

Rapidity-only TMD factorization

Ian Balitsky

TMDs' Synergy 23 June 2023

$$\frac{d\sigma}{d\eta d^2q_\perp} = \sum_{\text{flavors}} e_f^2 \int d^2k_\perp \mathcal{D}_{f/A}(x_A, k_\perp) \mathcal{D}_{f/B}(x_B, q_\perp - k_\perp) C(q, k_\perp) \\ + \text{power corrections} + \text{"Y - terms"}$$

The quantities $\mathcal{D}_{f/A}(x_A, k_\perp)$, $\mathcal{D}_{f/B}(x_B, q_\perp - k_\perp)$, and $C(q, k_\perp)$ are defined with cutoffs. The dependence on the cutoffs cancels in their product order by order in α_s .

At moderate x_A, x_B : CSS/SCET approach. The TMDs $\mathcal{D}_{f/A}(x_A, k_\perp)$ are defined with a combination of UV and rapidity cutoffs.

At $x_A, x_B \ll 1$: k_T -factorization approach. The TMDs are defined with rapidity-only cutoffs.

No CSS approach to small x (\Leftrightarrow nobody tried). Recently, there is a SCET activity in the small- x direction (LO BFKL level for now)

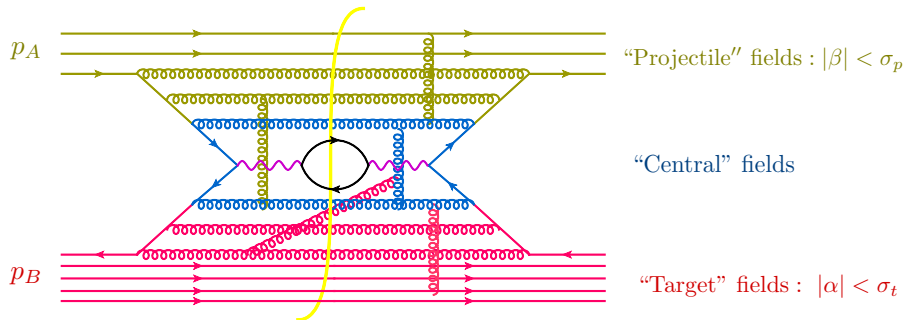
It is possible to study TMD factorization at moderate x using small- x methods (rapidity-only factorization etc.) (A. Tarasov, G. Chirilli, I.B, 2015-2023)

Example: power corrections $\sim \frac{1}{Q^2}$ for small- x DY hadronic tensor \Rightarrow EM gauge invariance of DY tensor.

TMD factorization from rapidity factorization (A. Tarasov and I.B.)

Sudakov variables: $\alpha = x_A$, $\beta = x_B$

$$p = \alpha p_1 + \beta p_2 + p_\perp, \quad p_1 \simeq p_A, \quad p_2 \simeq p_B, \quad p_1^2 = p_2^2 = 0$$



The result of the integration over “central” fields in the background of projectile and target fields is a series of TMD operators made from projectile (or target) fields multiplied by powers of $\frac{1}{Q^2} \Rightarrow$ **power corrections**

Result for $W_{\mu\nu}$ for unpolarized hadrons up to $\frac{1}{Q^2}$

Power corrections are \sim leading twist $\times \left(\frac{q_\perp}{Q}$ or $\frac{q_\perp^2}{Q^2}\right) \times \left(1 + \frac{1}{N_c} + \frac{1}{N_c^2}\right)$.

(Pleasant) surprise: all but one terms not suppressed by $\frac{1}{N_c}$ are determined by the leading-twist TMDs due to QCD equations of motion

Result:

$$W_{\mu\nu}^1(q) = W_{\mu\nu}^{1F}(q) + W_{\mu\nu}^{1H}(q),$$

$$W_{\mu\nu}^{1F}(q) = \sum_f e_f^2 W_{\mu\nu}^{fF}(q), \quad W_{\mu\nu}^{fF}(q) = \frac{1}{N_c} \int d^2k_\perp F^f(q, k_\perp) \mathcal{W}_{\mu\nu}^F(q, k_\perp),$$

$$W_{\mu\nu}^{1H}(q) = \sum_f e_f^2 W_{\mu\nu}^{fH}(q), \quad W_{\mu\nu}^{fH}(q) = \frac{1}{N_c} \int d^2k_\perp H^f(q, k_\perp) \mathcal{W}_{\mu\nu}^H(q, k_\perp)$$

where F^f and H^f are ($\alpha_q \equiv x_A, \beta_q \equiv x_B$)

$$F^f(q, k_\perp) = f_1^f(\alpha_q, k_\perp) \bar{f}_1^f(\beta_q, (q - k)_\perp) + f_1^f \leftrightarrow \bar{f}_1^f$$

$$H^f(q, k_\perp) = h_{1f}^\perp(\alpha_q, k_\perp) \bar{h}_{1f}^\perp(\beta_q, (q - k)_\perp) + h_{1f}^\perp \leftrightarrow \bar{h}_{1f}^\perp$$

$$\begin{aligned}
 & \mathcal{W}_{\mu\nu}^F(q, k_\perp) \\
 &= -g_{\mu\nu}^\perp + \frac{1}{Q_\parallel^2} (q_\mu^\parallel q_\nu^\perp + q_\nu^\parallel q_\mu^\perp) + \frac{q_\perp^2}{Q_\parallel^4} q_\mu^\parallel q_\nu^\parallel + \frac{\tilde{q}_\mu \tilde{q}_\nu}{Q_\parallel^2} [q_\perp^2 - 4(k, q - k)_\perp] \\
 & - \left[\frac{\tilde{q}_\mu}{Q_\parallel^2} \left(g_{\nu i}^\perp - \frac{q_\nu^\parallel q_i}{Q_\parallel^2} \right) (q - 2k)_\perp^i + \mu \leftrightarrow \nu \right] \quad \tilde{q} \equiv \alpha_q p_1 - \beta_q p_2
 \end{aligned}$$

$$\begin{aligned}
 & m^2 \mathcal{W}_{\mu\nu}^H(q, k_\perp) \\
 &= -k_\mu^\perp (q - k)_\nu^\perp - k_\nu^\perp (q - k)_\mu^\perp - g_{\mu\nu}^\perp (k, q - k)_\perp + 2 \frac{\tilde{q}_\mu \tilde{q}_\nu - q_\mu^\parallel q_\nu^\parallel}{Q_\parallel^4} k_\perp^2 (q - k)_\perp^2 \\
 & - \left(\frac{q_\mu^\parallel}{Q_\parallel^2} [k_\perp^2 (q - k)_\nu^\perp + k_\nu^\perp (q - k)_\perp^2] + \frac{\tilde{q}_\mu}{Q_\parallel^2} [k_\perp^2 (q - k)_\nu^\perp - k_\nu^\perp (q - k)_\perp^2] + \mu \leftrightarrow \nu \right) \\
 & - \frac{\tilde{q}_\mu \tilde{q}_\nu + q_\mu^\parallel q_\nu^\parallel}{Q_\parallel^4} [q_\perp^2 - 2(k, q - k)_\perp] (k, q - k)_\perp - \frac{q_\mu^\parallel \tilde{q}_\nu + \tilde{q}_\mu q_\nu^\parallel}{Q_\parallel^4} (2k - q, q)_\perp (k, q - k)_\perp
 \end{aligned}$$

Angular coefficients of Z-boson production

In CMS and ATLAS experiments $s = 8$ TeV, $Q = 80 - 100$ GeV and Q_{\perp} varies from 0 to 120 GeV.

Our analysis is valid at $Q_{\perp} = 10 - 30$ GeV and $Y \simeq 0$ ($x_A \sim x_B \sim 0.1$) so that power corrections are small but sizable.

Angular distribution of DY leptons in the Collins-Soper frame ($c_{\phi} \equiv \cos \phi$, $s_{\phi} \equiv \sin \phi$ etc.)

$$\frac{d\sigma}{dQ^2 dy d\Omega_l} = \frac{3}{16\pi} \frac{d\sigma}{dQ^2 dy} \left[(1 + c_{\theta}^2) + \frac{A_0}{2} (1 - 3c_{\theta}^2) + A_1 s_{2\theta} c_{\phi} + \frac{A_2}{2} s_{\theta}^2 c_{2\phi} \right. \\ \left. + A_3 s_{\theta} c_{\phi} + A_4 c_{\theta} + A_5 s_{\theta}^2 s_{2\phi} + A_6 s_{2\theta} s_{\phi} + A_7 s_{\theta} s_{\phi} \right]$$

Easy-to-do approximations

- Large N_c
- Only TMD f_1 in the factorization approximation: $f_1(x, k_{\perp}^2) \simeq f(x)g(k_{\perp}^2)$
- Log accuracy: $f_1(x, k_{\perp}^2) \simeq \frac{f(x)}{k_{\perp}^2}$ and $Q^2 \gg k_{\perp}^2 \gg q_{\perp}^2$

With this approximations, only A_0 and A_2 can be calculated

Comparison of A_0 with LHC results

Logarithmic estimate of A_0

$$A_0 = \frac{Q_{\perp}^2}{m_z^2} \frac{1 + 2 \frac{\ln m_z^2 / Q_{\perp}^2}{\ln Q_{\perp}^2 / m^2}}{1 + \frac{Q_{\perp}^2}{m_z^2} \frac{\ln m_z^2 / Q_{\perp}^2}{\ln Q_{\perp}^2 / m^2}} \quad (*)$$

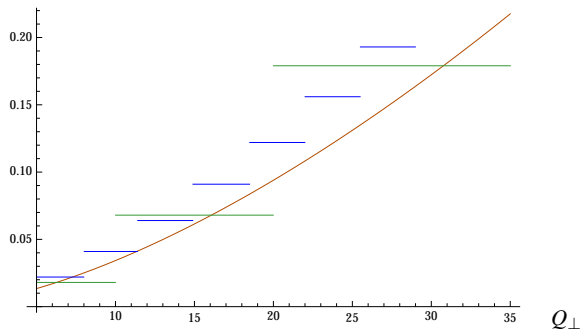


Figure: Comparison of prediction (*) with lines depicting angular coefficient A_0 in bins of Q_{\perp} and $Y < 1$ from [CMS \(arXiv:1504.03512\)](#) and [ATLAS \(arXiv:1606.00689\)](#)

Comparison of A_2 with LHC results

Logarithmic estimate of A_2

$$A_2 = \frac{Q_{\perp}^2}{m_z^2} \frac{1}{1 + \frac{Q_{\perp}^2 \ln m_z^2 / Q_{\perp}^2}{m_z^2 \ln Q_{\perp}^2 / m^2}} \quad (**)$$

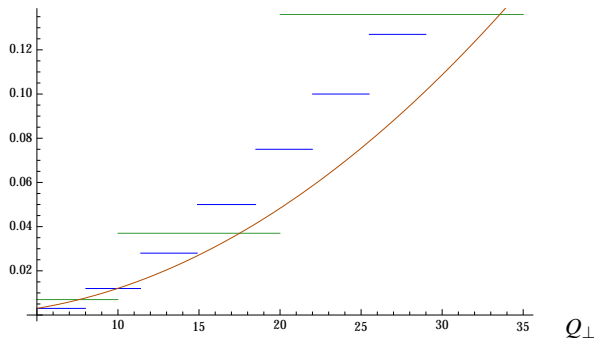
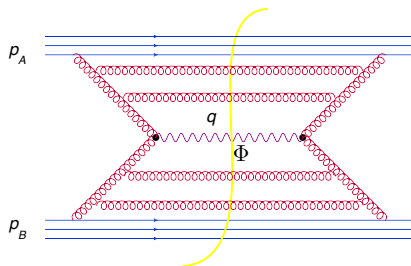


Figure: Comparison of prediction (**) with lines depicting angular coefficient A_2 in bins of Q_{\perp} and $Y < 1$ from **CMS** (arXiv:1504.03512) and **ATLAS** (arXiv:1606.00689)

Coefficient function for TMD factorization at one loop

Particle production by gluon-gluon fusion (point $gg\Phi$ vertex is a $\frac{m\mu}{m_t} \ll 1$ approximation for Higgs production.)



$$s \gtrsim Q^2 \gg Q_{\perp}^2 \gtrsim m^2$$

$$q^2 \equiv Q^2 = M_{\Phi}^2, \quad Q_{\perp}^2 \equiv q_{\perp}^2$$

Goal: one-loop TMD factorization formula for hadronic tensor.

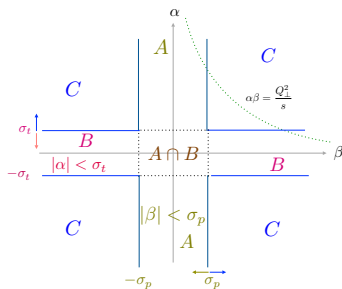
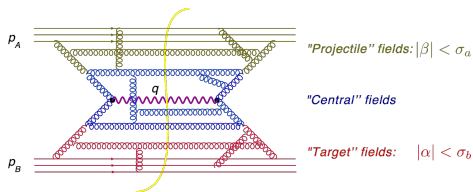
Result of calculation of one-loop coefficient function:

$$W(p_A, p_B; q) = \int db_{\perp} e^{i(q, b)_{\perp}} \mathcal{D}_{g/A}(x_A, b_{\perp}; \sigma_a) \mathcal{D}_{g/B}(x_B, b_{\perp}; \sigma_b)$$

$$\times \exp \left\{ \frac{\alpha_s N_c}{2\pi} \left[\ln^2 \frac{b_{\perp}^2 s \sigma_p \sigma_t}{4} - 2 \left(\ln \frac{\alpha_q}{\sigma_t} + \gamma \right) \left(\ln \frac{\beta_q}{\sigma_p} + \gamma \right) + \frac{\pi^2}{2} \right] \right\}$$

$$+ \text{NLO terms} \sim O(\alpha_s^2) + \text{power corrections}$$

Reminder: rapidity factorization of functional integral



Matching: $\ln \sigma_p$ in the projectile TMDs and $\ln \sigma_t$ in the target TMDs should cancel with $\ln \sigma_p$ and $\ln \sigma_t$ in the coefficient functions.

$A \cap B, k_\perp \sim m_\perp$:

Glauber gluons

$A \cap B, k_\perp \ll m_\perp$:

soft gluons

$A \cap B$ gluons \equiv soft/Glauber (sG) gluons

sG gluons cancel out

$\alpha_a \equiv x_A, \beta_b \equiv x_B$

Formal rescaling: $s = \zeta s_0, \zeta \rightarrow \infty, Q_\perp^2$ -fixed

Rapidity cutoffs: $\alpha_a \gg \sigma_t \gg \frac{Q_\perp^2}{\beta_b s} \sim \zeta^{-1}, \beta_b \gg \sigma_p \gg \frac{Q_\perp^2}{\alpha_a s} \sim \zeta^{-1}, \frac{\sigma_p \sigma_t s}{Q_\perp^2} \sim \zeta^{-1/2}$

Coefficient function in the functional-integral language

After integration over central fields

$$\begin{aligned}
 & \frac{1}{16} (N_c^2 - 1) \langle p'_A, p'_B | g^2 F_{\mu\nu}^a F^{a\mu\nu}(x_2) g^2 F_{\lambda\rho}^b F^{b\lambda\rho}(x_1) | p_A, p_B \rangle \\
 &= \int \mathcal{D}\Phi_{\mathcal{A}} \Psi_{p'_A}^*(t_i) \Psi_{p_A}(t_i) \Psi_{p'_B}^*(t_i) \Psi_{p_B}(t_i) \left[\mathcal{O}_{ij}^{\sigma_p}(x_2^-, x_{2\perp}; x_1^-, x_{1\perp}) \mathcal{O}^{ij;\sigma_t}(x_2^+, x_{2\perp}; x_1^+, x_{1\perp}) \right. \\
 & \quad + \int dz_1^- dz_{1\perp} dz_2^- dz_{2\perp} dw_1^+ dw_{1\perp} dw_2^+ dw_{2\perp} \frac{\alpha_s N_c}{2\pi} \mathfrak{C}_1(x_1, x_2; z_i^-, z_{i\perp}, w_i^+, w_{i\perp}; \sigma_p, \sigma_t) \\
 & \quad \left. \times \mathcal{O}_{ij}^{\sigma_p}(z_2^-, z_{2\perp}; z_1^-, z_{1\perp}) \mathcal{O}^{ij;\sigma_t}(z_2^+, z_{2\perp}; z_1^+, z_{1\perp}) + \dots \right]
 \end{aligned}$$

where $\mathcal{A} = \mathbf{A} + \mathbf{B} + s\mathbf{G}$

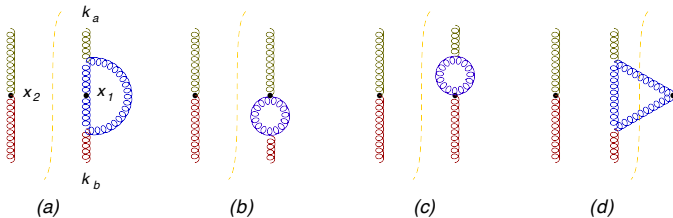
$$\text{and } \mathcal{O}(x^\pm, x_\perp; y^\pm, y_\perp) \equiv g^2 \check{F}^{\mp i}(x^\pm, x_\perp) [x, x - \infty^\pm] [-\infty^\pm + y, y] F^{\mp j}(y^\pm, y_\perp)$$

Calculation of coefficient function \mathfrak{C}_1 in the background field $\mathbb{A} = \bar{\mathbf{A}} + \bar{\mathbf{B}} + \bar{\mathbf{C}}$

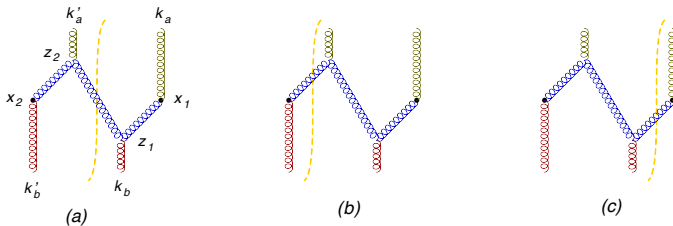
$$\begin{aligned}
 & \int dz_2^- dz_{2\perp} dz_1^- dz_{1\perp} dw_1^+ dw_{1\perp} dw_2^+ dw_{2\perp} \frac{\alpha_s N_c}{2\pi} \mathfrak{C}_1(x_1, x_2; z_i^-, z_{i\perp}, w_i^+, w_{i\perp}; \sigma_p, \sigma_t) \\
 & \quad \times \bar{\mathbf{A}}^{-i,a}(z_2^+, z_{2\perp}) \bar{\mathbf{A}}^{-j,a}(z_1^+, z_{1\perp}) \bar{\mathbf{B}}^{+i,a}(z_2^-, z_{2\perp}) \bar{\mathbf{B}}^{+j,a}(z_1^-, z_{1\perp}) \\
 &= \frac{N_c^2 - 1}{16} g^4 \langle \check{F}_{\mu\nu}^a \check{F}^{a\mu\nu}(x_2) F_{\lambda\rho}^b F^{b\lambda\rho}(x_1) \rangle_{\mathbb{A}} \\
 & \quad - \langle \check{\mathcal{O}}^{ij,\sigma_p}(x_2^-, x_{2\perp}; x_1^-, x_{1\perp}) \mathcal{O}^{ij;\sigma_t}(x_2^+, x_{2\perp}; x_1^+, x_{1\perp}) \rangle_{\mathbb{A}}
 \end{aligned}$$

Diagrams for $\langle \tilde{F}_{\mu\nu}^a \tilde{F}^{a\mu\nu}(x_2) F_{\lambda\rho}^b F^{b\lambda\rho}(x_1) \rangle_{\mathbb{A}}$ in background fields

“Virtual” diagrams



“Real” diagrams

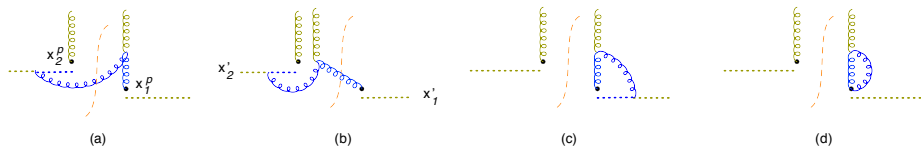


Diagrams for subtracted TMD matrix elements

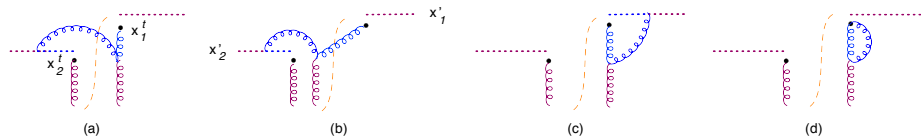
“Projectile” TMD matrix elements.

The rapidity-only $e^{-i\frac{\beta}{\sigma_p}}$ regularization is depicted by point splitting:

F^{+k} shown by dots stand at $x_1^p = x_{1\perp} + x_1^-$ and $x_2^p = x_{2\perp} + x_2^-$
 Wilson lines start from $x_1' = x_2 + \delta^+$ and $x_2' = x_1 + \delta^+$ where $\delta^+ = \frac{1}{\rho\sigma_p}$



“Target” TMD matrix elements. The rapidity-only $e^{-i\frac{\alpha}{\sigma_t}}$ regularization is depicted by point splitting.



Rapidity-only cutoff vs UV+rapidity regularization

Typical divergent integral ($\varepsilon = \frac{d}{2} - 2$, $\bar{d}^n p \equiv \frac{d^n p}{(2\pi)^n}$)

$$\begin{aligned}
 & -i\mu^{-2\varepsilon} \int \bar{d}\alpha \bar{d}\beta \bar{d}p_{\perp} \frac{1}{\beta - i\varepsilon} \frac{1}{\alpha\beta s - p_{\perp}^2 + i\varepsilon} \frac{s(\beta - \beta_B)}{\alpha(\beta - \beta_B)s - p_{\perp}^2 + i\varepsilon} (1 - e^{i(p,x)_{\perp}}) \\
 &= \mu^{-2\varepsilon} \int \frac{\bar{d}p_{\perp}}{p_{\perp}^2} (1 - e^{i(p,x)_{\perp}}) \int_0^{\beta_B} \frac{\bar{d}\beta}{\beta_B} \frac{\beta_B - \beta}{\beta - i\varepsilon} = -\frac{1}{8\pi^2} \frac{\Gamma(\varepsilon)}{(x_{\perp}^2 \mu^2)^{\varepsilon}} \int_0^{\beta_B} \frac{d\beta}{\beta_B} \frac{\beta_B - \beta}{\beta - i\varepsilon}
 \end{aligned}$$

δ -regularization with $A^-(z^+) \rightarrow A^-(z^+)e^{\pm\delta z^+}$

$$-\frac{1}{8\pi^2} \frac{\Gamma(\varepsilon)}{(x_{\perp}^2 \mu^2)^{\varepsilon}} \int_0^{\beta_B} \frac{d\beta}{\beta_B} \frac{\beta_B - \beta}{\beta - i\delta} \simeq \frac{1}{8\pi^2} \left(-\frac{1}{\varepsilon} + \ln \mu^2 \frac{x_{\perp}^2}{4} + \gamma_E \right) \left(\ln \frac{\beta_B}{-i\delta} - 1 \right)$$

Rapidity-only cutoff

$$\begin{aligned}
 & -i \int \bar{d}\alpha \bar{d}\beta \bar{d}p_{\perp} \frac{1}{\beta - i\varepsilon} \frac{e^{-i\frac{\alpha}{\sigma}}}{\alpha\beta s - p_{\perp}^2 + i\varepsilon} \frac{s(\beta - \beta_B)}{\alpha(\beta - \beta_B)s - p_{\perp}^2 + i\varepsilon} (1 - e^{i(p,x)_{\perp}}) \\
 &= \int \frac{\bar{d}p_{\perp}}{p_{\perp}^2} (1 - e^{i(p,x)_{\perp}}) \int_0^{\infty} \bar{d}\alpha \frac{\beta_B s}{\alpha\beta_B s + p_{\perp}^2} e^{-i\frac{\alpha}{\sigma}} = \frac{1}{16\pi^2} \ln^2 \left(-i\beta_B \sigma s \frac{x_{\perp}^2}{4} e^{\gamma_E} \right)
 \end{aligned}$$

(Intermediate) Result

$$\begin{aligned}
 & \mathcal{W}(x_1, x_2) - \mathcal{W}^{\text{tmd}}(x_1, x_2) \\
 &= \int \bar{d}\alpha'_a \bar{d}k'_{a\perp} \bar{d}\beta'_b \bar{d}k'_{b\perp} \bar{d}\alpha_a \bar{d}k'_{a\perp} \bar{d}\beta_b \bar{d}k'_{b\perp} e^{-i\alpha'_a \varrho x_2^- - i\alpha_a \varrho x_1^-} e^{-i\beta'_b \varrho x_2^+ - i\beta_b \varrho x_1^+} \\
 & \times e^{-i(k_a + k_b, x_1)_\perp - i(k'_a + k'_b, x_2)_\perp} \bar{A}_i^{+,b}(\alpha'_a, k'_{a\perp}) \bar{B}^{-i,a}(\beta'_b, k'_{b\perp}) \bar{A}_j^{+,b}(\alpha_a, k_{a\perp}) \bar{B}^{-j,a}(\beta_b, k_{b\perp}) \\
 & \times g^2 [I - I_{\text{tmd}}^{\sigma_p, \sigma_t}](\alpha_a, \alpha'_a, \beta_b, \beta'_b, k_{a\perp}, k'_{a\perp}, k_{b\perp}, k'_{b\perp}, x_1, x_2)
 \end{aligned}$$

with

$$\begin{aligned}
 & [I - I_{\text{tmd}}^{\sigma_p, \sigma_t}](\alpha'_a, \alpha_a, \beta'_b, \beta_b, k'_{a\perp}, k'_{a\perp}, k_{b\perp}, k'_{b\perp}, x_2, x_1) \\
 &= -\ln \frac{(-i\alpha'_a)k_{a\perp}^2}{(-i\alpha_a)k_{a\perp}^2} \ln \frac{(-i\beta'_b)k_{b\perp}^2}{(-i\beta_b)k_{b\perp}^2} + \ln^2 \frac{x_{12\perp}^2 s \sigma_p \sigma_t}{4} \\
 & \quad - \ln \frac{(-i\alpha'_a)e^\gamma}{\sigma_t} \ln \frac{(-i\beta'_b)e^\gamma}{\sigma_p} - \ln \frac{(-i\alpha_a)e^\gamma}{\sigma_t} \ln \frac{(-i\beta_b)e^\gamma}{\sigma_p} + \pi^2
 \end{aligned}$$

where $(-i\alpha_a) \equiv -i(\alpha_a + i\epsilon)$ etc. Power corrections $\sim \zeta^{-1}$ and $\sim \zeta^{-1/2}$ are neglected.

(Intermediate) Result

$$\begin{aligned}
 & \mathcal{W}(x_1, x_2) - \mathcal{W}^{\text{tmd}}(x_1, x_2) \\
 &= \int \bar{d}\alpha'_a \bar{d}k'_{a\perp} \bar{d}\beta'_b \bar{d}k'_{b\perp} \bar{d}\alpha_a \bar{d}k'_{a\perp} \bar{d}\beta_b \bar{d}k'_{b\perp} e^{-i\alpha'_a \varrho x_2^- - i\alpha_a \varrho x_1^-} e^{-i\beta'_b \varrho x_2^+ - i\beta_b \varrho x_1^+} \\
 & \times e^{-i(k_a + k_b, x_1)_\perp - i(k'_a + k'_b, x_2)_\perp} \bar{A}_i^{+,b}(\alpha'_a, k'_{a\perp}) \bar{B}^{-i,a}(\beta'_b, k'_{b\perp}) \bar{A}_j^{+,b}(\alpha_a, k_{a\perp}) \bar{B}^{-j,a}(\beta_b, k_{b\perp}) \\
 & \times g^2 [I - I_{\text{tmd}}^{\sigma_p, \sigma_t}](\alpha_a, \alpha'_a, \beta_b, \beta'_b, k_{a\perp}, k'_{a\perp}, k_{b\perp}, k'_{b\perp}, x_1, x_2)
 \end{aligned}$$

with

$$\begin{aligned}
 & [I - I_{\text{tmd}}^{\sigma_p, \sigma_t}](\alpha'_a, \alpha_a, \beta'_b, \beta_b, k'_{a\perp}, k'_{a\perp}, k_{b\perp}, k'_{b\perp}, x_2, x_1) \\
 &= -\ln \frac{(-i\alpha'_a)k_{a\perp}^2}{(-i\alpha_a)k_{a\perp}^2} \ln \frac{(-i\beta'_b)k_{b\perp}^2}{(-i\beta_b)k_{b\perp}^2} + \ln^2 \frac{x_{12\perp}^2 s \sigma_p \sigma_t}{4} \\
 & \quad - \ln \frac{(-i\alpha'_a)e^\gamma}{\sigma_t} \ln \frac{(-i\beta'_b)e^\gamma}{\sigma_p} - \ln \frac{(-i\alpha_a)e^\gamma}{\sigma_t} \ln \frac{(-i\beta_b)e^\gamma}{\sigma_p} + \pi^2
 \end{aligned}$$

where $(-i\alpha_a) \equiv -i(\alpha_a + i\epsilon)$ etc. Power corrections $\sim \zeta^{-1}$ and $\sim \zeta^{-1/2}$ are neglected.

This formula is not yet the final result for the coefficient function. The coefficient function was defined as a result of integration over C -fields with $\alpha > \sigma_t$ and $\beta > \sigma_p$. Since we did not impose these restrictions while calculating the loop integrals, we need to subtract sG contributions (with $\alpha < \sigma_t, \beta < \sigma_p$) to these integrals.

Result for the coefficient function

Result of sG subtraction:

term $-\ln \frac{(-i\alpha'_a)k_{a\perp}^2}{(-i\alpha_a)k_{a\perp}^2} \ln \frac{(-i\beta'_b)k_{b\perp}^2}{(-i\beta_b)k_{b\perp}^2}$ disappears \Rightarrow no dynamics in the transverse plane

$$\begin{aligned} & \mathcal{W}(x_1, x_2) - \mathcal{W}^{\text{tmd}}(x_1, x_2) - \mathcal{W}^{\text{sG}}(x_1, x_2) \\ &= \int d\alpha'_a d\beta'_b d\alpha_a d\beta_b e^{-i\alpha'_a \varrho x_2^- - i\alpha_a \varrho x_1^-} e^{-i\beta'_b \varrho x_2^+ - i\beta_b \varrho x_1^+} \\ & \quad \times U^{+,b}_i(\alpha'_a, x_{2\perp}) V^{-i,a}(\beta'_b, x_{2\perp}) U^{+,b}_j(\alpha_a, x_{1\perp}) V^{-j,a}(\beta_b, x_{1\perp}) \\ & \quad \times g^2 \mathfrak{C}_1(\alpha'_a, \alpha_a, \beta'_b, \beta_b; x_1, x_2) \end{aligned}$$

where

$$\begin{aligned} \mathfrak{C}_1(\alpha'_a, \alpha_a, \beta'_b, \beta_b; x_2, x_1) &= I - I_{\text{tmd}}^{\sigma_p, \sigma_t} - I_{\text{sG}}^{\sigma_p, \sigma_t} \\ &= \ln^2 \frac{x_{12\perp}^2 s \sigma_p \sigma_t}{4} - \ln \frac{(-i\alpha'_a) e^\gamma}{\sigma_t} \ln \frac{(-i\beta'_b) e^\gamma}{\sigma_p} - \ln \frac{(-i\alpha_a) e^\gamma}{\sigma_t} \ln \frac{(-i\beta_b) e^\gamma}{\sigma_p} + \pi^2 \end{aligned}$$

The coefficient function in the coordinate space is made of (+) - prescriptions since

$$\int d\alpha e^{i\alpha z} \left[\ln \left(-i \frac{\alpha}{\sigma} + \epsilon \right) = \frac{\theta(-z)}{z} + \delta(z) \int_0^{1/\sigma} \frac{dz'}{z'} \right]$$

Result for the coefficient function

Our formula

$$\begin{aligned}
 & \frac{1}{16}(N_c^2 - 1)\langle p'_A, p'_B | g^2 F_{\mu\nu}^a F^{a\mu\nu}(x_2) g^2 F_{\lambda\rho}^b F^{b\lambda\rho}(x_1) | p_A, p_B \rangle \\
 &= \int \mathcal{D}\Phi_{\mathcal{A}} \Psi_{p'_A}^*(t_i) \Psi_{p_A}(t_i) \Psi_{p'_B}^*(t_i) \Psi_{p_B}(t_i) \left[\mathcal{O}_{ij}^{\sigma_p}(x_2^-, x_{2\perp}; z_1^-, x_{1\perp}) \mathcal{O}^{ij;\sigma_t}(x_2^+, x_{2\perp}; x_1^+, x_{1\perp}) \right. \\
 & \quad \left. + \int dz_1^- dz_2^- dw_1^+ dw_2^+ \frac{\alpha_s N_c}{2\pi} \mathfrak{C}_1(x_1, x_2; z_i^-, w_i^+; \sigma_p, \sigma_t) \right. \\
 & \quad \left. \times \mathcal{O}_{ij}^{\sigma_p}(z_2^-, x_{2\perp}; z_1^-, x_{1\perp}) \mathcal{O}^{ij;\sigma_t}(z_2^+, x_{2\perp}; z_1^+, x_{1\perp}) + \mathcal{O}(\alpha_s^2) \right]
 \end{aligned}$$

is not yet TMD formula since $\mathcal{A} = A + B + sG$ and soft/Glauber gluons connect “projectile” and “target” gluons.

It is well known that Glauber gluons cancel and soft gluons form soft factors.

With rapidity-only cutoffs, soft factors are power corrections \Rightarrow TMD formula

$$\begin{aligned}
 & \frac{1}{16}(N_c^2 - 1)\langle p'_A, p'_B | g^2 F_{\mu\nu}^a F^{a\mu\nu}(x_2) g^2 F_{\lambda\rho}^b F^{b\lambda\rho}(x_1) | p_A, p_B \rangle \\
 &= \langle p'_A | \hat{\mathcal{O}}_{ij}^{\sigma_p}(x_2^-, x_{2\perp}; x_1^-, x_{1\perp}) | p_A \rangle \langle p'_B | \hat{\mathcal{O}}^{ij;\sigma_t}(x_2^+, x_{2\perp}; x_1^+, x_{1\perp}) | p_B \rangle \\
 & \quad + \int dz_1^- dz_2^- dw_1^+ dw_2^+ \frac{\alpha_s N_c}{2\pi} \mathfrak{C}_1(x_1, x_2; z_i^-, w_i^+; \sigma_p, \sigma_t) \\
 & \quad \times \langle p'_A | \hat{\mathcal{O}}_{ij}^{\sigma_p}(z_2^-, x_{2\perp}; z_1^-, x_{1\perp}) | p_A \rangle \langle p'_B | \hat{\mathcal{O}}^{ij;\sigma_t}(z_2^+, x_{2\perp}; z_1^+, x_{1\perp}) | p_B \rangle
 \end{aligned}$$

TMD evolution equations

$$\begin{aligned}
 & \sigma_p \frac{d}{d\sigma_p} \hat{\mathcal{O}}^{ij;\sigma_t}(\alpha'_a, \alpha_a, x_{2\perp}, x_{1\perp}) \\
 &= -\frac{\alpha_s N_c}{2\pi} \left[2 \ln \frac{sx_{12\perp}^2}{4} + \ln(-i\alpha'_a \sigma_p + \epsilon) + \ln(-i\alpha_a \sigma_p + \epsilon) + 2\gamma \right] \hat{\mathcal{O}}^{ij;\sigma_t}(\alpha'_a, \alpha_a, x_{2\perp}, x_{1\perp}) \\
 & \sigma_t \frac{d}{d\sigma_t} \hat{\mathcal{O}}^{ij;\sigma_t}(\beta'_b, \beta_b, x_{2\perp}, x_{1\perp}) \\
 &= -\frac{\alpha_s N_c}{2\pi} \left[2 \ln \frac{sx_{12\perp}^2}{4} + \ln(-i\beta'_b \sigma_t + \epsilon) + \ln(-i\beta_b \sigma_t + \epsilon) + 2\gamma \right] \hat{\mathcal{O}}^{ij;\sigma_t}(\beta'_b, \beta_b, x_{2\perp}, x_{1\perp})
 \end{aligned}$$

Matching of σ_p and σ_t evolutions \Rightarrow

$$\begin{aligned}
 \sigma_t \frac{d}{d\sigma_t} \mathfrak{C}(x_{1\perp}, x_{2\perp}; \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t) &= \frac{\alpha_s N_c}{2\pi} \left[2 \ln \frac{sx_{12\perp}^2}{4} \right. \\
 & \quad \left. + \ln(-i\beta'_b \sigma_t + \epsilon) + \ln(-i\beta_b \sigma_t + \epsilon) + 2\gamma \right] \mathfrak{C}(x_{1\perp}, x_{2\perp}; \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t) \\
 \sigma_p \frac{d}{d\sigma_p} \mathfrak{C}(x_{1\perp}, x_{2\perp}; \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t) &= \frac{\alpha_s N_c}{2\pi} \left[2 \ln \frac{sx_{12\perp}^2}{4} \right. \\
 & \quad \left. + \ln(-i\alpha'_a \sigma_p + \epsilon) + \ln(-i\alpha_a \sigma_p + \epsilon) + 2\gamma \right] \mathfrak{C}(x_{1\perp}, x_{2\perp}; \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t)
 \end{aligned}$$

Matching of coefficient function and TMDs

The solution of this equations compatible with our first-order result is

$$\mathfrak{C}(x_{1\perp}, x_{2\perp}; \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t) = e^{\frac{\alpha_s N_c}{2\pi}} \mathfrak{C}_1(x_{12\perp}, \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t)$$

⇒ hadronic tensor is

$$W(\alpha'_a, \alpha_a, \beta'_b, \beta_b, x_{1\perp}, x_{2\perp}) = \int \bar{d}\alpha'_a \bar{d}\alpha_a \bar{d}\beta'_b \bar{d}\beta_b e^{\frac{\alpha_s N_c}{2\pi}} \mathfrak{C}_1(x_{12\perp}, \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t) \\ \times \langle p'_A | \hat{O}_{ij}^{\sigma_p}(\alpha'_a, \alpha_a, x_{2\perp}, x_{1\perp}) | p_A \rangle \langle p'_B | \hat{O}^{ij; \sigma_t}(\beta'_b, \beta_b, x_{2\perp}, x_{1\perp}) | p_B \rangle + \dots$$

Reminder

$$\mathfrak{C}_1(\alpha'_a, \alpha_a, \beta'_b, \beta_b; x_1, x_2; \sigma_p, \sigma_t) \\ = \ln^2 \frac{x_{12\perp}^2 s \sigma_p \sigma_t}{4} - \ln \frac{(-i\alpha'_a) e^\gamma}{\sigma_t} \ln \frac{(-i\beta'_b) e^\gamma}{\sigma_p} - \ln \frac{(-i\alpha_a) e^\gamma}{\sigma_t} \ln \frac{(-i\beta_b) e^\gamma}{\sigma_p} + \pi^2$$

Forward case (\equiv particle production by gluon fusion)

$$W(p_A, p_B; q) = \int db_\perp e^{i(q, b)_\perp} W(p_A, p_B; \alpha_q, \beta_q, b_\perp),$$

$$\begin{aligned} W(p_A, p_B; \alpha_q, \beta_q, b_\perp) &= \frac{\pi^2}{2} \mathcal{Q}^2 \mathcal{G}_{ij}^{\sigma_p}(\alpha_q, b_\perp; p_A) \mathcal{G}^{ij; \sigma_t}(\beta_q, b_\perp; p_B) \\ &\times \exp \left\{ \frac{\alpha_s N_c}{2\pi} \left[\ln^2 \frac{b_\perp^2 s \sigma_p \sigma_t}{4} - 2 \left(\ln \frac{\alpha_q}{\sigma_t} + \gamma \right) \left(\ln \frac{\beta_q}{\sigma_p} + \gamma \right) + \frac{\pi^2}{2} \right] \right\} \\ &+ \text{NLO terms} \sim O(\alpha_s^2) + \text{power corrections} \quad (*) \end{aligned}$$

where $\mathcal{G}_{ij}^{\sigma_p}$, $\mathcal{G}_{ij}^{\sigma_t}$ are gluon TMDs:

$$\langle p_A | \hat{\mathcal{O}}_{ij}^{\sigma_p}(z^-, 0^-, b_\perp) | p_A \rangle = -g^2 \varrho^2 \int_0^1 du u \mathcal{G}_{ij}^{\sigma_p}(u, b_\perp) \cos u \varrho z^-,$$

$$\langle p_B | \hat{\mathcal{O}}_{ij}^{\sigma_t}(z^-, 0^-, b_\perp) | p_B \rangle = -g^2 \varrho^2 \int_0^1 du u \mathcal{G}_{ij}^{\sigma_t}(u, b_\perp) \cos u \varrho z^-,$$

Matching of coefficient function and TMDs

The r.h.s. of the evolution formula (*) does not depend on cutoffs σ_p and σ_t as long as $\sigma_p \geq \tilde{\sigma}_p = \frac{4b_\perp^{-2}}{\alpha_s s}$ and $\sigma_t \geq \tilde{\sigma}_t \equiv \frac{4b_\perp^{-2}}{\beta_q s}$. Thus, the result of double-log Sudakov evolution reads

$$W(p_A, p_B; \alpha_q, \beta_q, b_\perp) = \frac{\pi^2}{2} Q^2 \mathcal{G}_{ij}^{\tilde{\sigma}_p}(\alpha_q, b_\perp; p_A) \mathcal{G}^{ij; \tilde{\sigma}_t}(\beta_q, b_\perp; p_B) \\ \times \exp \left\{ -\frac{\alpha_s N_c}{2\pi} \left[\left(\ln \frac{Q^2 b_\perp^2}{4} + 2\gamma \right)^2 - 2\gamma^2 - \frac{\pi^2}{2} \right] \right\} + O(\alpha_s^2) \text{ terms} + \text{power corrections}$$

This result is universal for moderate x and small- x hadronic tensor. The difference lies in the continuation of the evolution beyond Sudakov region.

Double-log Sudakov evolution should stop at $\sigma_0 \simeq \frac{b_\perp^{-2}}{\beta_q s} \simeq \frac{q_\perp^2}{x_B s}$. After that:

- If $\beta_B \equiv x_B \sim 1$ - DGLAP-type evolution from $\sigma_0 = \frac{b_\perp^{-2}}{x_B s}$ to $\sigma_{\text{fin}} = \frac{m_N^2}{s}$:
summation of $\left(\alpha_s \ln \frac{b_\perp^{-2}}{m_N^2} \right)^n$
- If $\beta_B \equiv x_B \ll 1$ - BFKL-type evolution from $\sigma_0 = \frac{b_\perp^{-2}}{x_B s}$ to $\sigma_{\text{fin}} = \frac{b_\perp^{-2}}{s}$: summation of $\left(\alpha_s \ln x_B \right)^n$

1 Conclusion: rapidity-only TMD factorization works!

- Power corrections $\sim \frac{1}{Q^2}$ for DY hadronic tensor \Rightarrow EM gauge invariance of DY tensor.
- Back-of-the-envelope estimates of angular distributions for DY Z-boson production are in good agreement with LHC data.
- Rapidity factorization at the one-loop level gives Sudakov-type double logs for both small and intermediate x_B

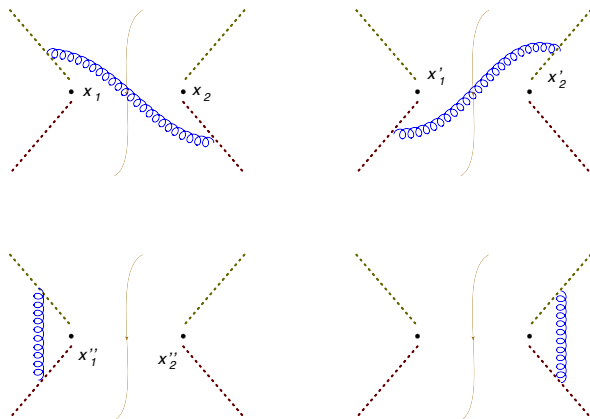
2 Outlook

- Power corrections $\sim \frac{1}{Q^2}$ for SIDIS.
- Matching to DGLAP and BFKL/BK evolutions
- Conformal invariance of rapidity-only factorization

Thank you for attention!

Backup slide: soft factor with rapidity-only cutoffs

Leading-order diagrams



Result of calculation: $\frac{1}{4\pi^2} \text{Li}_2\left(-\frac{x_{12\perp}^2}{2\delta+\delta^-}\right) \sim \mathcal{O}\left(\frac{\Delta_\perp^2}{2\delta+\delta^-}\right) \sim \mathcal{O}\left(\frac{\sigma_p\sigma_{rS}}{Q_\perp^2}\right) \sim \mathcal{O}(\zeta^{-1/2})$

Soft factor with rapidity-only regularization does not have perturbative contributions which can mix with the TMD evolution

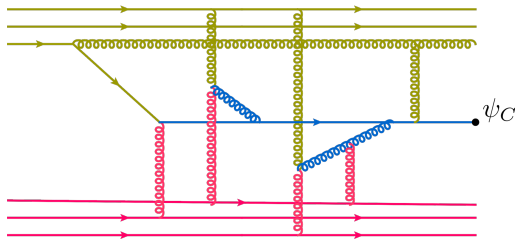
BACKUP SLIDES

In the tree approximation: classical YM field with sources

Tree approximation:

Projectile fields: $\beta = 0 \Rightarrow A(x^-, x_\perp), \psi_A(x^-, x_\perp)$

Target fields: $\alpha = 0 \Rightarrow B(x^+, x_\perp), \psi_B(x^-, x_\perp)$



ψ_C = sum of tree diagrams in external $A, \tilde{A}, \psi_A, \tilde{\psi}_A$ and $B, \tilde{B}, \psi_B, \tilde{\psi}_B$ fields with sources

$$J_\psi = (\not{P} + m)(\psi_A + \psi_B), \quad J_\nu = D^\mu F^{\mu\nu}(A + B)$$

and

$$\tilde{J}_\psi = (\not{P} + m)(\tilde{\psi}_A + \tilde{\psi}_B), \quad \tilde{J}_\nu = D^\mu F^{\mu\nu}(\tilde{A} + \tilde{B})$$

$\Sigma_X \Rightarrow$ Feynman diagrams with retarded propagators

The fields A, ψ and $\tilde{A}, \tilde{\psi}$ do not depend on x^+ \Rightarrow
if they coincide at $x^+ = \infty \Rightarrow$ they coincide everywhere.

Similarly,
 B, ψ_b and $\tilde{B}, \tilde{\psi}_b$ do not depend on $x^- \Rightarrow$
if they coincide at $x^- = \infty$ they should be equal.

Since $\tilde{A} = A$ and $\tilde{B} = B$ the sources and background fields are the same to the left and to the right of the cut

\Rightarrow

ψ_C and C_μ are given by the sum of tree diagrams with *retarded* Green functions
(F. Gelis, R. Venugopalan)

Classical solution

The sum of diagrams with retarded Green functions \Leftrightarrow solution of classical YM equations

$$(\not{P} + m_f)\psi^f = 0, \quad D^\nu F_{\mu\nu}^a = \sum_f g\bar{\psi}^f t^a \gamma_\mu \psi^f$$

Boundary conditions :

$$A_\mu(x) \stackrel{x^+ \rightarrow -\infty}{\cong} \bar{A}_\mu(x^-, x_\perp), \quad \psi(x) \stackrel{x^+ \rightarrow -\infty}{\cong} \psi_a(x^-, x_\perp)$$
$$A_\mu(x) \stackrel{x^- \rightarrow -\infty}{\cong} \bar{B}_\mu(x^+, x_\perp), \quad \psi(x) \stackrel{x^- \rightarrow -\infty}{\cong} \psi_b(x^+, x_\perp)$$

The projectile and target fields satisfy YM equations

$$(\not{P} + m_f)\psi_a^f = 0, \quad D^\nu F_{\mu\nu}^a = g\bar{\psi}_a^f t^a \gamma_\mu \psi_a^f$$
$$(\not{P} + m_f)\psi_b^f = 0, \quad D^\nu F_{\mu\nu}^a = g\bar{\psi}_b^f t^a \gamma_\mu \psi_b^f$$

Projectile partons: $k = \alpha p_1 + k_\perp$, target partons: $k = \beta p_1 + k_\perp \Rightarrow$ partons are *not* on the mass shell

Classical solution

The sum of diagrams with retarded Green functions \Leftrightarrow solution of classical YM equations

$$(\not{P} + m_f)\psi^f = 0, \quad D^\nu F_{\mu\nu}^a = \sum_f g\bar{\psi}^f t^a \gamma_\mu \psi^f$$

Boundary conditions :

$$A_\mu(x) \stackrel{x^+ \rightarrow -\infty}{\cong} \bar{A}_\mu(x^-, x_\perp), \quad \psi(x) \stackrel{x^+ \rightarrow -\infty}{\cong} \psi_a(x^-, x_\perp)$$
$$A_\mu(x) \stackrel{x^- \rightarrow -\infty}{\cong} \bar{B}_\mu(x^+, x_\perp), \quad \psi(x) \stackrel{x^- \rightarrow -\infty}{\cong} \psi_b(x^+, x_\perp)$$

The projectile and target fields satisfy YM equations

$$(\not{P} + m_f)\psi_a^f = 0, \quad D^\nu F_{\mu\nu}^a = g\bar{\psi}_a^f t^a \gamma_\mu \psi_a^f$$
$$(\not{P} + m_f)\psi_b^f = 0, \quad D^\nu F_{\mu\nu}^a = g\bar{\psi}_b^f t^a \gamma_\mu \psi_b^f$$

Projectile partons: $k = \alpha p_1 + k_\perp$, **target partons:** $k = \beta p_1 + k_\perp \Rightarrow$ partons are *not* on the mass shell

Method of solution:

- Start with $\psi_A + \psi_B$ and $\bar{A}_\mu + \bar{B}_\mu$ in the gauge $A^+ = 0, A^- = 0$
- Correct by computing Feynman diagrams (with retarded propagators) with sources $(\not{P} + m)(\psi_A + \psi_B)$ and $J_\nu = D^\mu F^{\mu\nu}(U + V)$

Classical fields in the leading order in $p_{\perp}^2/p_{\parallel}^2 \sim q_{\perp}^2/Q^2$

The solution of YM equations in general case (scattering of two “color glass condensates”) is yet unsolved problem.

Fortunately, for our case of particle production with $\frac{q_{\perp}}{Q} \ll 1$ we can use this small parameter and construct the approximate solution.

At the tree level transverse momenta are $\sim q_{\perp}^2$ and longitudinal are $\sim Q^2 \Rightarrow$

$$\psi, A = \text{series in } \frac{q_{\perp}}{Q} : \quad \psi = \psi^{(0)} + \psi^{(1)} + \dots, \quad A = A^{(0)} + A^{(1)} + \dots$$

NB: After the expansion

$$\frac{1}{p^2 + i\epsilon p_0} = \frac{1}{p_{\parallel}^2 - p_{\perp}^2 + i\epsilon p_0} = \frac{1}{p_{\parallel}^2} - \frac{1}{p_{\parallel}^2 + i\epsilon p_0} p_{\perp}^2 \frac{1}{p_{\parallel}^2 + i\epsilon p_0} + \dots$$

the dynamics in transverse space is trivial.

Fields are either at the point x_{\perp} or at the point $0_{\perp} \Rightarrow$ TMDs

Leading- N_c power corrections

Power corrections are \sim leading twist $\times \left(\frac{q_\perp}{Q}$ or $\frac{q_\perp^2}{Q^2}\right) \times \left(1 + \frac{1}{N_c} + \frac{1}{N_c^2}\right)$.

(Pleasant) surprise: most of the terms not suppressed by $\frac{1}{N_c}$ are determined by the leading-twist TMDs due to QCD equations of motion

Leading twist:

$$\frac{1}{8\pi^3 s} \int dx^- d^2x_\perp e^{-i\alpha x^- + i(k,x)_\perp} \langle A | \hat{\psi}_f(x^-, x_\perp) \not{p}_2 \hat{\psi}_f(0) | A \rangle = f_{1f}(\alpha, k_\perp^2)$$

Power correction:

$$\begin{aligned} & \frac{1}{8\pi^3 s} \int dx^- dx_\perp e^{-i\alpha_q x^- + i(k,x)_\perp} \\ & \times \langle A | \hat{\psi}^f(x^-, x_\perp) \not{p}_2 [\hat{U}_i(x^-, x_\perp) - i\gamma_5 \hat{U}_i(x^-, x_\perp)] \hat{\psi}^f(0) | A \rangle \\ & = -k_{if1}(\alpha_q, k_\perp) + \alpha_q k_i [f_\perp(\alpha_q, k_\perp) + g^\perp(\alpha_q, k_\perp)], \end{aligned}$$

(Mulders & Tangerman, 1996)

At small $\alpha_q \equiv x_A$ one can drop the second term

Gluon TMD operator

$$\mathcal{O}_g(x^+, x_\perp; y^+, y_\perp) \equiv g^2 F^{-i}(x^+, x_\perp)[x, x \pm \infty n][\pm \infty n + y, y] F^{-j}(y^+, y_\perp)$$

Rapidity-regularized operator

$$\mathcal{O}_g^\sigma(x^+, x_\perp; y^+, y_\perp) \equiv \tilde{\mathcal{F}}_i^{\sigma,a}(x_\perp, x^+) \mathcal{F}_i^{\sigma,a}(y_\perp, y^+),$$

$$\tilde{\mathcal{F}}_i^{a;\sigma}(x_\perp, x^+) = g(F^-_i)^b(x^+, x_\perp, -\delta^-)[x^+, -\infty]_x^{ba},$$

$$\delta^- = \frac{1}{\rho\sigma}$$

$$\mathcal{F}_i^{a;\sigma}(y_\perp, y^+) = [-\infty, y^+]_y^{ab} g(F^-_i)^b(x^+, x_\perp, -\delta'^-)$$

Approximation: $\beta_B \sigma s \gg (x - y)_\perp^{-2}$

Leading-order evolution equation is the same as in quark case with $c_f \rightarrow N_c$ replacement
(G.A. Chirilli and I.B., 2019)

Quark loop contribution to gluon TMD evolution

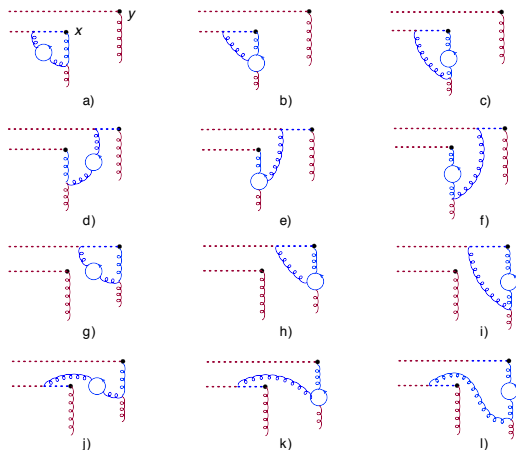


Figure: Quark loop correction to gluon TMD evolution

Quark loop contribution to gluon TMD evolution

Result of calculations: BLM scale is the same as in the quark case with $c_F \rightarrow N_c$ replacement \Rightarrow rapidity evolution is the same

$$\begin{aligned} \tilde{\mathcal{F}}_i^{a;s}(\beta'_B, x_\perp) \mathcal{F}^{a,i;s}(\beta_B, y_\perp) &= \tilde{\mathcal{F}}_i^{a;s_0}(\beta'_B, x_\perp) \mathcal{F}^{a,i;s_0}(\beta_B, y_\perp) \\ &\times e^{\frac{N_c}{4\pi} \left[\ln \frac{\alpha_s(\mu'_\zeta)}{\alpha_s(\mu'_{\zeta_0})} \left(\frac{1}{\alpha_s(\bar{b}_\perp^{-1})} + \ln[-i\tau'_B + \epsilon] \right) + \frac{1}{\alpha_s(\mu'_\zeta)} - \frac{1}{\alpha_s(\mu'_{\zeta_0})} \right]} \\ &\times e^{\frac{N_c}{4\pi} \left[\ln \frac{\alpha_s(\mu_\zeta)}{\alpha_s(\mu_{\zeta_0})} \left(\frac{1}{\alpha_s(\bar{b}_\perp^{-1})} + \ln[-i\tau_B + \epsilon] \right) + \frac{1}{\alpha_s(\mu_\zeta)} - \frac{1}{\alpha_s(\mu_{\zeta_0})} \right]} \end{aligned}$$

Double-log Sudakov evolution should stop at $\beta_B \sigma_0 s \simeq b_\perp^{-2}$. After that:

- If $\beta_B \equiv x_B \sim 1$ - DGLAP-type evolution from $\sigma_0 = \frac{b_\perp^{-2}}{x_B s}$ to $\sigma_{\text{fin}} = \frac{m_N^2}{s}$:
summation of $(\alpha_s \ln \frac{b_\perp^{-2}}{m_N^2})^n$
- If $\beta_B \equiv x_B \ll 1$ - BFKL-type evolution from $\sigma_0 = \frac{b_\perp^{-2}}{x_B s}$ to $\sigma_{\text{fin}} = \frac{b_\perp^{-2}}{s}$:
summation of $(\alpha_s \ln x_B)^n$

Matching: use general equation for TMD evolution at all x_B from papers with A. Tarasov.

Drawback: very complicated. MB conformal invariance will help?