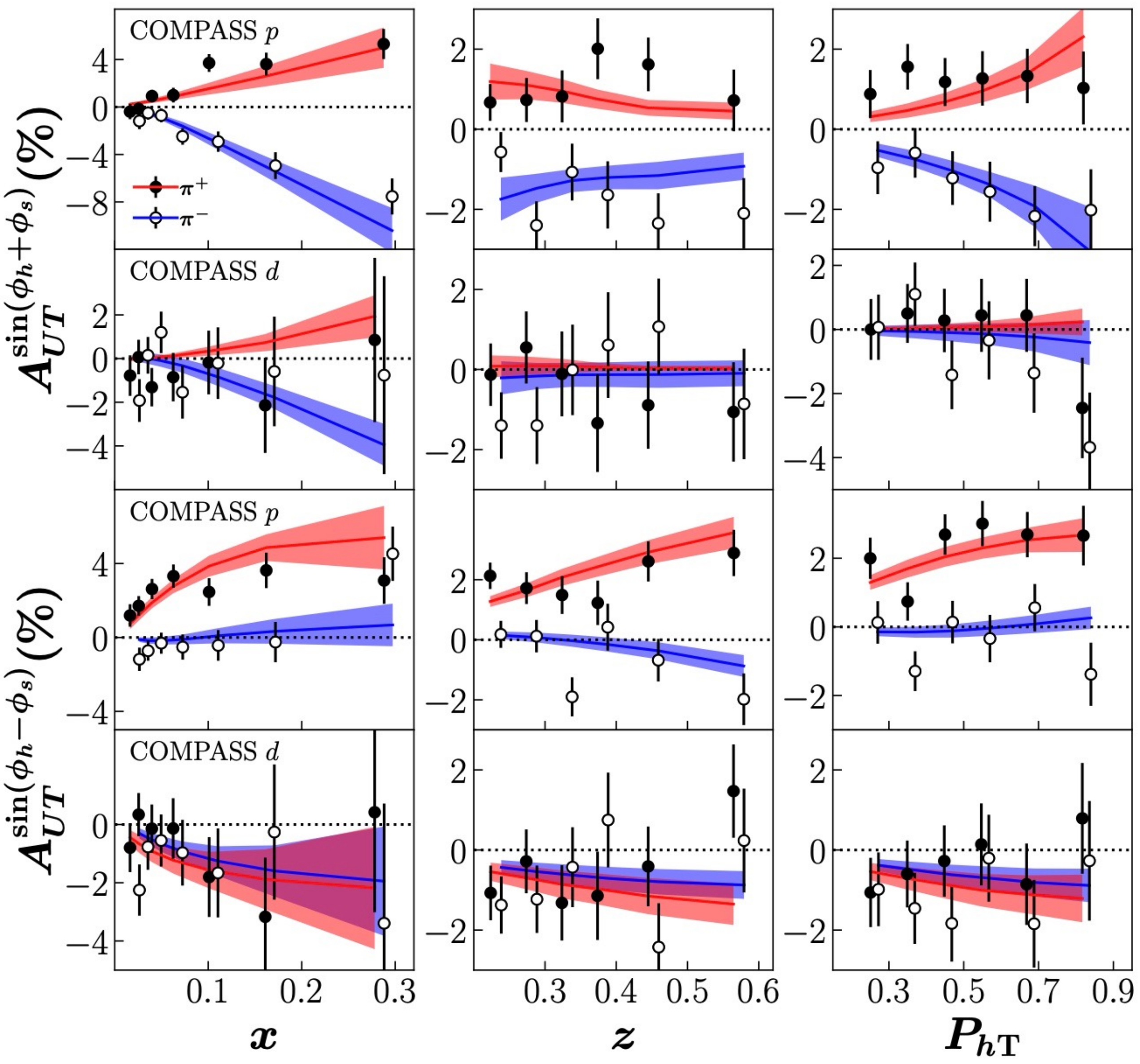

$A_N^\pi(\%)$
BRAHMS
$\theta = 2.3$
$\theta = 4.0$
π^+
π^-
x_F
HERMES p
$A_{UT}^{\sin\phi_s}(\%)$
x
z
STAR π^0
$\eta \in [3.3, 4.1]$
$\eta = 3.68$
$\eta = 3.3$
$\eta = 3.7$

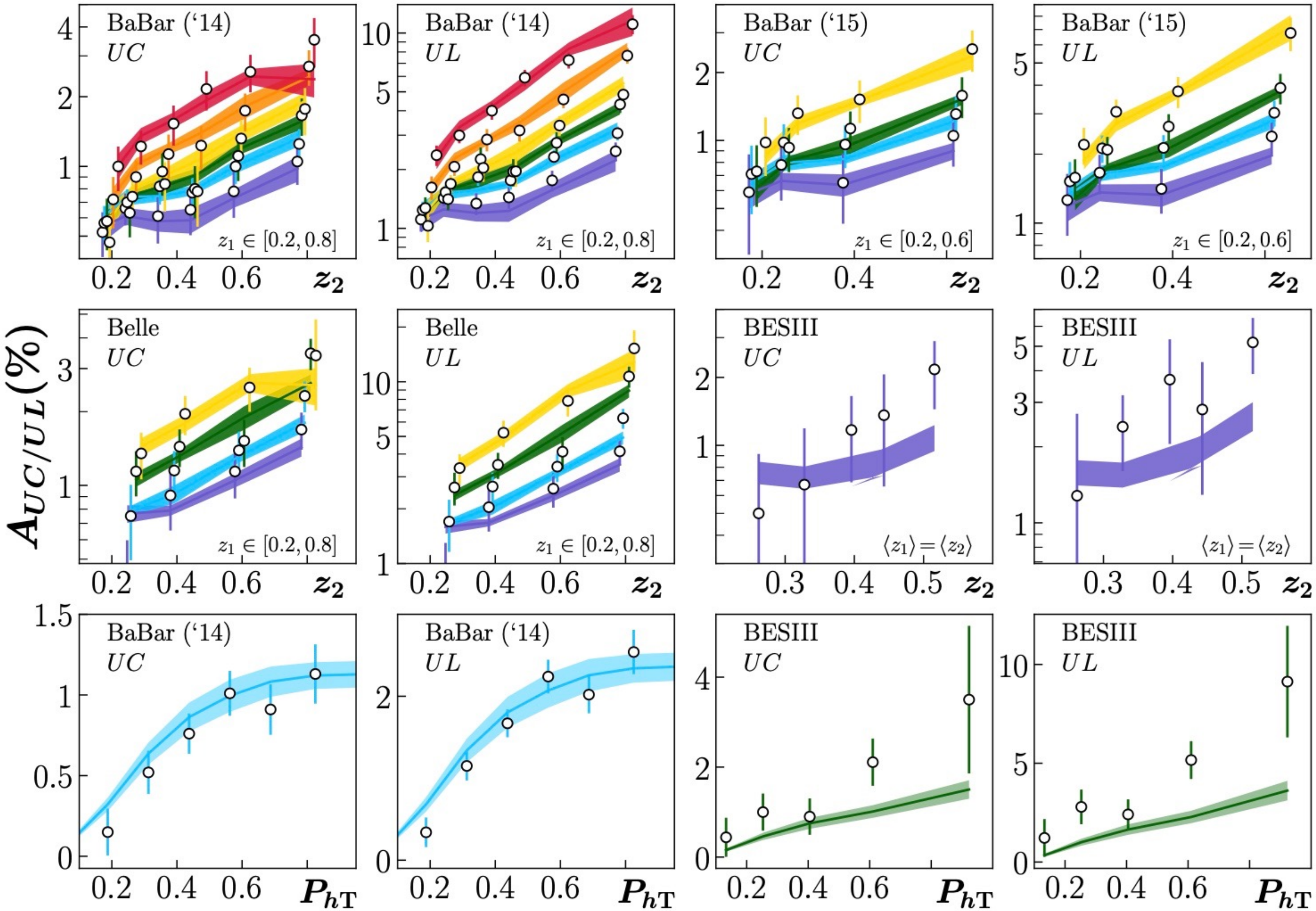

0.23 < P_hT < 0.36
0.36 < P_hT < 0.54
0.54 < P_hT < 2.00
HERMES π+
A_UT^sin(φ+φs) (%)
15
0
−15
0.1
0.2
x
0.20 < z < 0.28
0.28 < z < 0.37
0.37 < z < 0.49
0.49 < z < 0.70

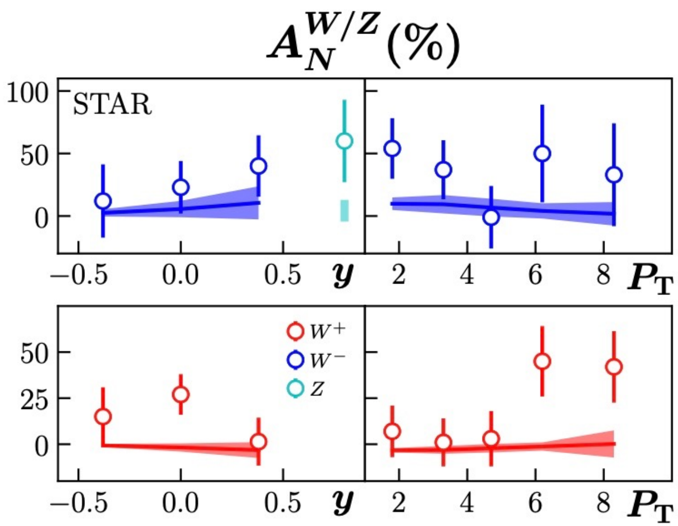

0.23 < P_{hT} < 0.36
0.36 < P_{hT} < 0.54
0.54 < P_{hT} < 2.00
HERMES π^-
$A_{UT}^{\sin(\phi+\phi_s)}$ (%)
15
0
−15
0.20 < z < 0.28
0.28 < z < 0.37
0.37 < z < 0.49
0.49 < z < 0.70
0.1
0.2
x

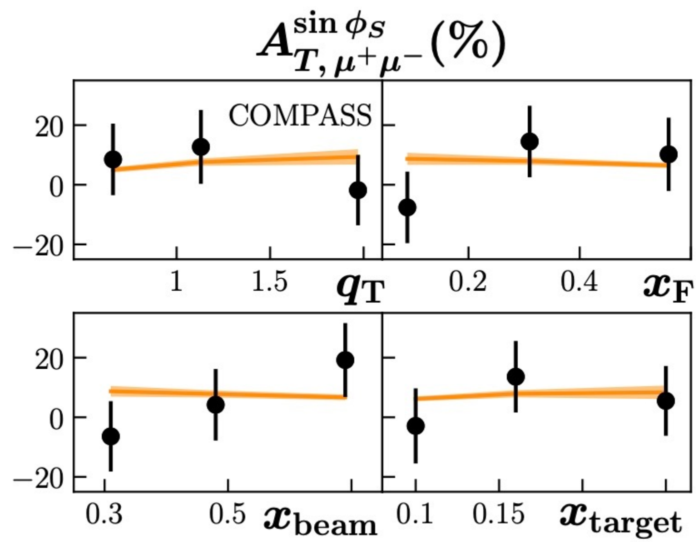

HERMES π^+
$0.23 < P_{hT} < 0.36$
$0.36 < P_{hT} < 0.54$
$0.54 < P_{hT} < 2.00$
$0.20 < z < 0.28$
$0.28 < z < 0.37$
$0.37 < z < 0.49$
$0.49 < z < 0.70$
$A_{UT}^{\sin(\phi-\phi_s)}(\%)$
x
10
0
−10
0.1
0.2

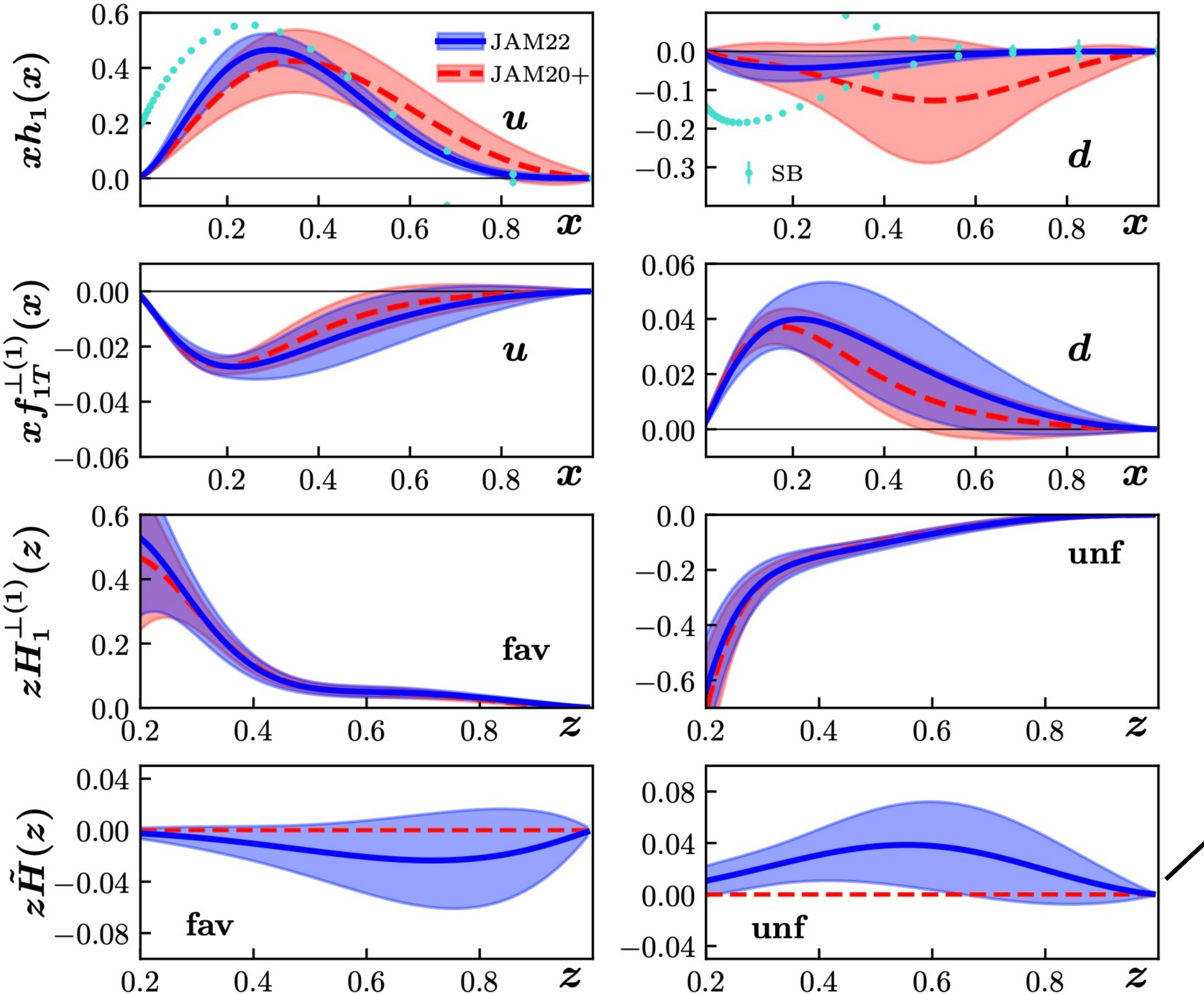

HERMES π^-
$0.23 < P_{hT} < 0.36$
$0.36 < P_{hT} < 0.54$
$0.54 < P_{hT} < 2.00$
$0.20 < z < 0.28$
$0.28 < z < 0.37$
$0.37 < z < 0.49$
$0.49 < z < 0.70$
$A_{UT}^{\sin(\phi-\phi_s)}$ (%)
x


COMPASS p
COMPASS d
COMPASS p
COMPASS d
π^+
π^-
$A_{UT}^{\sin(\phi_h+\phi_s)}$ (%)
$A_{UT}^{\sin(\phi_h-\phi_s)}$ (%)
x
z
P_{hT}


BaBar ('14)
UC
$z_1 \in [0.2, 0.8]$
BaBar ('14)
UL
$z_1 \in [0.2, 0.8]$
BaBar ('15)
UC
$z_1 \in [0.2, 0.6]$
BaBar ('15)
UL
$z_1 \in [0.2, 0.6]$
Belle
UC
$z_1 \in [0.2, 0.8]$
Belle
UL
$z_1 \in [0.2, 0.8]$
BESIII
UC
$\langle z_1 \rangle = \langle z_2 \rangle$
BESIII
UL
$\langle z_1 \rangle = \langle z_2 \rangle$
BaBar ('14)
UC
BaBar ('14)
UL
BESIII
UC
BESIII
UL
$A_{UC/UL}(\%)$
z_2
P_{hT}

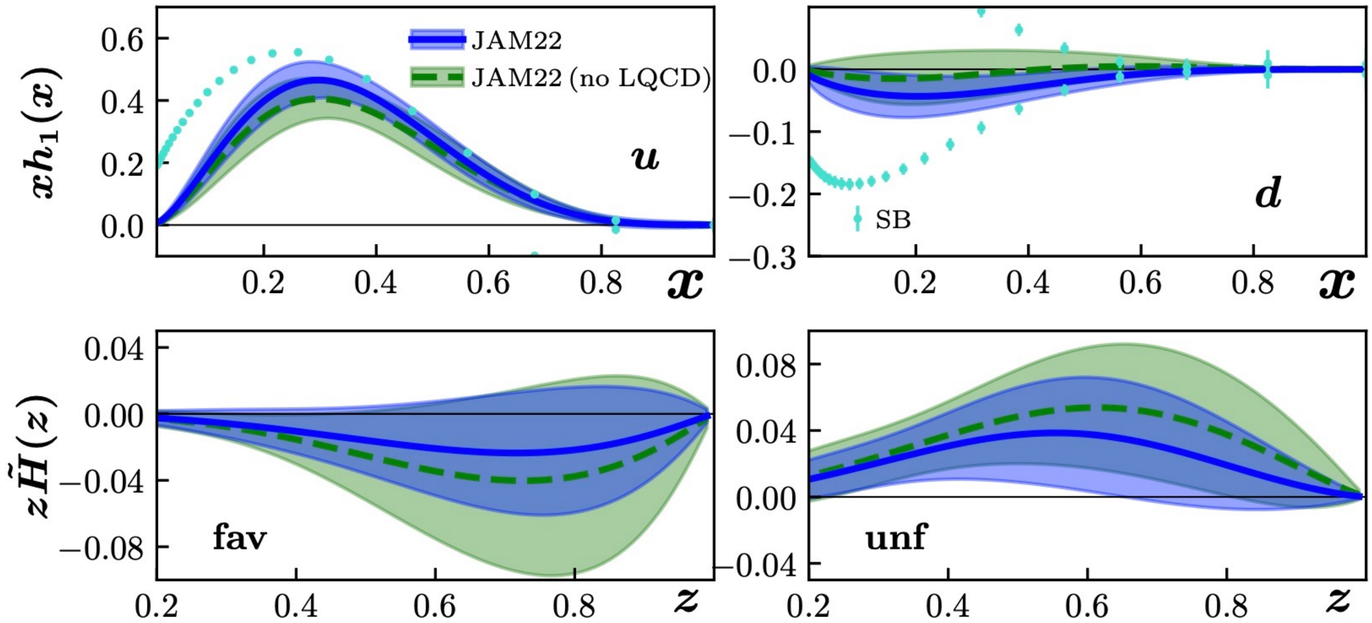

$A_N^{W/Z}$ (%)
STAR
100
50
0
−0.5
0.0
0.5
y
2
4
6
8
P_T
W^+
W^-
Z
25


$A_{T,\mu^+\mu^-}^{\sin\phi_S}$ (%)
COMPASS
20
0
−20
1
1.5
q_T
0.2
0.4
x_F
0.3
0.5
x_{beam}
0.1
0.15
x_{target}

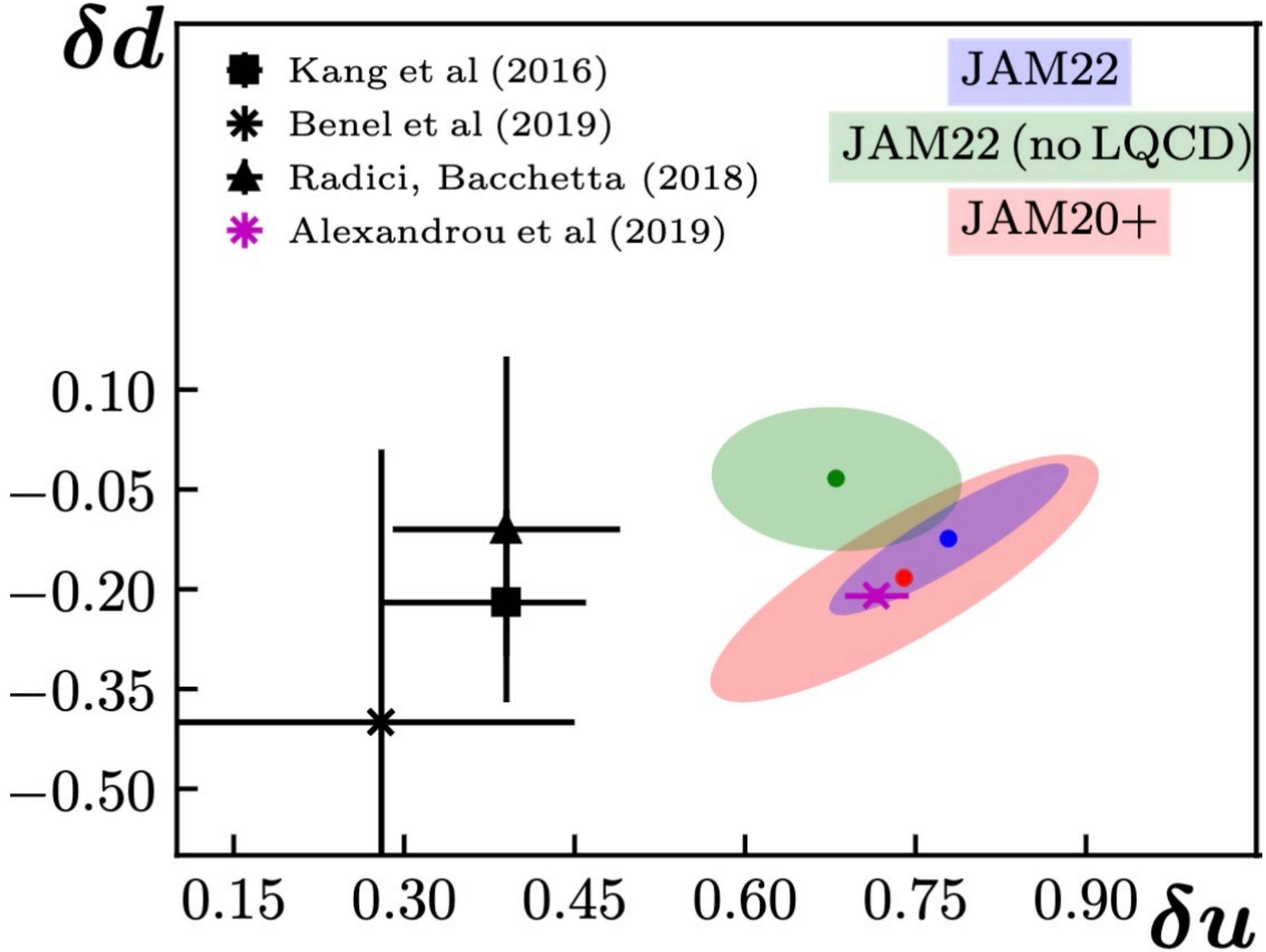


First direct information from experiment on $\tilde{H}(z)$

- Comments on the non-perturbative functions:
  - Transversity becomes much more tightly constrained by now imposing the SB and including the lattice g_T data point, in particular the latter



  - Collins and Sivers functions remain basically the same from JAM3D-20+
  - $\tilde{H}(z)$ behaves similar to the Collins function (favored and unfavored roughly equal in magnitude but opposite in sign) - expected since both are derived from the same underlying quark-gluon-quark FF (Kanazawa, et al. (2016))


δd
Kang et al (2016)
Benel et al (2019)
Radici, Bacchetta (2018)
Alexandrou et al (2019)
JAM22
JAM22 (no LQCD)
JAM20+
0.10
−0.05
−0.20
−0.35
−0.50
0.15
0.30
0.45
0.60
0.75
0.90
δu

- Dihadron (e.g., Radici, Bacchetta (2018); Benel, Courtoy, Ferro-Hernandez (2019)) and TMD analyses that only include e^+e^- and SIDIS Collins effect data (e.g., Kang, et al. (2016)), are generally below the lattice values for g_T and δu

- Note that because of the SB, one initially finds JAM3D-22 has more tension with lattice, but this does *not* imply phenomenology and lattice are incompatible – one can only fully answer this by including lattice data in the analysis

- **Once the the lattice g_T data point is included, we find the non-perturbative functions can accommodate it *and still describe the experimental data well***

➢ Slight update (JAM3D-22*) relevant for the next talk by Chris Cocuzza:

- "turn on" transversity antiquarks with $\bar{u} = -\bar{d}$

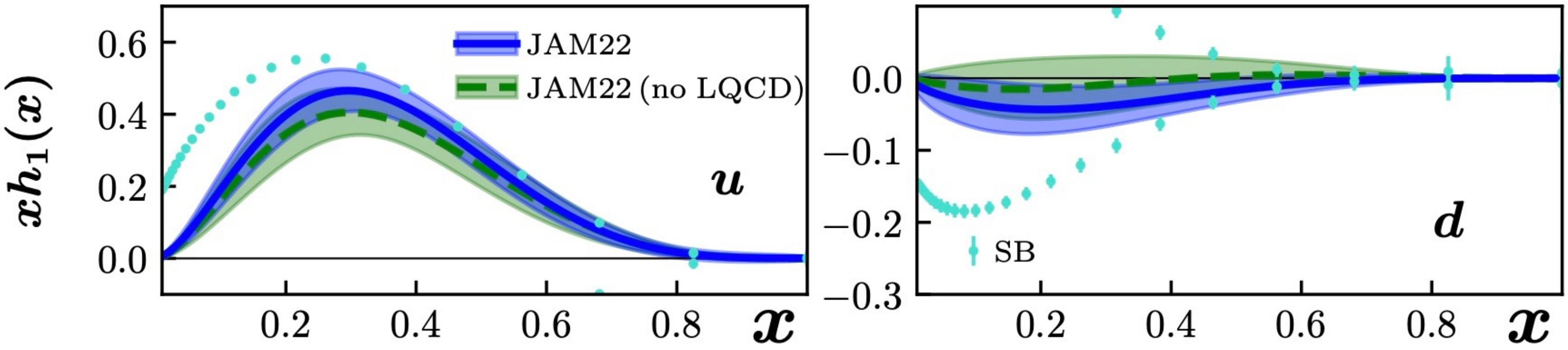
- $\delta u, \delta d$ from ETMC and PNDME are both included in the with lattice fit (rather than just g_T from ETMC)

- incorporate constraint on the "a" parameter from the small-x asymptotic behavior of transversity (Kovchegov, Sievert (2019))

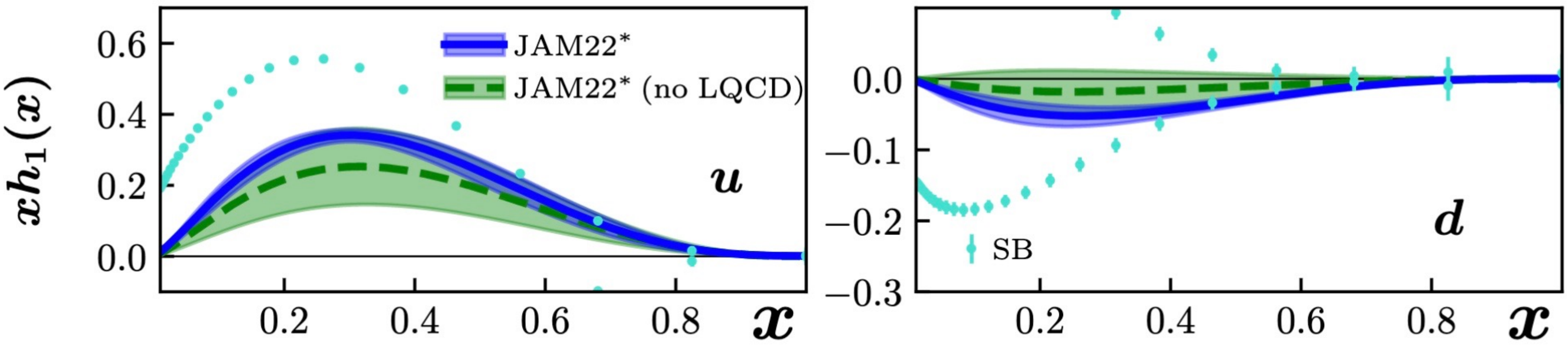
$$a \xrightarrow{x \to 0} 1 - 2\sqrt{\frac{\alpha_s N_c}{2\pi}} \quad \Longrightarrow \quad a = 0.24 \pm 0.12$$

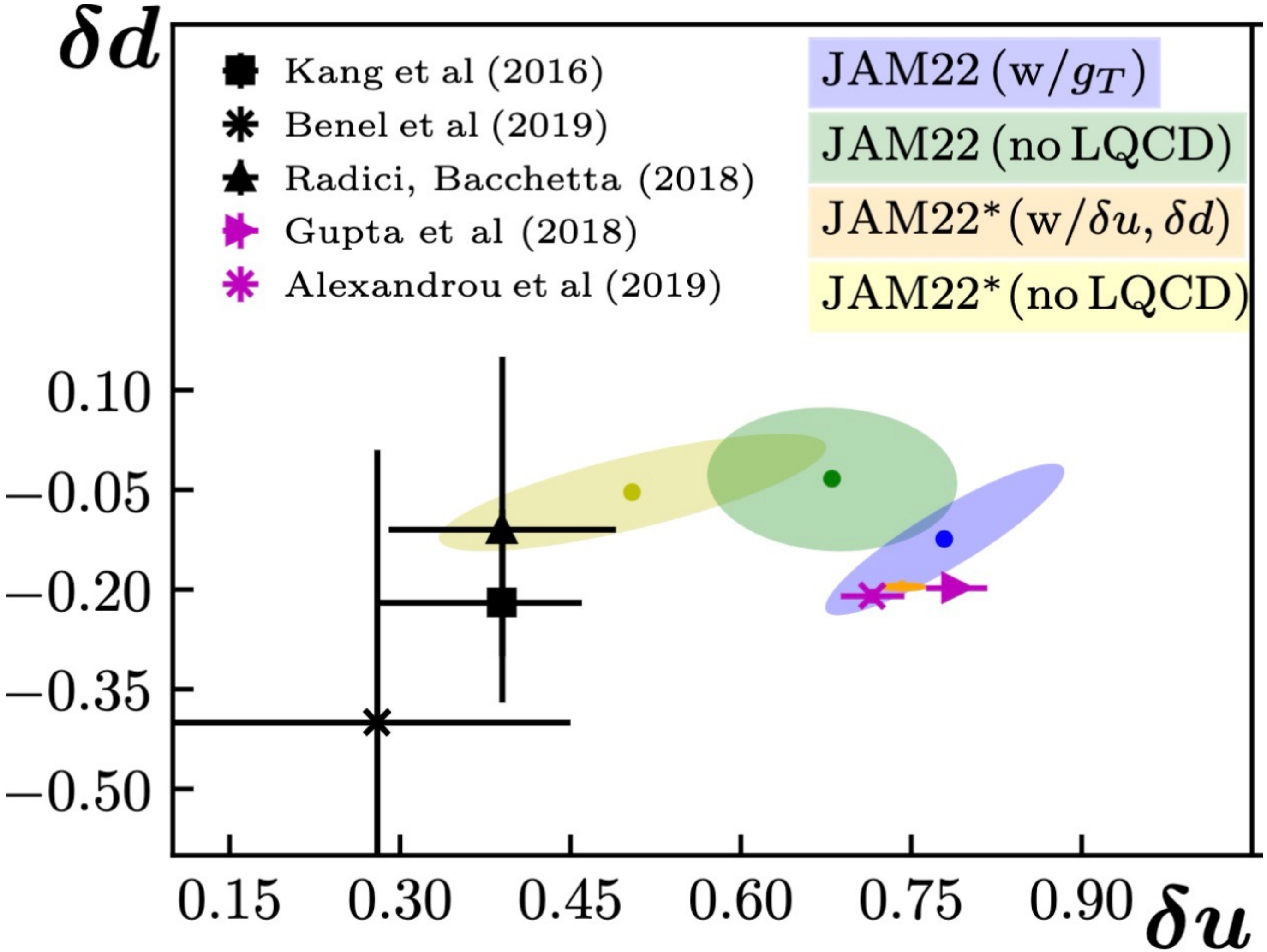
50% uncertainty due to unaccounted for $1/N_c$ and NLO corrections

JAM3D-22



JAM3D-22*





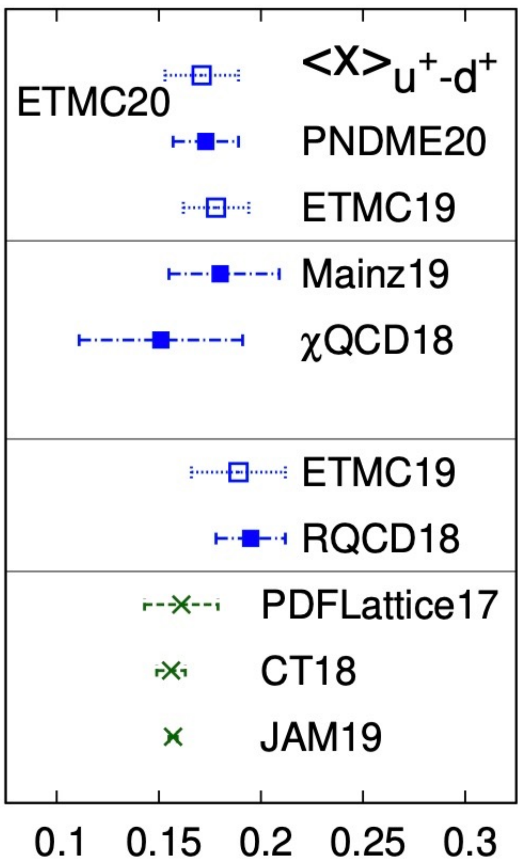
NB: **The experimental data is still described very well even when including $\boldsymbol{\delta u, \delta d}$ from lattice in the fit**

New DiFF Theory Developments

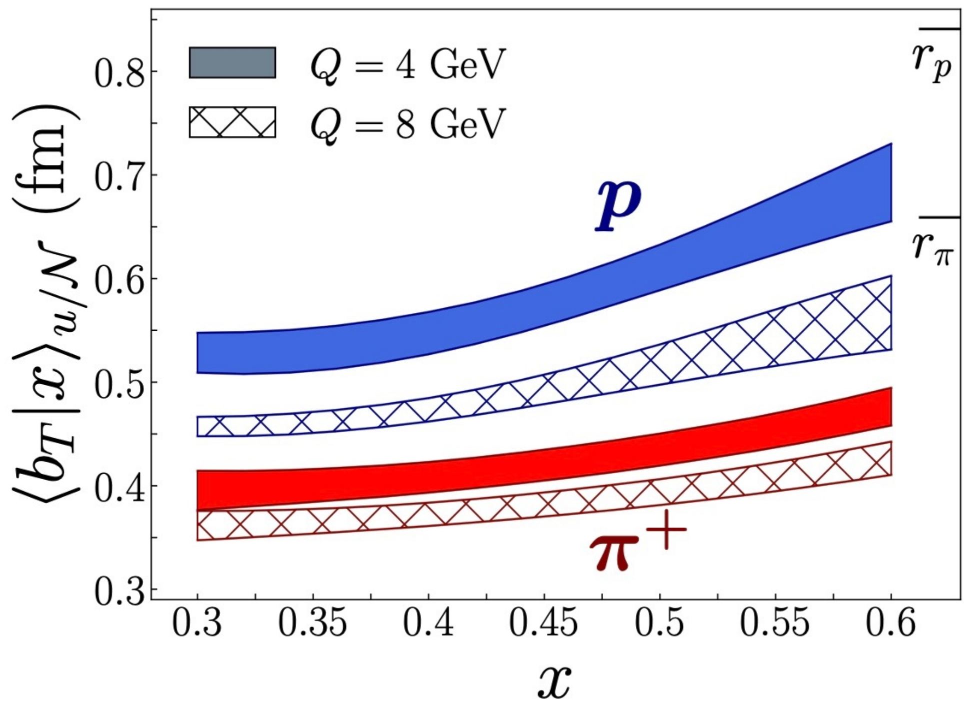
DP, Cocuzza, Metz, Prokudin, Sato, arXiv:2305.11995

- (TMD) PDFs and (single-hadron) FFs are defined in a way so that they are number densities (before renormalization)
  - *crucial property for our ability to use them to understand hadronic structure (e.g., calculating expectation values)*

Constantinou, et al., PDFLattice Report (2020)



Barry, et al. (2023) - JAM Collaboration



- (TMD) PDFs and (single-hadron) FFs are defined in a way so that they are number densities (before renormalization)
  - *certain sum rules are satisfied that can be used to constrain/cross-check phenomenological extractions of PDFs/FFs and model calculations*

<u>Number sum rules</u>

$$\sum_{i=u,d,s,..}\int_0^1 dx\,[f_1^{i/N}(x) - f_1^{\bar{i}/N}(x)] = \mathcal{B}$$

($\mathcal{B}$ is the baryon number, e.g.,= 3 for a proton)

$$\sum_h \int_0^1 dz\, D_1^{h/i}(z) = \mathcal{N}$$

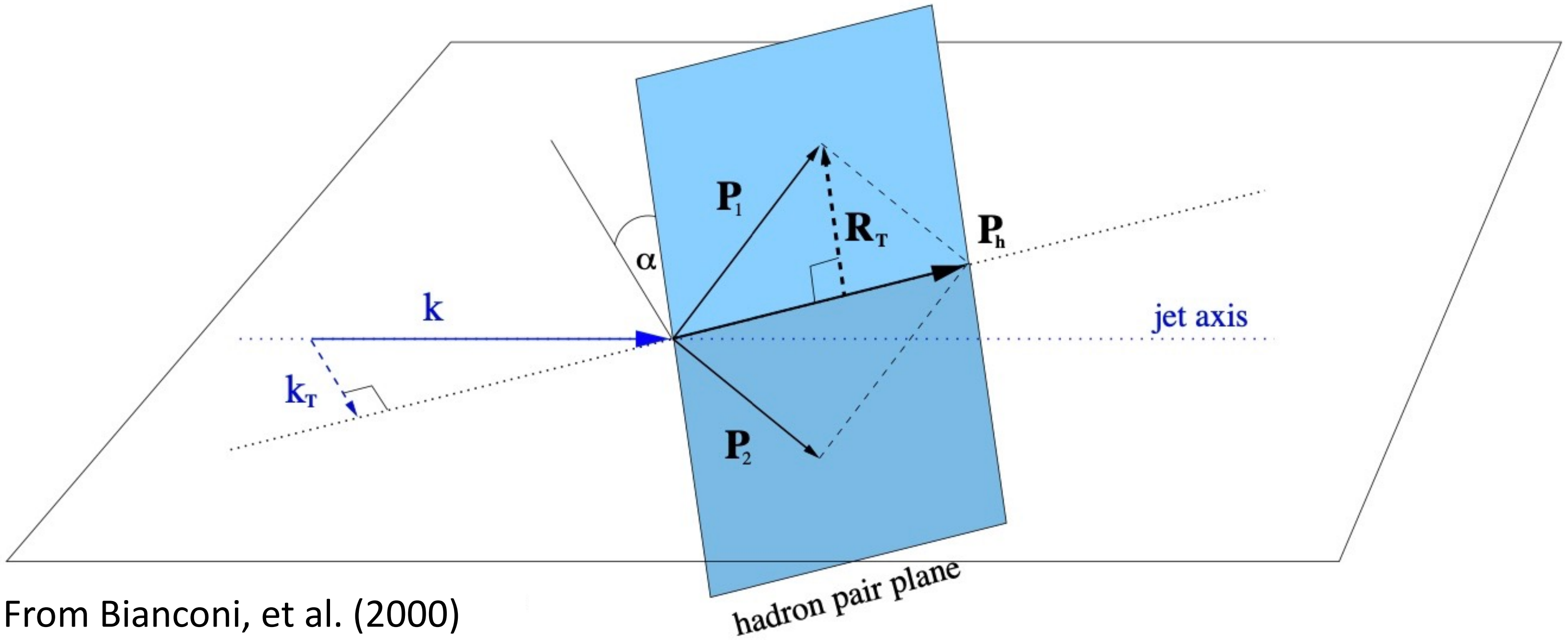
($\mathcal{N}$ is the total number of hadrons produced when the parton fragments)

<u>Momentum sum rules</u>

$$\sum_{\text{all } i}\int_0^1 dx\, x\, f_1^{i/N}(x) = 1 \qquad \sum_h \int_0^1 dz\, z\, D_1^{h/i}(z) = 1$$

NB: sum rules hold under renormalization

$$P_h = P_1 + P_2 \qquad R = (P_1 - P_2)/2 \qquad z = z_1 + z_2 \qquad \zeta = (z_1 - z_2)/z$$



From Bianconi, et al. (2000)

Single-hadron FF

$$D_1^{h/q}(z,\vec{P}_\perp^2) = \frac{1}{N_c}\frac{\mathbf{1}}{\mathbf{4z}}\sum_X\int\frac{d\xi^+ d^2\vec{\xi}_\perp}{(2\pi)^3}\,e^{ik^-\xi^+}\mathrm{Tr}\Big[\langle 0|\mathcal{W}(\infty,\xi)\psi_q(\xi^+,0^-,\vec{\xi}_\perp)|P;X\rangle \times\,\langle P;X|\bar{\psi}_q(0^+,0^-,\vec{0}_\perp)\mathcal{W}(0,\infty)|0\rangle\,\gamma^-\Big]$$

Dihadron FF

$$\frac{\mathbf{1}}{\mathbf{64\pi^3 z_1 z_2}}\mathrm{Tr}\Big[\Delta^{h_1h_2/q}(z_1,z_2,\vec{P}_{1\perp},\vec{P}_{2\perp})\gamma^-\Big] = D_1^{h_1h_2/q}(z_1,z_2,\vec{P}_{1\perp}^2,\vec{P}_{2\perp}^2,\vec{P}_{1\perp}\cdot\vec{P}_{2\perp})$$

$$\frac{1}{64\pi^3 z_1 z_2}\mathrm{Tr}\left[\Delta^{h_1 h_2/q}(z_1, z_2, \vec{P}_{1\perp}, \vec{P}_{2\perp})\gamma^-\right] = D_1^{h_1 h_2/q}(z_1, z_2, \vec{P}_{1\perp}^2, \vec{P}_{2\perp}^2, \vec{P}_{1\perp}\cdot\vec{P}_{2\perp})$$

This ***prefactor is key to the number density interpretation*** of dihadron FFs (see also Majumder, Wang (2004)) because in order to prove a number sum rule we need to introduce the number operator separately for each hadron (*j* = 1, 2)

$$\hat{N}_{h_j} \equiv \int \frac{dP_j^- d^2\vec{P}_{j\perp}}{(2\pi)^3\, 2P_j^-}\, \hat{a}_{h_j}^\dagger \hat{a}_{h_j} = \int \frac{dz_j d^2\vec{P}_{j\perp}}{(2\pi)^3\, 2z_j}\, \hat{a}_{h_j}^\dagger \hat{a}_{h_j}$$

$$\frac{1}{64\pi^3 z_1 z_2}\mathrm{Tr}\left[\Delta^{h_1 h_2/q}(z_1, z_2, \vec{P}_{1\perp}, \vec{P}_{2\perp})\gamma^-\right] = D_1^{h_1 h_2/q}(z_1, z_2, \vec{P}_{1\perp}^2, \vec{P}_{2\perp}^2, \vec{P}_{1\perp}\cdot\vec{P}_{2\perp})$$

⋮

$$\sum_{h_1}\int_0^1 dz_1 \int d^2\vec{P}_{1\perp} D_1^{h_1 h_2/q}(z_1, z_2, \vec{P}_{1\perp}^2, \vec{P}_{2\perp}^2, \vec{P}_{1\perp}\cdot\vec{P}_{2\perp}) = (\mathcal{N}-1)\, D_1^{h_2/q}(z_2, \vec{P}_{2\perp}^2)$$

$$\sum_{h_1}\sum_{h_2}\int_0^1 dz_2 \int_0^{1-z_2} dz_1 \int d^2\vec{P}_{1\perp}\int d^2\vec{P}_{2\perp} D_1^{h_1 h_2/q}(z_1, z_2, \vec{P}_{1\perp}^2, \vec{P}_{2\perp}^2, \vec{P}_{1\perp}\cdot\vec{P}_{2\perp}) = \mathcal{N}(\mathcal{N}-1)$$

Total number of *hadron pairs* produced when the parton fragments

$$\frac{1}{64\pi^3 z_1 z_2}\mathrm{Tr}\left[\Delta^{h_1 h_2/q}(z_1, z_2, \vec{P}_{1\perp}, \vec{P}_{2\perp})\gamma^-\right] = D_1^{h_1 h_2/q}(z_1, z_2, \vec{P}_{1\perp}^2, \vec{P}_{2\perp}^2, \vec{P}_{1\perp}\cdot\vec{P}_{2\perp})$$

⋮

$$\sum_{h_1}\int_0^1 dz_1 \int d^2\vec{P}_{1\perp} D_1^{h_1 h_2/q}(z_1, z_2, \vec{P}_{1\perp}^2, \vec{P}_{2\perp}^2, \vec{P}_{1\perp}\cdot\vec{P}_{2\perp}) = (\mathcal{N}-1)\, D_1^{h_2/q}(z_2, \vec{P}_{2\perp}^2)$$

$$\sum_{h_1}\sum_{h_2}\int_0^1 dz_2 \int_0^{1-z_2} dz_1 \int d^2\vec{P}_{1\perp}\int d^2\vec{P}_{2\perp} D_1^{h_1 h_2/q}(z_1, z_2, \vec{P}_{1\perp}^2, \vec{P}_{2\perp}^2, \vec{P}_{1\perp}\cdot\vec{P}_{2\perp}) = \mathcal{N}(\mathcal{N}-1)$$

Total number of *hadron pairs* produced when the parton fragments

The proof is not possible if a prefactor of 1/(4z) = 1/(4(z_1+z_2)) is used!

Number sum rule

$$\sum_{h_1}\sum_{h_2}\int_0^1 dz_2 \int_0^{1-z_2} dz_1 \int d^2\vec{P}_{1\perp}\int d^2\vec{P}_{2\perp}\, D_1^{h_1 h_2/i}(z_1, z_2, \vec{P}_{1\perp}^2, \vec{P}_{2\perp}^2, \vec{P}_{1\perp}\cdot\vec{P}_{2\perp}) = \mathcal{N}(\mathcal{N}-1)$$

$$\Longrightarrow \boldsymbol{D_1^{h_1 h_2/i}(w, x, \vec{Y}^2, \vec{Z}^2, \vec{Y}\cdot\vec{Z})} \equiv \mathcal{J}\cdot D_1^{h_1 h_2/i}(z_1, z_2, \vec{P}_{1\perp}^2, \vec{P}_{2\perp}^2, \vec{P}_{1\perp}\cdot\vec{P}_{2\perp})$$

is a number density

Jacobian for the variable transformation

Number sum rule

$$\sum_{h_1}\sum_{h_2}\int_0^1 dz_2\int_0^{1-z_2} dz_1\int d^2\vec{P}_{1\perp}\int d^2\vec{P}_{2\perp}\, D_1^{h_1h_2/i}(z_1,z_2,\vec{P}_{1\perp}^2,\vec{P}_{2\perp}^2,\vec{P}_{1\perp}\cdot\vec{P}_{2\perp}) = \mathcal{N}(\mathcal{N}-1)$$

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is a number density

Jacobian for the variable transformation

Using our new definition, DiFFs can now be interpreted as densities in any momentum variables of choice for the number of hadron pairs ($h_1 h_2$) fragmenting from the parton

***NB*: number density interpretation holds not only for unpolarized quarks (γ^- projection) but also for longitudinally ($\gamma^-\gamma^5$ projection) and transversely ($i\sigma^{i-}\gamma^5$ projection) polarized quarks**

Number sum rule

$$\sum_{h_1}\sum_{h_2}\int_0^1 dz_2 \int_0^{1-z_2} dz_1 \int d^2\vec{P}_{1\perp}\int d^2\vec{P}_{2\perp}\, D_1^{h_1h_2/i}(z_1, z_2, \vec{P}_{1\perp}^2, \vec{P}_{2\perp}^2, \vec{P}_{1\perp}\cdot\vec{P}_{2\perp}) = \mathcal{N}(\mathcal{N}-1)$$

$\Longrightarrow$ $\boldsymbol{D_1^{h_1h_2/i}(w, x, \vec{Y}^2, \vec{Z}^2, \vec{Y}\cdot\vec{Z})} \equiv \mathcal{J}\cdot D_1^{h_1h_2/i}(z_1, z_2, \vec{P}_{1\perp}^2, \vec{P}_{2\perp}^2, \vec{P}_{1\perp}\cdot\vec{P}_{2\perp})$

is a number density

Momentum sum rule

$$\sum_{h_1}\int_0^{1-z_2} dz_1 \int d^2\vec{P}_{1\perp}\; z_1\, D_1^{h_1h_2/i}(z_1, z_2, \vec{P}_{1\perp}^{\,2}, \vec{P}_{2\perp}^{\,2}, \vec{P}_{1\perp}\cdot\vec{P}_{2\perp}) = (1-z_2)\, D_1^{h_2/i}(z_2, \vec{P}_{2\perp}^{\,2})$$

NB: $D_1^{h_1h_2/i}(z_1, z_2, \vec{P}_{1\perp}^{\,2}, \vec{P}_{2\perp}^{\,2}, \vec{P}_{1\perp}\cdot\vec{P}_{2\perp})/D_1^{h_2/i}(z_2, \vec{P}_{2\perp}^{\,2})$ is a conditional number density

- Connection to phenomenology - work in a frame where the dihadron has no transverse momentum

$$P_h = P_1 + P_2 \qquad R = (P_1 - P_2)/2 \qquad z = z_1 + z_2 \qquad \zeta = (z_1 - z_2)/z$$

$$(z_1, z_2, \vec{P}_{1\perp}, \vec{P}_{2\perp}) \longrightarrow (z, \zeta, \vec{k}_T, \vec{R}_T) : \quad \mathcal{J} = z^3/2$$

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$$D_1^{h_1 h_2/q}(z, \zeta, \vec{k}_T^2, \vec{R}_T^2, \vec{k}_T \cdot \vec{R}_T) = \frac{\boldsymbol{z}}{\boldsymbol{32\pi^3(1-\zeta^2)}} \mathrm{Tr}\Big[\Delta^{h_1 h_2/q}(z_1, z_2, \vec{P}_{1\perp}, \vec{P}_{2\perp})\gamma^-\Big]$$

is a number density in $(z, \zeta, \vec{k}_T, \vec{R}_T)$

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is a number density in $(z, \zeta, \vec{k}_T, \vec{R}_T)$

Compare this to the original definition of Bianconi, Boffi, Jakob, Radici (2000) that has been the basis for all dihadron research (sensitive to R_T) for the last 20+ years

$$D_1^{h_1 h_2/q,\mathrm{BBJR}}(z, \zeta, \vec{k}_T^2, \vec{R}_T^2, \vec{k}_T \cdot \vec{R}_T) = \frac{\boldsymbol{1}}{\boldsymbol{4z}} \mathrm{Tr}\left[\Delta^{h_1 h_2/q}(z_1, z_2, \vec{P}_{1\perp}, \vec{P}_{2\perp})\gamma^-\right]$$

Does ***not*** allow sum rules to be derived that justify a number density interpretation

- Experimental measurements are sensitive to the so-called "extended" DiFFs where k_T (and usually $\boldsymbol{\zeta}$) is integrated out

$$\frac{z}{32\pi^3(1-\zeta^2)}\int d^2\vec{k}_T\,\mathrm{Tr}\Big[\Delta^{h_1h_2/q}(z_1,z_2,\vec{P}_{1\perp},\vec{P}_{2\perp})\gamma^-\Big]=D_1^{h_1h_2/q}(z,\zeta,\vec{R}_T^2)$$

$$\frac{z}{32\pi^3(1-\zeta^2)}\int d^2\vec{k}_T\,\mathrm{Tr}\Big[\Delta^{h_1h_2/q}(z_1,z_2,\vec{P}_{1\perp},\vec{P}_{2\perp})i\sigma^{i-}\gamma_5\Big]=-\frac{\epsilon_T^{ij}R_T^j}{M_h}H_1^{\sphericalangle\,h_1h_2/q}(z,\zeta,\vec{R}_T^2)$$

are number densities in $(z,\zeta,\vec{R}_T)$

chiral-odd "interference" FF (IFF) that can couple to transversity

- Experimental measurements are sensitive to the so-called "extended" DiFFs where k_T (and usually $\boldsymbol{\zeta}$) is integrated out

$$\frac{z}{32\pi^3(1-\zeta^2)}\int d^2\vec{k}_T\, \mathrm{Tr}\Big[\Delta^{h_1h_2/q}(z_1,z_2,\vec{P}_{1\perp},\vec{P}_{2\perp})\gamma^-\Big] = D_1^{h_1h_2/q}(z,\zeta,\vec{R}_T^2)$$

$$\frac{z}{32\pi^3(1-\zeta^2)}\int d^2\vec{k}_T\, \mathrm{Tr}\Big[\Delta^{h_1h_2/q}(z_1,z_2,\vec{P}_{1\perp},\vec{P}_{2\perp})i\sigma^{i-}\gamma_5\Big] = -\frac{\epsilon_T^{ij}R_T^j}{M_h}H_1^{\sphericalangle\, h_1h_2/q}(z,\zeta,\vec{R}_T^2)$$

are number densities in $(z,\zeta,\vec{R}_T)$

chiral-odd "interference" FF (IFF) that can couple to transversity

NB: Experiments report measurements in terms of M_h

$$\vec{R}_T^2 = \frac{1-\zeta^2}{4}M_h^2 - \frac{1-\zeta}{2}M_1^2 - \frac{1+\zeta}{2}M_2^2$$

One *cannot* simply replace the R_T dependence in the DiFF with an M_h and still maintain a number density interpretation in M_h

$$(z_1,z_2,\vec{P}_{1\perp},\vec{P}_{2\perp}) \longrightarrow (z,\zeta,\vec{k}_T,M_h,\phi_{R_T}):\quad \mathcal{J} = z^3(1-\zeta^2)/8$$

- Experimental measurements are sensitive to the so-called "extended" DiFFs where k_T (and usually ζ) is integrated out

$$D_1^{h_1 h_2/i}(z, M_h) \equiv \frac{\pi}{2} M_h \int_{-1}^{1} d\zeta \, (1-\zeta^2) D_1^{h_1 h_2/i}(z, \zeta, \vec{R}_T^2)$$

is a number density in (z, M_h)

$$\langle \mathcal{O}(z, M_h) \rangle^{h_1 h_2/i} = \int dz \, dM_h \, \mathcal{O}(z, M_h) \, D_1^{h_1 h_2/i}(z, M_h)$$

Can meaningfully calculate expectation values!

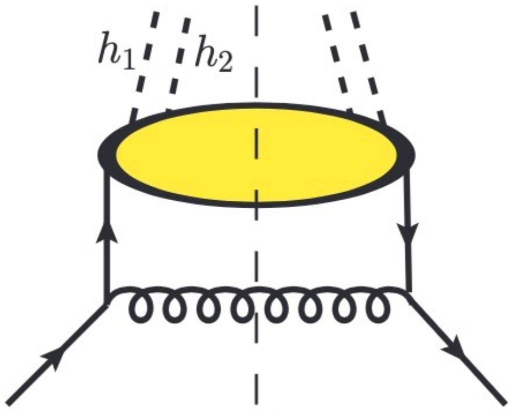
- Experimental measurements are sensitive to the so-called "extended" DiFFs where k_T (and usually ζ) is integrated out

$$D_1^{h_1h_2/i}(z, M_h) \equiv \frac{\pi}{2} M_h \int_{-1}^{1} d\zeta \,(1-\zeta^2) D_1^{h_1h_2/i}(z, \zeta, \vec{R}_T^2)$$

$$H_1^{\sphericalangle\, h_1h_2/i}(z, M_h) \equiv \frac{\pi}{2} M_h \int_{-1}^{1} d\zeta \, \frac{|\vec{R}_T|}{M_h}\,(1-\zeta^2) H_1^{\sphericalangle\, h_1h_2/i}(z, \zeta, \vec{R}_T^2)$$

The analytical formulas in the literature need to be modified to account for these new definitions

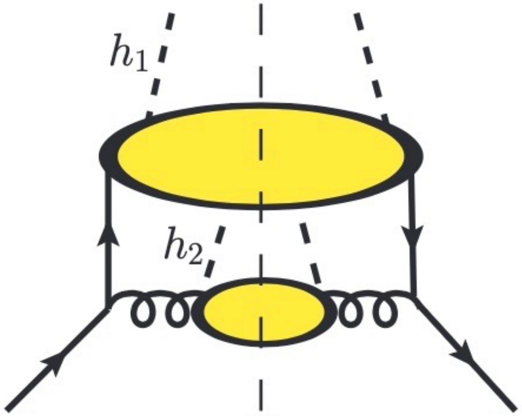
- Evolution equations for extended DiFFs



(a)

$$D_1^{h_1 h_2/i} \to D_1^{h_1 h_2/j}$$

"Homogeneous term"

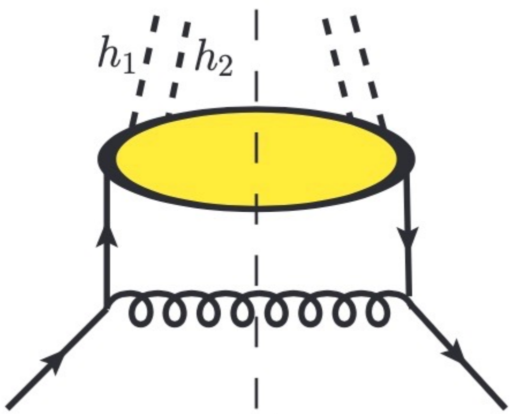


(b)

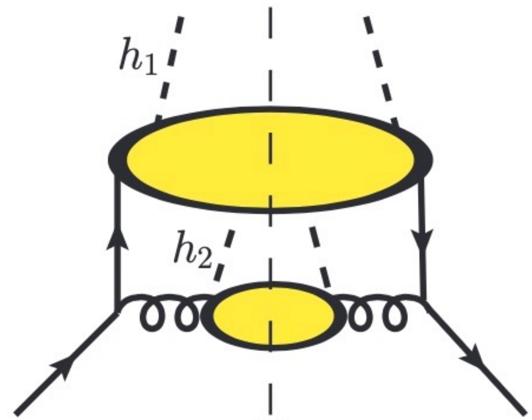
$$D_1^{h_1 h_2/i} \to D_1^{h_1/j} \otimes D_1^{h_2/k}$$

"Inhomogeneous term"

- Evolution equations for extended DiFFs



(a)



(b)

$$D_1^{h_1 h_2/i} \to D_1^{h_1 h_2/j}$$

"Homogeneous term"

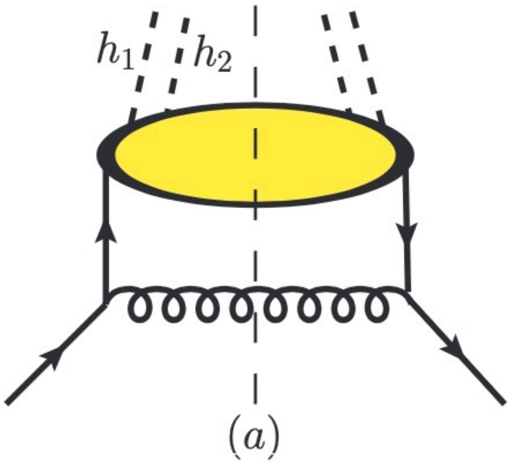
$$D_1^{h_1 h_2/i} \to D_1^{h_1/j} \otimes D_1^{h_2/k}$$

"Inhomogeneous term"

$$D_1^{h_1 h_2/q,(\mathrm{b})}(z_1, z_2, \vec{R}_T^2; \mu) = \frac{1}{\vec{R}_T^2} \frac{C_F \alpha_s}{2\pi^2} \mu^{2\epsilon} \int_{z_1}^{1-z_2} \frac{dw}{w(1-w)} D_1^{h_1/q}(z_1/w) D_1^{h_2/g}(z_2/(1-w)) \frac{1+w^2}{1-w}$$

The inhomogeneous terms are *not* UV divergent when one keeps the dependence on R_T (see also Ceccopieri, et al. (2007))

- Evolution equations for extended DiFFs



The evolution equations of the extended DiFFs are the same as single-hadron collinear FFs

$$\frac{\partial \mathcal{D}^{h_1 h_2/i}(z,\zeta,\vec{R}_T^2;\mu)}{\partial \ln \mu^2} = \sum_{i'} \int_z^1 \frac{dw}{w} \mathcal{D}^{h_1 h_2/i'}\Big(\frac{z}{w},\zeta,\vec{R}_T^2;\mu\Big) P_{i\to i'}(w)$$

where $\mathcal{D} = D_1 \text{ or } H_1^{\sphericalangle}$

- D_1: use unpolarized splitting kernels
- $H_1^{\sphericalangle}$: use transversely polarized splitting kernels

Summary

- We have updated our JAM3D-20 analysis using new data from HERMES (3D-binned Collins and Sivers effects and $A_{UT}^{\sin\phi_S}$) as well as constraints from lattice QCD (tensor charge g_T) and the Soffer bound on transvesity.
- Our JAM3D-22 results show it is still possible to accommodate these data/constraints and describe all TSSAs. The newly extracted transversity function and associated tensor charges are much more precise. We also have the first direct information from experiment on $\tilde{H}(z)$.
- We can eventually include x-dependent lattice data on transversity into phenomenology (more constraining than the tensor charge data).
- We have introduced a ***new definition of dihadron fragmentation functions that is consistent with a number density interpretation***, giving these functions a clear physical meaning.