BES Theory:
Search for the QCD critical point

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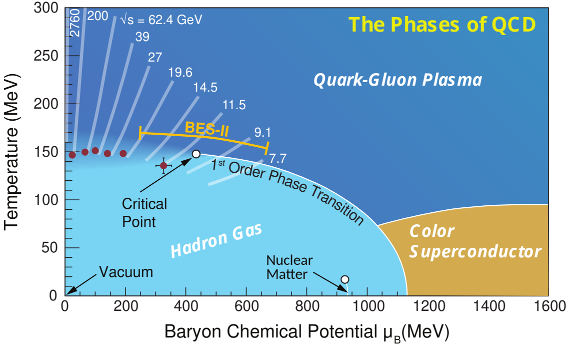
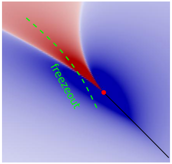
- New opportunities to learn about onset of deconfinement, CME, vorticity-spin interaction, thermalization and hydrodynamics in smaller systems, etc.

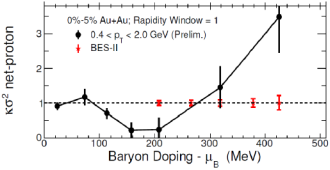
Critical point: intriguing hints

Where on the QCD phase boundary is the CP?



Equilibrium κ_4
vs T and μ_B:





“intriguing hint” (2015 LRPNS)

Motivation for phase II of BES at RHIC and BEST topical collaboration.

BEST Framework

The goal of BES theory: connect QCD phase diagram to observables.

BEST framework: An et al (40+ authors, 100+ pp, 369 refs) 2108.13867

- Lattice EOS + CP $\rightarrow$ parametric EOS
- Initial conditions and transition to viscous hydrodynamics.
- Hydrodynamics with (critical, non-gaussian) fluctuations.
- Freezeout, including fluctuations.
- “Particlization” and hadronic phase evolution.
- Comparison with experiment. Bayesian analysis.

Theory: beyond equilibrium

Predictions of CP signatures assume equilibrium,
but in heavy-ion collisions

> non-equilibrium physics is essential,
> especially near the critical point.

Critical slowing down: certain slow degrees of freedom are further away from equilibrium. These degrees of freedom are directly related to fluctuations.

Challenge: develop hydrodynamics *with fluctuations* capable of describing *non-equilibrium* effects on fluctuation signatures of CP.

Randomness in hydrodynamics

Stochastic description

Random hydro variables: $\breve{\psi}$

$$\partial_t \breve{\psi} = -\nabla \cdot \Big(\text{Flux}[\breve{\psi}] + \text{Noise} \Big)$$

$-$ cutoff dependence (infinite noise)

Landau-Lifshits, Kapusta et al, Gale et al, Nahrgang et al, Schaefer-Skokov, . . .

Deterministic description

$\psi \equiv \langle \breve{\psi} \rangle$, $G \equiv \langle \breve{\psi}\breve{\psi} \rangle$, etc.

$$\partial_t \psi = -\nabla \cdot \text{Flux}[\psi; G];$$

$$\partial_t G = -2\Gamma(G - \bar{G}[\psi]);$$

$+$ no cutoff dependence after renormalization

Andreev, Akamatsu et al, Yin et al, An et al, Martinez et al, . . .

Hydro+

Yin, MS, 1712.10305

- Hydro+ extends Hydro with new *non-hydrodynamic* d.o.f..
- At the CP, the *slowest* (i.e., most out of equilibrium) new d.o.f. is the 2-pt function $\langle \delta m \delta m \rangle$ of the slowest hydro variable $m \equiv s/n$:

$$\phi_{\boldsymbol{Q}}(\boldsymbol{x}) = \int_{\Delta \boldsymbol{x}} \langle \delta m\,(\boldsymbol{x}_+)\ \delta m\,(\boldsymbol{x}_-) \rangle\ e^{i\boldsymbol{Q}\cdot\Delta\boldsymbol{x}}$$

  where $\boldsymbol{x} = (\boldsymbol{x}_+ + \boldsymbol{x}_-)/2$ and $\Delta\boldsymbol{x} = \boldsymbol{x}_+ - \boldsymbol{x}_-$.
- $\phi_{\boldsymbol{Q}}$ quantifies the magnitude of fluctuation harmonics with wavevector $\boldsymbol{Q}$.

Relaxation of fluctuations towards equilibrium

- As usual, equilibration maximizes entropy $S = \sum_i p_i \log(1/p_i)$:

$$s_{(+)}(\epsilon, n, \phi_{\boldsymbol{Q}}) = s(\epsilon, n) + \frac{1}{2}\int_{\boldsymbol{Q}} \left(\log \frac{\phi_{\boldsymbol{Q}}}{\bar{\phi}_{\boldsymbol{Q}}} - \frac{\phi_{\boldsymbol{Q}}}{\bar{\phi}_{\boldsymbol{Q}}} + 1 \right)$$

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- The equation for $\phi_{\boldsymbol{Q}}$ is a relaxation equation with rate

$$\Gamma(\boldsymbol{Q}) \approx 2DQ^2 \quad \text{for} \quad Q \ll \xi^{-1}, \quad D \sim 1/\xi.$$

- Impact on fluctuation observables: “memory” effects

(Berdnikov-Rajagopal, Mukherjee-Venugopalan-Yin, . . .)

non-Gaussian fluctuations are sensitive signatures of the critical point

Deterministic approach to non-Gaussian fluctuations

An et al 2009.10742, PRL

- *Infinite* hierarchy of coupled equations

  for cumulants $G^{\mathsf{c}}_n \equiv \langle \underbrace{\delta\breve{\psi}\ldots\delta\breve{\psi}}_{n} \rangle^{\mathsf{c}}$:

$$\partial_t \psi = -\nabla \cdot \mathsf{Flux}[\psi, G, G^{\mathsf{c}}_3, G^{\mathsf{c}}_4, \ldots];$$

$$\partial_t G = \mathsf{F}[\psi, G, G^{\mathsf{c}}_3, G^{\mathsf{c}}_4, \ldots];$$

$$\partial_t G^{\mathsf{c}}_3 = \mathsf{F}_3[\psi, G, G^{\mathsf{c}}_3, G^{\mathsf{c}}_4, \ldots];$$

$$\vdots$$

Controlled perturbation theory

- *Small* fluctuations are *almost* Gaussian
- Introduce expansion parameter ε, so that $\delta\breve{\psi} \sim \sqrt{\varepsilon}$.

  Then $G_n^{\mathsf{c}} \equiv \varepsilon^{n-1}$ and to leading order in ε:

$$\partial_t \psi = -\nabla \cdot \mathsf{Flux}[\psi] + \mathcal{O}(\varepsilon);$$

$$\partial_t G = -2\Gamma(G - \bar{G}[\psi]) + \mathcal{O}(\varepsilon^2);$$

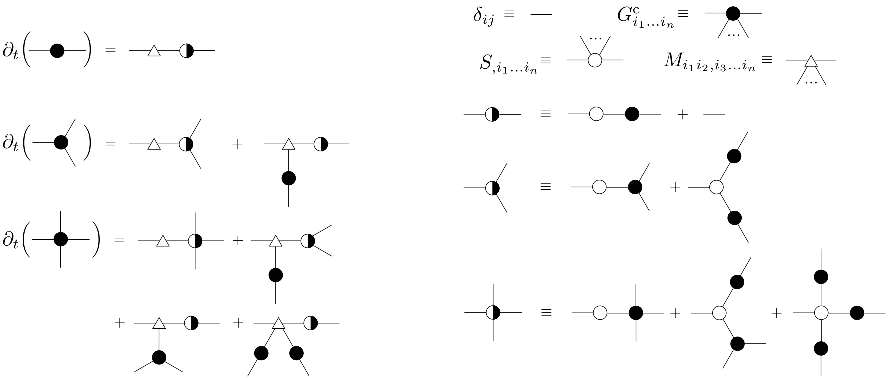
$$\vdots$$

$$\partial_t G_n^{\mathsf{c}} = -n\Gamma(G_n^{\mathsf{c}} - \bar{G}_n^{\mathsf{c}}[\psi, G, \ldots, G_{n-1}^{\mathsf{c}}]) + \mathcal{O}(\varepsilon^n);$$

  To leading order, the equations are iterative and "linear".
- In hydrodynamics the small parameter is $(q/\Lambda)^3$, i.e., fluctuation wavelength $1/q \gg$ size of hydro cell $1/\Lambda$ (UV cutoff).

Diagrammatic representation

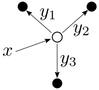
Systematically expand in ε and truncate at leading order:



- Leading order in ε $\Leftrightarrow$ tree diagrams.
- In higher-orders, loops describe feedback of fluctuations (e.g., long-time tails).

Generalizing Wigner transform

- Definition:



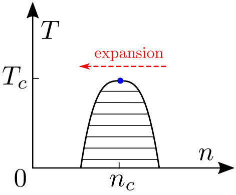
$$W_n(\boldsymbol{x}; \boldsymbol{q}_1, \ldots, \boldsymbol{q}_n) \equiv \int d\boldsymbol{y}_1^3 \ldots \int d\boldsymbol{y}_n^3 \, G_n\left(\boldsymbol{x}+\boldsymbol{y}_1, \ldots, \boldsymbol{x}+\boldsymbol{y}_n\right)$$
$$\delta^{(3)}\left(\frac{\boldsymbol{y}_1+\ldots+\boldsymbol{y}_n}{n}\right) e^{-i(\boldsymbol{q}_1\cdot\boldsymbol{y}_1+\ldots+\boldsymbol{q}_n\cdot\boldsymbol{y}_n)};$$

$$G_n\left(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\right) = \int \frac{d\boldsymbol{q}_1^3}{(2\pi)^3} \ldots \int \frac{d\boldsymbol{q}_n^3}{(2\pi)^3} \, W_n(\boldsymbol{x}, \boldsymbol{q}_1, \ldots, \boldsymbol{q}_n)$$
$$\delta^{(3)}\left(\frac{\boldsymbol{q}_1+\ldots+\boldsymbol{q}_n}{2\pi}\right) e^{i(\boldsymbol{q}_1\cdot\boldsymbol{x}_1+\ldots+\boldsymbol{q}_n\cdot\boldsymbol{x}_n)}.$$

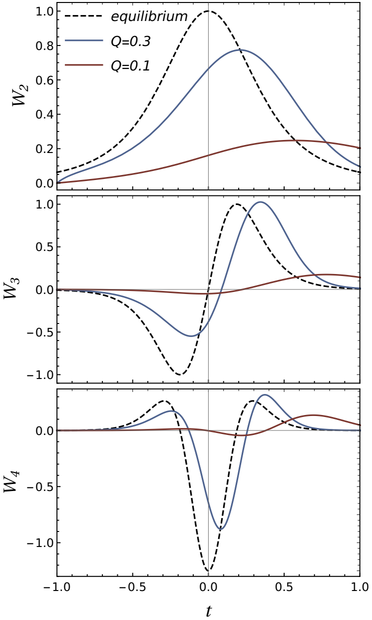
- W_n's quantify magnitude and non-gaussianity of fluctuation harmonics with wave-vectors $\boldsymbol{q}_i$.

Example: expansion through a critical region

An et al 2009.10742, PRL



- Two main features:
  - Lag, "memory".
  - Smaller Q – slower evolution. Conservation laws.
- Critical point signatures depend on the scale of fluctuations probed.



Experiments measure particles, not hydro variables

Freezing out (critical) hydrodynamic fluctuations

Pradeep et al, 2204.00639, PRD

- Cooper-Frye deals with with 1-particle observables.
  We need 2-particle (and n-particle) *correlations*.
- Critical contribution to fluctuations of $f(x,p)$:

$$\delta f = \frac{\partial f}{\partial \sigma}\delta\sigma, \quad \text{via} \quad \delta m = g\delta\sigma.$$

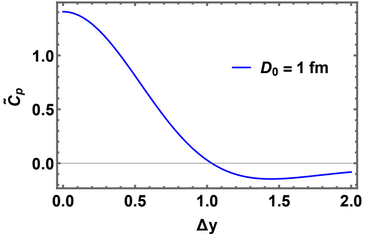
$$\langle \delta\sigma\delta\sigma\rangle \sim \text{ F.T. } \phi_{\boldsymbol{Q}}$$

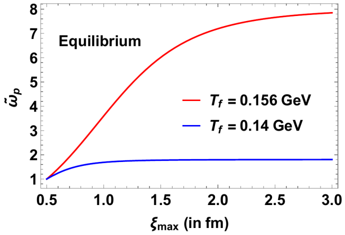
- Critical contribution to observables

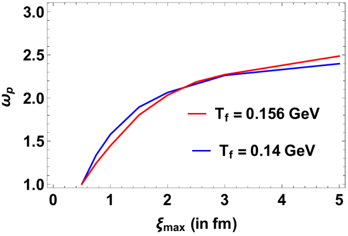
$$\langle \delta N^2 \rangle = \int_{\boldsymbol{p}_1}\int_{\boldsymbol{p}_2}\langle \delta f_1 \delta f_2\rangle \sim \int_{\boldsymbol{p}_1}\int_{\boldsymbol{p}_2}\frac{\partial f_1}{\partial\sigma}\frac{\partial f_2}{\partial\sigma} \text{ F.T. } \phi_{\boldsymbol{Q}}$$

Freeze out in Hydro+: model calculation and lessons

- Effect of conservation laws:
  - particle (anti)correlations
  - suppression relative to equilibrium critical expectations







($\xi_{\rm max}$ – how close fireball gets to CP; $T_{\rm f}$ – how long it evolves after passing CP.)

Signal less sensitive to $T_{\rm freezeout}$ due to noneq. effects.

How do the *non-gaussian* fluctuations freezeout?

Maximum entropy freezeout of fluctuations

Pradeep, MS, 2211.09142, PRL

- Freezeout: translation of correlators of hydrodynamic fluctuations (n-point functions) $H_n = \langle \delta\epsilon \ldots \delta\epsilon \rangle$ to particle correlators $G_n = \langle \delta f \ldots \delta f \rangle$.
- Conservation laws relate momentum space integrals of G_n to H_n, but there are ∞ many possibilities/solutions for G_n matching these constraints. Because f and G_n are functions of $\boldsymbol{p}$'s in addition to $\boldsymbol{x}$'s.

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- There is a special solution which maximizes the entropy!
  - for $n = 1$ equivalent to Cooper-Frye
  - for critical fluctuations equivalent to the σ field coupling
  - but applies much more generally
- Work in progress – implement in a hydro model and estimate *nonequilibrium* expectations for multiplicity cumulants in BES

(Karthein, Pradeep, MS, Rajagopal, Yin)

Work in progress and outlook

- *Non-Gaussian* fluctuations in *full* relativistic hydrodynamics.

  An et al 2212.14029 and work in progress

- Connect *fluctuating* hydro with kinetics via Maximum Entropy. Particlize and implement in full hydrodynamic code and event generator.

- First-order transition in *fluctuating* hydrodynamics.

- Compare with experiment.