Rapidity-only TMD factorization at one loop

Ian Balitsky

Surge-1 28 June 2023

Long-term project: TMD factorization valid at small and moderate x

$$\frac{d\sigma}{d\eta d^2q_\perp} = \sum_{\text{flavors}} e_f^2 \int d^2k_\perp \mathcal{D}_{f/A}(x_A,k_\perp)\mathcal{D}_{f/B}(x_B,q_\perp-k_\perp)C(q,k_\perp)$$
$$+ \text{ power corrections } + \text{ “Y} - \text{terms''}$$

The quantities $\mathcal{D}_{f/A}(x_A,k_\perp)$, $\mathcal{D}_{f/B}(x_B,q_\perp-k_\perp)$, and $C(q,k_\perp)$ are defined with cutoffs. The dependence on the cutoffs cancels in their product order by order in α_s.

At moderate x_A,x_B: CSS/SCET approach. The TMDs $\mathcal{D}_{f/A}(x_A,k_\perp)$ are defined with a combination of UV and rapidity cutoffs.

At $x_A,x_B \ll 1$: k_T-factorization approach. The TMDs are defined with rapidity-only cutoffs.

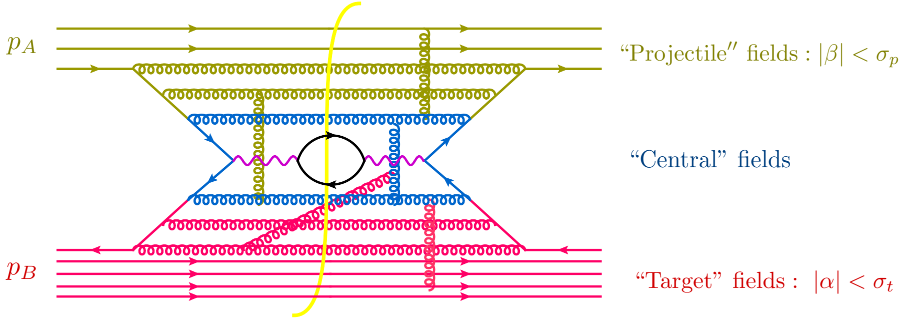
No CSS approach to small x ($\Leftrightarrow$ nobody tried). Recently, there is a SCET activity in the small-x direction (LO BFKL level for now)

It *is* possible to study TMD factorization at moderate x using small-x methods (rapidity-only factorization etc.) (A. Tarasov, G. Chirilli, I.B, 2015-2023)

Example: power corrections $\sim \frac{1}{Q^2}$ for small-x DY hadronic tensor $\Rightarrow$ EM gauge invariance of DY tensor.

Sudakov variables: $\alpha = x_A,\ \beta = x_B$

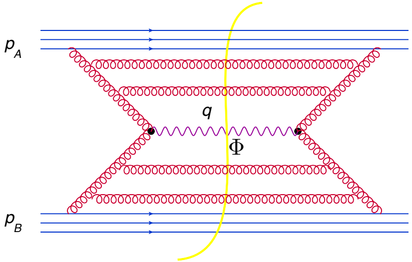
$$p\ =\ \alpha p_1 + \beta p_2 + p_\perp, \qquad p_1 \simeq p_A,\ \ p_2 \simeq p_B,\ \ \ p_1^2 = p_2^2 = 0$$



The result of the integration over "central" fields in the background of projectile and target fields is a series of TMD operators made from projectile (or target) fields multiplied by powers of $\frac{1}{Q^2} \Rightarrow$ power corrections

Coefficient function for TMD factorization at one loop

Particle production by gluon-gluon fusion (point $gg\Phi$ vertex is a $\frac{m_H}{m_t} \ll 1$ approximation for Higgs production.)

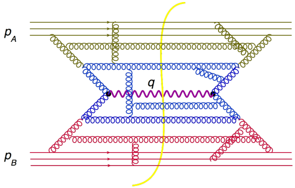


$$s \gtrsim Q^2 \quad \gg \quad Q_\perp^2 \gtrsim m^2$$
$$q^2 \equiv Q^2 = M_\Phi^2, \quad Q_\perp^2 \equiv q_\perp^2$$

Goal: one-loop TMD factorizaton formula for hadronic tensor.

Result of calculation of one-loop coefficient function:

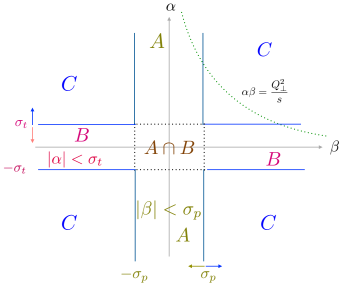
$$\begin{aligned} W(p_A, p_B; q) \;&=\; \int db_\perp\; e^{i(q,b)_\perp} \mathcal{D}_{g/A}(x_A, b_\perp; \sigma_a) \mathcal{D}_{g/B}(x_B, b_\perp; \sigma_b) \\ &\times\; \exp\Big\{\frac{\alpha_s N_c}{2\pi}\Big[\ln^2 \frac{b_\perp^2 s\sigma_p\sigma_t}{4} - 2\big(\ln\frac{\alpha_q}{\sigma_t} + \gamma\big)\big(\ln\frac{\beta_q}{\sigma_p} + \gamma\big) + \frac{\pi^2}{2}\Big]\Big\} \\ &\qquad + \text{NLO terms} \sim O\big(\alpha_s^2\big) \;+\; \text{power corrections} \end{aligned}$$



"Projectile" fields: $|\beta| < \sigma_a$

"Central" fields

"Target" fields: $|\alpha| < \sigma_b$



Matching: $\ln\sigma_p$ in the projectile TMDs and $\ln\sigma_t$ in the target TMDs should cancel with $\ln\sigma_p$ and $\ln\sigma_t$ in the coefficient functions.

$A \cap B$, $k_\perp \sim m_\perp$: Glauber gluons

$A \cap B$, $k_\perp \ll m_\perp$: soft gluons

$A \cap B$ gluons $\equiv$ soft/Glauber (sG) gluons

sG gluons cancel out

Formal rescaling: $s = \zeta s_0,\ \zeta \to \infty$, $Q_\perp^2$-fixed

$\alpha_a \equiv x_A,\ \beta_b \equiv x_B$

$$\text{Rapidity cutoffs}:\ \ \alpha_a \gg \sigma_t \gg \frac{Q_\perp^2}{\beta_b s} \sim \zeta^{-1},\quad \beta_b \gg \sigma_p \gg \frac{Q_\perp^2}{\alpha_a s} \sim \zeta^{-1},\quad \frac{\sigma_p\sigma_t s}{Q_\perp^2} \sim \zeta^{-1/2}$$

Coefficient function in the "double OPE" approach

After integration over central fields

$$
\begin{aligned}
&\frac{1}{16}(N_c^2-1)\langle p'_A,p'_B|g^2F^a_{\mu\nu}F^{a\mu\nu}(x_2)g^2F^b_{\lambda\rho}F^{b\lambda\rho}(x_1)|p_A,p_B\rangle \\
&= \int \mathscr{D}\Phi_{\mathscr{A}}\ \Psi^*_{p'_A}(t_i)\Psi_{p_A}(t_i)\Psi^*_{p'_B}(t_i)\Psi_{p_B}(t_i)\Big[\mathcal{O}^{\sigma_p}_{ij}(x_2^-,x_{2_\perp};x_1^-,x_{1_\perp})\mathcal{O}^{ij;\sigma_t}(x_2^+,x_{2_\perp};x_1^+,x_{1_\perp}) \\
&\quad + \int dz_1^-dz_{1_\perp}dz_2^-dz_{2_\perp}dw_1^+dw_{1_\perp}dw_2^+dw_{2_\perp}\frac{\alpha_sN_c}{2\pi}\mathfrak{C}_1(x_1,x_2;z_i^-,z_{i_\perp},w_i^+,w_{i_\perp};\sigma_p,\sigma_t) \\
&\qquad \times\mathcal{O}^{\sigma_p}_{ij}(z_2^-,z_{2_\perp};z_1^-,z_{1_\perp})\mathcal{O}^{ij;\sigma_t}(z_2^+,z_{2_\perp};z_1^+,z_{1_\perp})\ +\ldots\Big]
\end{aligned}
$$

where $\mathscr{A}=A+B+sG$

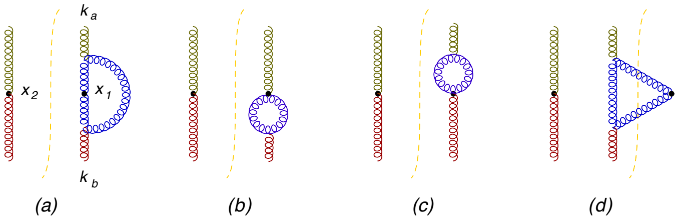
$$
\text{and}\qquad \mathcal{O}^{ij}(x^\pm,x_\perp;y^\pm,y_\perp)\ \equiv\ g^2F^{\mp i}(x^\pm,x_\perp)[x,x-\infty^\pm][-\infty^\pm+y,y]F^{\mp j}(y^\pm,y_\perp)
$$

Calculation of coefficient function $\mathfrak{C}_1$ in the classical background field $\mathbb{A}$
($\mathbb{A}\ =\ \bar{A}+\bar{B}$ + correction field)

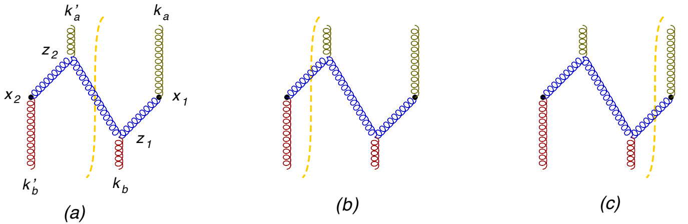
$$
\begin{aligned}
&\int dz_2^-dz_{2_\perp}dz_1^-dz_{1_\perp}dw_1^+dw_{1_\perp}dw_2^+dw_{2_\perp}\frac{\alpha_sN_c}{2\pi}\mathfrak{C}_1(x_1,x_2;z_i^-,z_{i_\perp},w_i^+,w_{i_\perp};\sigma_p,\sigma_t) \\
&\qquad \times\bar{A}^{-i,a}(z_2^+,z_{2_\perp}\bar{A}^{-j,a}(z_1^+,z_{1_\perp})\bar{B}^{+i,a}(z_2^-,z_{2_\perp})\bar{B}^{+j,a}(z_1^-,z_{1_\perp}) \\
&= \frac{N_c^2-1}{16}g^4\langle\tilde{F}^a_{\mu\nu}\tilde{F}^{a\mu\nu}(x_2)F^b_{\lambda\rho}F^{b\lambda\rho}(x_1)\rangle_{\mathbb{A}} \\
&\qquad -\langle\breve{\mathcal{O}}^{ij,\sigma_p}(x_2^-,x_{2_\perp};x_1^-,x_{1_\perp})\mathcal{O}^{ij;\sigma_t}(x_2^+,x_{2_\perp};x_1^+,x_{1_\perp})\rangle_{\mathbb{A}}
\end{aligned}
$$

Diagrams for $\langle \tilde{F}^a_{\mu\nu}\tilde{F}^{a\mu\nu}(x_2)F^b_{\lambda\rho}F^{b\lambda\rho}(x_1)\rangle_{\mathbb{A}}$ in background fields

"Virtual" diagrams



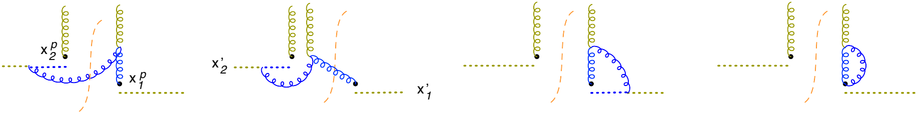
"Real" diagrams



Diagrams for subtracted TMD matrix elements

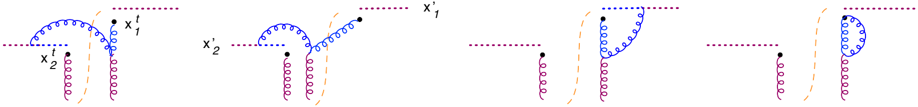
"Projectile" TMD matrix elements.

The rapidity-only $e^{-i\frac{\beta}{\sigma_p}}$ regularization is depicted by point splitting:
F^{+k} shown by dots stand at $x_1^p = x_{1_\perp} + x_1^-$ and $x_2^p = x_{2_\perp} + x_2^-$
Wilson lines start from $x_1' = x_2 + \delta^+$ and $x_2' = x_1 + \delta^+$ where $\delta^+ = \frac{1}{\varrho\sigma_p}$



(a) (b) (c) (d)

"Target" TMD matrix elements. The rapidity-only $e^{-i\frac{\alpha}{\sigma_t}}$ regularization is depicted by point splitting.



(a) (b) (c) (d)

Rapidity-only cutoff *vs* UV+rapidity regularization

Typical divergent integral ($\varepsilon = \frac{d}{2} - 2$, $\text{đ}^n p \equiv \frac{d^n p}{(2\pi)^n}$)

$$-i\mu^{-2\varepsilon}\int \text{đ}\alpha \text{đ}\beta \text{đ}p_\perp \frac{1}{\beta - i\epsilon}\frac{1}{\alpha\beta s - p_\perp^2 + i\epsilon}\frac{s(\beta-\beta_B)}{\alpha(\beta-\beta_B)s - p_\perp^2 + i\epsilon}\big(1 - e^{i(p,x)_\perp}\big)$$

$$= \mu^{-2\varepsilon}\int \frac{\text{đ}p_\perp}{p_\perp^2}\big(1 - e^{i(p,x)_\perp}\big)\int_0^{\beta_B}\frac{\text{đ}\beta}{\beta_B}\frac{\beta_B - \beta}{\beta - i\epsilon} = -\frac{1}{8\pi^2}\frac{\Gamma(\varepsilon)}{(x_\perp^2\mu^2)^\varepsilon}\int_0^{\beta_B}\frac{d\beta}{\beta_B}\frac{\beta_B-\beta}{\beta - i\epsilon}$$

δ-regularization with $A^-(z^+) \to A^-(z^+)e^{\pm\delta z^+}$

$$-\frac{1}{8\pi^2}\frac{\Gamma(\varepsilon)}{(x_\perp^2\mu^2)^\varepsilon}\int_0^{\beta_B}\frac{d\beta}{\beta_B}\frac{\beta_B-\beta}{\beta - i\delta} \simeq \frac{1}{8\pi^2}\Big(-\frac{1}{\varepsilon} + \ln\mu^2\frac{x_\perp^2}{4} + \gamma_E\Big)\Big(\ln\frac{\beta_B}{-i\delta} - 1\Big)$$

Rapidity-only cutoff

$$-i\int \text{đ}\alpha \text{đ}\beta \text{đ}p_\perp \frac{1}{\beta - i\epsilon}\frac{e^{-i\frac{\alpha}{\sigma}}}{\alpha\beta s - p_\perp^2 + i\epsilon}\frac{s(\beta-\beta_B)}{\alpha(\beta-\beta_B)s - p_\perp^2 + i\epsilon}\big(1 - e^{i(p,x)_\perp}\big)$$

$$= \int \frac{\text{đ}p_\perp}{p_\perp^2}\big(1 - e^{i(p,x)_\perp}\big)\int_0^{\infty}\text{đ}\alpha\frac{\beta_B s}{\alpha\beta_B s + p_\perp^2}e^{-i\frac{\alpha}{\sigma}} = \frac{1}{16\pi^2}\ln^2\Big(-i\beta_B\sigma s\frac{x_\perp^2}{4}e^{\gamma_E}\Big)$$

(Intermediate) Result

$$
\begin{aligned}
&\mathcal{W}(x_1,x_2) \;-\; \mathcal{W}^{\text{tmd}}(x_1,x_2)\\
&= \int \bar{d}\alpha'_a \bar{d}k'_{a_\perp} \bar{d}\beta'_b \bar{d}k_{b_\perp} \bar{d}\alpha_a \bar{d}k'_{a_\perp} \bar{d}\beta_b \bar{d}k'_{b_\perp}\, e^{-i\alpha'_a\varrho x_2^- - i\alpha_a\varrho x_1^-} e^{-i\beta'_b\varrho x_2^+ - i\beta_b\varrho x_1^+}\\
&\quad\times e^{-i(k_a+k_b,x_1)_\perp - i(k'_a+k'_b,x_2)_\perp} \bar{A}^{+,b}_i(\alpha'_a, k'_{a_\perp}) \bar{B}^{-i,a}(\beta'_b, k'_{b_\perp}) \bar{A}^{+,b}_j(\alpha_a, k_{a_\perp}) \bar{B}^{-j,a}(\beta_b, k_{b_\perp})\\
&\quad\times g^2[I - I^{\sigma_p,\sigma_t}_{\text{tmd}}](\alpha_a, \alpha'_a, \beta_b, \beta'_b, k_{a_\perp}, k'_{a_\perp}, k_{b_\perp}, k'_{b_\perp}, x_1, x_2)
\end{aligned}
$$

with

$$
\begin{aligned}
&[I - I^{\sigma_p,\sigma_t}_{\text{tmd}}](\alpha'_a, \alpha_a, \beta'_b, \beta_b, k'_{a_\perp}, k'_{a_\perp}, k_{b_\perp}, k'_{b_\perp}, x_2, x_1)\\
&\quad = -\ln\frac{(-i\alpha'_a)k'^2_{a_\perp}}{(-i\alpha_a)k'^2_{a_\perp}} \ln\frac{(-i\beta'_b)k'^2_{b_\perp}}{(-i\beta_b)k^2_{b_\perp}} + \ln^2\frac{x^2_{12_\perp} s\sigma_p\sigma_t}{4}\\
&\qquad - \ln\frac{(-i\alpha'_a)e^\gamma}{\sigma_t}\ln\frac{(-i\beta'_b)e^\gamma}{\sigma_p} - \ln\frac{(-i\alpha_a)e^\gamma}{\sigma_t}\ln\frac{(-i\beta_b)e^\gamma}{\sigma_p} + \pi^2
\end{aligned}
$$

where $(-i\alpha_a) \equiv -i(\alpha_a + i\epsilon)$ etc. Power corrections $\sim \zeta^{-1}$ and $\sim \zeta^{-1/2}$ are neglected.

(Intermediate) Result

$$
\begin{aligned}
&\mathcal{W}(x_1,x_2) \;-\; \mathcal{W}^{\rm tmd}(x_1,x_2)\\
&= \int \not{d}\alpha'_a \not{d}k'_{a_\perp} \not{d}\beta'_b \not{d}k_{b_\perp} \not{d}\alpha_a \not{d}k'_{a_\perp} \not{d}\beta_b \not{d}k'_{b_\perp}\, e^{-i\alpha'_a \varrho x_2^- - i\alpha_a \varrho x_1^-} e^{-i\beta'_b \varrho x_2^+ - i\beta_b \varrho x_1^+}\\
&\quad \times e^{-i(k_a+k_b,x_1)_\perp - i(k'_a+k'_b,x_2)_\perp} \bar{A}^{+,b}_i(\alpha'_a, k'_{a_\perp}) \bar{B}^{-i,a}(\beta'_b, k'_{b_\perp}) \bar{A}^{+,b}_j(\alpha_a, k_{a_\perp}) \bar{B}^{-j,a}(\beta_b, k_{b_\perp})\\
&\quad \times g^2[I - I^{\sigma_p,\sigma_t}_{\rm tmd}](\alpha_a, \alpha'_a, \beta_b, \beta'_b, k_{a_\perp}, k'_{a_\perp}, k_{b_\perp}, k'_{b_\perp}, x_1, x_2)
\end{aligned}
$$

with

$$
\begin{aligned}
&[I - I^{\sigma_p,\sigma_t}_{\rm tmd}](\alpha'_a, \alpha_a, \beta'_b, \beta_b, k'_{a_\perp}, k'_{a_\perp}, k_{b_\perp}, k'_{b_\perp}, x_2, x_1)\\
&\qquad = -\ln\frac{(-i\alpha'_a)k'^2_{a_\perp}}{(-i\alpha_a)k'^2_{a_\perp}} \ln\frac{(-i\beta'_b)k'^2_{b_\perp}}{(-i\beta_b)k^2_{b_\perp}} + \ln^2\frac{x^2_{12_\perp} s\sigma_p\sigma_t}{4}\\
&\qquad\quad - \ln\frac{(-i\alpha'_a)e^\gamma}{\sigma_t} \ln\frac{(-i\beta'_b)e^\gamma}{\sigma_p} - \ln\frac{(-i\alpha_a)e^\gamma}{\sigma_t} \ln\frac{(-i\beta_b)e^\gamma}{\sigma_p} + \pi^2
\end{aligned}
$$

where $(-i\alpha_a) \equiv -i(\alpha_a + i\epsilon)$ etc. Power corrections $\sim \zeta^{-1}$ and $\sim \zeta^{-1/2}$ are neglected.

This formula is not yet the final result for the coefficient function. The coefficient function was defined as a result of integration over C-fields with $\alpha > \sigma_t$ and $\beta > \sigma_p$. Since we did not impose these restrictions while calculating the loop integrals, we need to subtract sG contributions (with $\alpha < \sigma_t, \beta < \sigma_p$) to these integrals.

Result for the coefficient function

Result of sG subtraction:

term "$-\ln\frac{(-i\alpha'_a)k'^2_{a_\perp}}{(-i\alpha_a)k'^2_{a_\perp}}\ln\frac{(-i\beta'_b)k'^2_{b_\perp}}{(-i\beta_b)k^2_{b_\perp}}$" disappears $\Rightarrow$ no dynamics in the transverse plane

$$
\begin{aligned}
&\mathcal{W}(x_1,x_2)\ -\ \mathcal{W}^{\rm tmd}(x_1,x_2)\ -\ \mathcal{W}^{\rm sG}(x_1,x_2)\\
&=\int đ\alpha'_a đ\beta'_b đ\alpha_a đ\beta_b e^{-i\alpha'_a\varrho x_2^- - i\alpha_a\varrho x_1^-}e^{-i\beta'_b\varrho x_2^+ - i\beta_b\varrho x_1^+}\\
&\quad\times\bar{A}^{+,b}_i(\alpha'_a,x_{2_\perp})\bar{B}^{-i,a}(\beta'_b,x_{2_\perp})\bar{A}^{+,b}_j(\alpha_a,x_{1_\perp})\bar{B}^{-j,a}(\beta_b,x_{1_\perp})\\
&\quad\times g^2\mathfrak{C}_1(\alpha'_a,\alpha_a,\beta'_b,\beta_b;x_1,x_2)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathfrak{C}_1(\alpha'_a,\alpha_a,\beta'_b,\beta_b;x_2,x_1)\ &=\ I-I^{\sigma_p,\sigma_t}_{\rm tmd}-I^{\sigma_p,\sigma_t}_{\rm sG}\\
=\ \ln^2\frac{x^2_{12_\perp}s\sigma_p\sigma_t}{4}&-\ln\frac{(-i\alpha'_a)e^\gamma}{\sigma_t}\ln\frac{(-i\beta'_b)e^\gamma}{\sigma_p}-\ln\frac{(-i\alpha_a)e^\gamma}{\sigma_t}\ln\frac{(-i\beta_b)e^\gamma}{\sigma_p}+\pi^2
\end{aligned}
$$

The coefficient function in the coordinate space is made of $(+)$ - prescriptions since

$$
\int d\alpha\ e^{i\alpha z}\Big[\ln\Big(-i\frac{\alpha}{\sigma}+\epsilon\Big)\ =\ \frac{\theta(-z)}{z}+\delta(z)\int_0^{1/\sigma}\frac{dz'}{z'}
$$

Result for the coefficient function

Our formula

$$
\begin{aligned}
&\frac{1}{16}(N_c^2-1)\langle p'_A,p'_B|g^2F^a_{\mu\nu}F^{a\mu\nu}(x_2)g^2F^b_{\lambda\rho}F^{b\lambda\rho}(x_1)|p_A,p_B\rangle\\
&=\int\mathscr{D}\Phi_{\mathscr{A}}\ \Psi^*_{p'_A}(t_i)\Psi_{p_A}(t_i)\Psi^*_{p'_B}(t_i)\Psi_{p_B}(t_i)\Big[\mathcal{O}^{\sigma_p}_{ij}(x_2^-,x_{2_\perp};z_1^-,x_{1_\perp})\mathcal{O}^{ij;\sigma_t}(x_2^+,x_{2_\perp};x_1^+,x_{1_\perp})\\
&\quad+\int dz_1^-dz_2^-dw_1^+dw_2^+\frac{\alpha_sN_c}{2\pi}\mathfrak{C}_1(x_1,x_2;z_i^-,w_i^+;\sigma_p,\sigma_t)\\
&\qquad\times\mathcal{O}^{\sigma_p}_{ij}(z_2^-,x_{2_\perp};z_1^-,x_{1_\perp})\mathcal{O}^{ij;\sigma_t}(z_2^+,x_{2_\perp};z_1^+,x_{1_\perp})\ +\ O(\alpha_s^2)\Big]
\end{aligned}
$$

is not yet TMD formula since $\mathscr{A}=A+B+sG$ and soft/Glauber gluons connect "projectile" and "target" gluons.

It is well known that Glauber gluons cancel and soft gluons form soft factors.

With rapidity-only cutoffs, soft factors are power corrections $\Rightarrow$ TMD formula

$$
\begin{aligned}
&\frac{1}{16}(N_c^2-1)\langle p'_A,p'_B|g^2F^a_{\mu\nu}F^{a\mu\nu}(x_2)g^2F^b_{\lambda\rho}F^{b\lambda\rho}(x_1)|p_A,p_B\rangle\\
&=\ \langle p'_A|\hat{\mathcal{O}}^{\sigma_p}_{ij}(x_2^-,x_{2_\perp};x_1^-,x_{1_\perp})|p_A\rangle\langle p'_B|\hat{\mathcal{O}}^{ij;\sigma_t}(x_2^+,x_{2_\perp};x_1^+,x_{1_\perp})|p_B\rangle\\
&\quad+\int dz_1^-dz_2^-dw_1^+dw_2^+\frac{\alpha_sN_c}{2\pi}\mathfrak{C}_1(x_1,x_2;z_i^-,w_i^+;\sigma_p,\sigma_t)\\
&\qquad\times\ \langle p'_A|\hat{\mathcal{O}}^{\sigma_p}_{ij}(z_2^-,x_{2_\perp};z_1^-,x_{1_\perp})|p_A\rangle\langle p'_B|\hat{\mathcal{O}}^{ij;\sigma_t}(z_2^+,x_{2_\perp};z_1^+,x_{1_\perp})|p_B\rangle
\end{aligned}
$$

Matching of coefficient function and TMDs

TMD evolution equations

$$\sigma_p \frac{d}{d\sigma_p} \hat{\mathcal{O}}^{ij;\sigma_t}(\alpha'_a, \alpha_a, x_{2_\perp}, x_{1_\perp})$$
$$= -\frac{\alpha_s N_c}{2\pi}\Big[2\ln\frac{s x^2_{12_\perp}}{4} + \ln(-i\alpha'_a\sigma_p + \epsilon) + \ln(-i\alpha_a\sigma_p + \epsilon) + 2\gamma\Big]\hat{\mathcal{O}}^{ij;\sigma_t}(\alpha'_a, \alpha_a, x_{2_\perp}, x_{1_\perp})$$
$$\sigma_t \frac{d}{d\sigma_t} \hat{\mathcal{O}}^{ij;\sigma_t}(\beta'_b, \beta_b, x_{2_\perp}, x_{1_\perp})$$
$$= -\frac{\alpha_s N_c}{2\pi}\Big[2\ln\frac{s x^2_{12_\perp}}{4} + \ln(-i\beta'_b\sigma_t + \epsilon) + \ln(-i\beta_b\sigma_t + \epsilon) + 2\gamma\Big]\hat{\mathcal{O}}^{ij;\sigma_t}(\beta'_b, \beta_b, x_{2_\perp}, x_{1_\perp})$$

Matching of σ_p and σ_t evolutions $\Rightarrow$

$$\sigma_t \frac{d}{d\sigma_t}\mathfrak{C}(x_{1_\perp}, x_{2_\perp}; \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t) = \frac{\alpha_s N_c}{2\pi}\Big[2\ln\frac{s x^2_{12_\perp}}{4}$$
$$+ \ln(-i\beta'_b\sigma_t + \epsilon) + \ln(-i\beta_b\sigma_t + \epsilon) + 2\gamma\Big]\mathfrak{C}(x_1, x_2; \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t)$$
$$\sigma_p \frac{d}{d\sigma_p}\mathfrak{C}(x_{1_\perp}, x_{2_\perp}; \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t) = \frac{\alpha_s N_c}{2\pi}\Big[2\ln\frac{s x^2_{12_\perp}}{4}$$
$$+ \ln(-i\alpha'_a\sigma_p + \epsilon) + \ln(-i\alpha_a\sigma_p + \epsilon) + 2\gamma\Big]\mathfrak{C}(x_{1_\perp}, x_{2_\perp}; \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t)$$

Matching of coefficient function and TMDs

The solution of this equations compatible with our first-order result is

$$\mathfrak{C}(x_{1_\perp}, x_{2_\perp}; \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t) \;=\; e^{\frac{\alpha_s N_c}{2\pi}\mathfrak{C}_1(x_{12_\perp}, \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t)}$$

$\Rightarrow$ hadronic tensor is

$$W(\alpha'_a, \alpha_a, \beta'_b, \beta_b, x_{1_\perp}, x_{2_\perp}) \;=\; \int đ\alpha'_a đ\alpha_a đ\beta'_b đ\beta_b \; e^{\frac{\alpha_s N_c}{2\pi}\mathfrak{C}_1(x_{12_\perp}, \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t)}$$
$$\times \langle p'_A|\hat{\mathcal{O}}^{\sigma_p}_{ij}(\alpha'_a, \alpha_a, x_{2_\perp}, x_{1_\perp})|p_A\rangle \langle p'_B|\hat{\mathcal{O}}^{ij;\sigma_t}(\beta'_b, \beta_b, x_{2_\perp}, x_{1_\perp})|p_B\rangle \;+ \ldots$$

Reminder

$$\mathfrak{C}_1(\alpha'_a, \alpha_a, \beta'_b, \beta_b; x_1, x_2; \sigma_p, \sigma_t)$$
$$= \ln^2 \frac{x^2_{12_\perp} s\sigma_p\sigma_t}{4} - \ln\frac{(-i\alpha'_a)e^\gamma}{\sigma_t}\ln\frac{(-i\beta'_b)e^\gamma}{\sigma_p} - \ln\frac{(-i\alpha_a)e^\gamma}{\sigma_t}\ln\frac{(-i\beta_b)e^\gamma}{\sigma_p} + \pi^2$$

Forward case ($\equiv$ particle production by gluon fusion)

$$
\begin{aligned}
W(p_A,p_B;q) &= \int db_\perp\, e^{i(q,b)_\perp} W(p_A,p_B;\alpha_q,\beta_q,b_\perp),\\
W(p_A,p_B;\alpha_q,\beta_q,b_\perp) &= \frac{\pi^2}{2} Q^2 \mathcal{G}_{ij}^{\sigma_p}(\alpha_q,b_\perp;p_A)\,\mathcal{G}^{ij;\sigma_t}(\beta_q,b_\perp;p_B)\\
&\times\ \exp\Big\{\frac{\alpha_s N_c}{2\pi}\Big[\ln^2\frac{b_\perp^2 s\sigma_p\sigma_t}{4} - 2\Big(\ln\frac{\alpha_q}{\sigma_t}+\gamma\Big)\Big(\ln\frac{\beta_q}{\sigma_p}+\gamma\Big)+\frac{\pi^2}{2}\Big]\Big\}\\
&+\ \text{NLO terms} \sim O\big(\alpha_s^2\big)\ +\ \text{power corrections} \qquad (*)
\end{aligned}
$$

where $\mathcal{G}_{ij}^{\sigma_p}$, $\mathcal{G}_{ij}^{\sigma_t}$ are gluon TMDs:

$$
\begin{aligned}
\langle p_A|\hat{\mathcal{O}}_{ij}^{\sigma_p}(z^-,0^-,b_\perp)|p_A\rangle &= -g^2\varrho^2\int_0^1 du\, u\,\mathcal{G}_{ij}^{\sigma_p}(u,b_\perp)\cos u\varrho z^-,\\
\langle p_B|\hat{\mathcal{O}}_{ij}^{\sigma_t}(z^-,0^-,b_\perp)|p_B\rangle &= -g^2\varrho^2\int_0^1 du\, u\,\mathcal{G}_{ij}^{\sigma_t}(u,b_\perp)\cos u\varrho z^-,
\end{aligned}
$$

Matching of coefficient function and TMDs

The r.h.s. of the evolution formula $(*)$ does not depend on cutoffs σ_p and σ_t as long as $\sigma_p \geq \tilde{\sigma}_p = \frac{4b_\perp^{-2}}{\alpha_q s}$ and $\sigma_t \geq \tilde{\sigma}_t \equiv \frac{4b_\perp^{-2}}{\beta_q s}$. Thus, the result of double-log Sudakov evolution reads

$$
\begin{aligned}
W(p_A, p_B; \alpha_q, \beta_q, b_\perp) \;&=\; \frac{\pi^2}{2} Q^2 \mathcal{G}_{ij}^{\tilde{\sigma}_p}(\alpha_q, b_\perp; p_A) \mathcal{G}^{ij;\tilde{\sigma}_t}(\beta_q, b_\perp; p_B) \\
\times \;\exp\Big\{ -\frac{\alpha_s N_c}{2\pi}\Big[\Big(\ln\frac{Q^2 b_\perp^2}{4} + 2\gamma\Big)^2 - 2\gamma^2 - \frac{\pi^2}{2}\Big]\Big\} &+ \; O\big(\alpha_s^2\big) \text{ terms } + \text{ power corrections}
\end{aligned}
$$

This result is universal for moderate x and small-x hadronic tensor. The difference lies in the continuation of the evolution beyond Sudakov region.

Double-log Sudakov evolution should stop at $\sigma_0 \simeq \frac{b_\perp^{-2}}{\beta_q s} \simeq \frac{q_\perp^2}{x_B s}$. After that:

- If $\beta_B \equiv x_B \sim 1$ - DGLAP-type evolution from $\sigma_0 = \frac{b_\perp^{-2}}{x_B s}$ to $\sigma_{\text{fin}} = \frac{m_N^2}{s}$: summation of $\big(\alpha_s \ln \frac{b_\perp^{-2}}{m_N^2}\big)^n$
- If $\beta_B \equiv x_B \ll 1$ - BFKL-type evolution from $\sigma_0 = \frac{b_\perp^{-2}}{x_B s}$ to $\sigma_{\text{fin}} = \frac{b_\perp^{-2}}{s}$: summation of $\big(\alpha_s \ln x_B\big)^n$

Conclusions

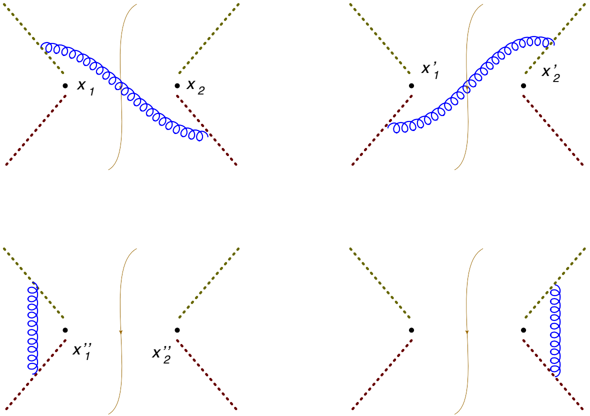
1. Conclusion: rapidity-only TMD factorization works!
   - Power corrections $\sim \frac{1}{Q^2}$ for DY hadronic tensor $\Rightarrow$ EM gauge invariance of DY tensor.
   - Back-of-the-envelope estimates of angular distributions for DY Z-boson production are in good agreement with LHC data.
   - Rapidity factorization at the one-loop level gives Sudakov-type double logs for both small and intermediate x_B

2. Outlook
   - Power corrections $\sim \frac{1}{Q^2}$ for SIDIS.
   - Matching to DGLAP and BFKL/BK evolutions
   - Conformal invariance of rapidity-only factorization

Thank you for attention!

Backup slide: soft factor with rapidity-only cutoffs

Leading-order diagrams



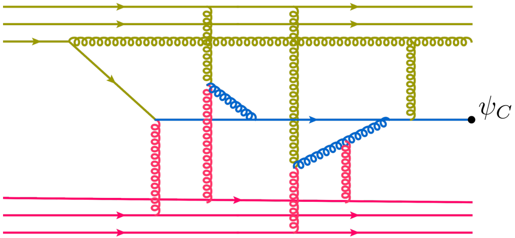
Result of calculation: $\frac{1}{4\pi^2}\mathrm{Li}_2\left(-\frac{x^2_{12\perp}}{2\delta^+\delta^-}\right) \sim O\left(\frac{\Delta^2_\perp}{2\delta^+\delta^-}\right) \sim O\left(\frac{\sigma_p\sigma_t s}{Q^2_\perp}\right) \sim O(\zeta^{-1/2})$

Soft factor with rapidity-only regularization does not have perturbative contributions which can mix with the TMD evolution

BACKUP SLIDES

In the tree approximation: classical YM field with sources

Tree approximation:
Projectile fields: $\beta = 0 \Rightarrow A(x^-, x_\perp),\ \psi_A(x^-, x_\perp)$
Target fields: $\alpha = 0 \Rightarrow B(x^+, x_\perp),\ \psi_B(x^-, x_\perp)$



ψ_C = sum of tree diagrams in external $A, \tilde{A}, \psi_A, \tilde{\psi}_A$ and $B, \tilde{B}, \psi_B, \tilde{\psi}_B$ fields with sources

$J_\psi = (\not{P} + m)(\psi_A + \psi_B)$, $\qquad J_\nu = D^\mu F^{\mu\nu}(A + B)$

and

$\tilde{J}_\psi = (\not{P} + m)(\tilde{\psi}_A + \tilde{\psi}_B)$, $\qquad \tilde{J}_\nu = D^\mu F^{\mu\nu}(\tilde{A} + \tilde{B})$

$\sum_X \Rightarrow$ Feynman diagrams with retarded propagators

The fields A, ψ and $\tilde{A}, \tilde{\psi}$ do not depend on $x^+ \Rightarrow$
if they coincide at $x^+ = \infty \Rightarrow$ they coincide everywhere.

Similarly,
B, ψ_b and $\tilde{B}, \tilde{\psi}_b$ do not depend on $x^- \Rightarrow$
if they coincide at $x^- = \infty$ they should be equal.

Since $\tilde{A} = A$ and $\tilde{B} = B$ the sources and background fields are the same to the left and to the right of the cut

$\Rightarrow$

ψ_C and C_μ are given by the sum of tree diagrams with *retarded* Green functions

(F. Gelis, R. Venugopalan)

Classical solution

The sum of diagrams with retarded Green functions ⇔ solution of classical YM equations

$$(\not{P}+m_f)\psi^f \ = \ 0, \quad D^\nu F^a_{\mu\nu} \ = \ \sum_f g\bar{\psi}^f t^a\gamma_\mu\psi^f$$

Boundary conditions :
$$A_\mu(x) \overset{x^+\to-\infty}{=} \bar{A}_\mu(x^-,x_\perp), \quad \psi(x) \overset{x^+\to-\infty}{=} \psi_a(x^-,x_\perp)$$
$$A_\mu(x) \overset{x^-\to-\infty}{=} \bar{B}_\mu(x^+,x_\perp), \quad \psi(x) \overset{x^-\to-\infty}{=} \psi_b(x^+,x_\perp)$$

The projectile and target fields satisfy YM equations

$$(\not{P}+m_f)\psi^f_a \ = \ 0, \quad D^\nu F^a_{\mu\nu} \ = \ g\bar{\psi}^f_a t^a\gamma_\mu\psi^f_a$$
$$(\not{P}+m_f)\psi^f_b \ = \ 0, \quad D^\nu F^a_{\mu\nu} \ = \ g\bar{\psi}^f_b t^a\gamma_\mu\psi^f_b$$

Projectile partons: $k=\alpha p_1+k_\perp$, target partons: $k=\beta p_1+k_\perp$ ⇒ partons are *not* on the mass shell

Classical solution

The sum of diagrams with retarded Green functions $\Leftrightarrow$ solution of classical YM equations

$$(\not{P}+m_f)\psi^f = 0, \quad D^\nu F^a_{\mu\nu} = \sum_f g\bar{\psi}^f t^a \gamma_\mu \psi^f$$

Boundary conditions :

$$A_\mu(x) \stackrel{x^+\to-\infty}{=} \bar{A}_\mu(x^-,x_\perp), \quad \psi(x) \stackrel{x^+\to-\infty}{=} \psi_a(x^-,x_\perp)$$
$$A_\mu(x) \stackrel{x^-\to-\infty}{=} \bar{B}_\mu(x^+,x_\perp), \quad \psi(x) \stackrel{x^-\to-\infty}{=} \psi_b(x^+,x_\perp)$$

The projectile and target fields satisfy YM equations

$$(\not{P}+m_f)\psi^f_a = 0, \quad D^\nu F^a_{\mu\nu} = g\bar{\psi}^f_a t^a \gamma_\mu \psi^f_a$$
$$(\not{P}+m_f)\psi^f_b = 0, \quad D^\nu F^a_{\mu\nu} = g\bar{\psi}^f_b t^a \gamma_\mu \psi^f_b$$

Projectile partons: $k=\alpha p_1+k_\perp$, target partons: $k=\beta p_1+k_\perp$ $\Rightarrow$ partons are *not* on the mass shell

Method of solution:

- Start with $\psi_A+\psi_B$ and $\bar{A}_\mu+\bar{B}_\mu$ in the gauge $A^+=0$, $A^-=0$
- Correct by computing Feynman diagrams (with retarded propagators) with sources $(\not{P}+m)(\psi_A+\psi_B)$ and $J_\nu = D^\mu F^{\mu\nu}(U+V)$

Classical fields in the leading order in $p_\perp^2/p_\parallel^2 \sim q_\perp^2/Q^2$

The solution of YM equations in general case (scattering of two "color glass condensates") is yet unsolved problem.

Fortunately, for our case of particle production with $\frac{q_\perp}{Q} \ll 1$ we can use this small parameter and construct the approximate solution.

At the tree level transverse momenta are $\sim q_\perp^2$ and longitudinal are $\sim Q^2 \Rightarrow$

$$\psi, A \;=\; \text{series in } \frac{q_\perp}{Q}: \qquad \psi = \psi^{(0)} + \psi^{(1)} + ..., \qquad A = A^{(0)} + A^{(1)} + ...$$

NB: After the expansion

$$\frac{1}{p^2 + i\epsilon p_0} \;=\; \frac{1}{p_\parallel^2 - p_\perp^2 + i\epsilon p_0} \;=\; \frac{1}{p_\parallel^2} - \frac{1}{p_\parallel^2 + i\epsilon p_0} p_\perp^2 \frac{1}{p_\parallel^2 + i\epsilon p_0} \;+ ...$$

the dynamics in transverse space is trivial.

Fields are either at the point $x_\perp$ or at the point $0_\perp \Rightarrow$ TMDs

Leading-N_c power corrections

Power corrections are $\sim$ leading twist $\times\Big(\frac{q_\perp}{Q}\text{ or }\frac{q_\perp^2}{Q^2}\Big)\times\big(1+\frac{1}{N_c}+\frac{1}{N_c^2}\big)$.

(Pleasant) surprise: most of the terms not suppressed by $\frac{1}{N_c}$ are determined by the leading-twist TMDs due to QCD equations of motion

Leading twist:

$$\frac{1}{8\pi^3 s}\int dx^- d^2x_\perp\ e^{-i\alpha x^- + i(k,x)_\perp}\ \langle A|\hat{\bar{\psi}}_f(x^-,x_\perp)\ \not{p}_2\hat{\psi}_f(0)|A\rangle\ =\ f_{1f}(\alpha,k_\perp^2)$$

Power correction:

$$\frac{1}{8\pi^3 s}\int dx^- dx_\perp\ e^{-i\alpha_q x^- + i(k,x)_\perp}$$
$$\times\ \langle A|\hat{\bar{\psi}}^f(x^-,x_\perp)\ \not{p}_2[\hat{U}_i(x^-,x_\perp) - i\gamma_5\hat{\tilde{U}}_i(x^-,x_\perp)]\hat{\psi}^f(0)|A\rangle$$
$$=\ -k_i f_1(\alpha_q,k_\perp) + \alpha_q k_i\big[f_\perp(\alpha_q,k_\perp) + g^\perp(\alpha_q,k_\perp)\big],$$

(Mulders & Tangerman, 1996)

At small $\alpha_q \equiv x_A$ one can drop the second term

Evolution of gluon TMDs

Gluon TMD operator

$$\mathcal{O}_g(x^+,x_\perp;y^+,y_\perp) \ \equiv\ g^2F^{-i}(x^+,x_\perp)[x,x\pm\infty n][\pm\infty n+y,y]F^{-j}(y^+,y_\perp)$$

Rapidity-regularized operator

$$\begin{aligned}
\mathcal{O}_g^\sigma(x^+,x_\perp;y^+,y_\perp) \ &\equiv\ \tilde{\mathscr{F}}_i^{\sigma,a}(x_\perp,x^+)\mathscr{F}_i^{\sigma,a}(y_\perp,y^+),\\
\tilde{\mathscr{F}}_i^{a;\sigma}(x_\perp,x^+) \ &=\ g(F^-{}_i)^b(x^+,x_\perp,-\delta^-)[x^+,-\infty]_x^{ba},\\
\mathscr{F}_i^{a;\sigma}(y_\perp,y^+) \ &=\ [-\infty,y^+]_y^{ab}g(F^-{}_i)^b(x^+,x_\perp,-\delta'^-)
\end{aligned}$$

$$\delta^- = \frac{1}{\varrho\sigma}$$

Approximation: $\beta_B\sigma s \gg (x-y)_\perp^{-2}$

Leading-order evolution equation is the same as in quark case with $c_f \to N_c$ replacement (G.A. Chirilli and I.B., 2019)

Quark loop contribution to gluon TMD evolution

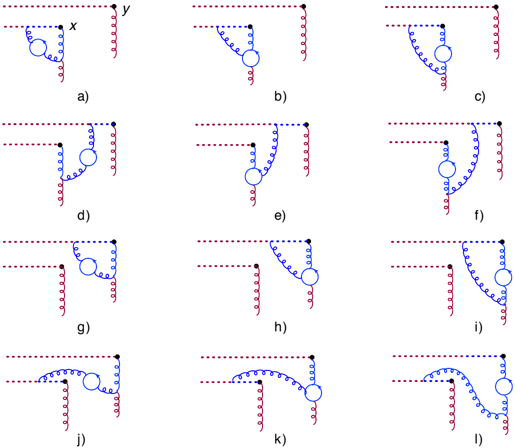


Figure: Quark loop correction to gluon TMD evolution

Quark loop contribution to gluon TMD evolution

Result of calculations: BLM scale is the same as in the quark case with $c_F \to N_c$ replacement $\Rightarrow$ rapidity evolution is the same

$$\tilde{\mathscr{F}}_i^{a;\varsigma}(\beta'_B, x_\perp)\mathscr{F}^{a,i;\varsigma}(\beta_B, y_\perp) \;=\; \tilde{\mathscr{F}}_i^{a;\varsigma_0}(\beta'_B, x_\perp)\mathscr{F}^{a,i;\varsigma_0}(\beta_B, y_\perp)$$
$$\times\, e^{\frac{N_c}{4\pi}\left[\ln\frac{\alpha_s(\mu'_\varsigma)}{\alpha_s(\mu'_{\varsigma_0})}\left(\frac{1}{\alpha_s(\tilde{b}_\perp^{-1})}+\ln[-i\tau'_B+\epsilon]\right)+\frac{1}{\alpha_s(\mu'_\varsigma)}-\frac{1}{\alpha_s(\mu'_{\varsigma_0})}\right]}$$
$$\times\, e^{\frac{N_c}{4\pi}\left[\ln\frac{\alpha_s(\mu_\varsigma)}{\alpha_s(\mu_{\varsigma_0})}\left(\frac{1}{\alpha_s(\tilde{b}_\perp^{-1})}+\ln[-i\tau_B+\epsilon]\right)+\frac{1}{\alpha_s(\mu_\varsigma)}-\frac{1}{\alpha_s(\mu_{\varsigma_0})}\right]}$$

Double-log Sudakov evolution should stop at $\beta_B\sigma_0 s \simeq b_\perp^{-2}$. After that:

- If $\beta_B \equiv x_B \sim 1$ - DGLAP-type evolution from $\sigma_0 = \frac{b_\perp^{-2}}{x_B s}$ to $\sigma_{\text{fin}} = \frac{m_N^2}{s}$: summation of $\left(\alpha_s \ln\frac{b_\perp^{-2}}{m_N^2}\right)^n$
- If $\beta_B \equiv x_B \ll 1$ - BFKL-type evolution from $\sigma_0 = \frac{b_\perp^{-2}}{x_B s}$ to $\sigma_{\text{fin}} = \frac{b_\perp^{-2}}{s}$: summation of $\left(\alpha_s \ln x_B\right)^n$

Matching: use general equation for TMD evolution at all x_B from papers with A. Tarasov.
Drawback: very complicated. MB conformal invariance will help?