Revealing the fundamental character of the strong force

From PDFs to the underlying QCD

Fred Olness
SMU

Thanks for substantial input
from my friends & colleagues

nCTEQ
nuclear parton distribution functions

xFitter

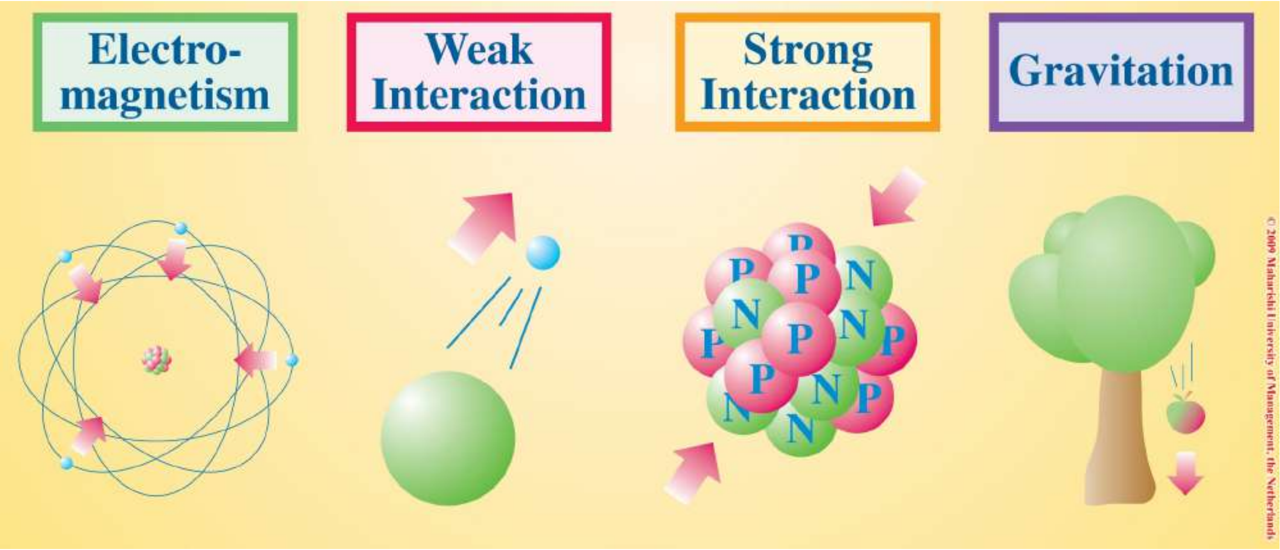
SURGE COLLABORATION

CTEQ

BNL

7 April 2023

QCD:
Quantum
ChromoDynamics
… the strong nuclear force

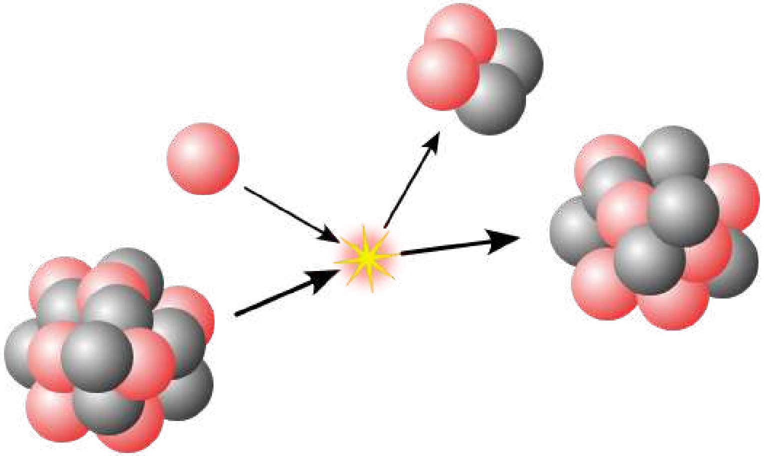


Nobel Prizes: EW '79,'95,'99,'02,'08,'15 QCD '04 Gravity '11, '17, '20 Higgs '13

QCD is our most perfect physical theory

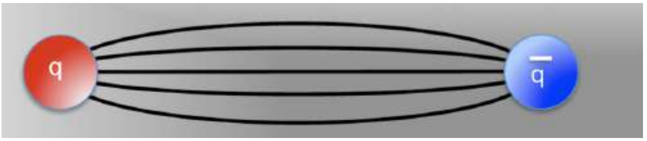
What QCD Tells Us About Nature – and Why We Should Listen. *Frank Wilczek*

In many respects, our most complex
- asymptotic freedom
- strong color confinement
- … associated manifestations



Lessons: The Nature of Nature

"… alien, simple, beautiful, weird, & comprehensible"

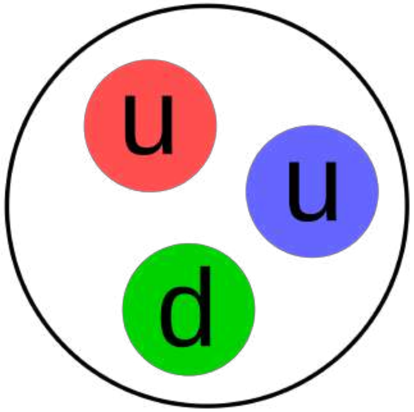


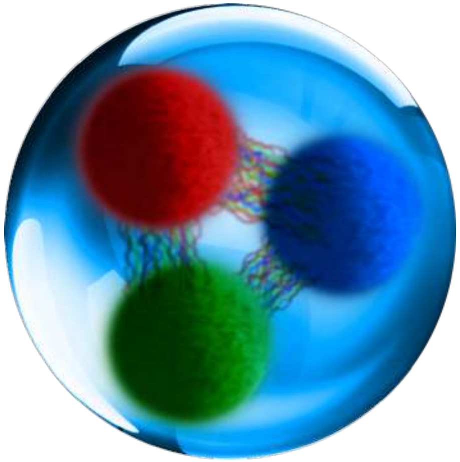
Quark Confinement
(Asymptotic Freedom)

Proton PDFs

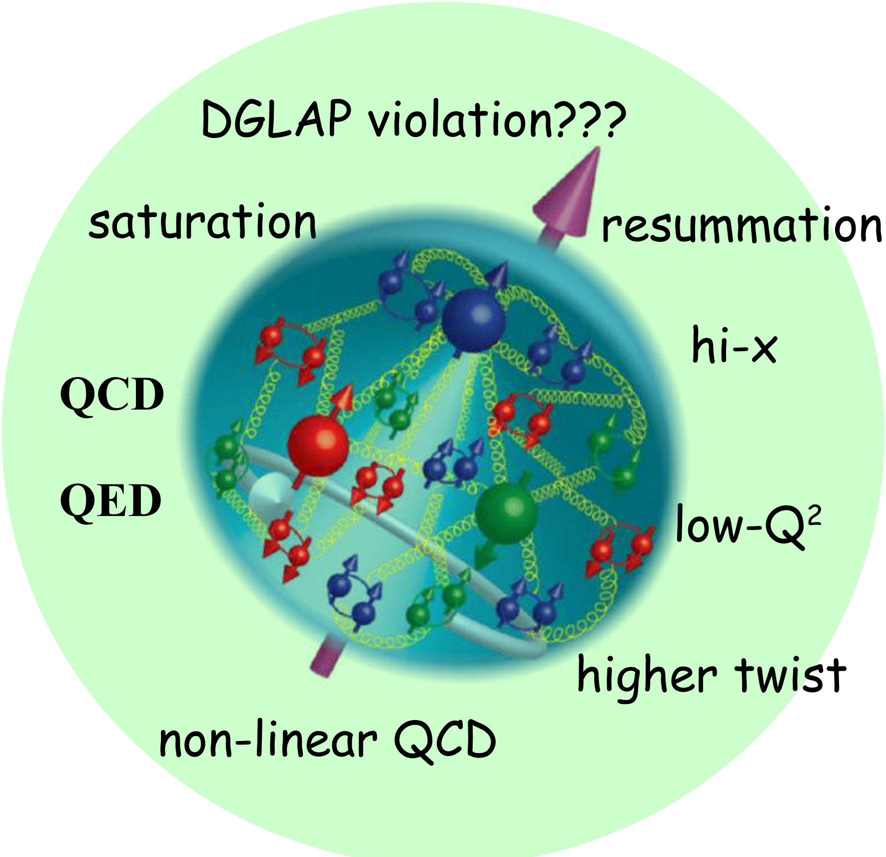
Parton Distribution Functions

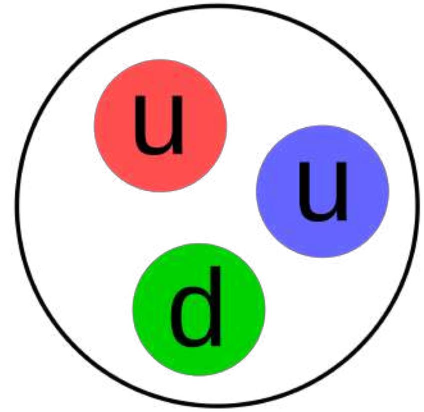
... a probabilistic description

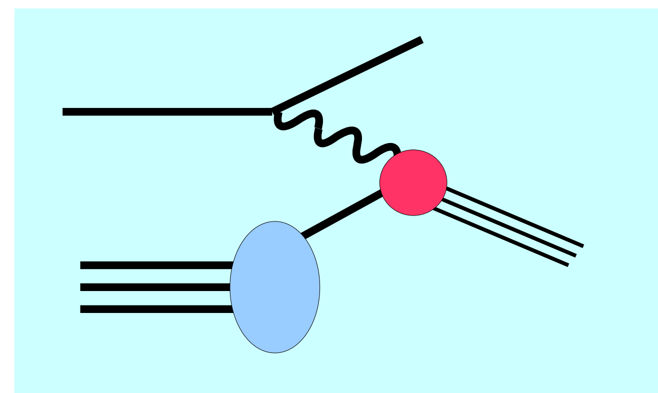




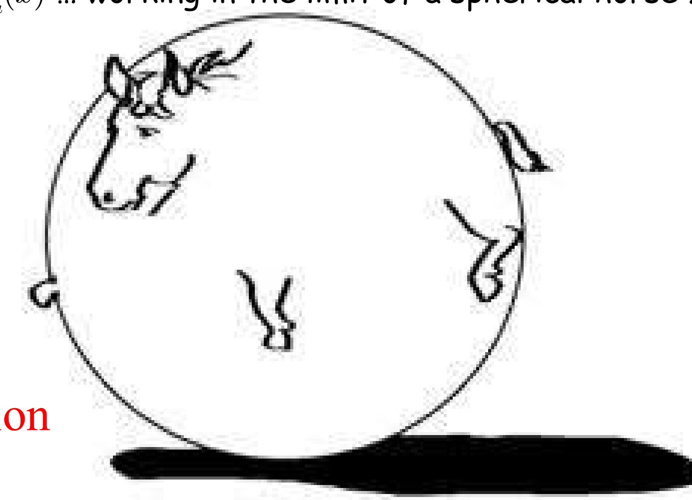
PDFs & the Parton Model ... *a few simpifications*







$f_a(x)$ ... working in the limit of a spherical horse ...



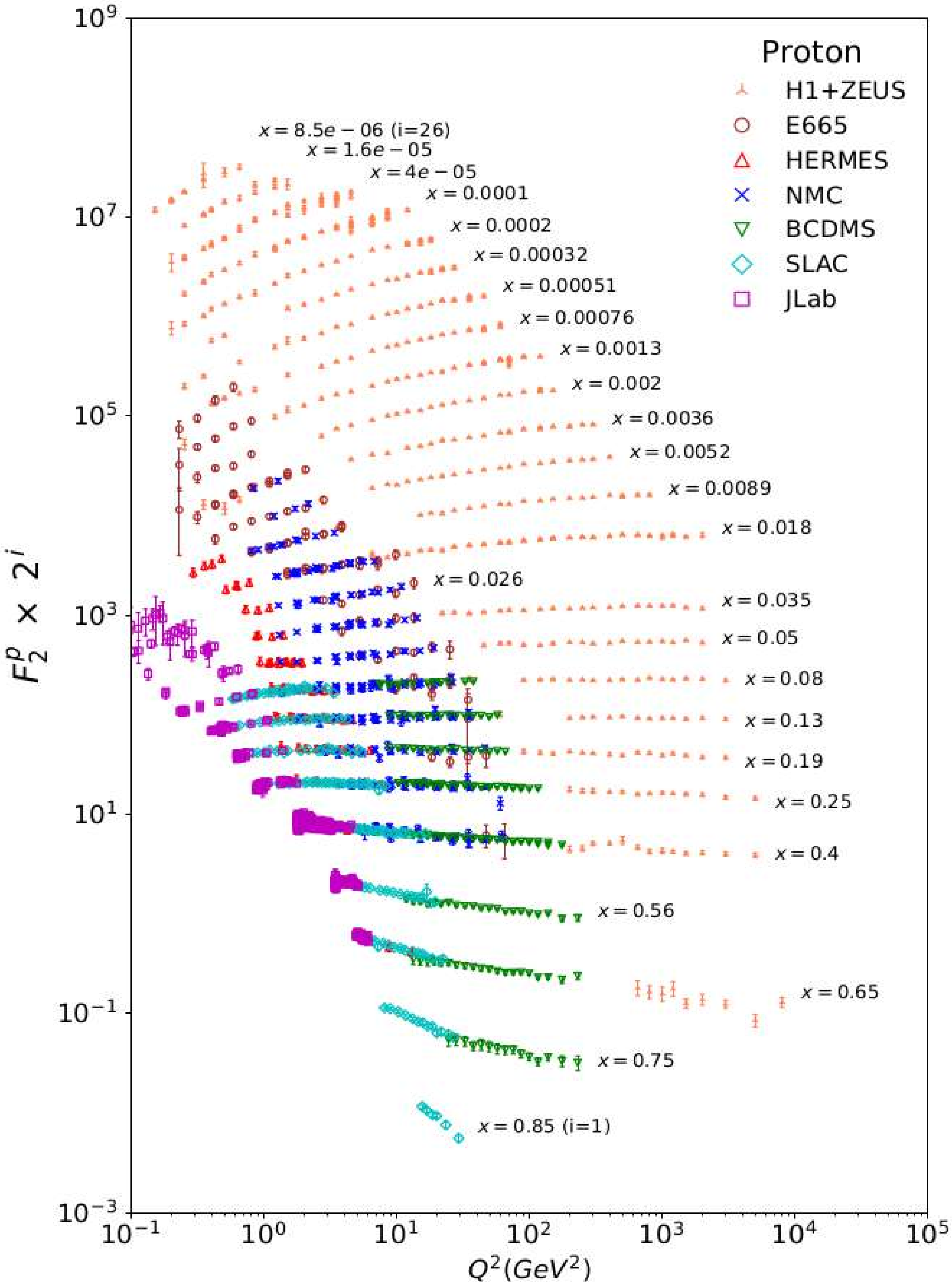
The QCD Parton Model

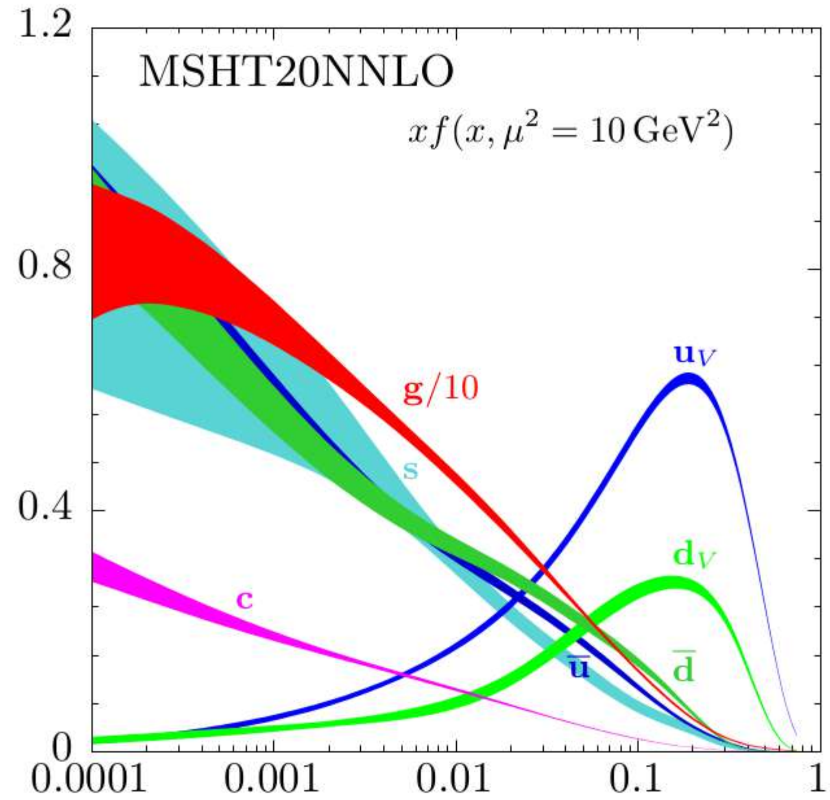
$$d\sigma = f_a(x) \otimes \hat{\sigma}$$

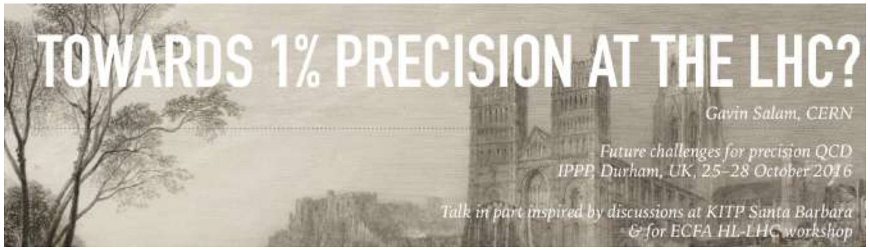
Parameterized in terms of a single variable x, the momentum fraction

... use DGLAP to determine Q dependence

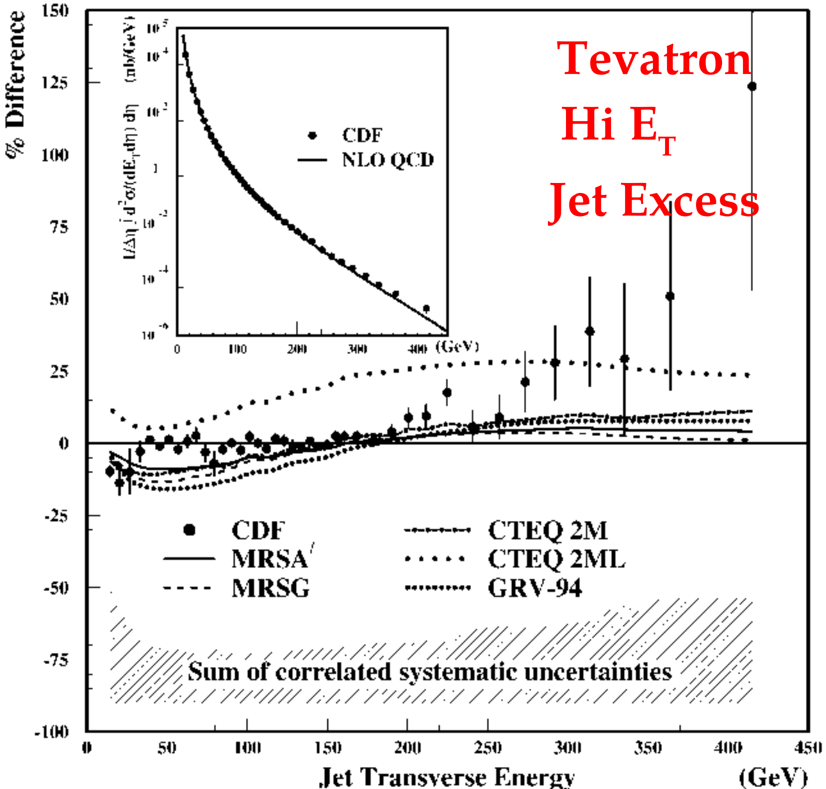
The incredible success of the QCD Parton Model Framework



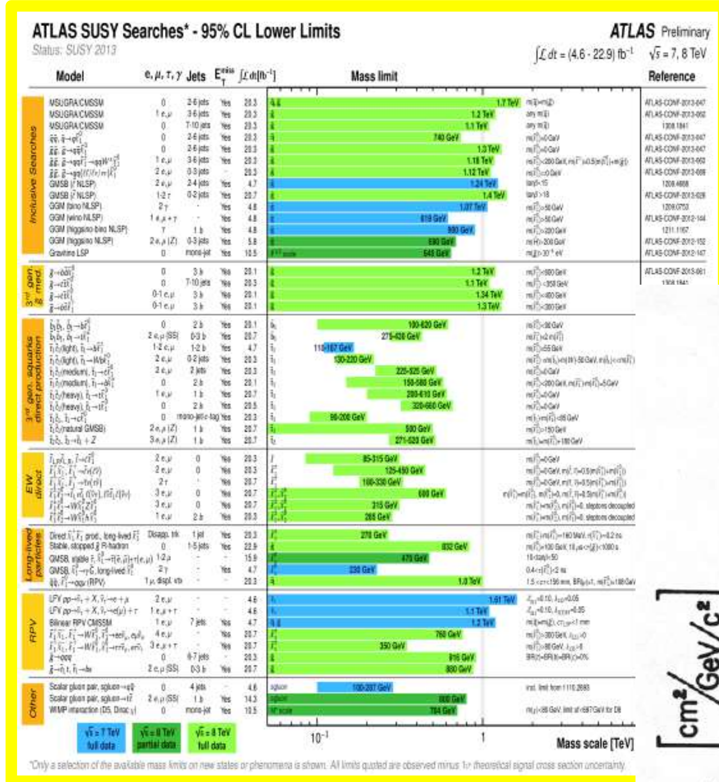


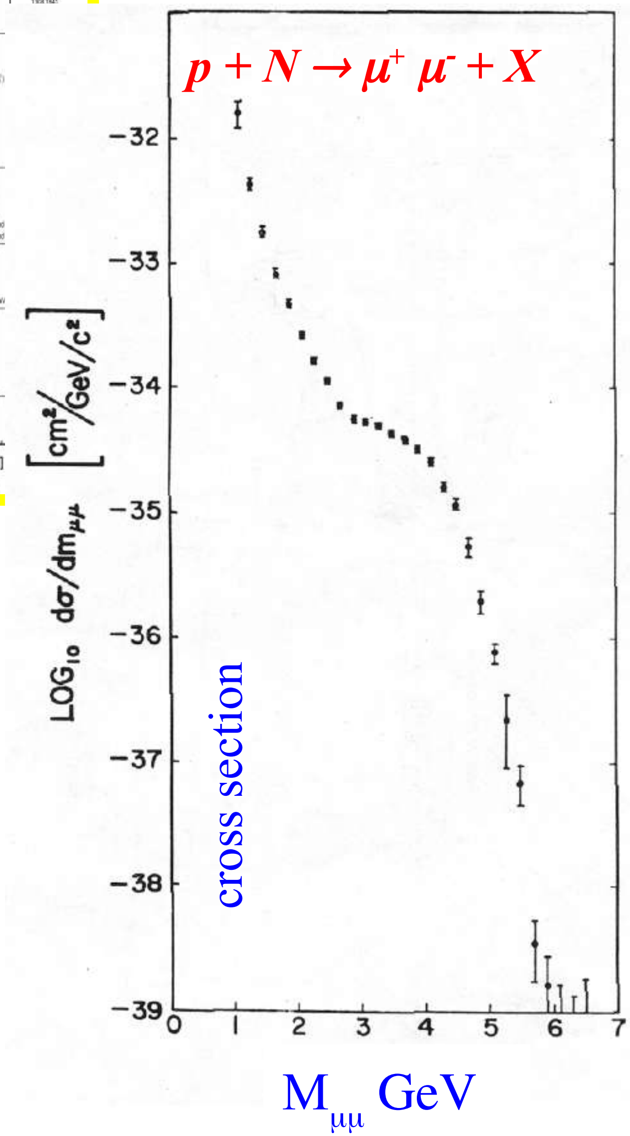


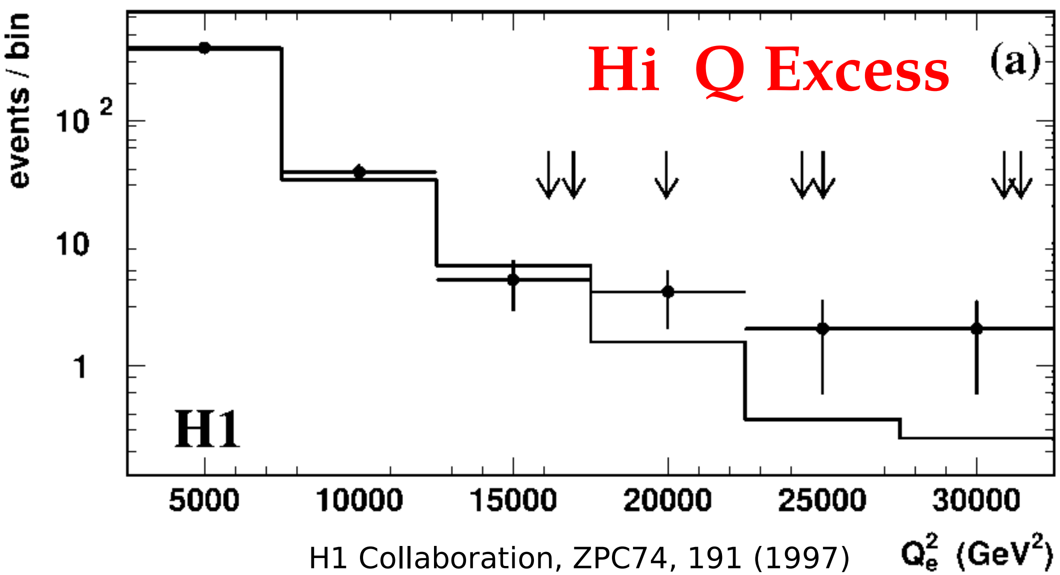
CAUTION: Just because you can absorb an effect in to the PDF ...



CDF Collaboration, PRL 77, 438 (1996)



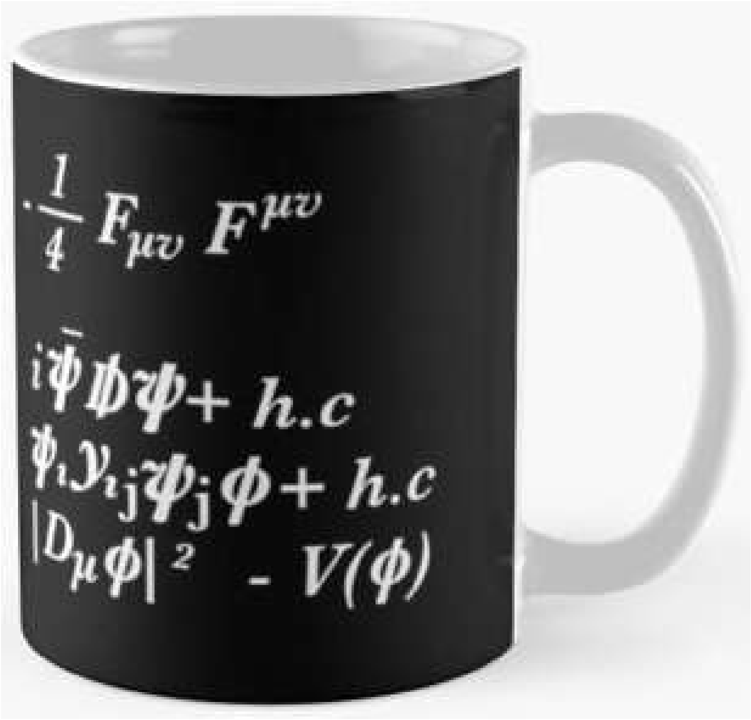


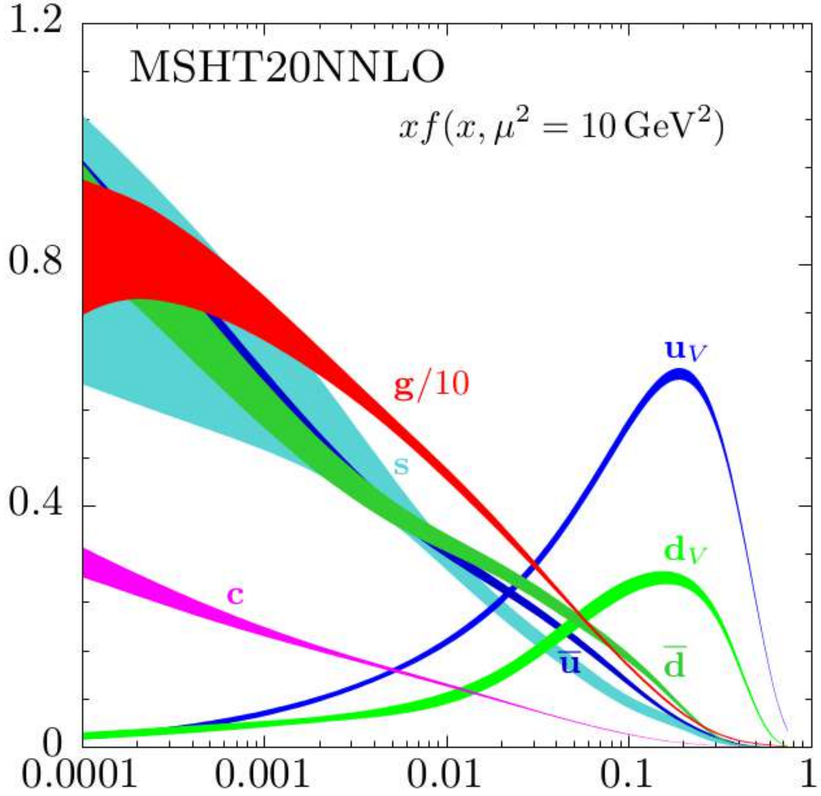


beware circular arguments

H1 Collaboration, ZPC74, 191 (1997)
ZEUS Collaboration, ZPC74, 207 (1997)

How do we go further with QCD?





The difference between knowing the name of something, and knowing something …

Richard Feynman

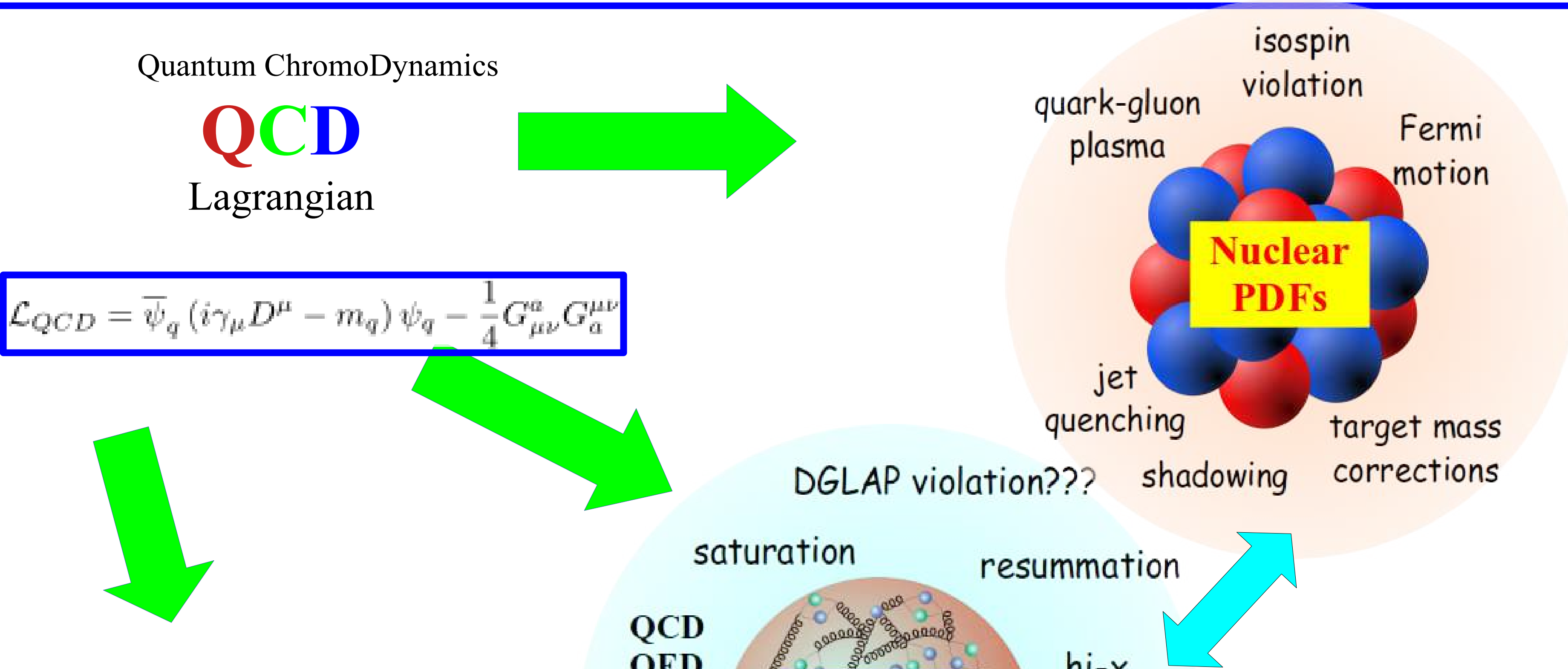
A Deeper Understanding of the strong nuclear force

Quantum ChromoDynamics

QCD

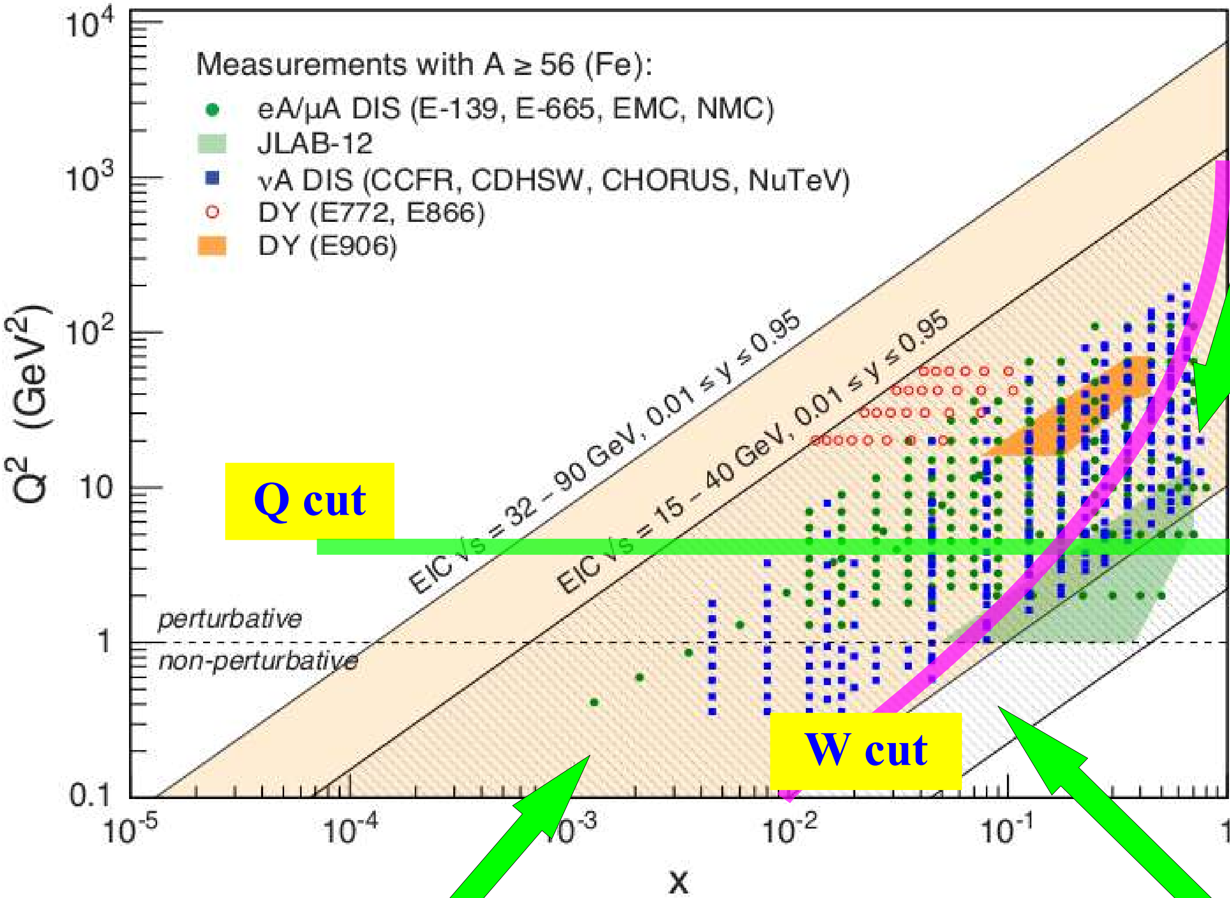
Lagrangian

$$\mathcal{L}_{QCD} = \overline{\psi}_q \left(i\gamma_\mu D^\mu - m_q\right)\psi_q - \frac{1}{4} G^a_{\mu\nu} G_a^{\mu\nu}$$



Conjecture: A theory can't be fundamental unless it fits on a coffee mug.

Do we really understand QCD … push to extreme {x,Q}



High-x:

Nuclear PDFs: x>1 allowed;

impacts F_2^{Nuc}/F_2^{Iso} in Fermi region

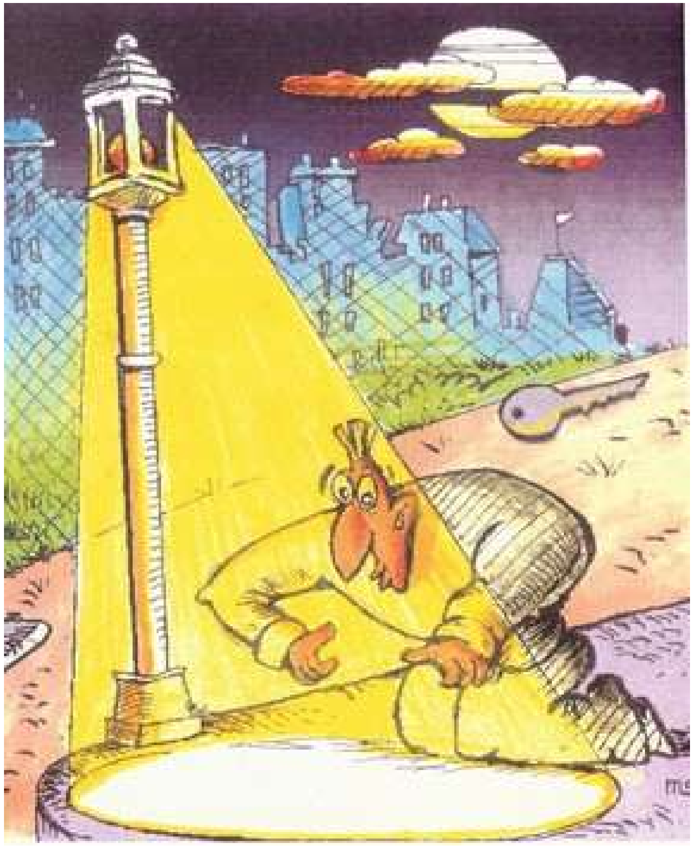
Target Mass Corrections

pick up M^2/Q^2 higher twist

Deuteron Corrections

impacts $F_2^{Nuc}/F_2^{Deuteron}$ ratio

Are we just looking under the lamppost



Low-x:

Shadowing

Recombination

Resummation

BFKL

Saturation

Low-Q^2:

Non-Perturbative inteface

collective effects

Target Mass Corrections

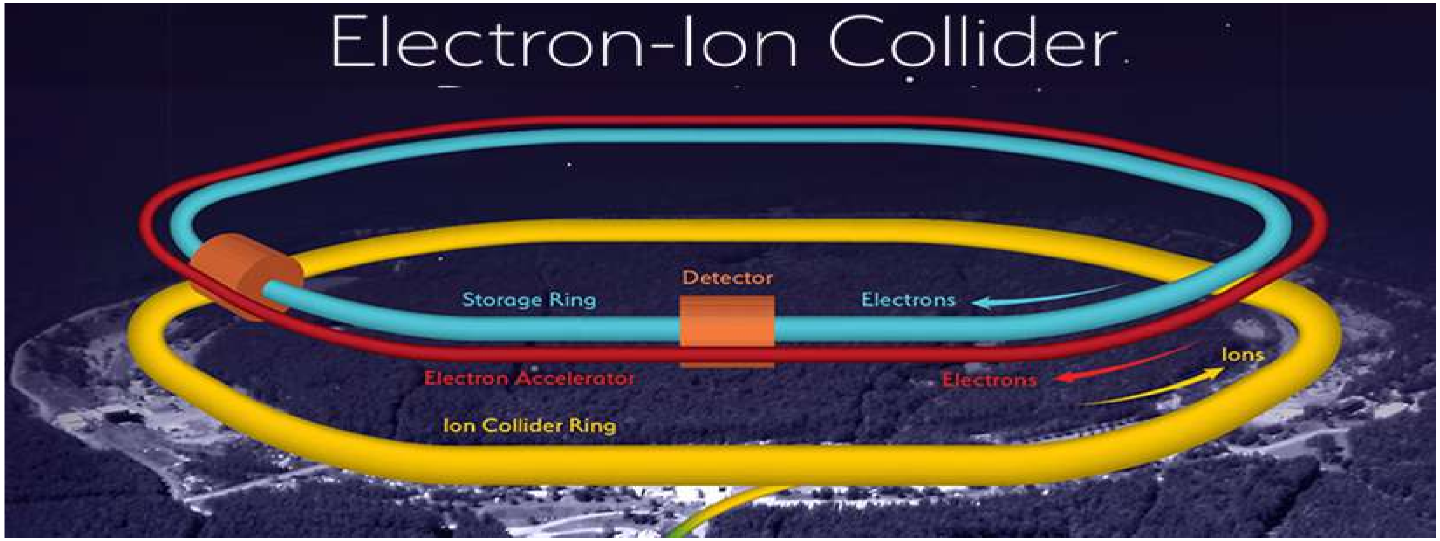
pick up M^2/Q^2 higher twist

F_L at low Q^2 access to g(x)

Need theoretical guidance in these regions

These are hard problems

… we need good ideas & data



“EIC would unlock scientific mysteries” ***NAP Report***

Ideally suited to “ ... glean the fundamental insights into QCD”

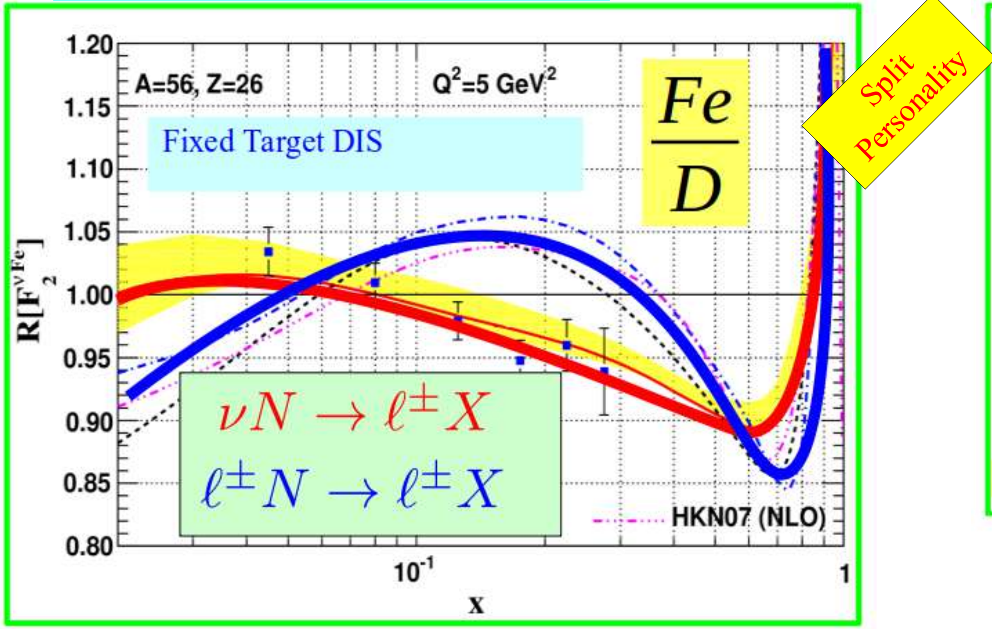
nCTEQ
nuclear parton distribution functions

xFitter

SURGE COLLABORATION

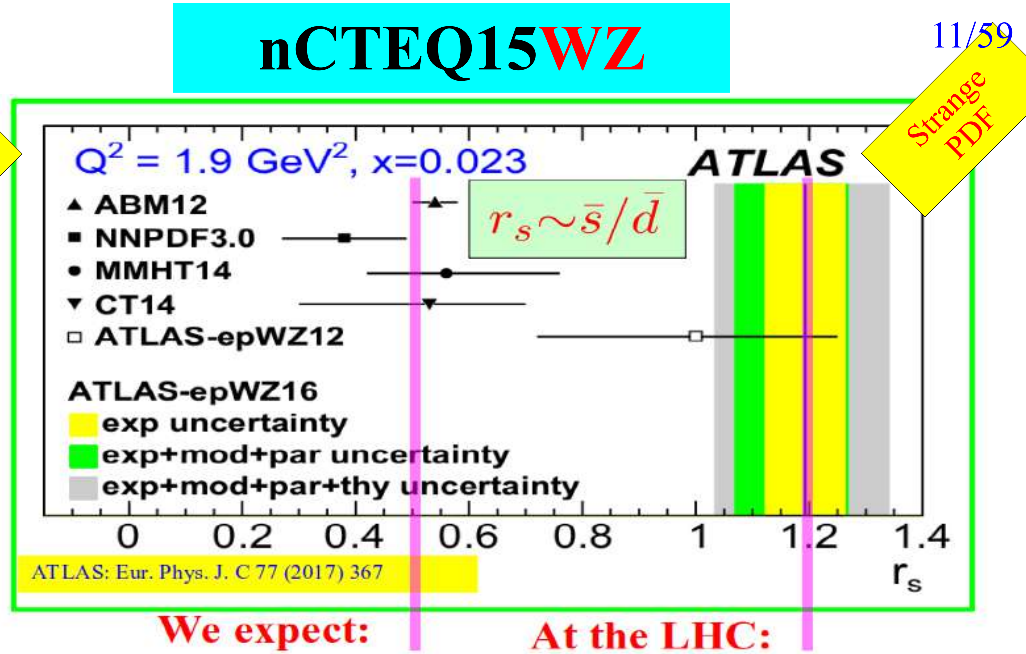
Need theoretical guidance

nCTEQ15ν



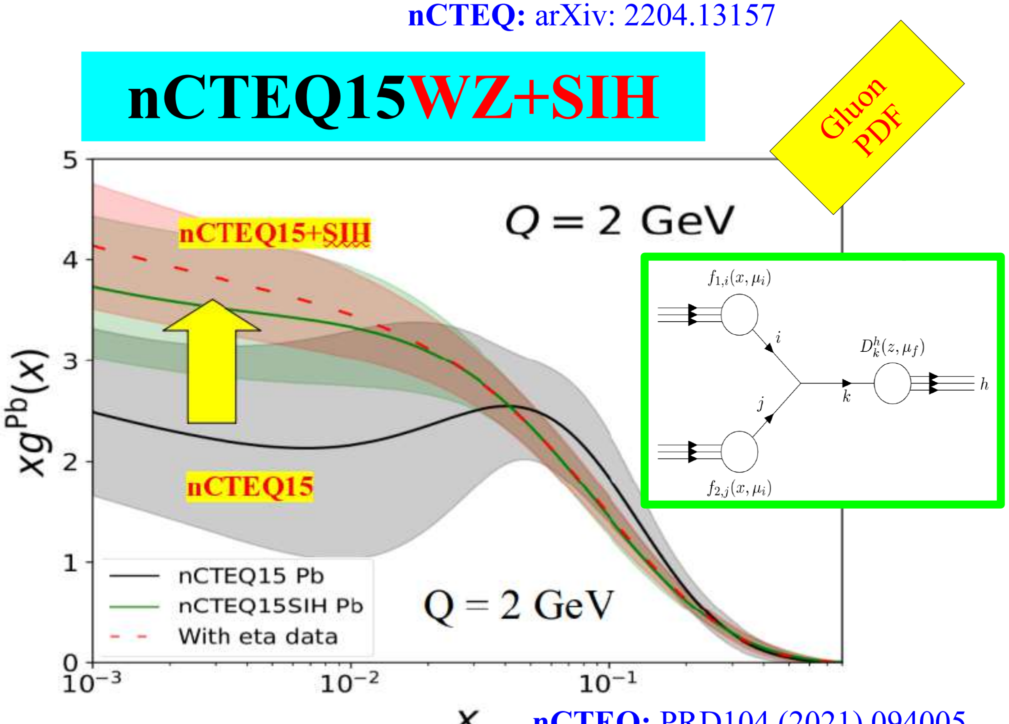
nCTEQ: arXiv: 2204.13157

nCTEQ15WZ



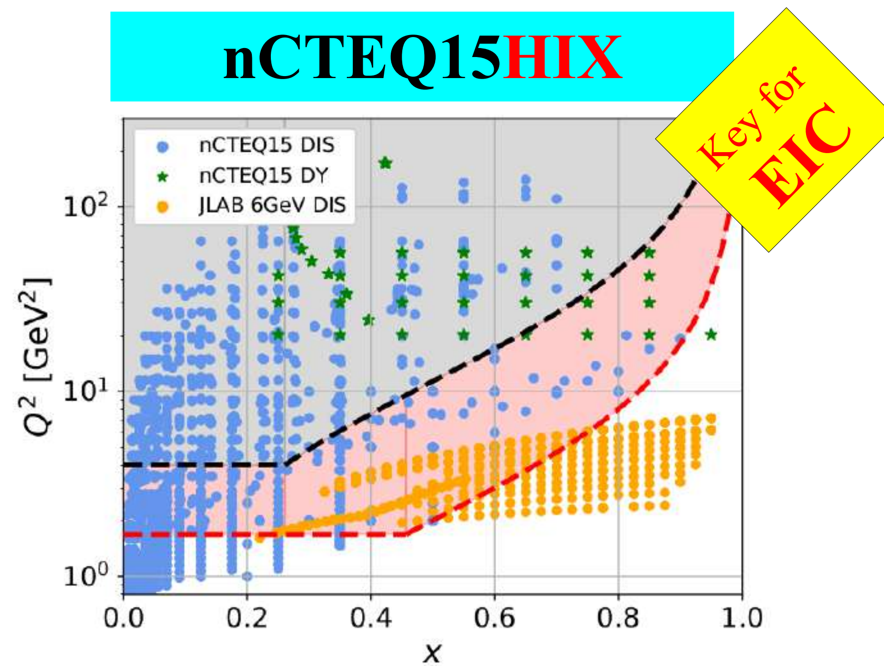
nCTEQ: Phys.Rev.D 104 (2021) 094005

nCTEQ15WZ+SIH



nCTEQ: PRD104 (2021) 094005.

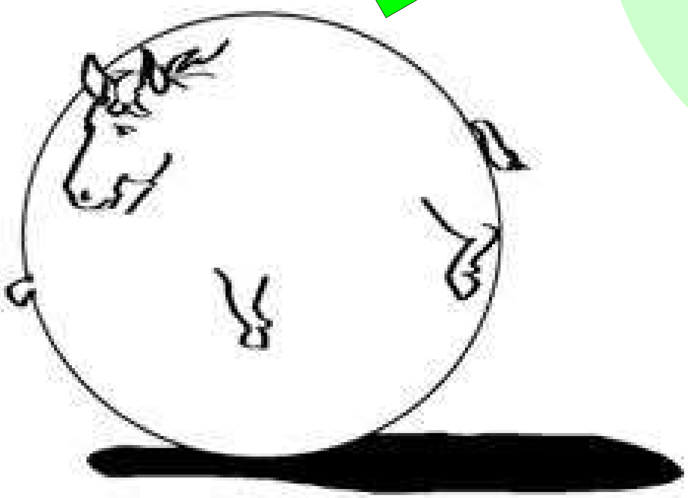
nCTEQ15HIX

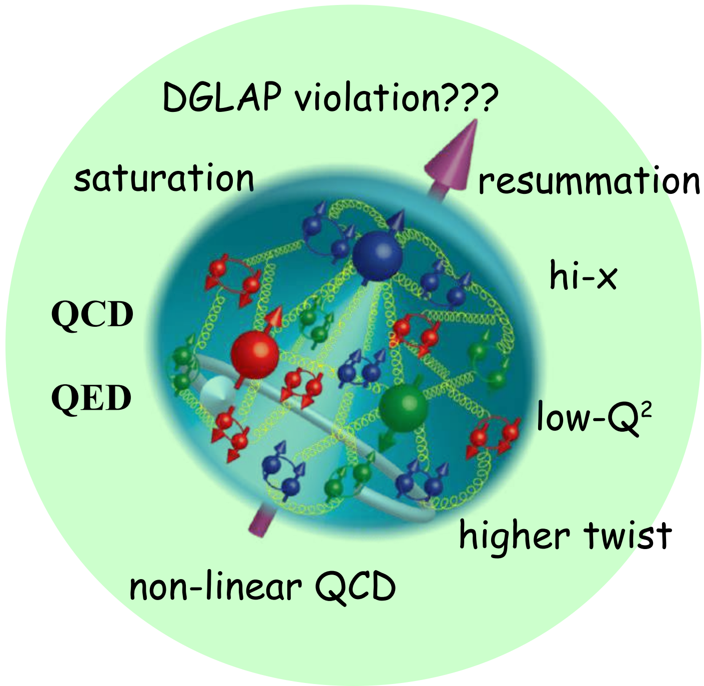


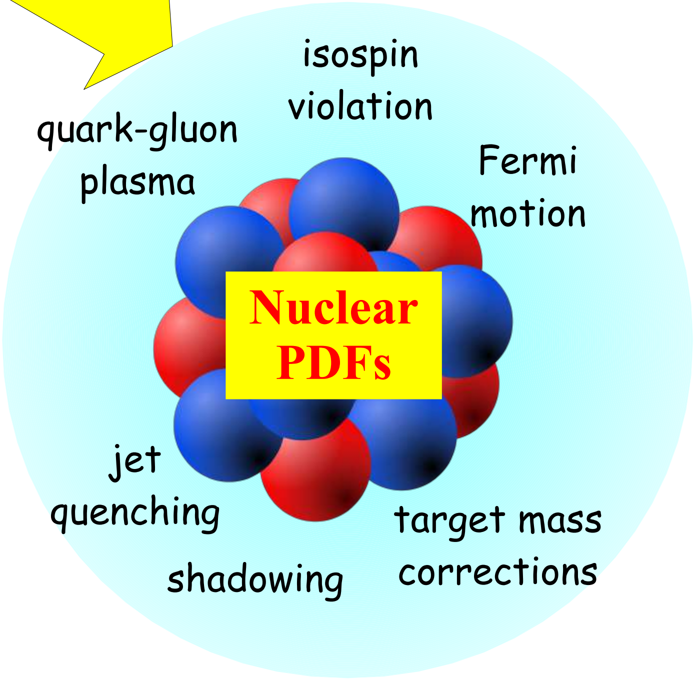
nCTEQ: Phys.Rev.D 103 (2021) 11, 114015

precision $f_A(x,Q)$ can serve as Boundary Condition for $f_A(x,Q,k_T,b_T,\sigma)$

Need theoretical guidance







xFitter

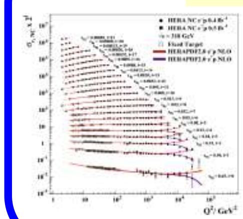
www.xFitter.org

PROTON **NUCLEON** **MESON**

Sample data files:

- **LHC:** ATLAS, CMS, LHCb
- **Tevatron:** CDF, D0
- **HERA:** H1, ZEUS, Combined
- **Fixed Target:** …
- **User Supplied:** ...

Experimental Data



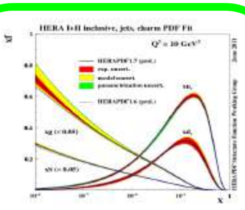
Data: HERA, Tevatron, LHC, fixed target experiments

Processes:
Inclusive DIS, Jets, Drell-Yan, Diffraction, Top production
W and Z production

Theory Calculations

HQ Schemes:	MSTW, NNPDF, ABM, ACOT
Jets, W, Z:	FastNLO, ApplGrid
Top:	Hathor
Evolution:	QCDNUM, APFEL, k_T
Other:	NNPDF reweighting TMDs, Dipole Model, ...

xFitter



Parton Distribution Functions:
PDF, Updf, TMD

$\alpha_S(M_Z)$, m_c, m_b, m_t ...

Theoretical Cross Sections

Comparisons to other PDFs (LHAPDF)

extensions include nuclear PDFs

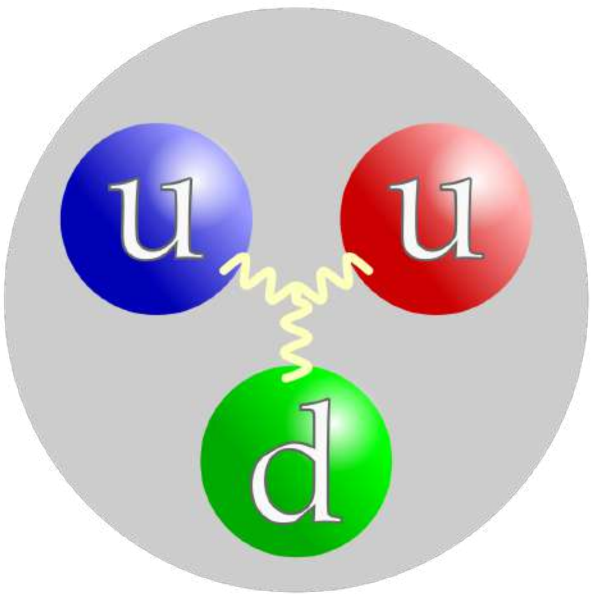
Features & Recent Updates:

- **NNLO DGLAP**
- Photon PDF & **QED**
- Pole & MS-bar masses
- Profiling and Re-Weighting
- **BFKL interface**
- Heavy Quark Variable Treshold
- Improvements in χ^2 and correlations
- **TMD** PDFs (uPDFs)
- *... and many other*

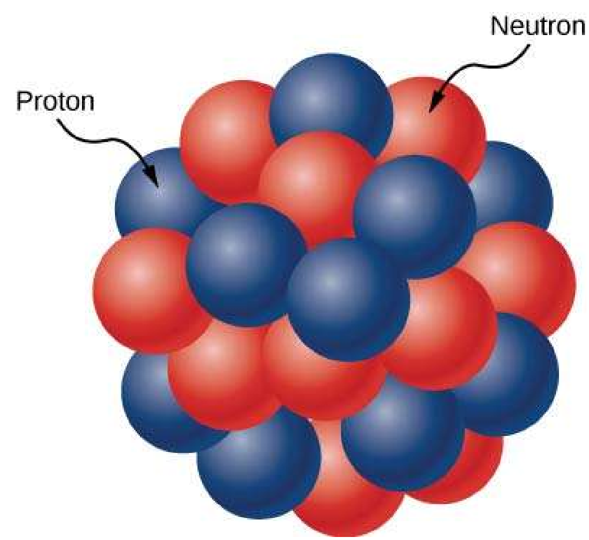
xFitter 2.2.0
Future Freeze

Nuclear PDFs

Parton Distribution Functions



. . .



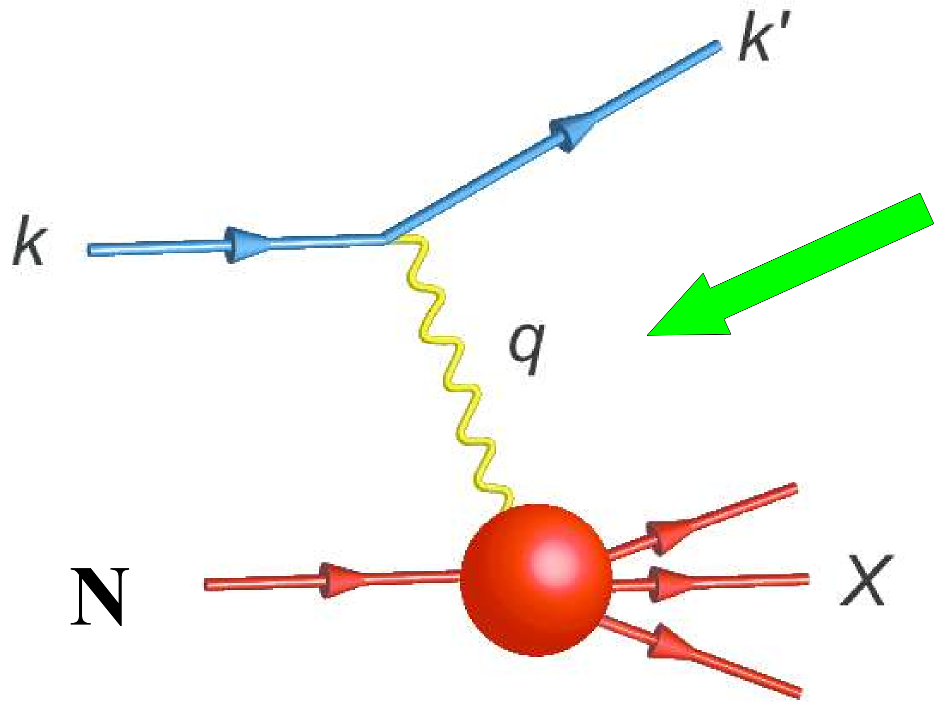
Proton

Nuclei

is it just a bag of protons & neutrons ???

Deep Inelastic Scattering (DIS)

nCTEQ: Faiq Muzakka, Karol Kovarik, ...



Could be:

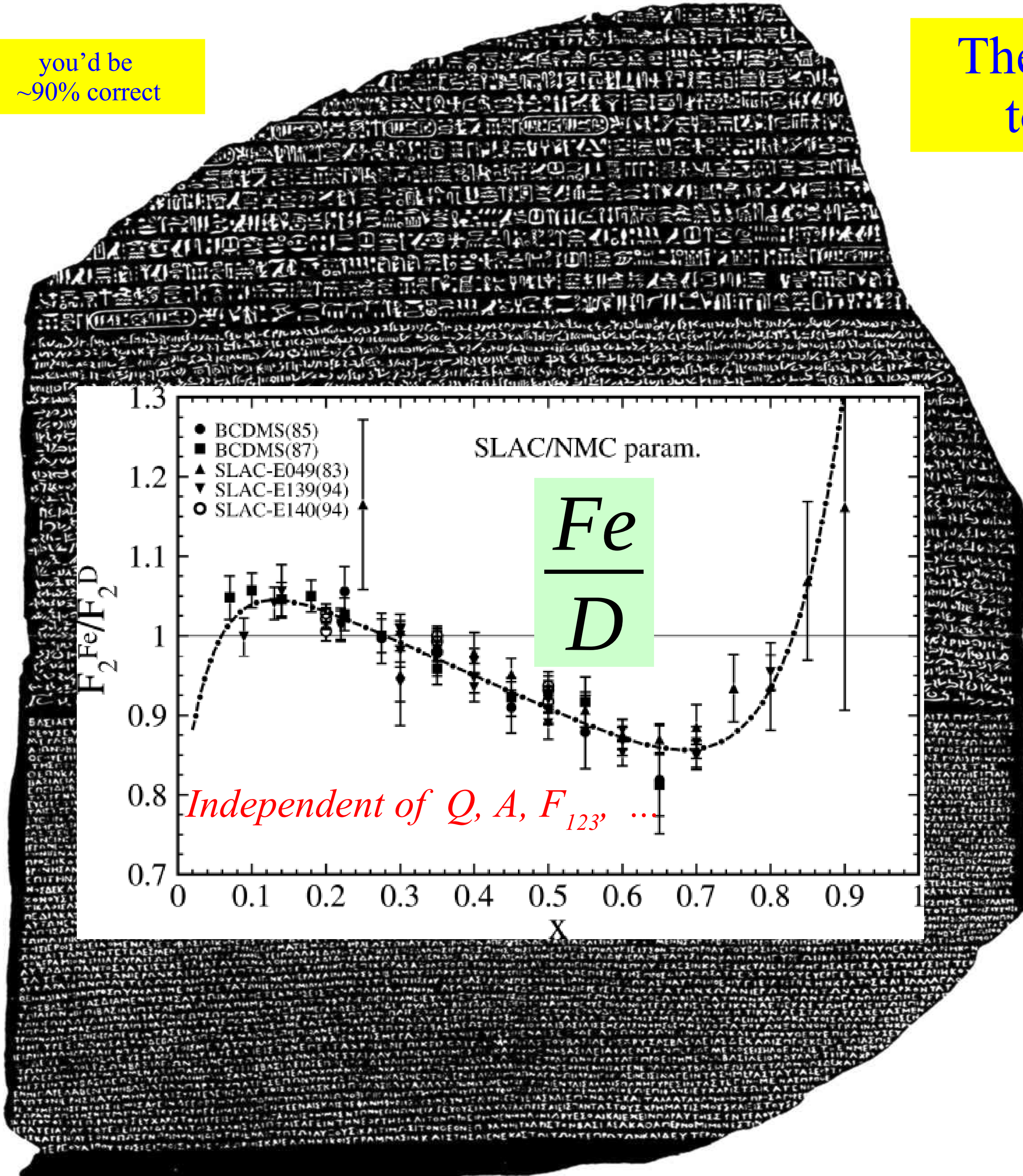
neutral photon γ

or charged $W^{\pm}$

… is it just a bag of protons & neutrons ???

~ 90% correct

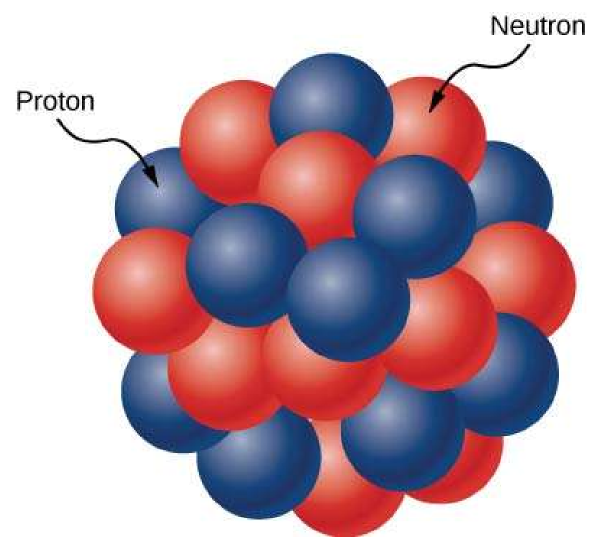
you'd be
~90% correct



The ratio of iron (Fe) to Deuterium (D)

$$\frac{Fe}{D}$$

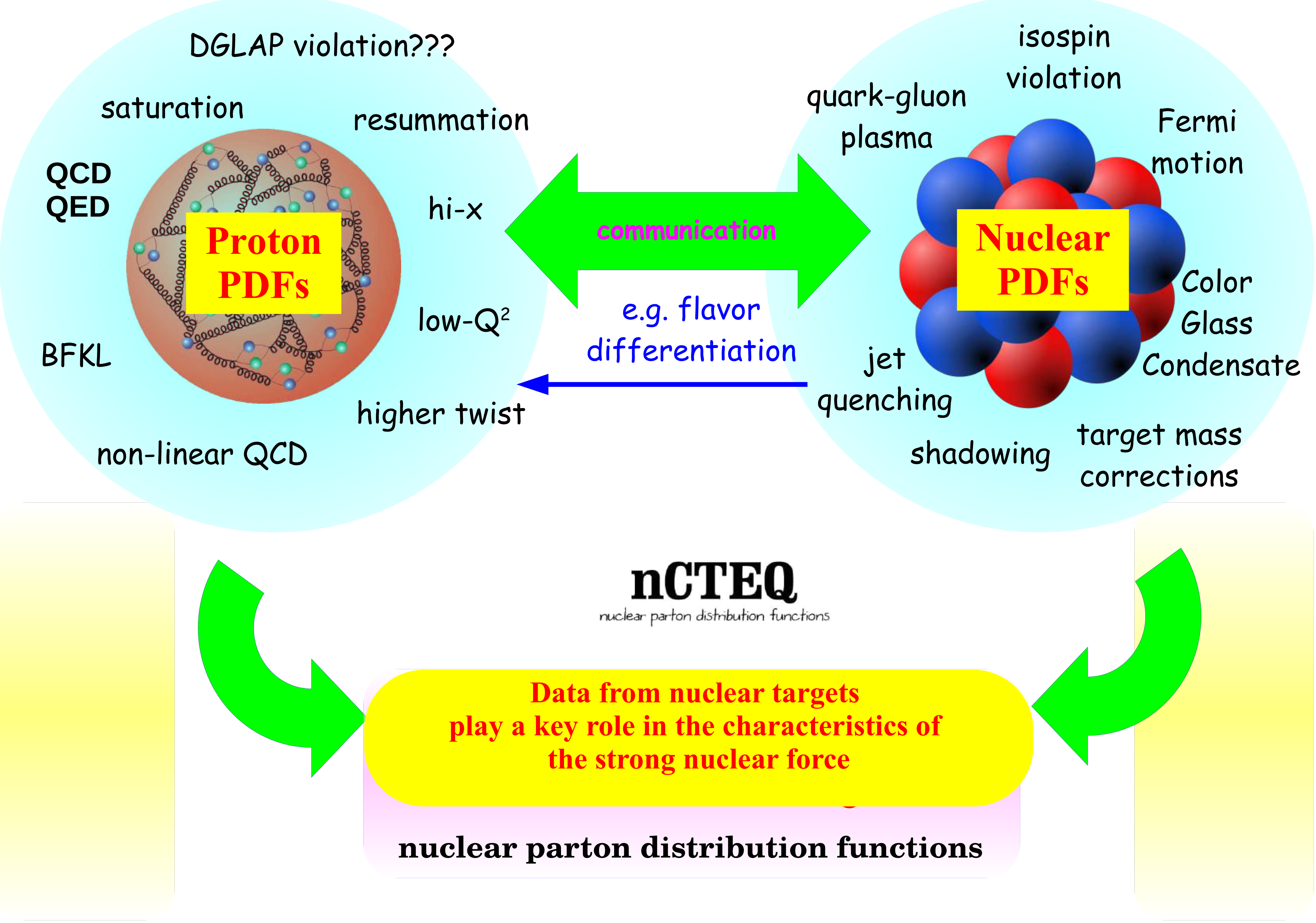
$$\frac{Iron}{(proton + neutron)}$$



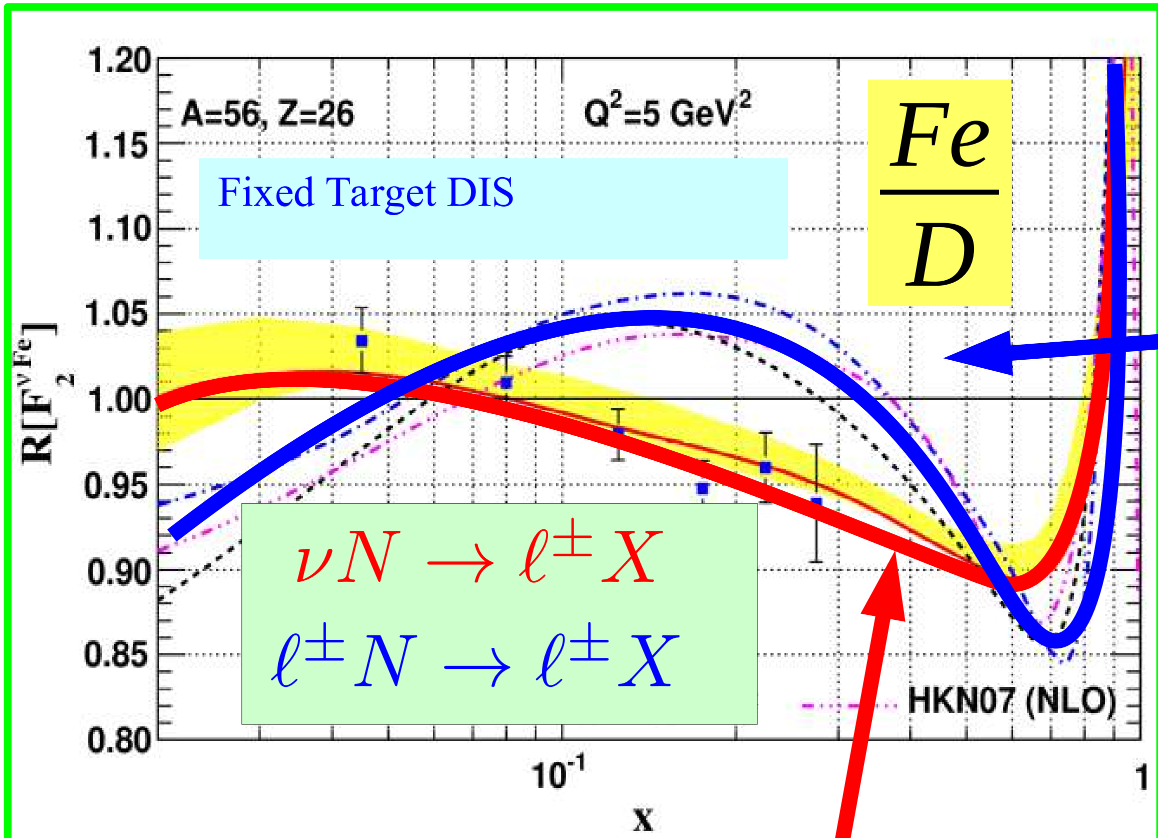
But ...

Discovered by the French in 1799 at Rosetta, a harbor on the Mediterranean coast in Egypt. Comparative translation of the stone assisted in understanding many previously undecipherable examples of hieroglyphics.

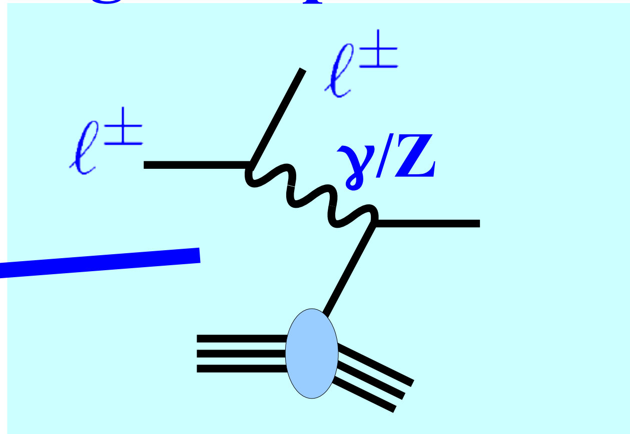
… the ultimate goal for nPDFs



Puzzle: **Split Personality …** *What is the correct Nuclear ratio*



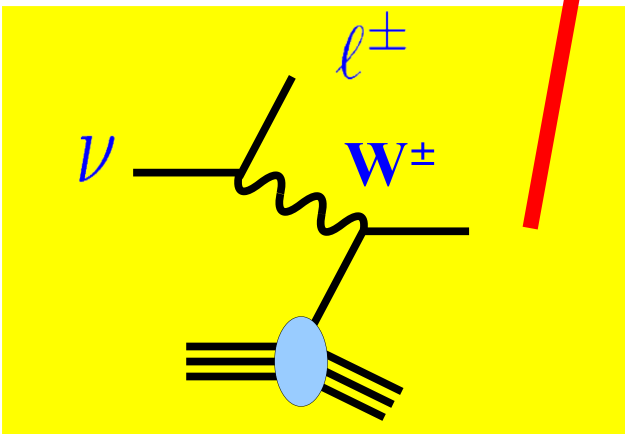
Charged Lepton DIS



some caveats
… correlated errors

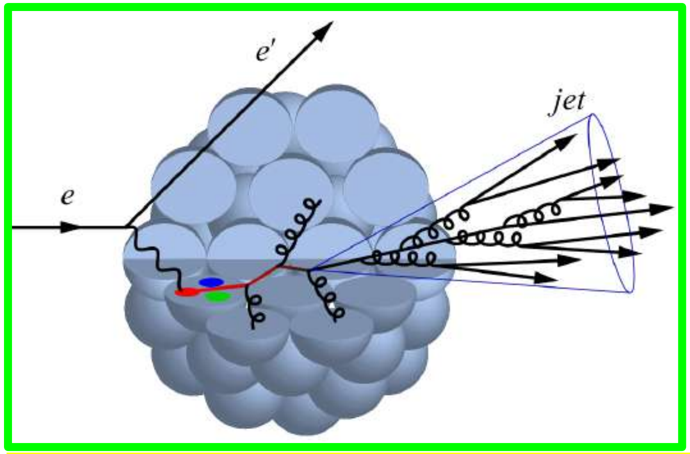
Ingo Schienbein, … (2007)

Karol Kovarik, … (2010)



Neutrino DIS

Depends on nuclear corrections

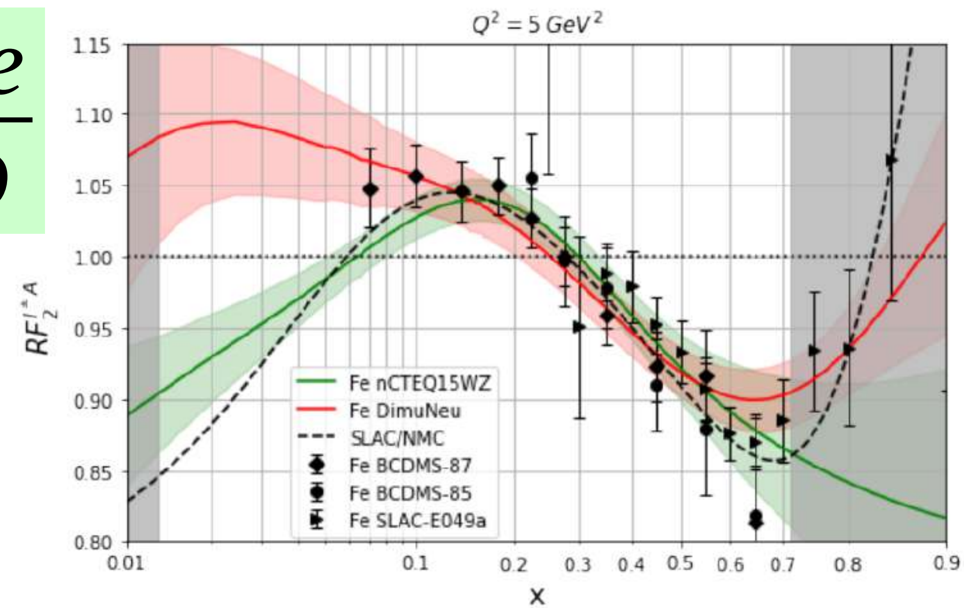


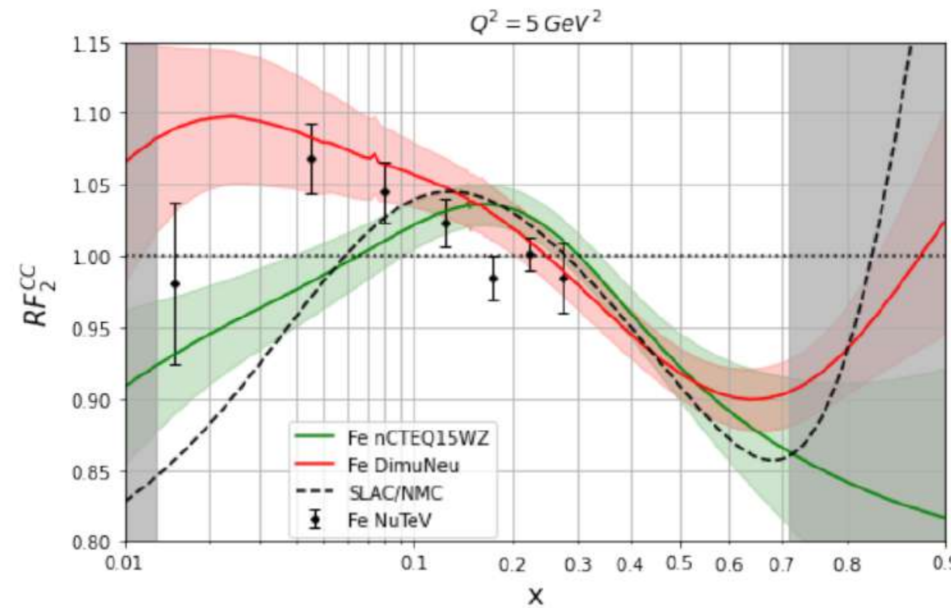
Propagation of γ/W thru nuclei

Puzzle: **Split Personality …** *What is the correct Nuclear ratio*

nCTEQ: Faiq Muzakka, Karol Kovarik, ...

$$\frac{Fe}{D}$$





$$\frac{Iron}{(proton+neutron)}$$

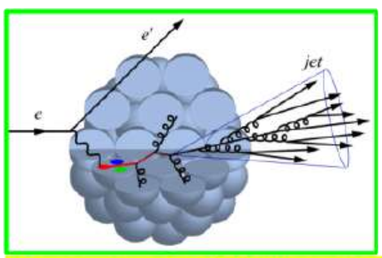
What is the correct nuclear correction ???

Are these data sets compatible???

nCTEQ: K.F. Muzakka, ...
Phys.Rev.D 106 (2022) 7, 074004

Compatibility of neutrino DIS data and its impact on nuclear parton distribution functions

K.F. Muzakka,[1,*] P. Duwentäster,[1,†] T.J. Hobbs,[2,3,4] T. Ježo,[5,‡] M. Klasen,[1,§] K. Kovařík,[1,¶] A. Kusina,[6,**] J.G. Morfín,[7,††] F. I. Olness,[2,‡‡] R. Ruiz,[6] I. Schienbein,[8,§§]
[1]Institut für Theoretische Physik, Westfälische Wilhelms-Universität Münster,



Propagation of γ/W thru nuclei

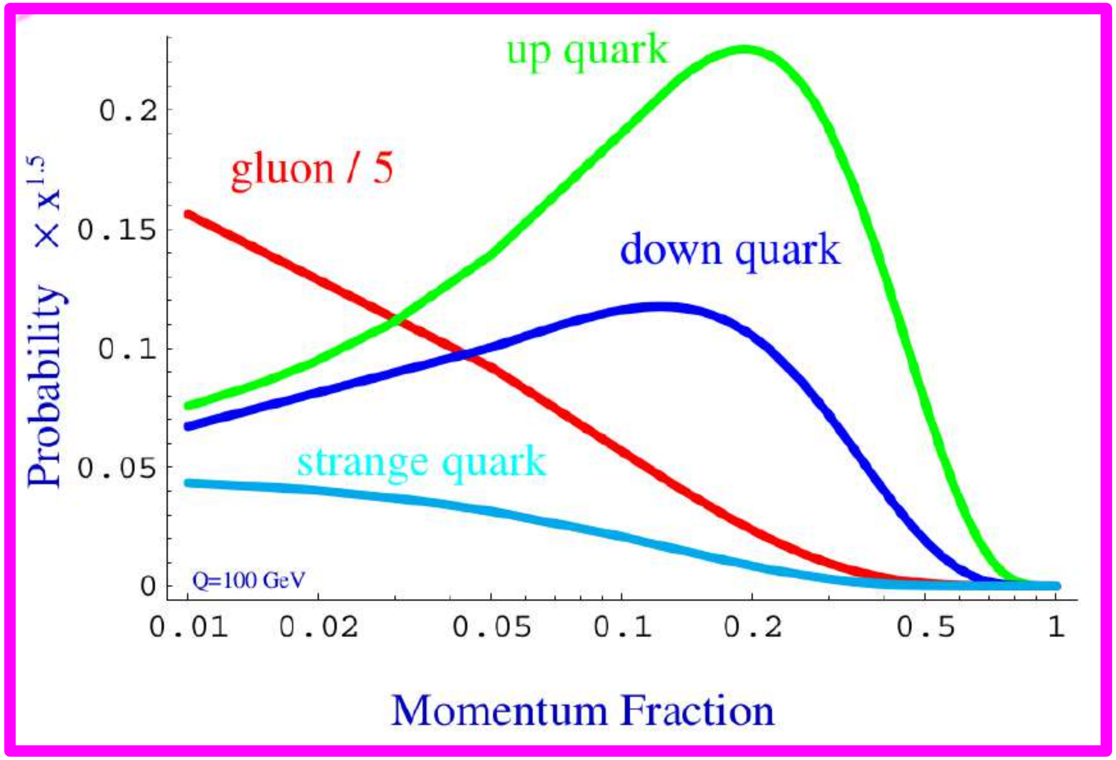
Strange PDF

Parton Distribution Functions

νDIS ... has a significant impact on the strange quark PDF

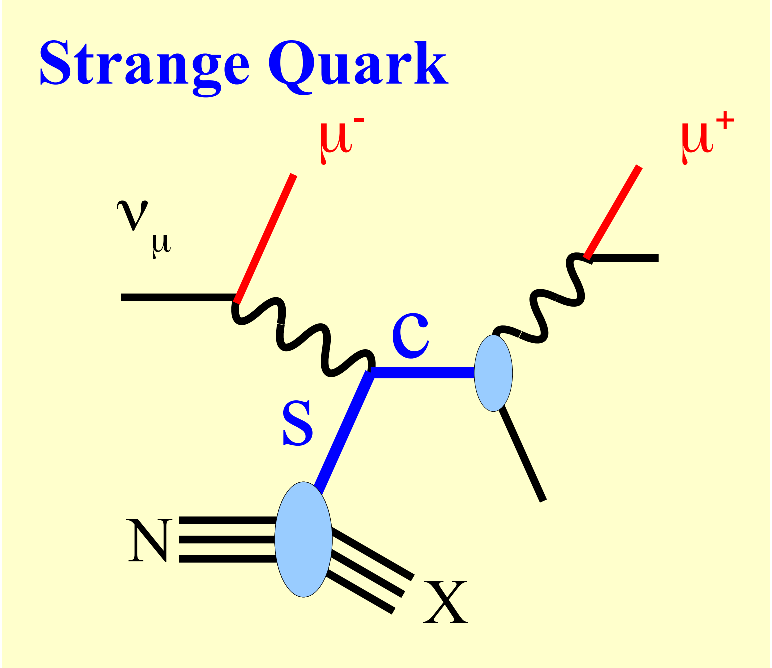
Strange PDF: ν Nucleon di-muon Production

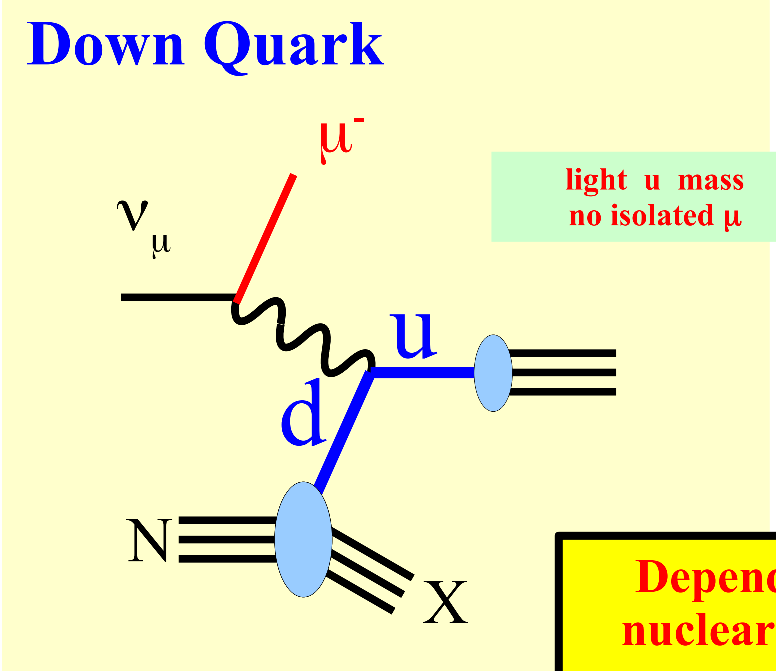
Need to "dig out" s(x) underneath d(x)



Result:

$$\bar{s}(x) \sim \frac{1}{2}\,\bar{d}(x)$$



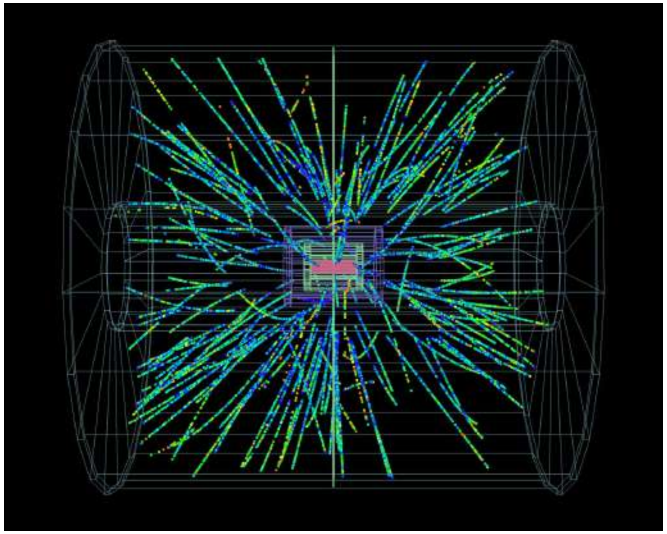


light u mass
no isolated μ

Depends on nuclear PDFs

W and Z Boson Production at the Large Hadron Collider (LHC)

nCTEQ: Tomas Jezo, Aleksander Kusina, Fred Olness, ...



$$p\ p \rightarrow W, Z$$
$$p\,Pb \rightarrow W, Z$$

LHC Heavy Ion

… there's another way to measure the strange quark

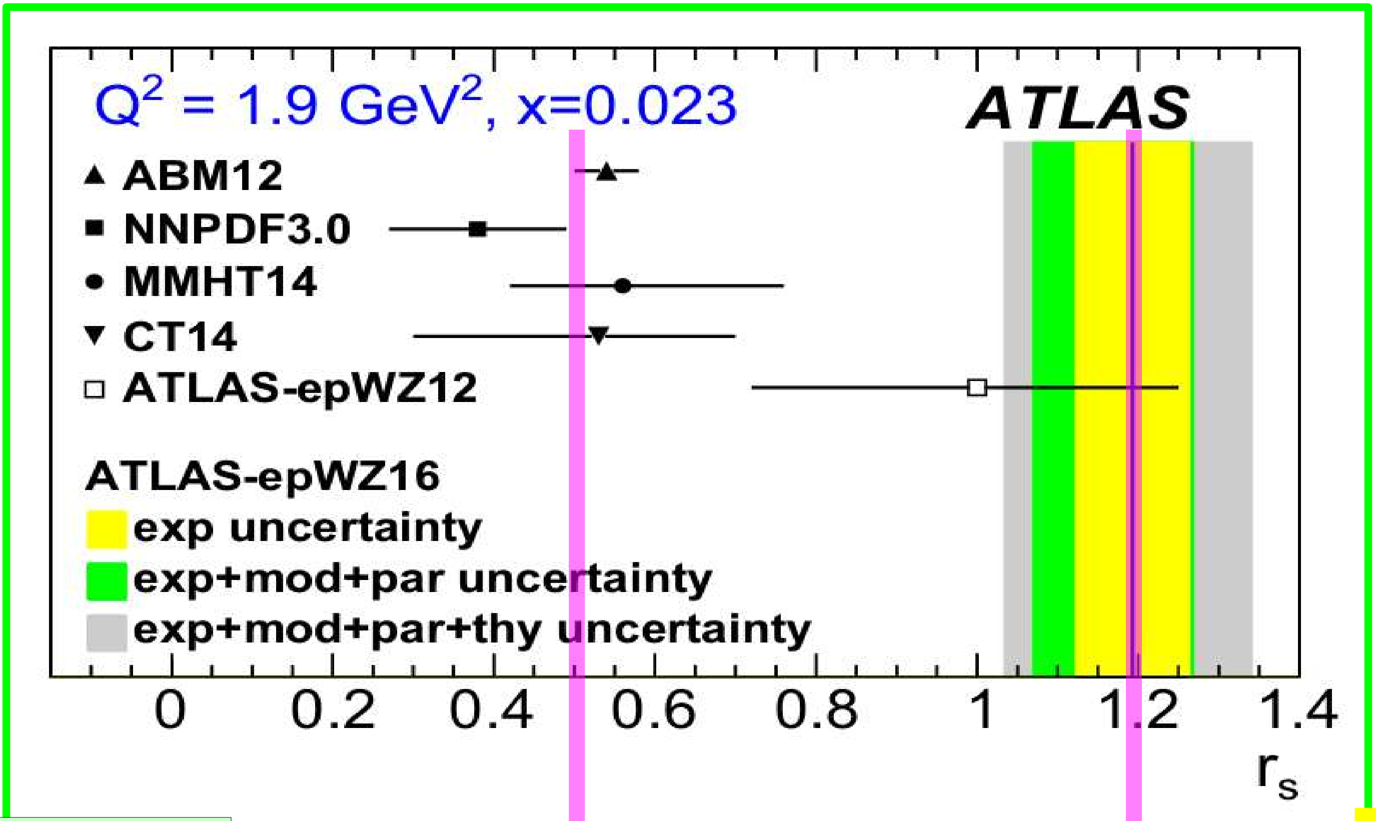
Surprise: … LHC sees more strange than expected

$$u\bar{d} + u\bar{s} + c\bar{d} + c\bar{s} \rightarrow W^+$$
$$\bar{u}d + \bar{u}s + \bar{c}d + \bar{c}s \rightarrow W^-$$
$$u\bar{u} + d\bar{d} + s\bar{s} + c\bar{c} \rightarrow Z$$

Surprise:

We expected $r_s = 1/2$

LHC finds $r_s > 1$



Proton case

$$r_s \sim \bar{s}/\bar{d}$$

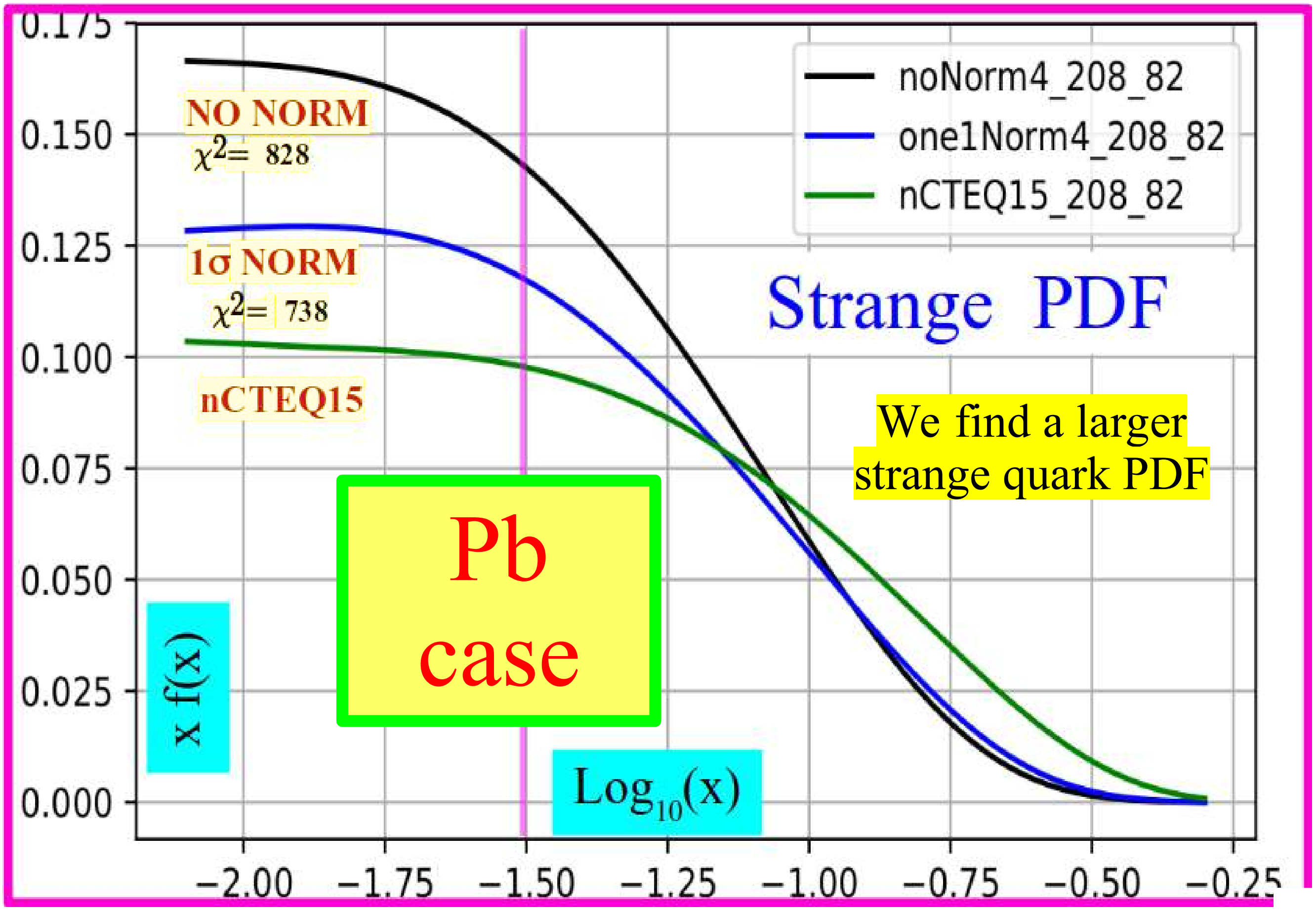
We expect:

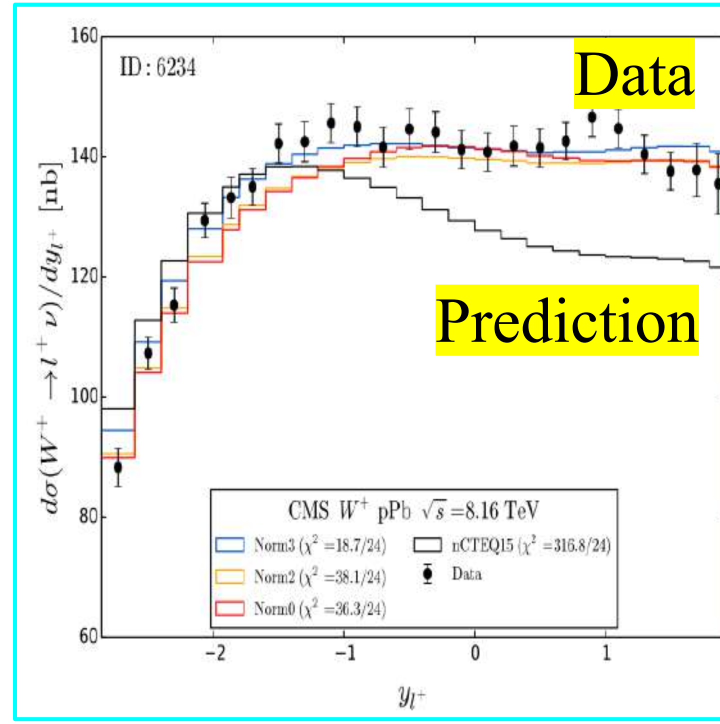
At the LHC:

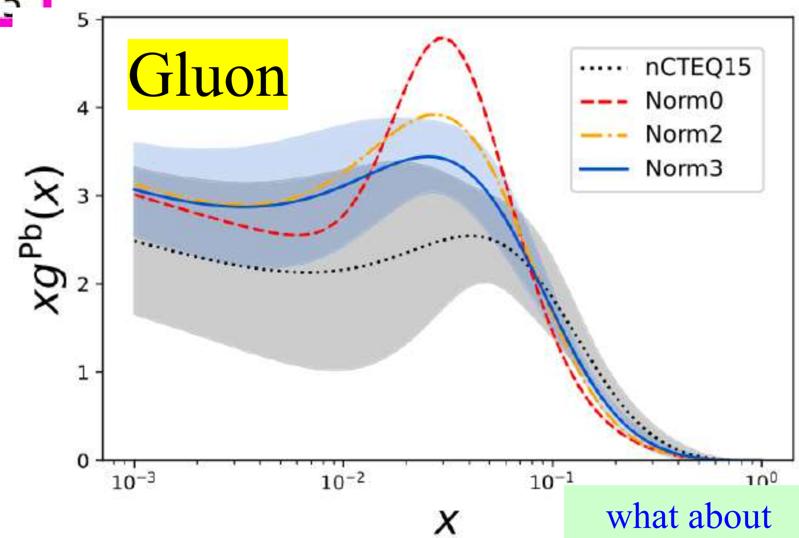
$$r_s = \frac{\bar{s} + s}{2\bar{d}}$$

ATLAS: Eur. Phys. J. C 77 (2017) 367

pPb Heavy Ion Case: … LHC STILL sees more strange than expected







Is the strange PDF driving the data ...

Or is the data driving the strange ???

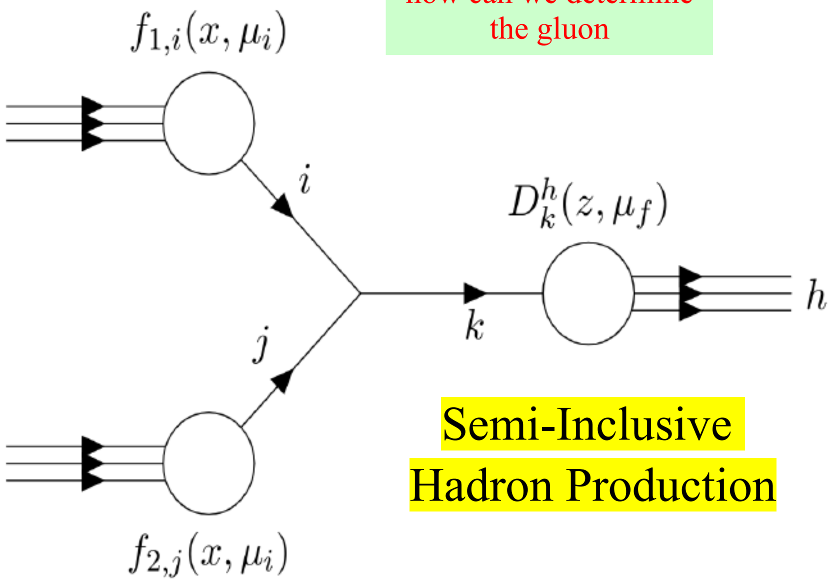
what about the gluon???

nCTEQ: Tomas Jezo, Aleksander Kusina, Fred Olness ...
Eur.Phys.J.C 80 (2020) 10, 968

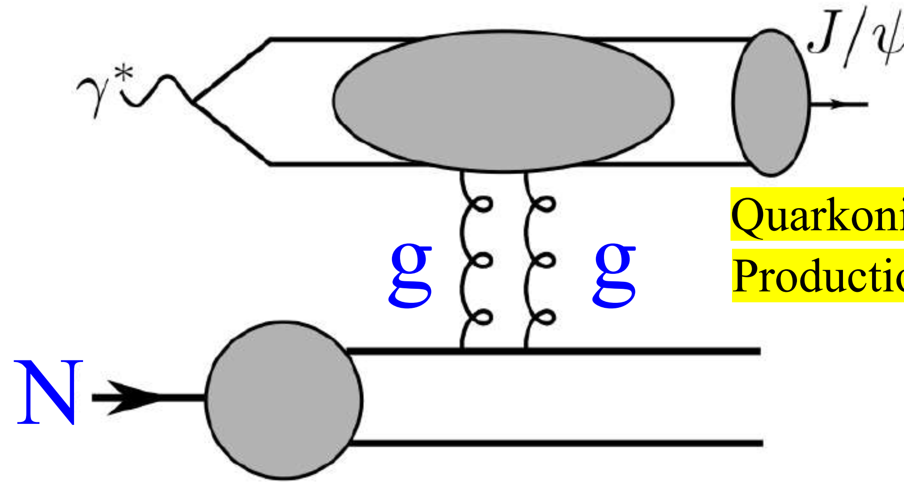
Measuring the nuclear Gluon PDF

Parton Distribution Functions

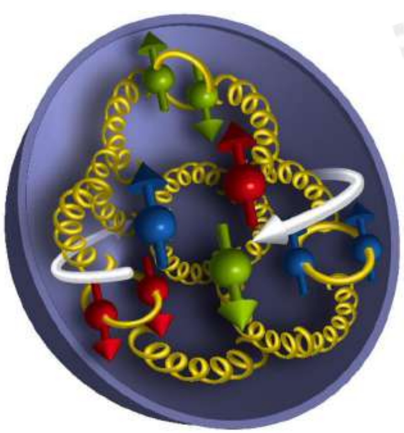
how can we determine the gluon

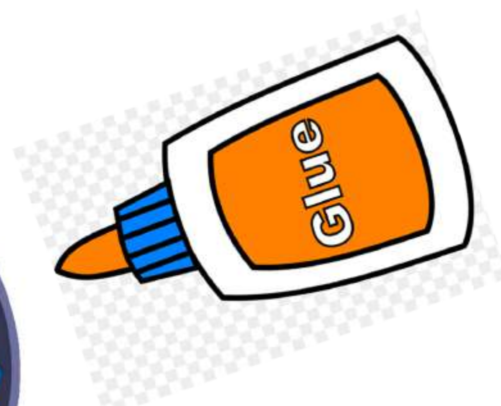


Semi-Inclusive Hadron Production

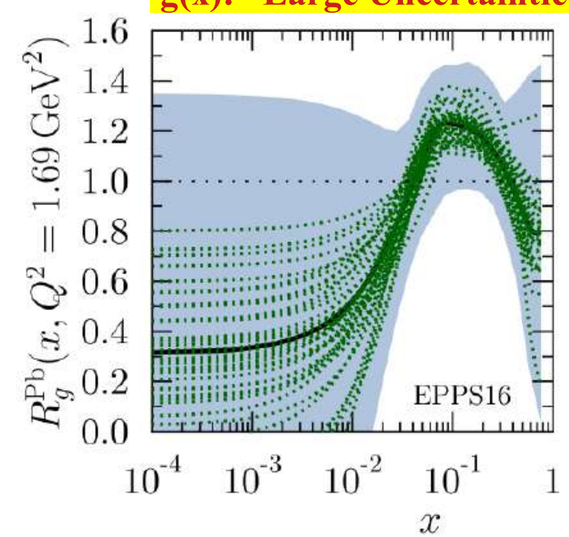


Quarkonia Production

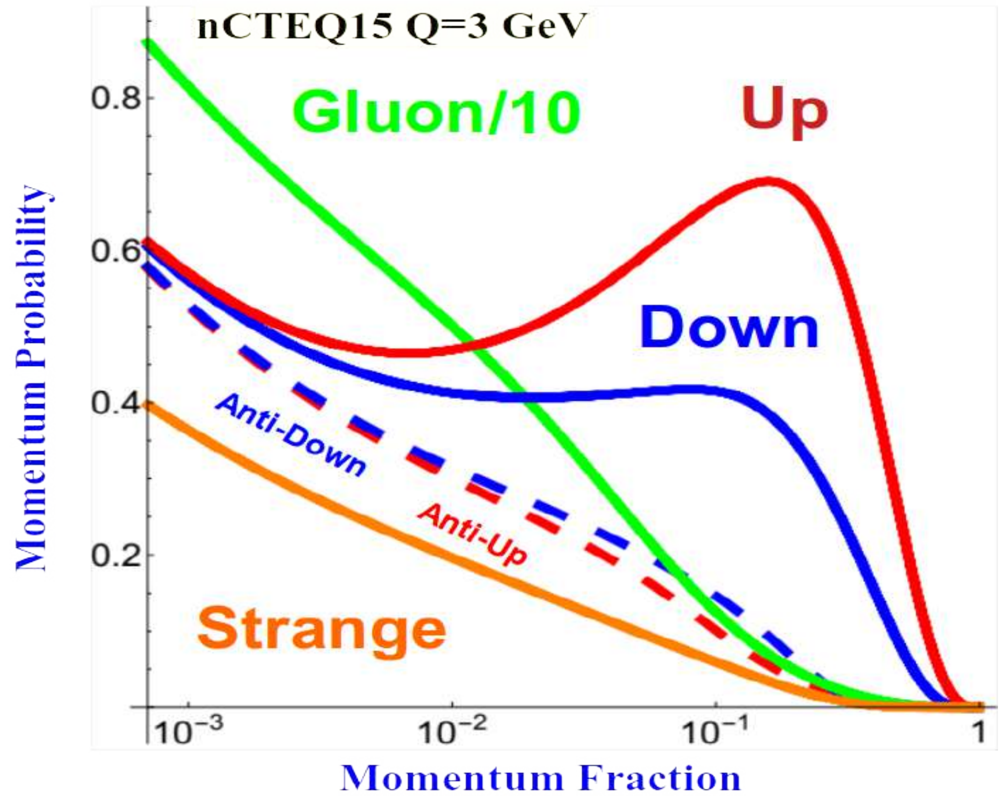


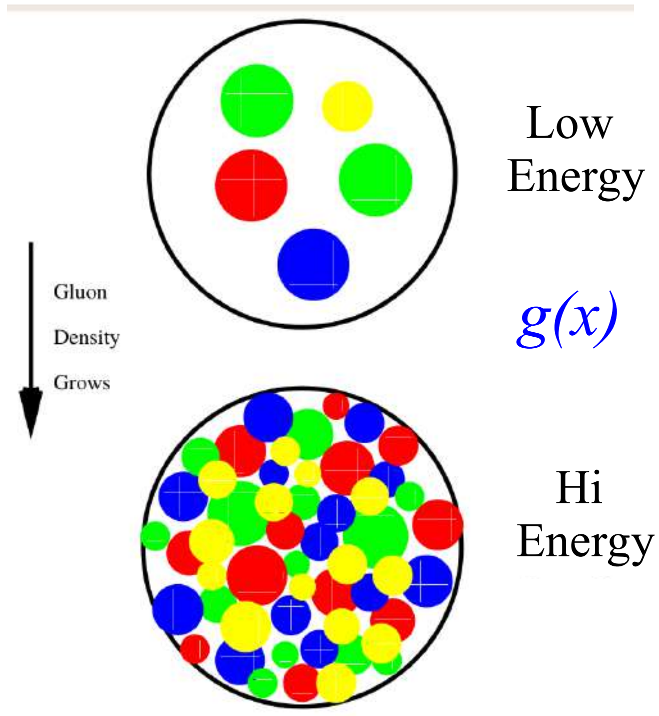


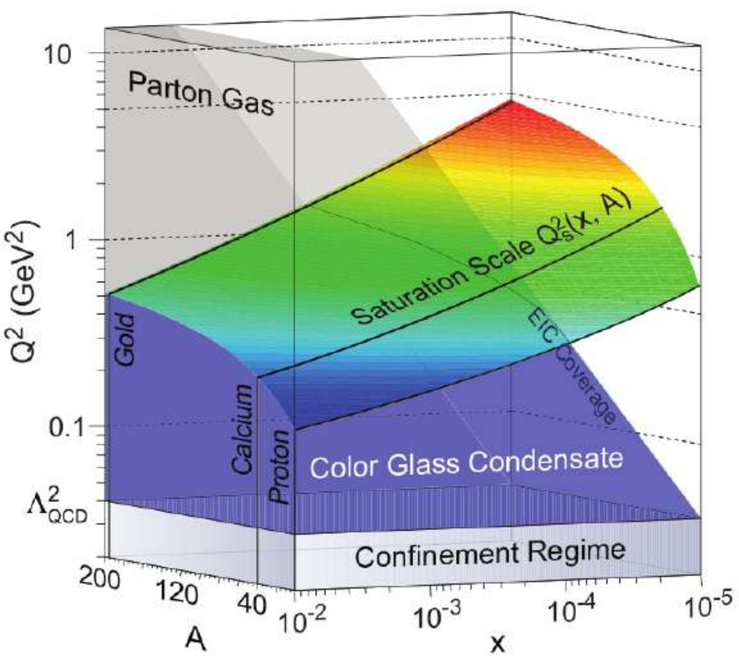
g(x): Large Uncertainties



Gluon: Nuclear Medium Effects at small momentum fraction (x)



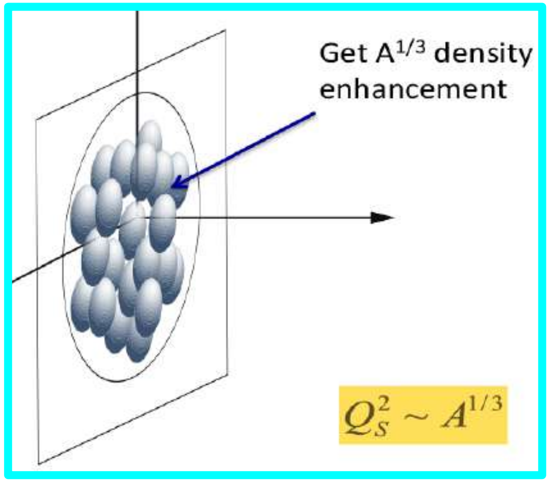




Nuclear medium effects:

- Quark Gluon Plasma
- Color Glass Condensate
- Recombination
- Saturation
- Resummation
- … *your theory here*

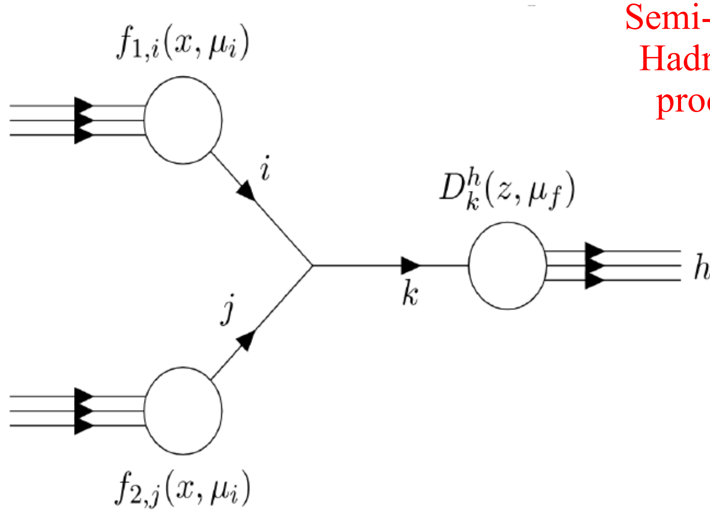
We gain a geometric factor of $A^{1/3}$



Review: Edmond Iancu & Raju Venugopalan: arXiv:0303204

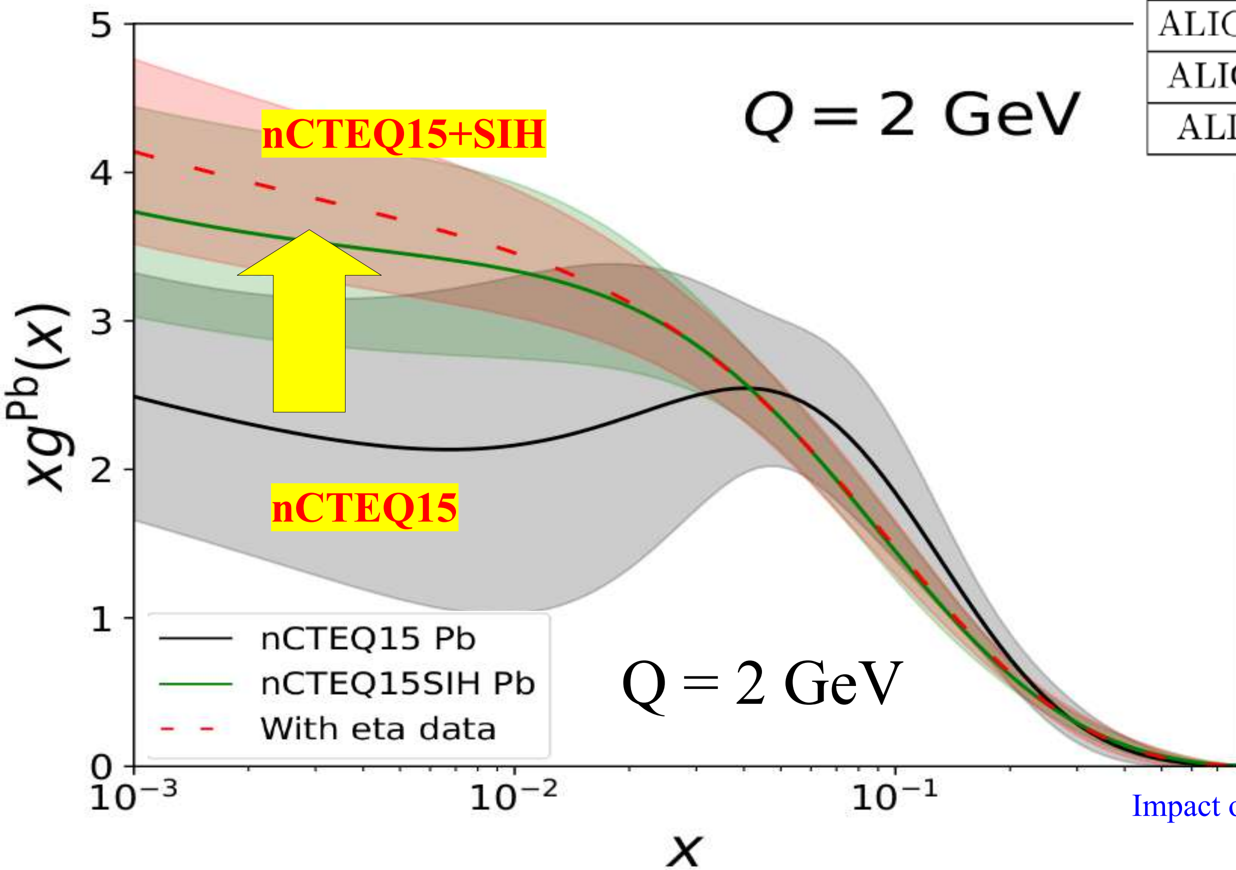
Precision Gluon can help study nuclear medium effects

nCTEQ: Pit Duwentaster, Michael Klasen, ...



Semi-Inclusive Hadron (SIH) production

Data set	$\sqrt{s_{NN}}$ [GeV]	Observ.	No. points
PHENIX π^0	200	R_{dAu}	21
PHENIX η	200	R_{dAu}	12
PHENIX $\pi^\pm$	200	R_{dAu}	20
PHENIX $K^\pm$	200	R_{dAu}	15
STARπ^0	200	R_{dAu}	13
STAR η	200	R_{dAu}	7
STAR $\pi^\pm$	200	R_{dAu}	23
ALICE 5 TeV π^0	5020	R_{pPb}	31
ALICE 5 TeV η	5020	R_{pPb}	16
ALICE 5 TeV $\pi^\pm$	5020	R_{pPb}	58
ALICE 5 TeV $K^\pm$	5020	R_{pPb}	58
ALICE 8 TeV π^0	8160	R_{pPb}	30
ALICE 8 TeV η	8160	R_{pPb}	14



Semi-Inclusive Hadron (SIH) production

Determines gluon in small x region

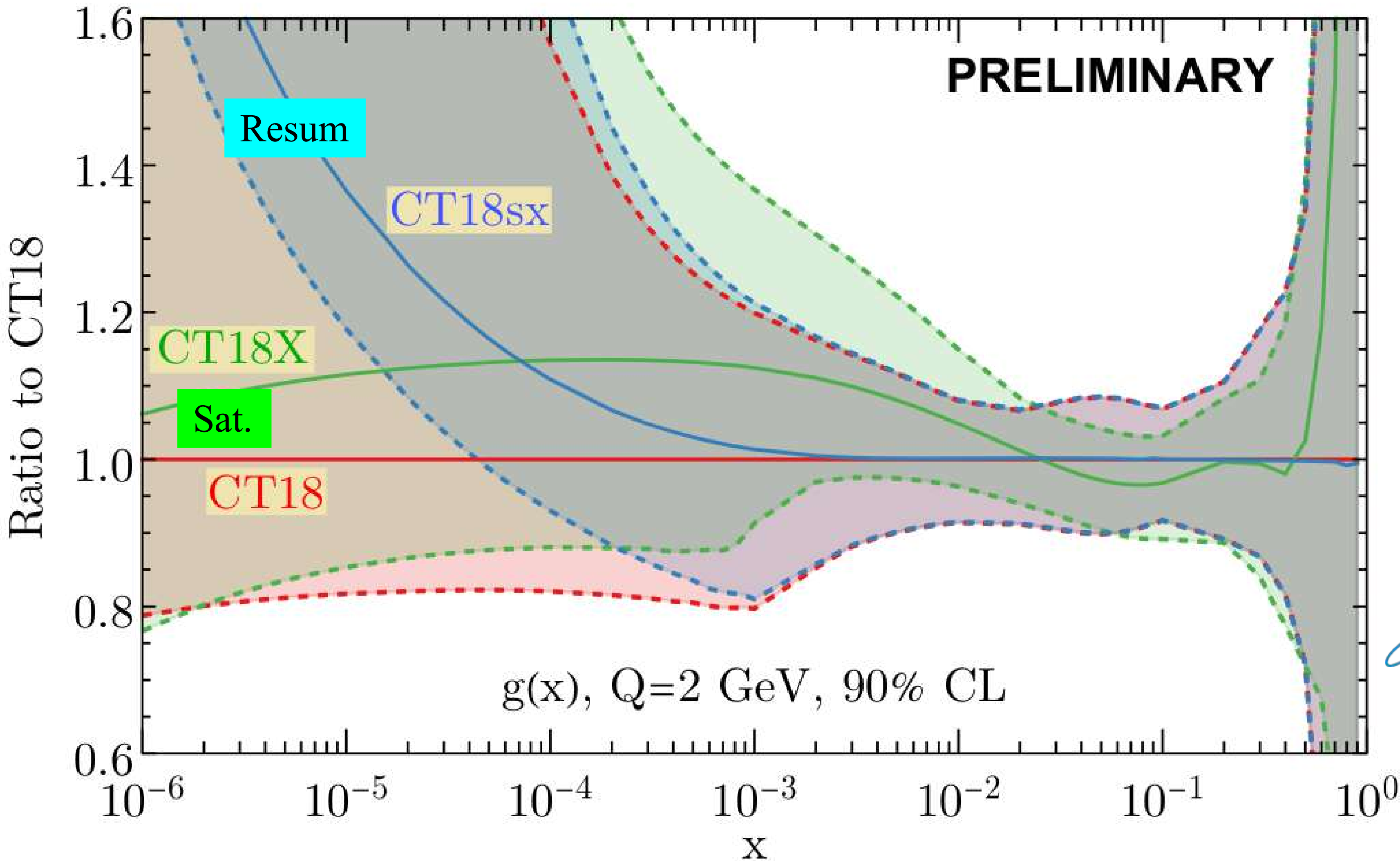
Impact of inclusive hadron production data on nuclear gluon PDFs
nCTEQ: P. Duwentäster, et al., PRD104 (2021) 094005.

Gluon is sensitive to small-x treatment

CT18x: Saturation inspired μ modification
CT18sx: w/ HELL small-x resummation code

Saturation inspired x-dependent

$$\mu^2 = a_1 \left(Q^2 + \frac{a_2}{x^{a_3}} \right)$$



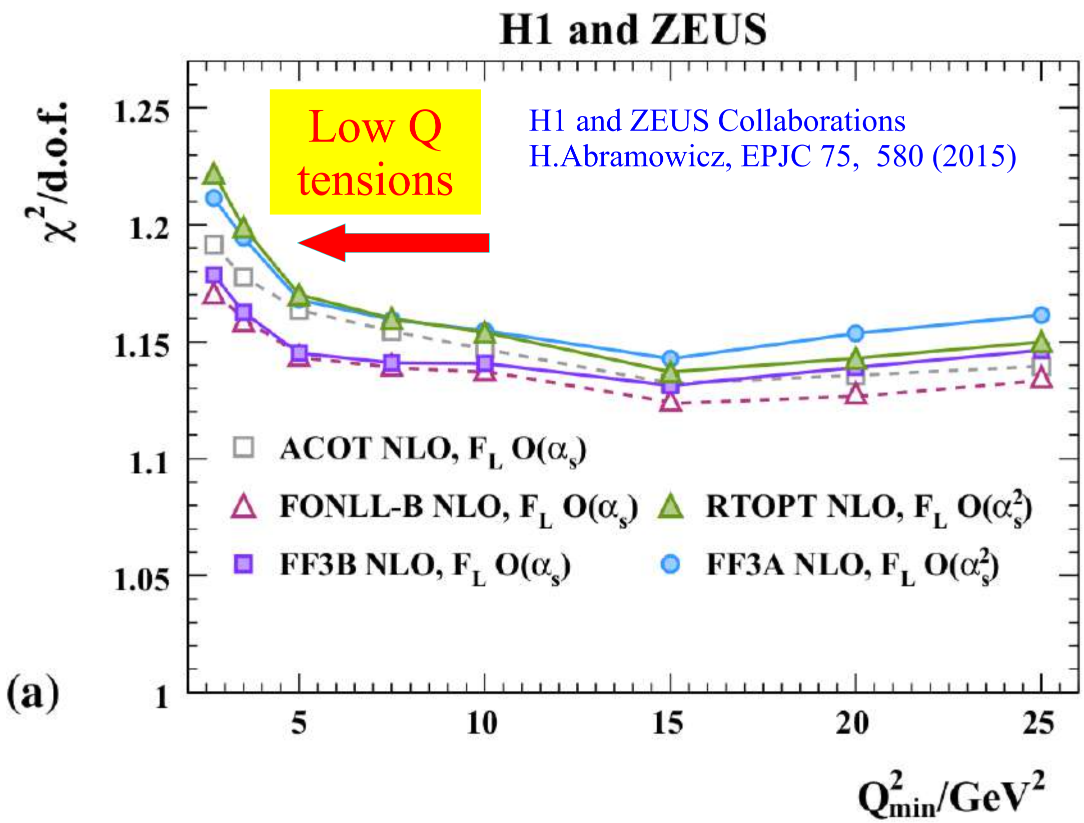
BFKL & Saturation differ at very small x

Small-x resummation from HELL
Marco Bonvini, et al., Eur.Phys.J.C 76 (2016) 11, 597

CT Collaboration: 2108.06596 [hep-ph]

Motivation for Improved Treatment

Small x *(Low Q)*: need to improve fits
NNLO: "fits at NNLO do not improve agreement"



HERAPDF2.0 shows tensions between data and fit, independent of the heavy-flavour scheme used, at low Q^2, i.e. below $Q^2 = 15\,\mathrm{GeV}^2$, and at high Q^2, i.e. above $Q^2 = 150\,\mathrm{GeV}^2$. Comparisons between the behaviour of the fits with different Q^2_{min} values indicate that the NLO theory evolves faster than the data towards lower Q^2 and x. Fits at NNLO do not improve the agreement. HERAPDF2.0 NNLO and NLO have a similar fit quality.

NNLO vs. NLO

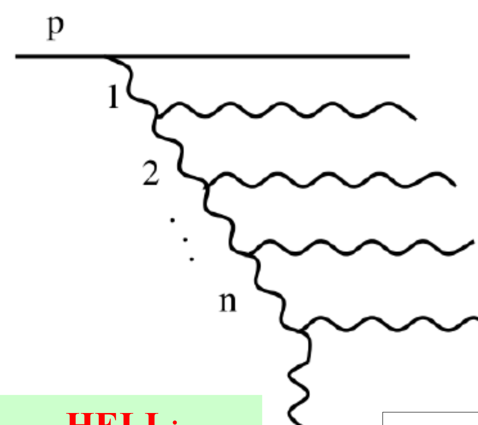
Eur. Phys. J. C (2018) 78:621
https://doi.org/10.1140/epjc/s10052-018-6090-8

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article - Theoretical Physics

Impact of low-*x* resummation on QCD analysis of HERA data

xFitter Developers' team, Hamed Abdolmaleki[1], Valerio Bertone[2,3,a], Daniel Britzger[4], Stefano



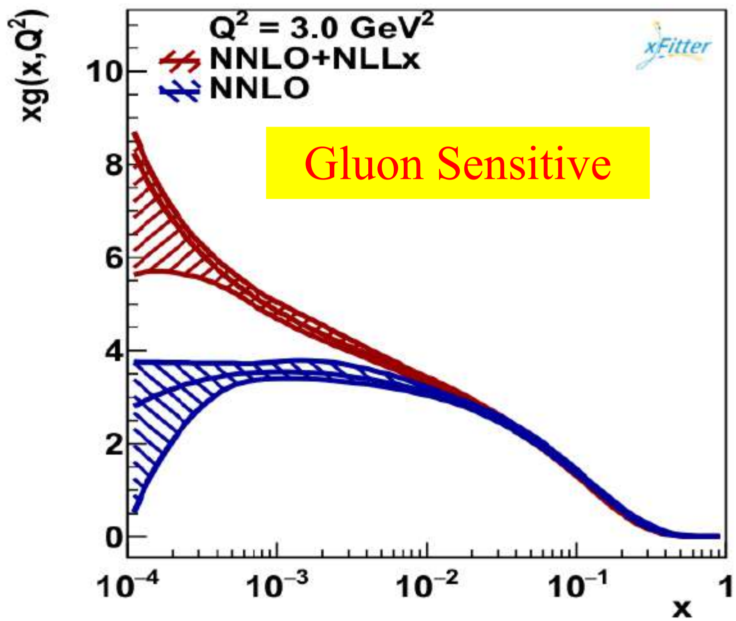
resum logs

$$\alpha_S^n \frac{\ln^k(x)}{x}$$

HELL:
High Energy
Leading Logs

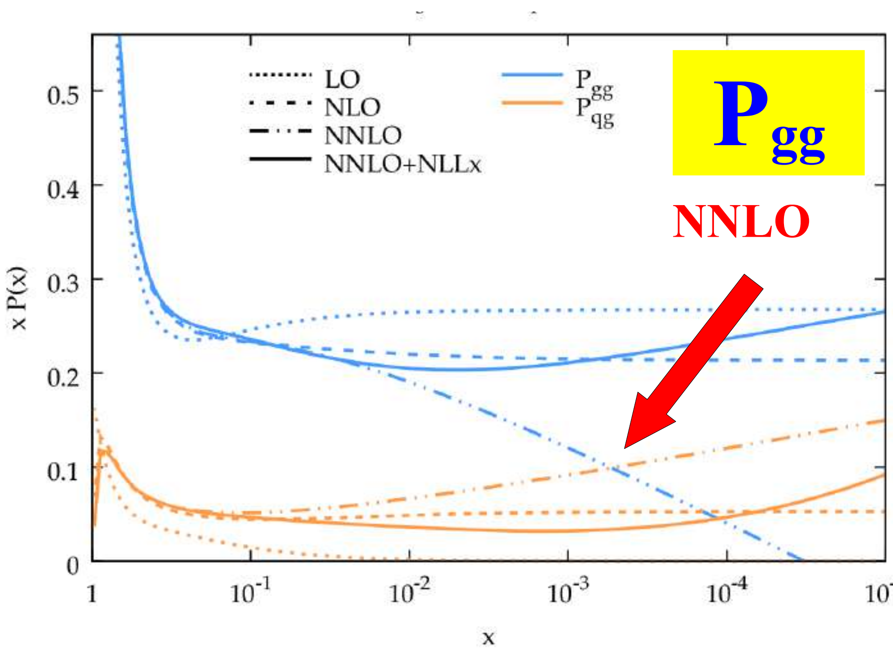
Small-x resummation from HELL
Marco Bonvini, et al.,
Eur.Phys.J.C 76 (2016) 11, 597

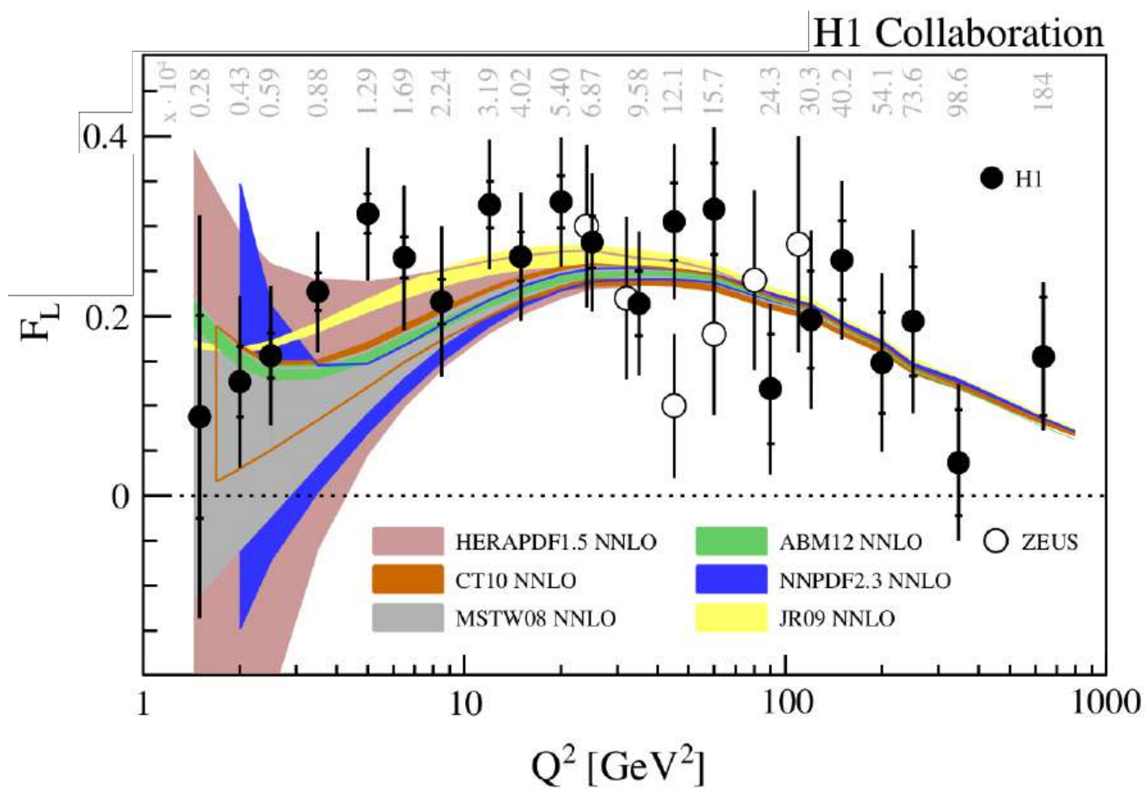
xFitter Resummation Study



F_L is senstive to Gluon and Small-x Treatments

Eur. Phys. J. C (2014) 74:2814
DOI 10.1140/epjc/s10052-014-28





Impact of low-x resummation on QCD analysis of HERA data
xFitter Developers' Team: Eur.Phys.J.C 78 (2018) 8, 621

Short Range Correlations (SRC)

nCTEQ with

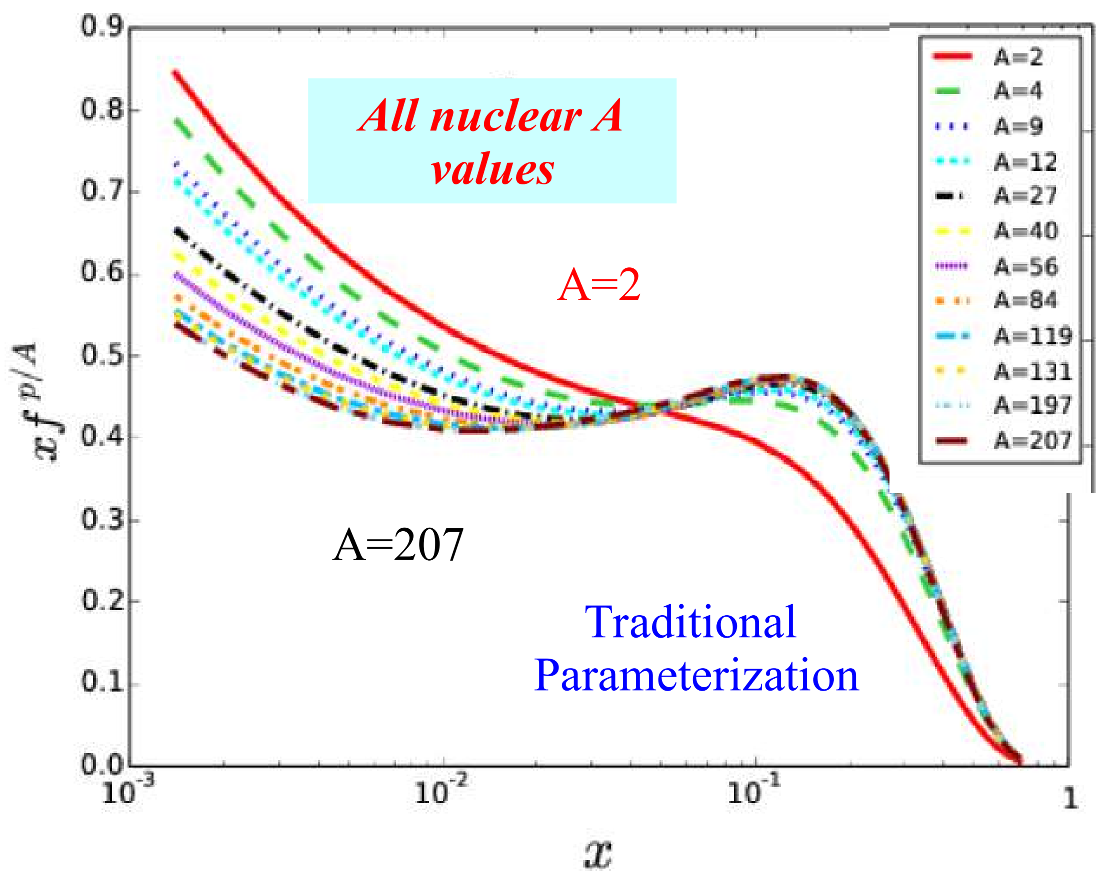
Andrew Denniston & Or Hen

Hen Lab

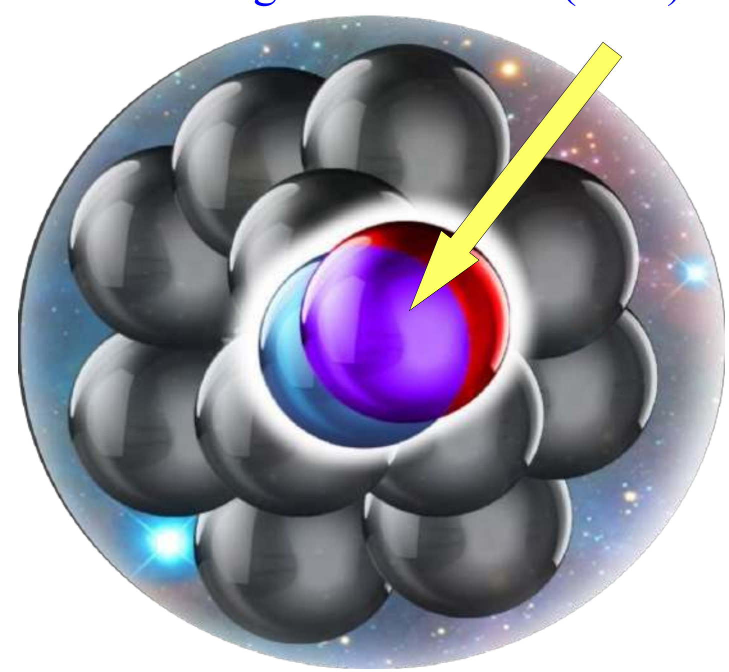
Laboratory for Nuclear Science @ MIT

$$f_A = (1\text{-}c)\ f_p + c\ f_{SRC}$$

nuclear PDF — normal proton PDF — SRC modified PDF

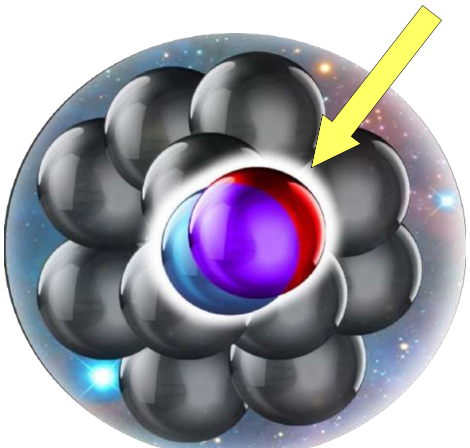


Short Range Correlations (SRC)



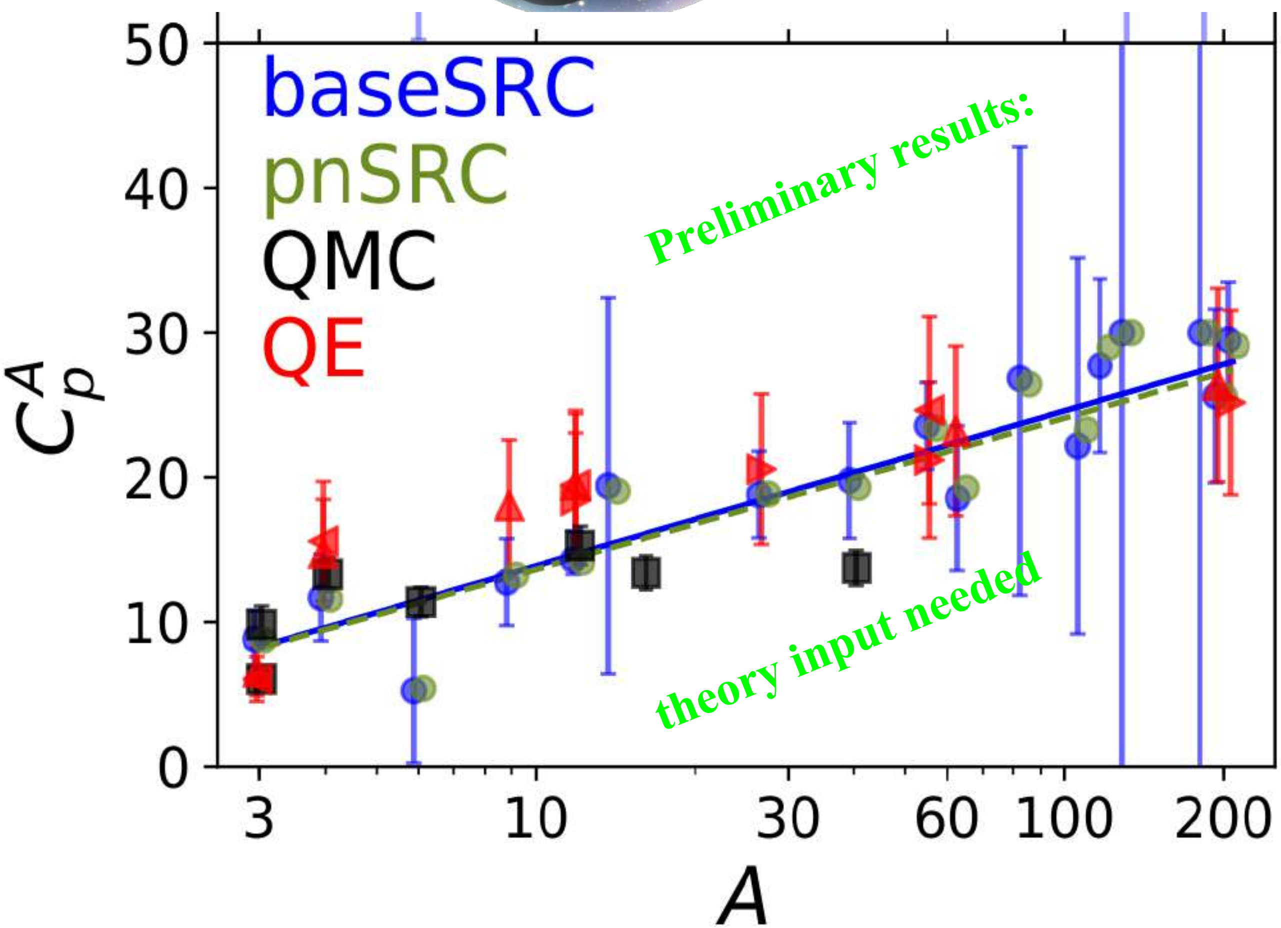
Short Range Correlations (SRC)

Short Range Correlations (SRC)



$$f_A = (1\text{-}c)\ f_p + c\ f_{SRC}$$

nuclear PDF — normal proton PDF — SRC modified PDF



Use relaxed {Q,W} cuts

Preliminary results:

- Yields good fit: $\chi^2/N \sim 0.80$ vs. 0.85
- Compatible with (pn) SRC pairs

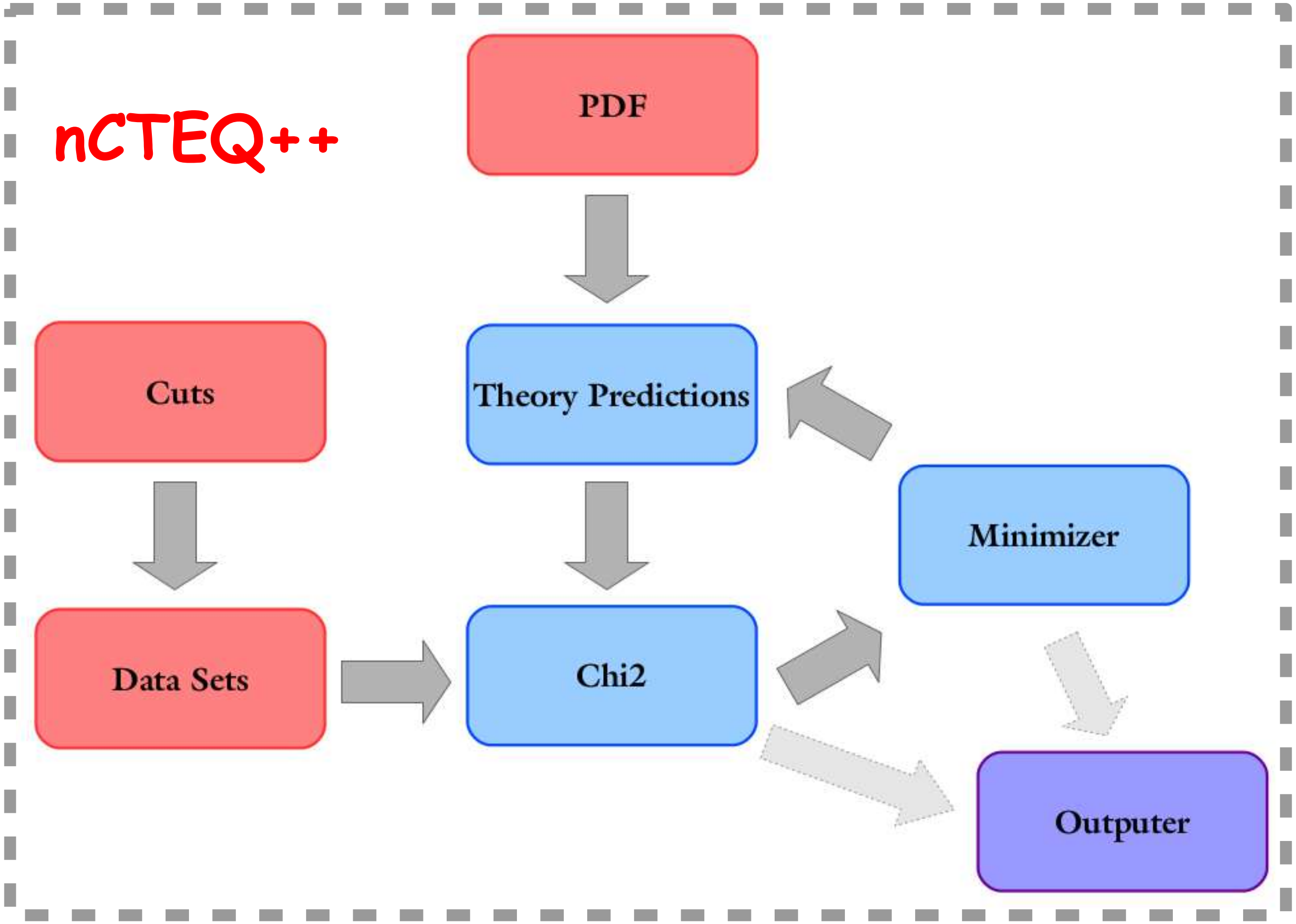
New Tools

nCTEQ++

a modern, modular code base

nCTEQ++ … a complete rewrite in C++

Top level C++, modular structure, output to YAML & Python scripts



Use external programs

- Minuit
- HOPPET
- MCFM
- APPLgrid

Pre-Computed Grids

Tremendous speed-up for higher order calculation

… for example …

High order DIS processes (Peter Risse)

Special thanks to:
Tomas Jezo
Eric Godat
Florian Lyonnet
Aleksander Kusina

nCTEQ++ … a complete rewrite in C++

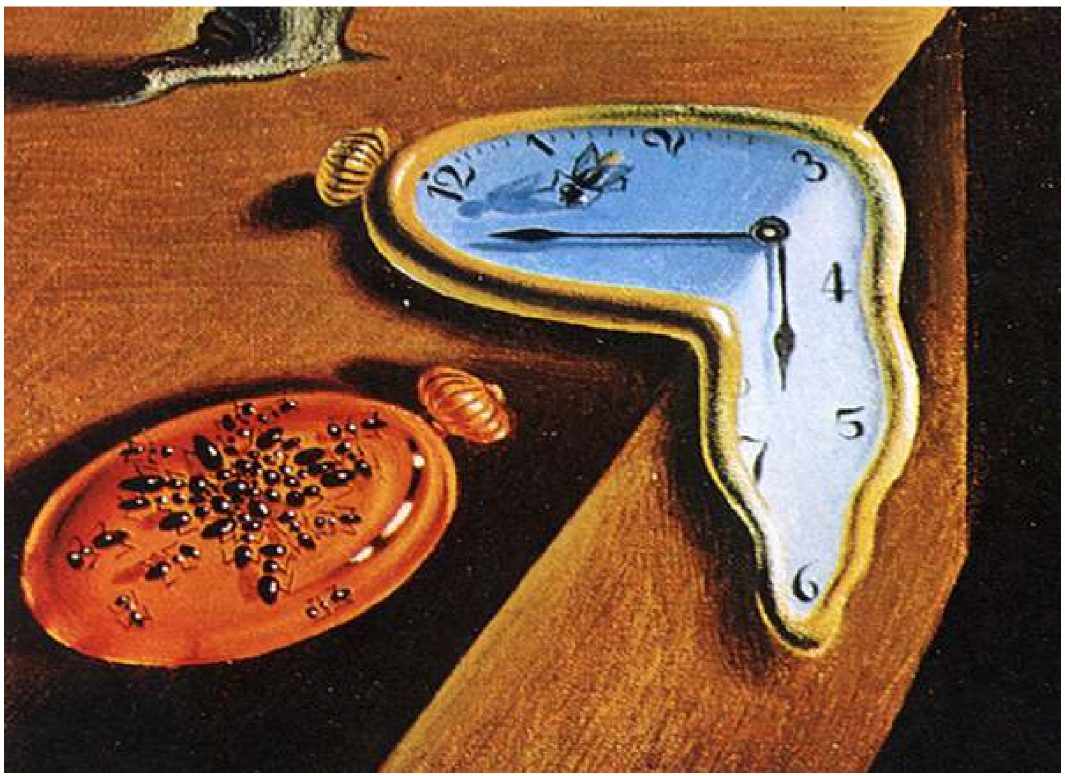
Code benchmark timings:

Original Fortran Code

contains multiple levels of integrals

New C++ Code

using modern grid techniques



Typical fits current run a few days to a week.
This will be reduced to a few hours.

High order DIS processes
(Peter Risse)

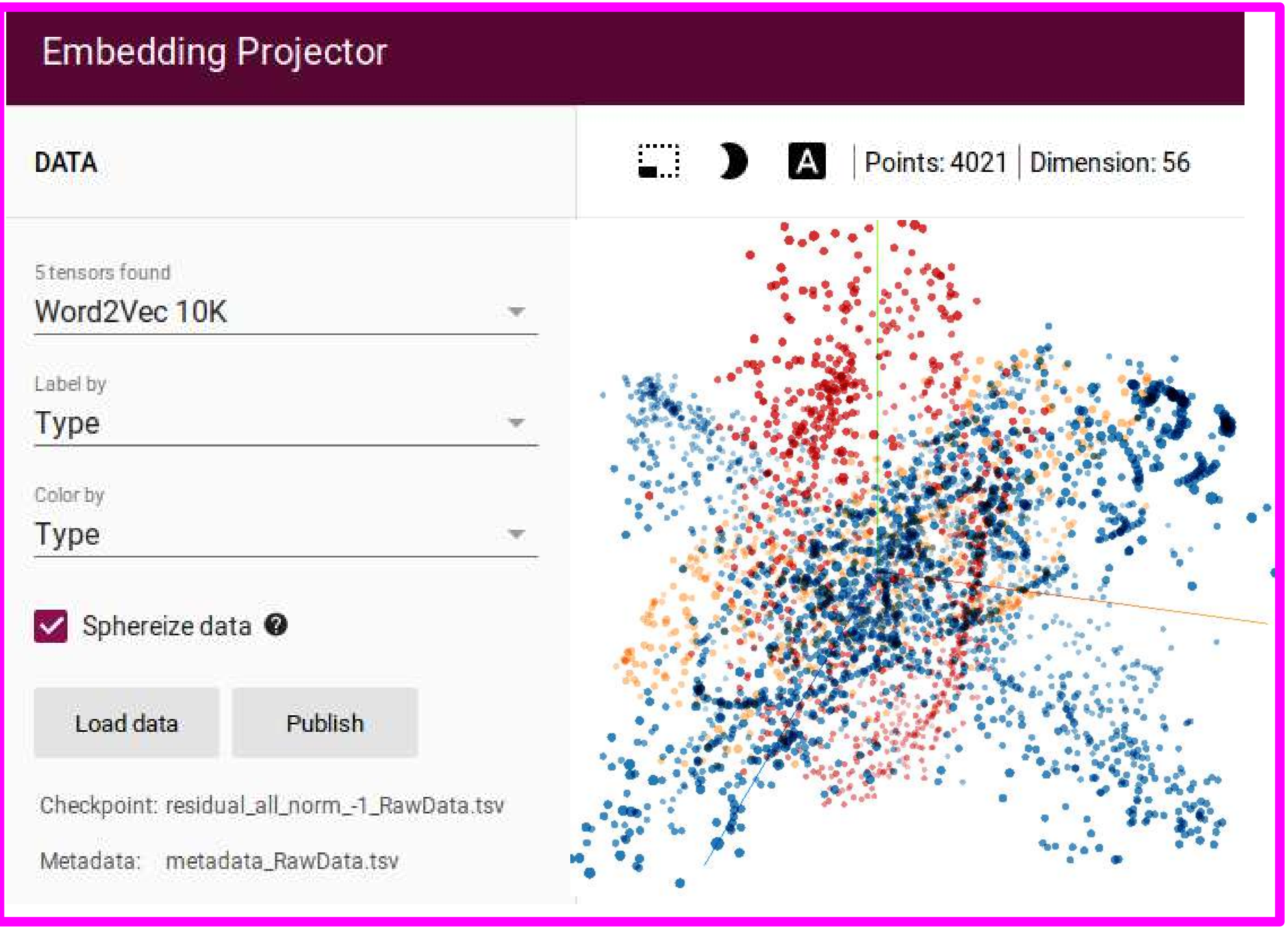
New Tools

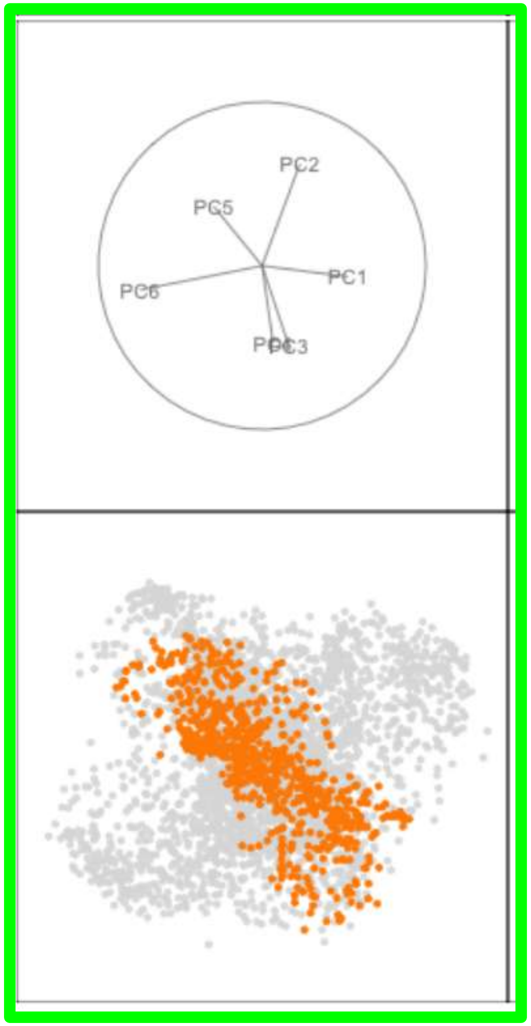
PDFSense
&
… borrowing from AI

Pavel Nadolsky, Boting Wang … (SMU)

Applications of PDFSense:

Artificial Intelligence Tools: Projector tool of Google TensorFlow





Dynamical projections for the visualization of PDFSense data
Dianne Cook, Ursula Laa, German Valencia arXiv:1806.09742

Pavel Nadolsky, Boting Wang … (SMU)

CONCLUSIONS

QCD: From Parameterization to a Deeper Understanding

Proton PDF: $f_p(x,Q)$

generally NNLO; approaching ~1% precision; Boundary Conditions for nuclear PDF

Nuclear PDF: $f_A(x,Q)$

generally NLO; leverage proton PDF tools; recent progress encouraging *(e.g., PDG)*

with EIC, evolve from parameterizing to deeper understanding of QCD

Extend kinematic {x,Q} range: … probe extreme regions of QCD

Low Q: non-perturbative region; correlation effects …

Low x: resummation; saturation; BFKL; …

Low W: resonance region; duality; …

Need theoretical guidance in these regions

Extend Unpolarized Colinear to Spin, TMD & GPD

… explore full tomographic nuclear structure in spin, k_T, b_T

precision $f_A(x,Q)$ can serve as Boundary Condition for $f_A(x,Q,k_T,b_T,\sigma)$

include Lattice QCD info on moments and quasi-PDFs

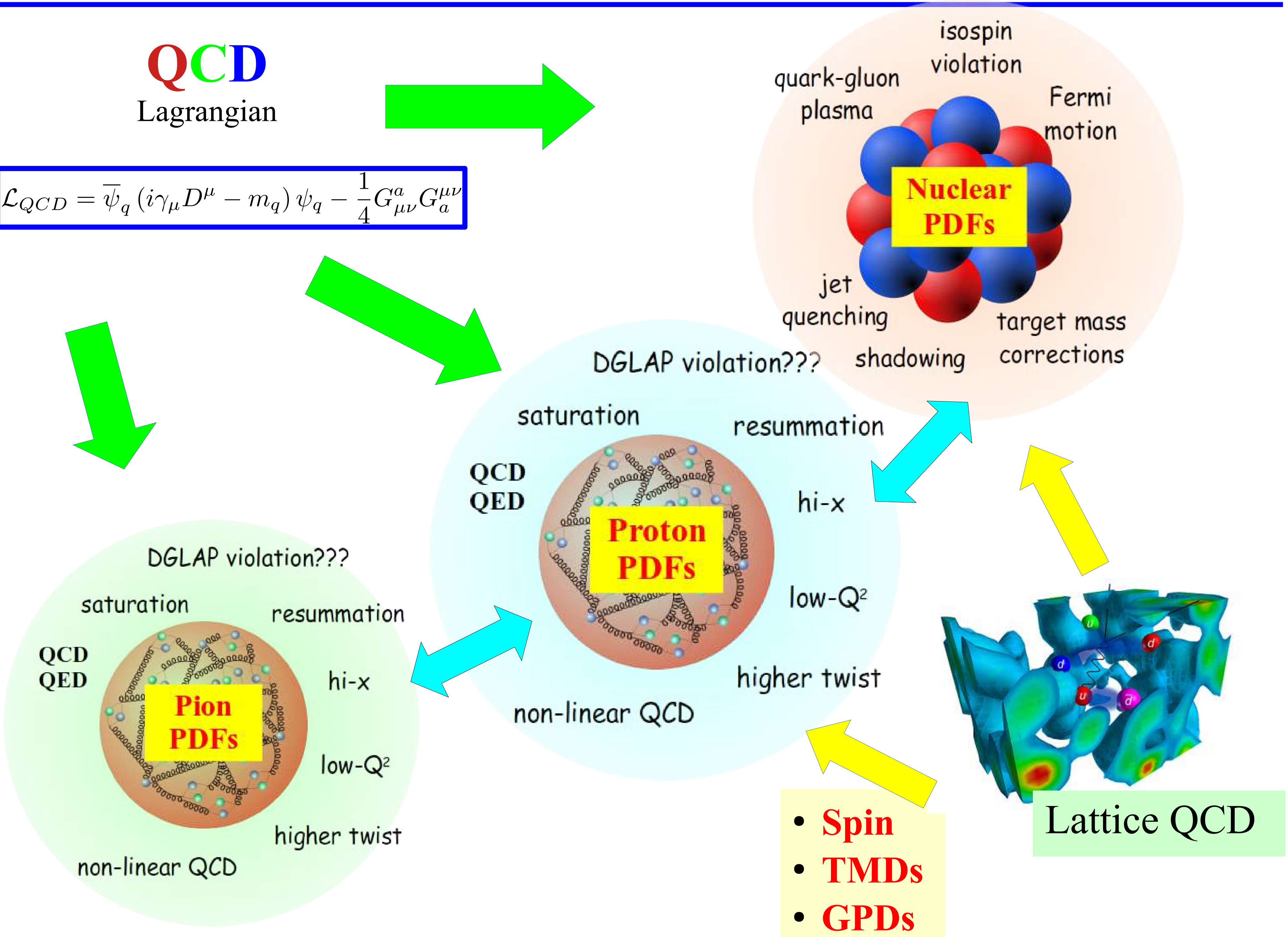
Need coordination/communication between efforts

QCD: From Parameterization to a Deeper Understanding

QCD

Lagrangian

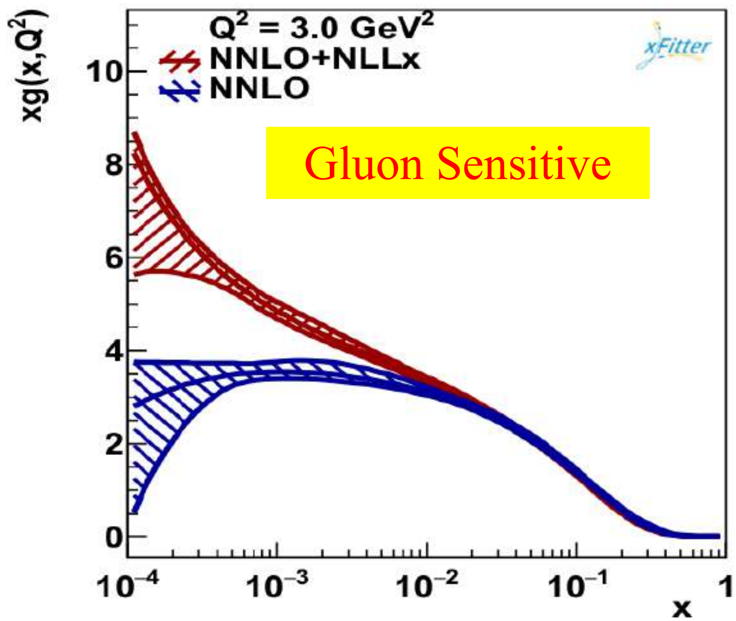
$$\mathcal{L}_{QCD} = \overline{\psi}_q \left(i\gamma_\mu D^\mu - m_q\right)\psi_q - \frac{1}{4}G^a_{\mu\nu}G_a^{\mu\nu}$$

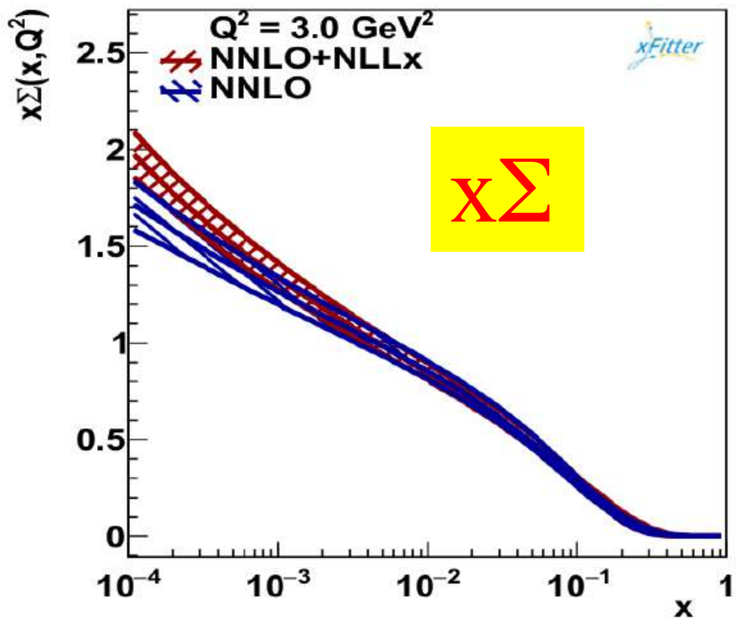


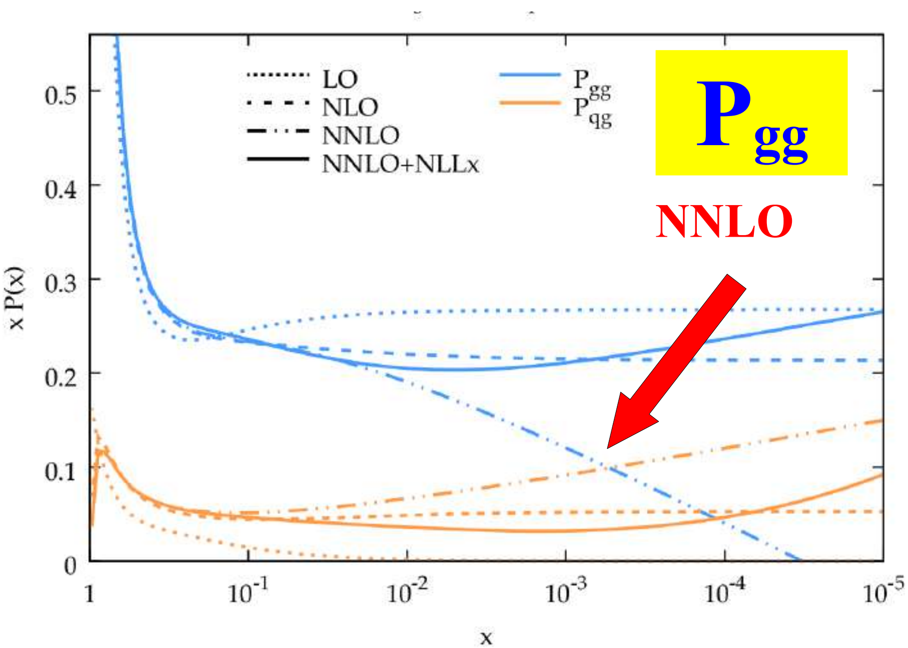
- **Spin**
- **TMDs**
- **GPDs**

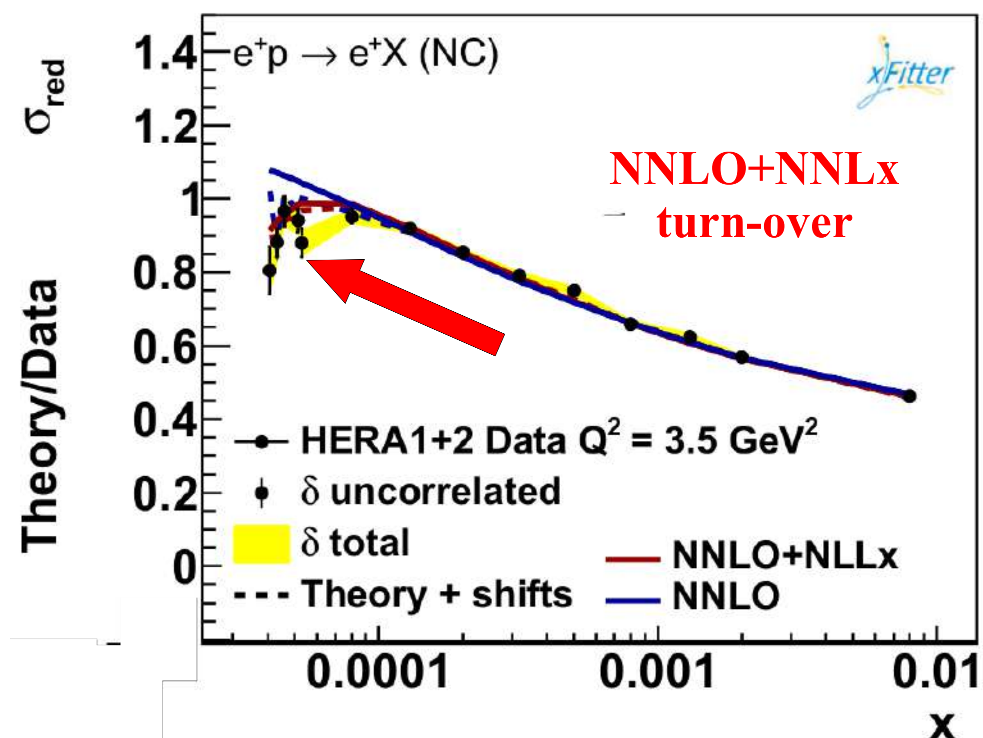
EXTRAS

xFitter Resummation Study





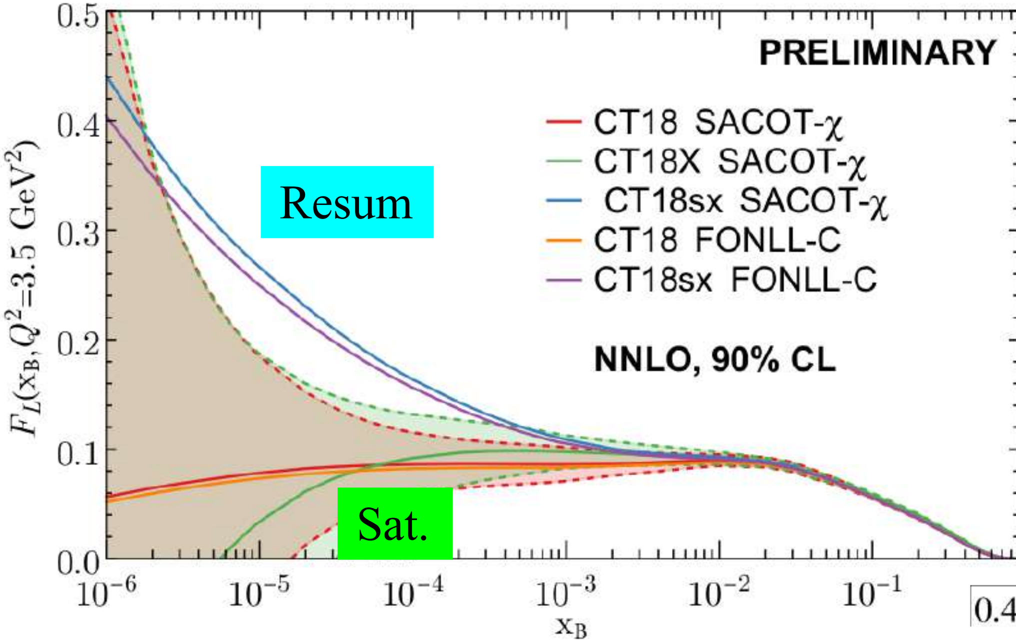




Impact of low-x resummation on QCD analysis of HERA data
xFitter Developers' Team: Eur.Phys.J.C 78 (2018) 8, 621

F_L is senstive to Gluon and Small-x Treatments

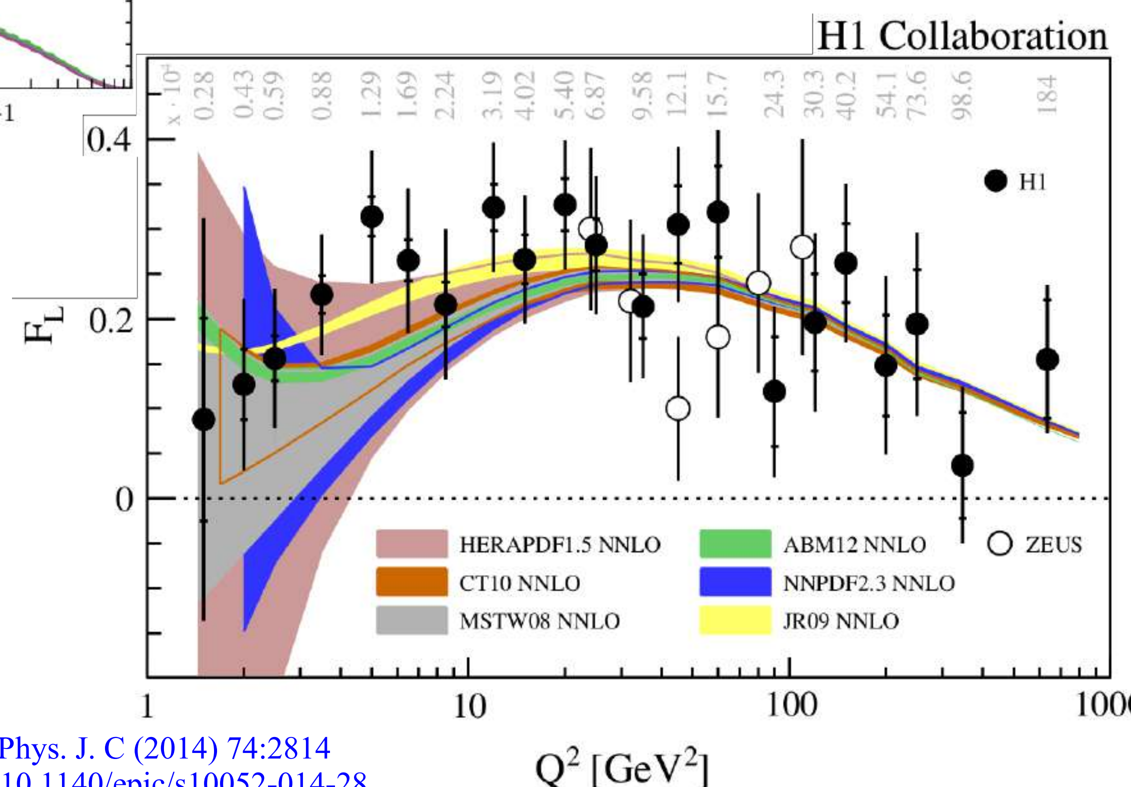
Still, large uncertainty at small x *(low Q^2)*



CT Collaboration: 2108.06596 [hep-ph]

Saturation & Resummation:

again
different behavior
at very small x



Eur. Phys. J. C (2014) 74:2814
DOI 10.1140/epjc/s10052-014-28

F_L Challenge:

Large Uncertainties at small x

nCTEQ Wish List

nCTEQ
nuclear parton distribution functions

www.ncteq.org

PDF General Issues:

- Proton PDF; nuclear corrections for interpreting heavy target DIS (Ar, Fe, Pb).

Strange quark & Gluon PDF:

- Resolve tension between fixed-target (νN, ℓN) and collider expectations ($W^{\pm}$,Z)

Charm & Bottom: c(x) & b(x)

- Multi-scale & resummation issues: $\text{Log}(m_{c,b}/Q)$
- "Fitted" charm: $c(x) \neq 0$ at m_c
- Intrinsic heavy flavors: $c(x) \neq 0$ at $Q < m_c$

Neutrino cross sections on heavy targets (Ar, Fe, Pb)

- Universality of Neutral Current (γ) & Charged Current ($W^{\pm}$) processes

Expanded $\{x,Q^2\}$ Kinematic Regime

- Small-x saturation, resummation: $\text{Log}[1/x]$
- Large-x higher twist: (M^2/Q^2)
- Low Q^2 non-perturbative effects

Compilation by Fred Olness with helpful feedback from: Alberto Accardi, Tim Hobbs, Tomas Jezo, Thia Keppel, Michael Klasen, Karol Kovarik, Aleksander Kusina, Jorge Morfin, Pavel Nadolsky, Jeff Owens, Ingo Schienbein, Efrain Segarra, Steve Sekula, Ji-Young Yu

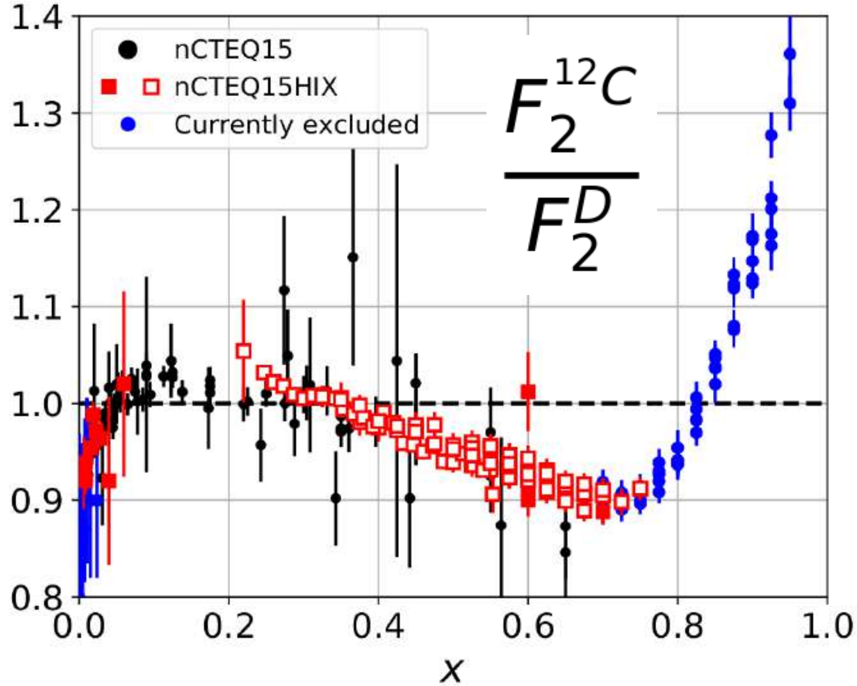
SHADOWING
&
ANTI-SHADOWING

An exercise at large x & low Q

...

extensible to mid-x region

Challenges at Large x & Low Q^2: JLab data ⇒ EIC



Nuclear PDFs: x>1 allowed;

impacts F_2^{Nuc}/F_2^{Iso} in Fermi region

Target Mass Corrections

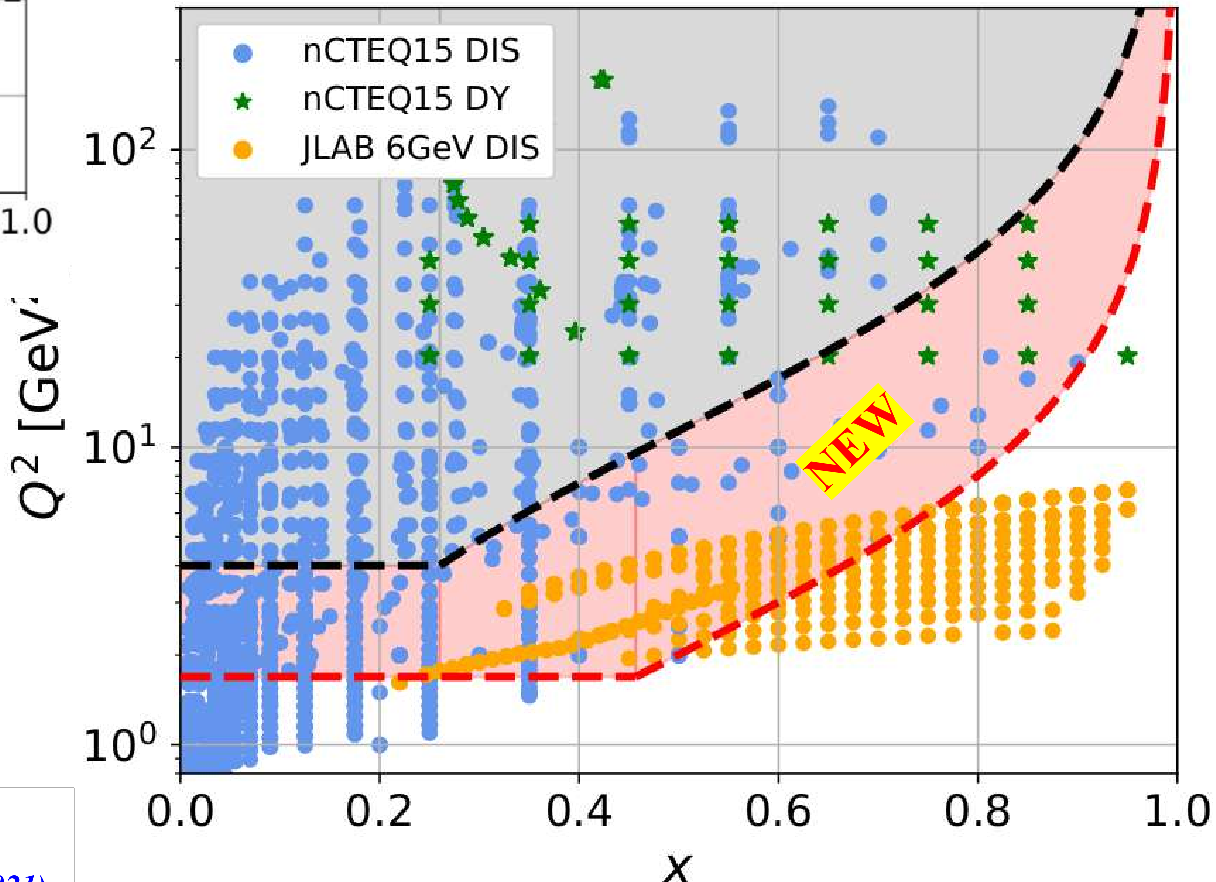
pick up M^2/Q^2 higher twist contributions

Deuteron Corrections

impacts $F_2^{Nuc}/F_2^{Deuteron}$ ratio

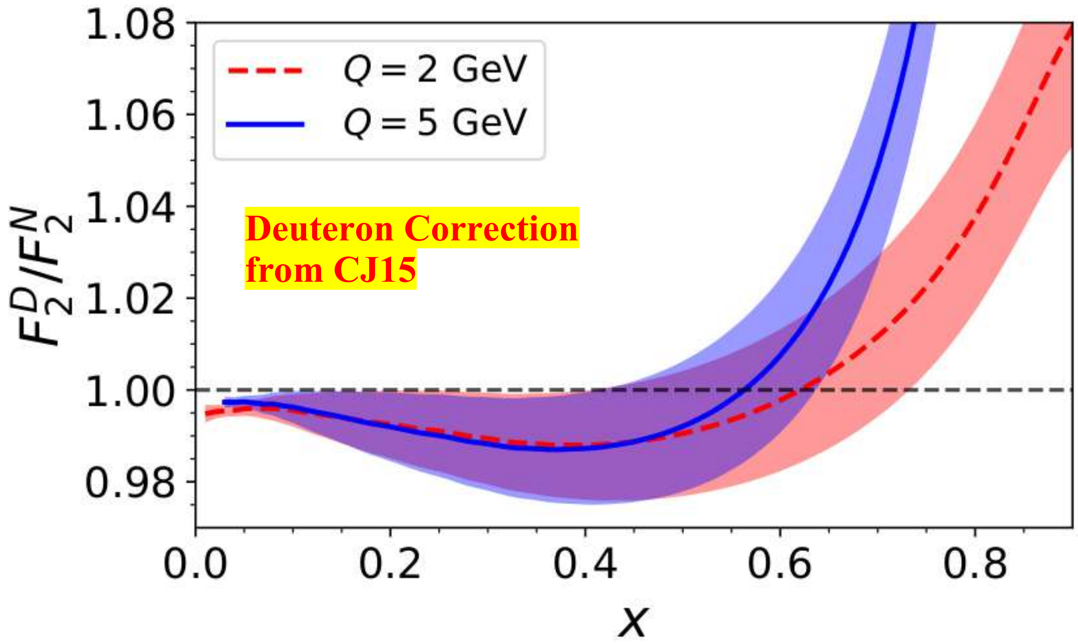
JLab Data @ Hi-X Low-Q^2

extend nCTEQ framework
to accommodate this region
& prepare for EIC

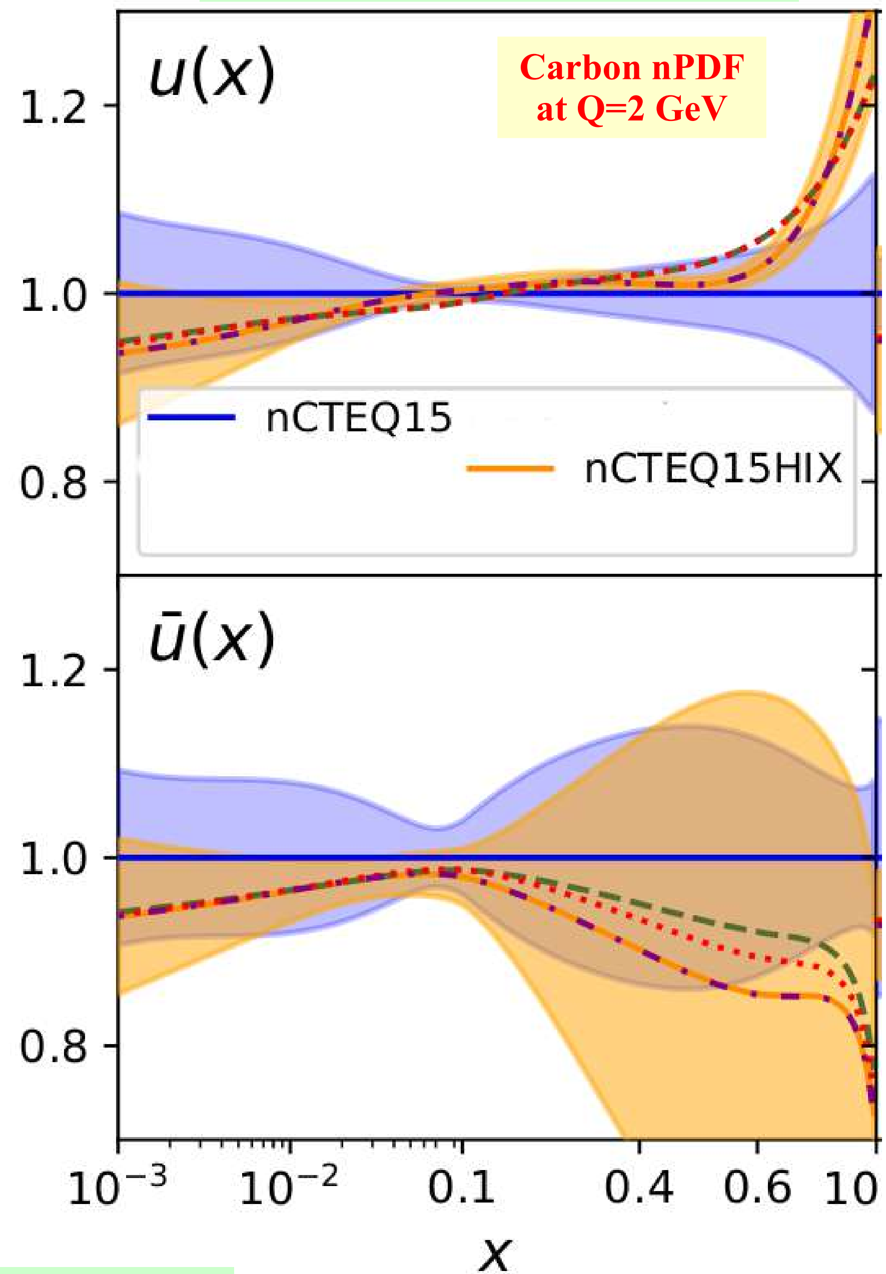


nCTEQ15HIX -- Extending nPDF Analyses into the High-x, Low Q2 Region
E.P. Segarra, T. Ježo, et al., ***PRD 103, 114015 (2021)***

nCTEQ15HIX include large x JLab data



JLab data: Shifts valence PDFs from low to hi-x



Deuteron Corrections Important!!!

Overall χ^2/N_{dof} ~0.83

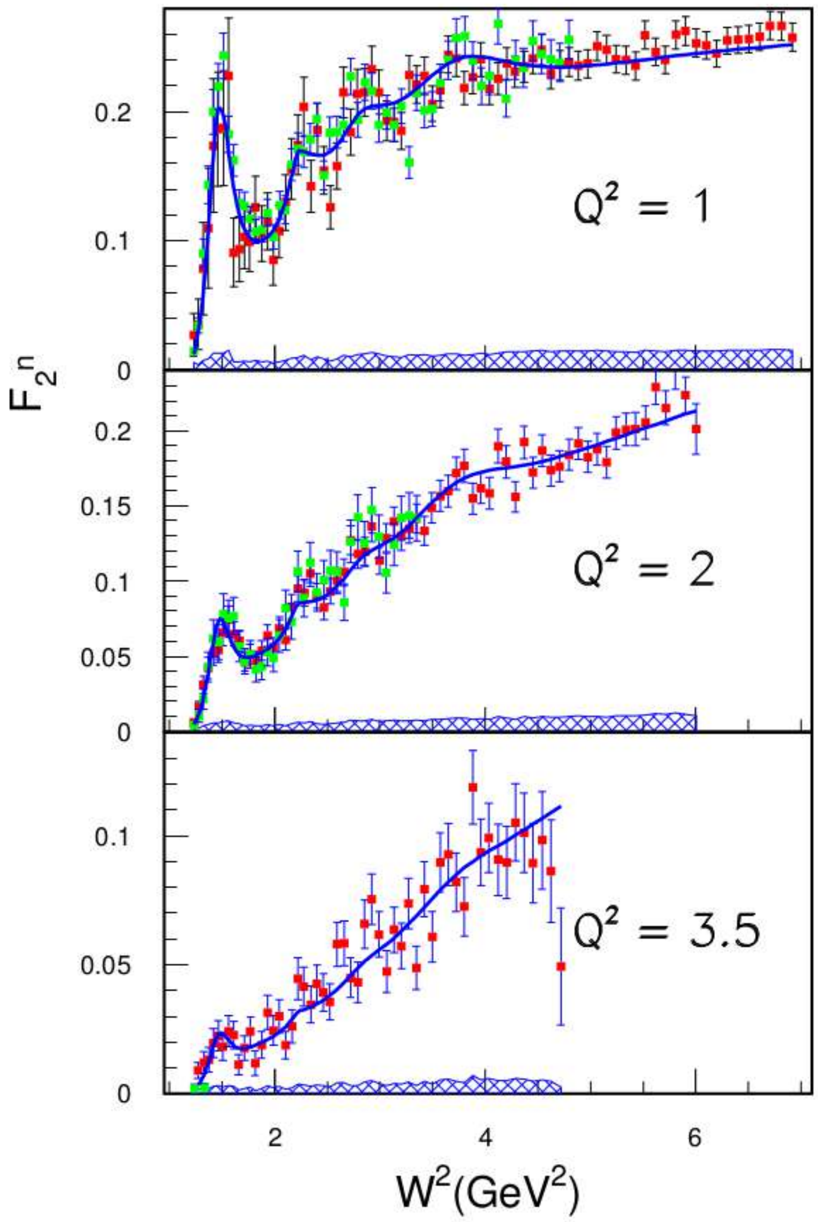
Fit	χ^2	N_{data}	χ^2/N_{dof}	Q_{cut}	W_{cut}
nCTEQ15	587	740	0.81	2.0	3.5
nCTEQ15*	2664	1564	1.70	1.3	1.7
nCTEQ15HIX	1291	1564	0.83	1.3	1.7

We can extend our kinematic reach in $\{x,Q^2\}$

what about mid x region

How Low in W^2 and Q^2 can we go???

BoNuS data



Overall χ^2/N_{dof} ~0.83

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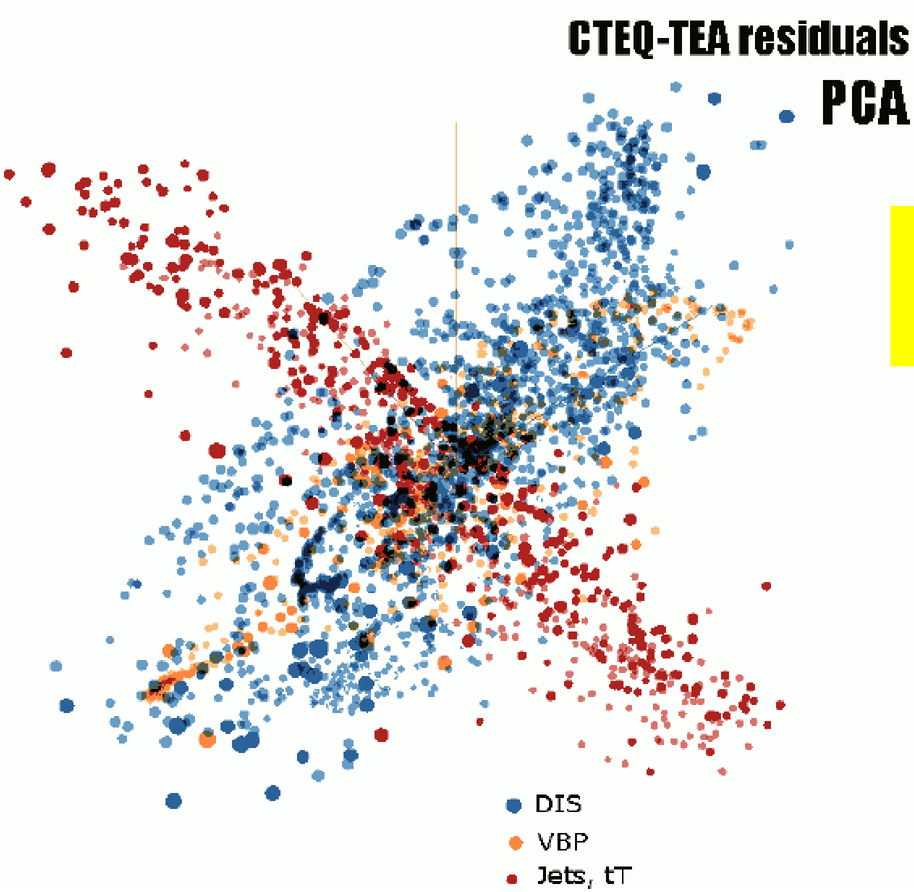
nCTEQ15*	2664	1564	1.70	1.3	1.7
nCTEQ15HIX	1291	1564	0.83	1.3	1.7

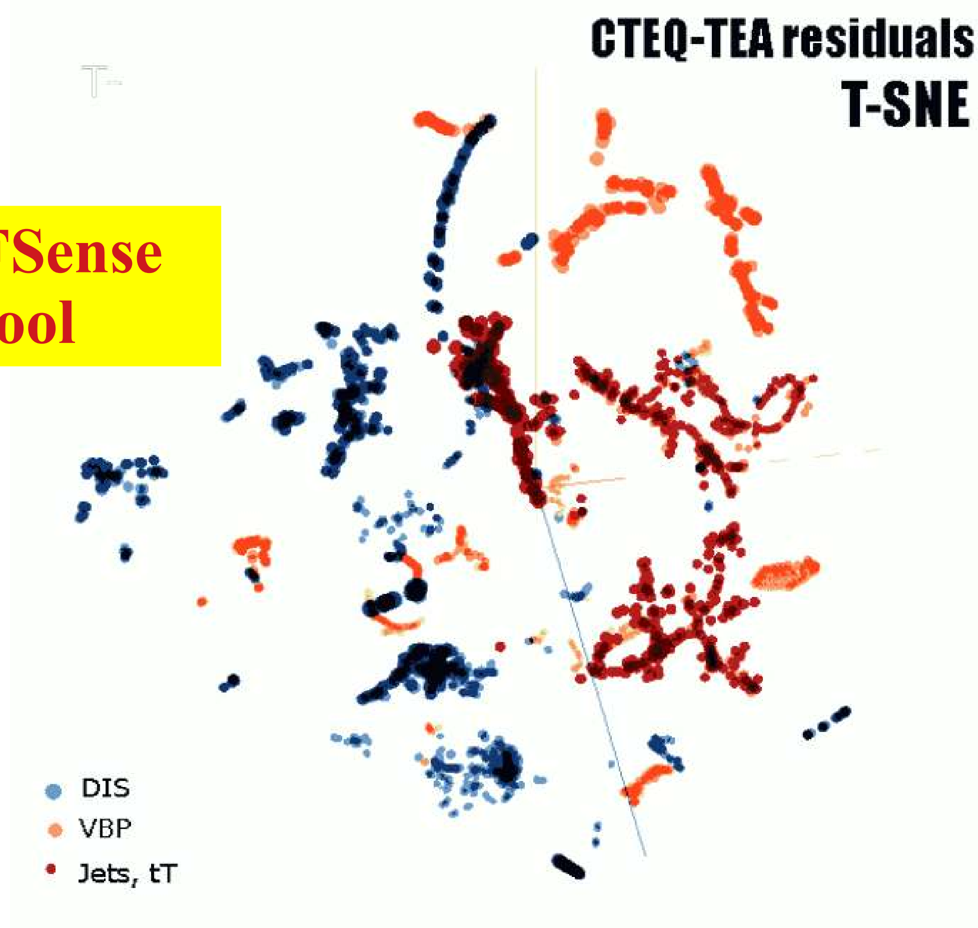
Can we push into the resonance region?

TensorFlow Embedding Projector

https://metapdf.hepforge.org/PDFSense/



PDFSense tool



Principal Component Analysis (PCA) visualizes the 56-dim. manifold by reducing it to 10 dimensions (à la META PDFs)

t-distributed stochastic neighbor embedding (**t-SNE**) sorts vectors according to their similarity

$$r_i(\vec{a}) = \frac{1}{s_i}\left(T_i(\vec{a}) - D_{i,sh}(\vec{a})\right)$$

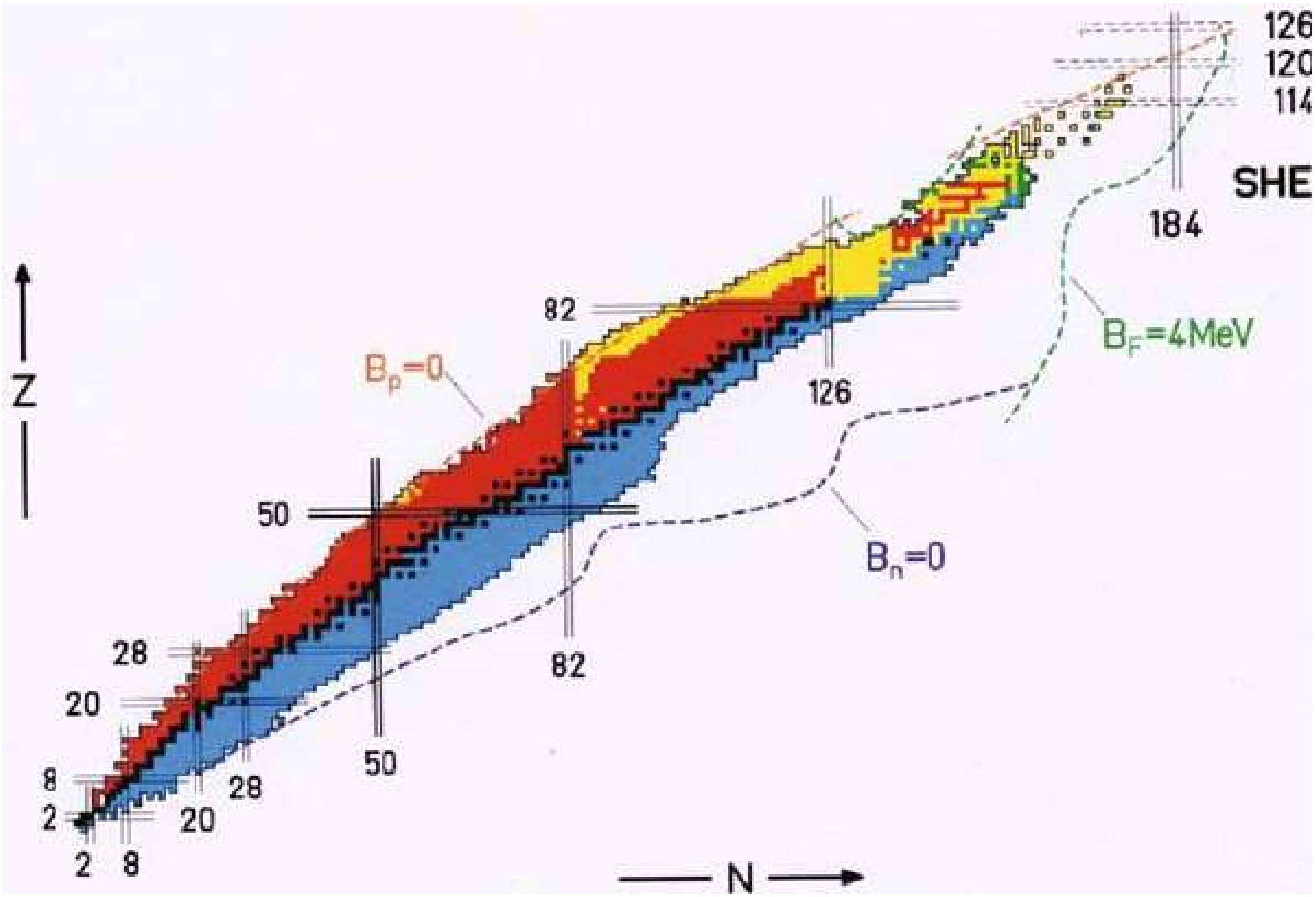
http://projector.tensorflow.org

Target Mass Corrections (TMC)

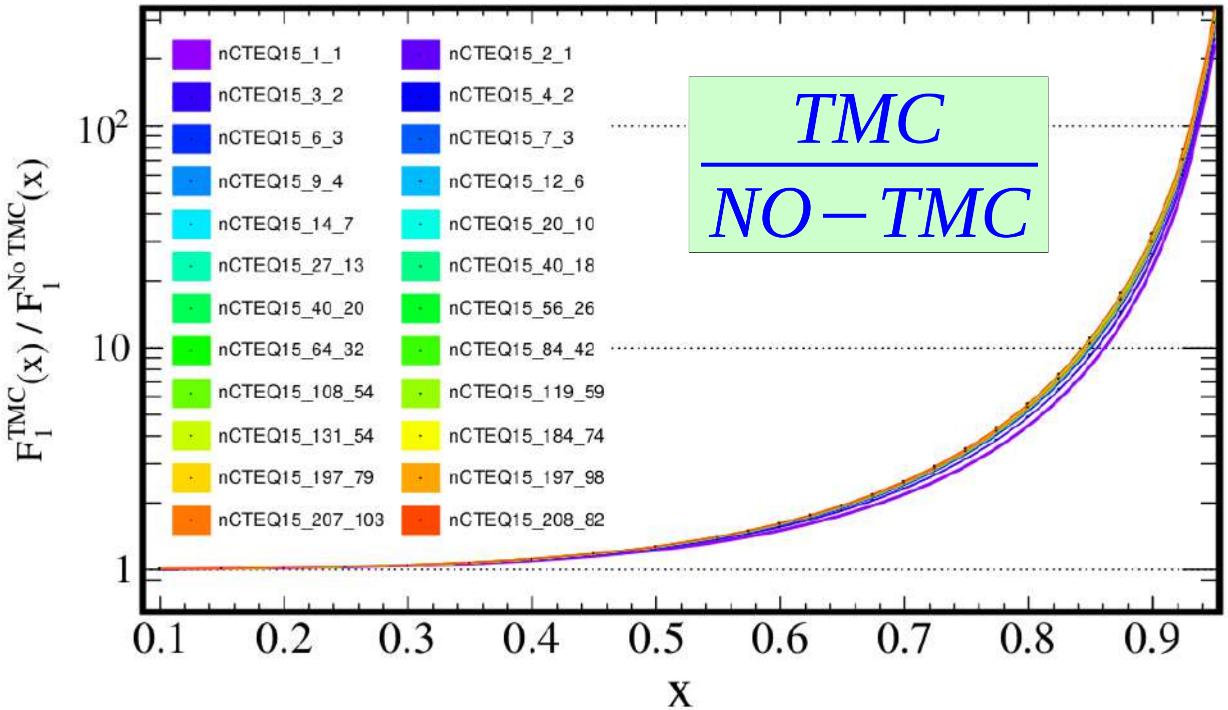
The challenge of a multi-scale problem

...

Ingo Schienbein, Chloe Leger, Richard Ruiz ...

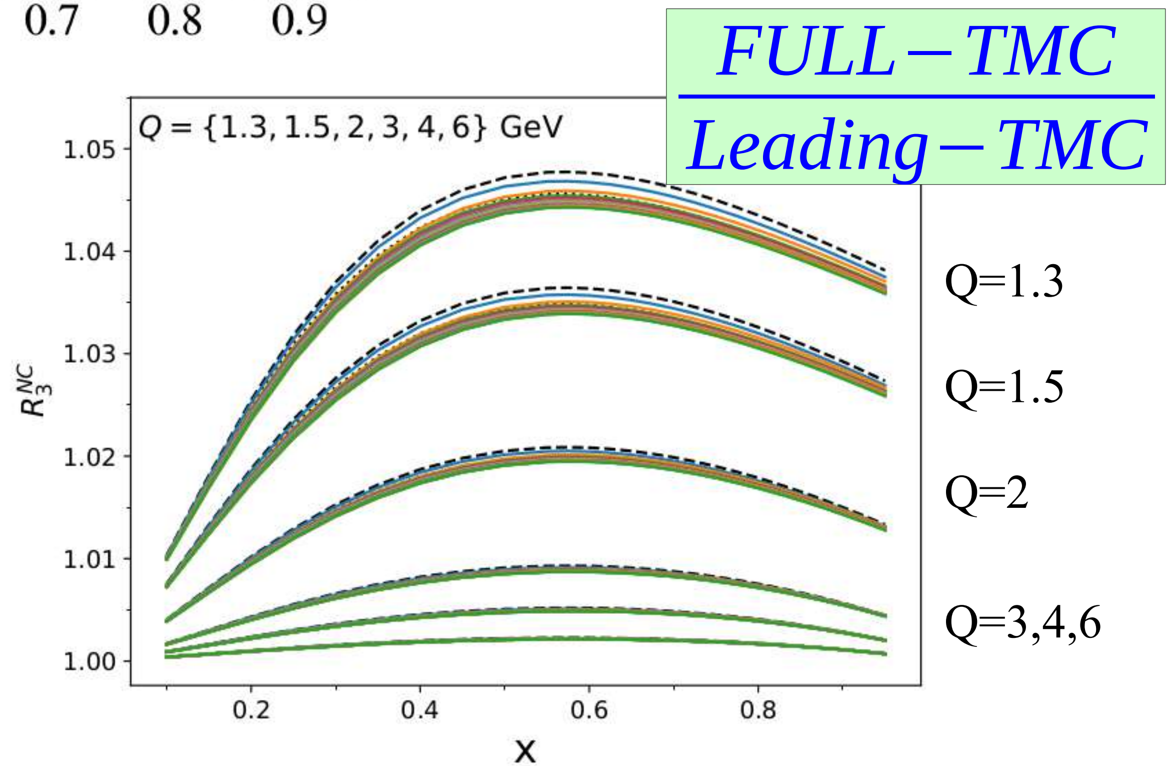


Target Mass Corrections (TMC) for nuclear processes



TMCs can be large

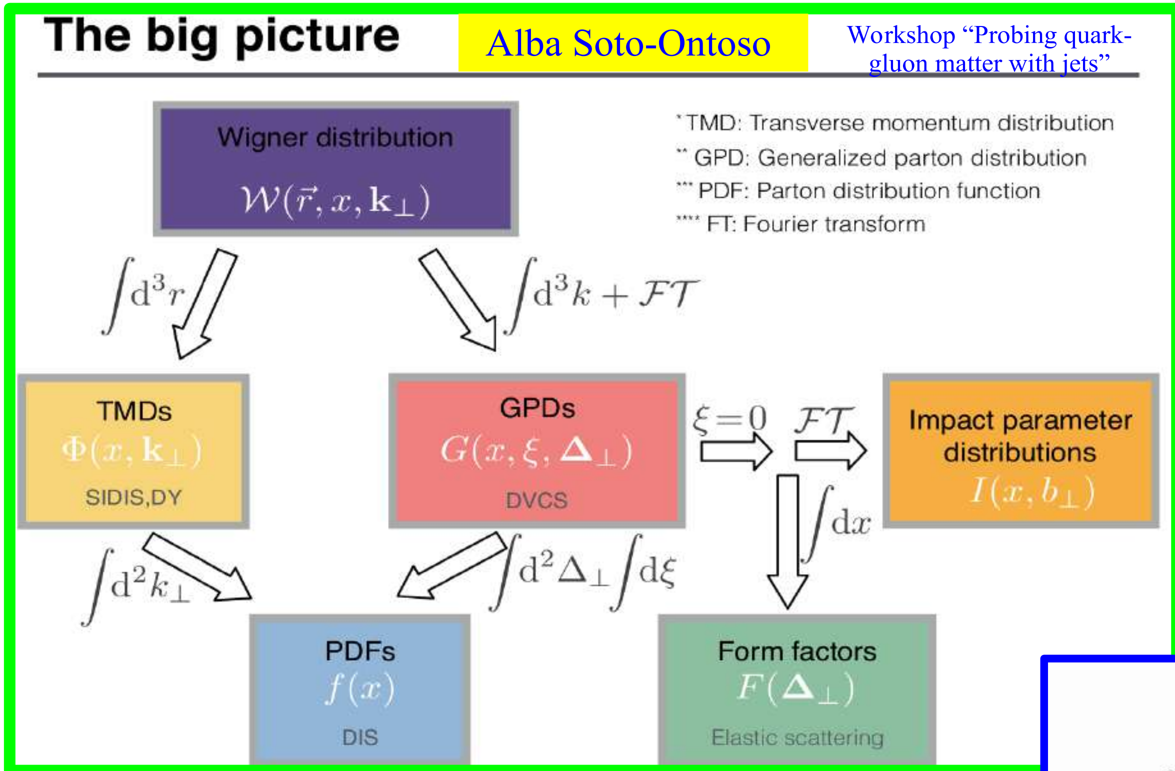
Corrections are nearly universal

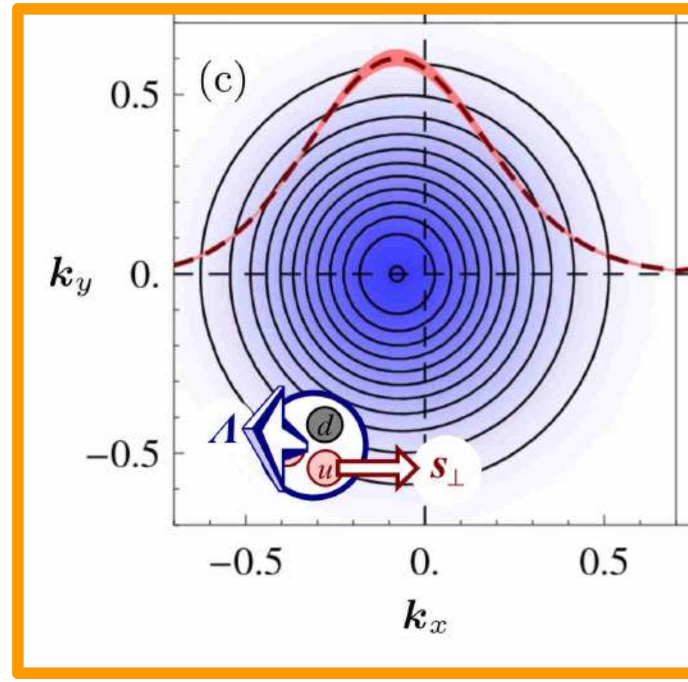


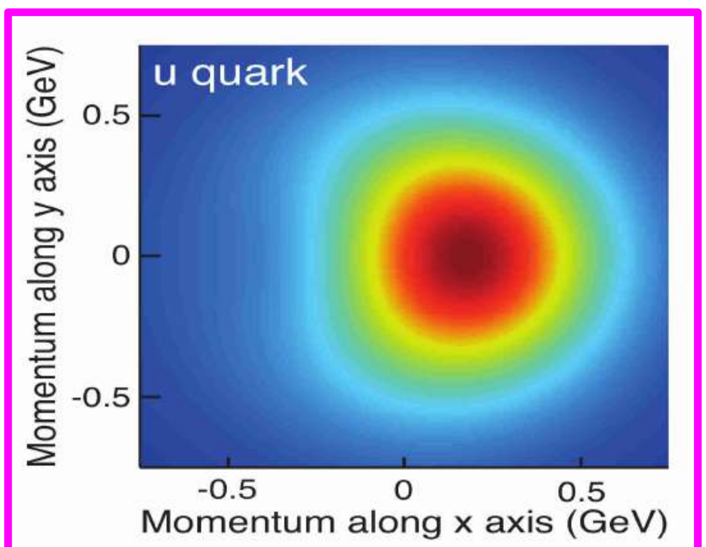
Ingo Schienbein, Chloe Leger, Richard Ruiz ...

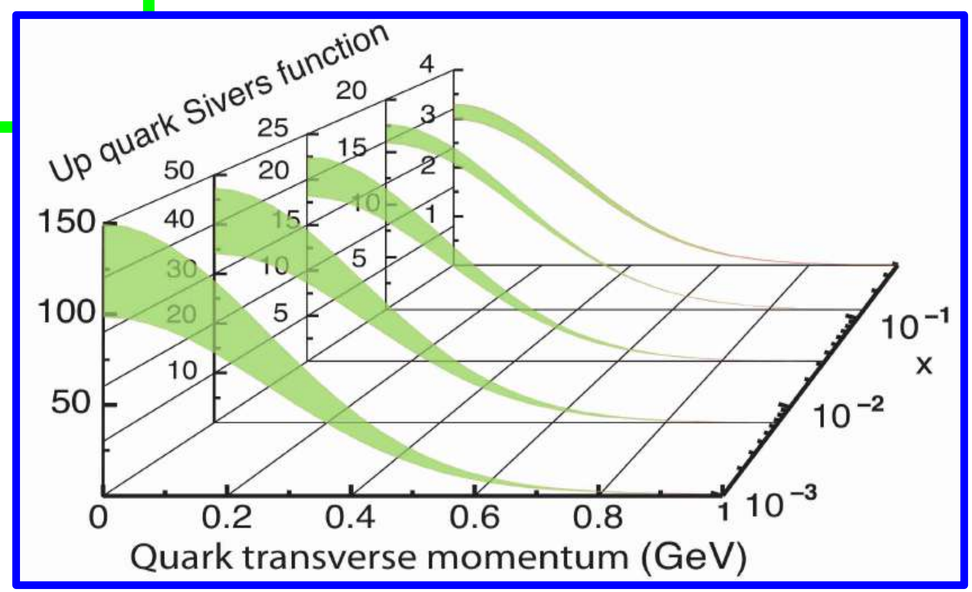
Nuclear Structure

Hadron structure is much richer than f(x) conveys





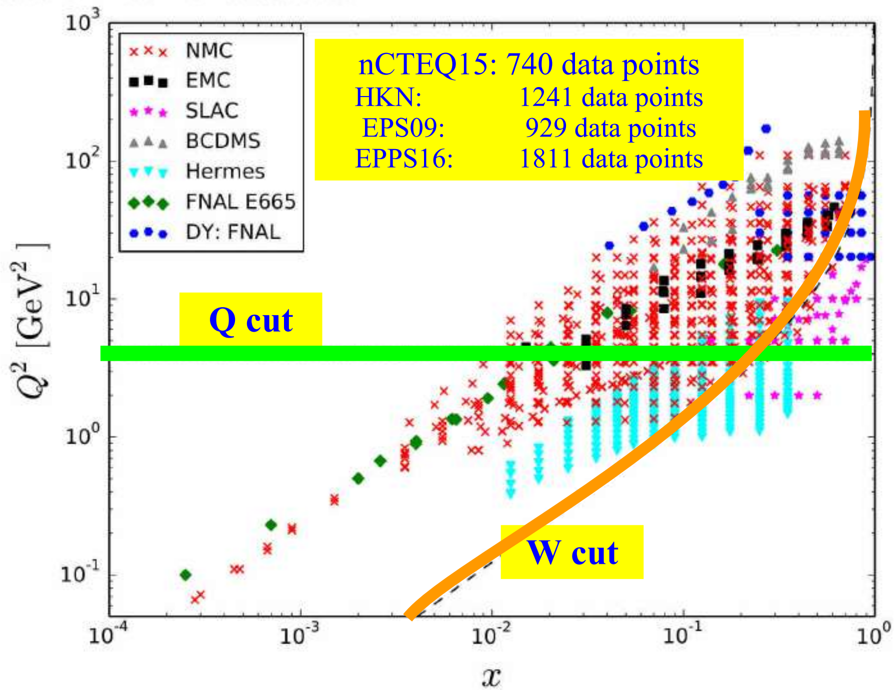


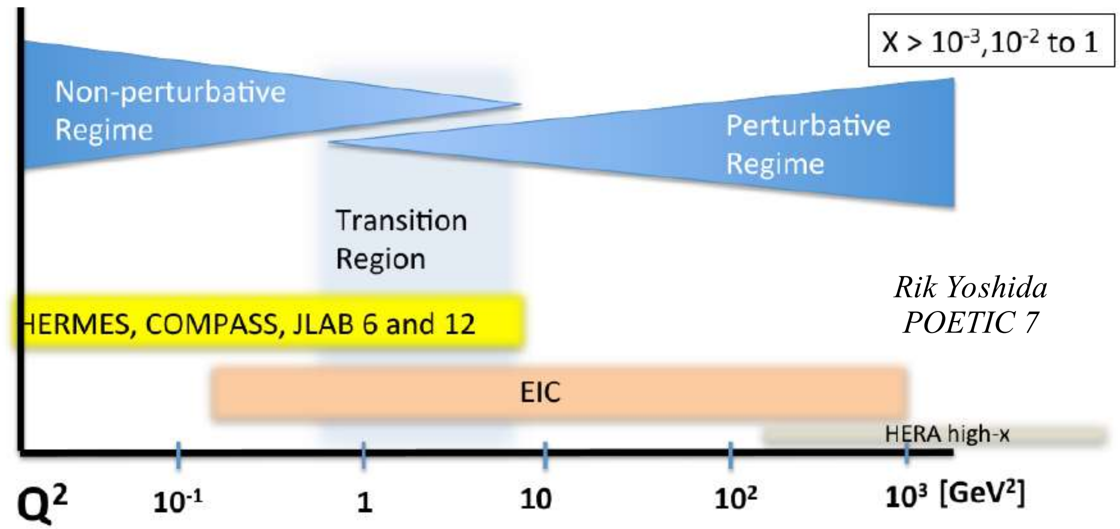


Conventional PDFs f(x) are the Boundary Conditions

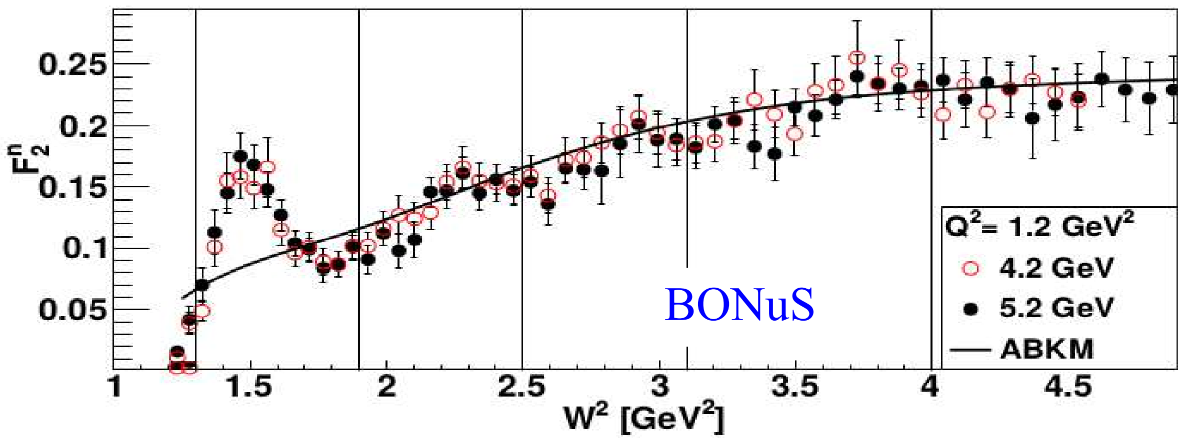
Hadron / Parton Transition:

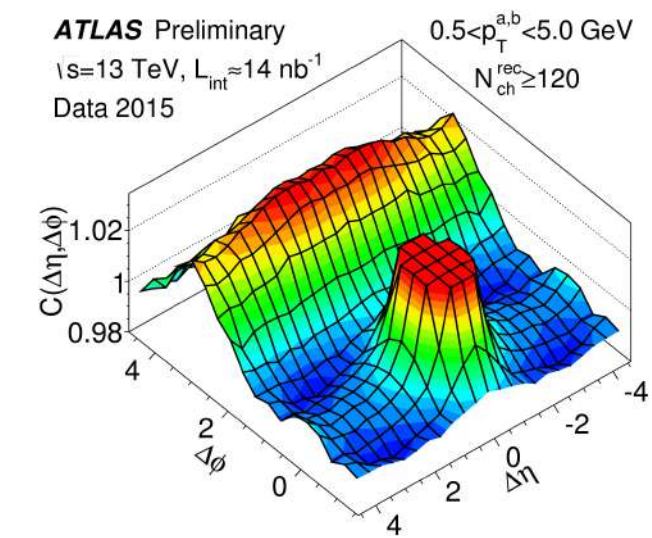
Higher Twist, many body problem, duality, hi-x, mass corrections ...





Rik Yoshida
POETIC 7





EIC & LHeC can push these boundaries

... not all the new discoveries are at the energy frontier

nCTEQ

nuclear parton distribution functions

… the collaboration

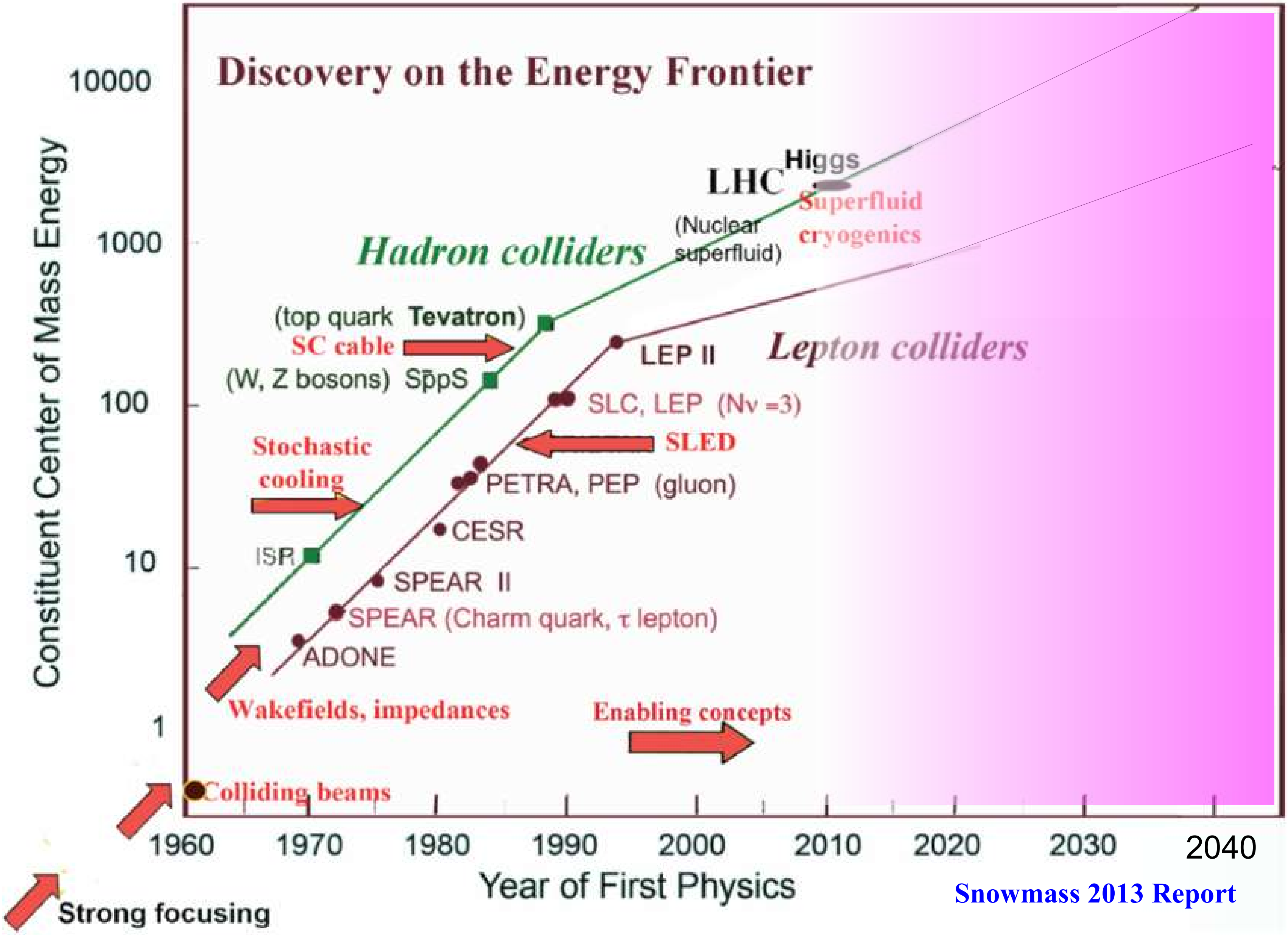
USA:
Hobbs, Keppel, Morfin, Olness, Owens

Muenster:
Klasen, Kovarik, Jezo, Duwentaster, Muzakka, Risse Wissmann

Krakow:
Kusina, Ruiz Derakhshanian

Grenoble:
Schienbein, J.Y.Yu Leger

Landscape of the Energy Frontier & New Physics Searches



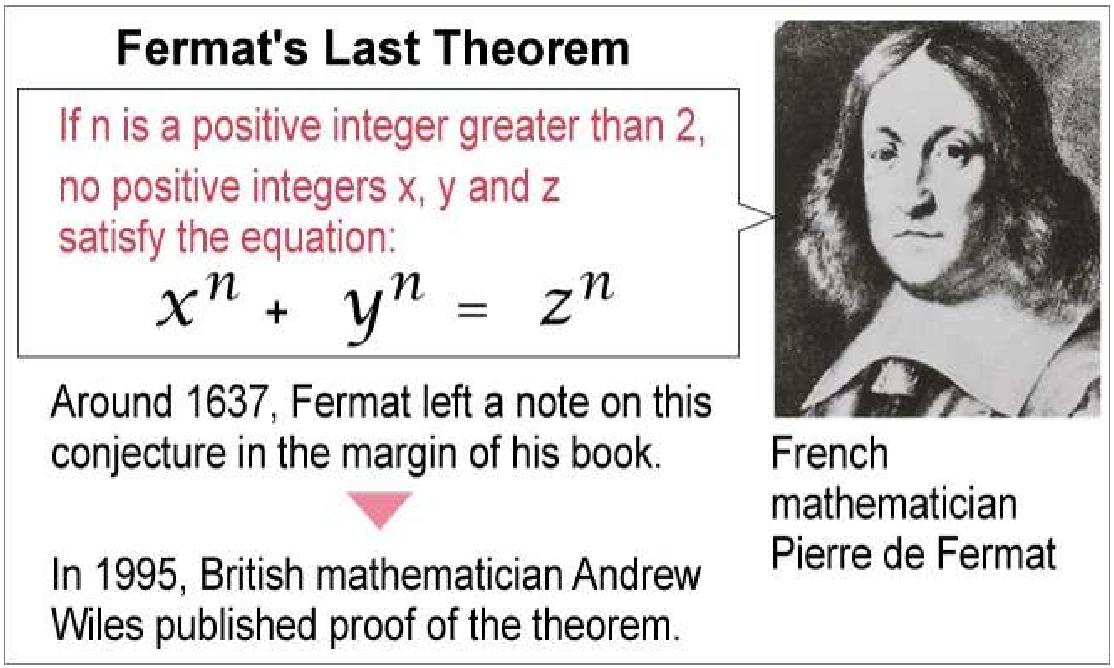
We've reached the peak energy. Future searches require precision!!!

" PDF uncertainties are among the leading uncertainties in the first LHC precision measurements by CMS" *Jan Kretzschmar*

Solutions may not fit on an envelope ...

Fermat's Last Theorem

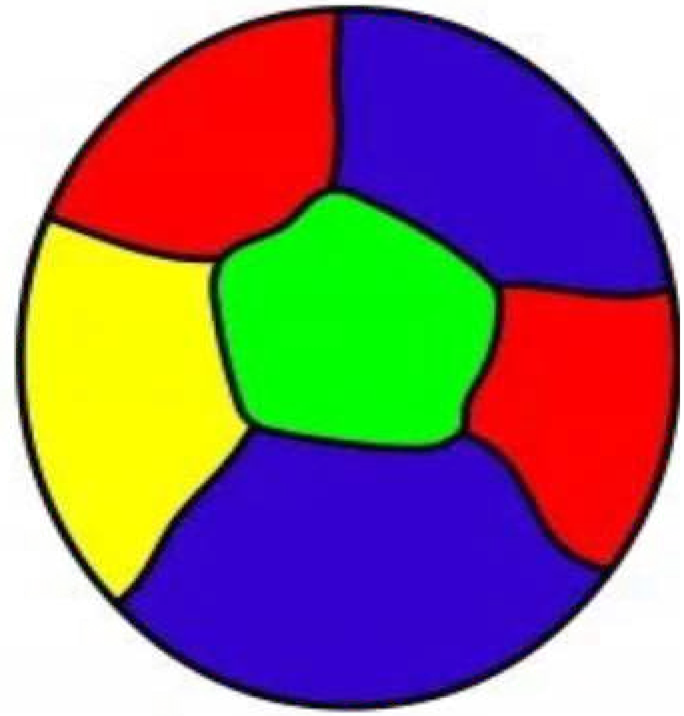
1637-1995



Two papers totaling 129 pages long, using modern algebraic geometry & seven years of Wiles's research time.

Four Color Map Theorem

1852-1976



1976 Appel and Haken: 1,834 reducible configurations checked one by one by computer and took over a thousand hours.