Resurgence in QED

Probing the Schwinger effect in inhomogeneous fields

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The Schwinger effect

The Schwinger effect Borel summation in physics Inhomogeneous field: Exact effective action Finite order extrapolation Conclusion

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The vacuum in QED is characterized by virtual fluctuations



These virtual processes have interesting consequences:

- The Casimir effect
- Light-by-light scattering
- The Schwinger effect

Status of experiments



Light-by-light scattering: evidence from quasi-real photons²

Schwinger effect: related positron production from multi-photon light-by-light scattering³

• Upcoming: LUXE at DESY and FACET-II at SLAC (lasers $+ e^-$ beams)

¹Mohideen, Umar, and Anushree Roy. Physical Review Letters 81.21 (1998)

²ATLAS collaboration. Nature physics 13, no. 9 (2017)

³Burke, D., et al. Physical Review Letters 79.9 (1997)

The Schwinger effect

The vacuum is unstable in the presence of an electric field



The field strengths required for this are astronomical

$$2eE\frac{\hbar}{mc}\approx 2mc^2 \quad \Rightarrow \quad E\approx 10^{18}\,{\rm V/m}$$

- Experimental fields will be strongly inhomogeneous (XFEL facilities at SLAC, DESY), and so the effects of inhomogeneities must be understood.
- Prevailing computational techniques rely on approximations (e.g. LCFA) which are known to be insufficient.

This is an open problem that requires new ideas.

Constant fields

These quantum corrections to Maxwell theory are encoded in the effective action, $\Gamma[A]$

$$\langle 0_{\rm out} | 0_{\rm in} \rangle = \exp\left(\frac{i}{\hbar}\Gamma[A]\right)$$

The imaginary part gives the pair-production rate

$$P_{e^+e^-} = 1 - \left| \langle 0_{\rm out} | 0_{\rm in} \rangle \right|^2 \approx \frac{2}{\hbar} \, \mathrm{Im} \, \Gamma$$

Euler, Heisenberg (1936): constant electromagnetic fields

$$\frac{\Gamma_{\rm EH}(B)}{V_4} = -\frac{e^2 B^2}{8\pi^2} \int_0^\infty \frac{\mathrm{d}s}{s^2} \left(\coth s - \frac{1}{s} - \frac{s}{3}\right) e^{-m^2 s/(eB)}$$
$$\frac{\Gamma_{\rm EH}(E)}{V_4} = -\frac{e^2 E^2}{8\pi^2} \int_0^\infty \frac{\mathrm{d}s}{s^2} \left(\cot s - \frac{1}{s} + \frac{s}{3}\right) e^{-m^2 s/(eE)}$$

Pair-production rate

The residues at the poles give the imaginary part

$$\frac{\mathrm{Im}\,\Gamma_{\mathrm{EH}}(E)}{V_4} = \frac{e^2 E^2}{8\pi^3} \sum_{k=1}^{\infty} \frac{1}{k^2} \exp\left(-\frac{k\pi m^2}{eE}\right)$$



This is a non-perturbative phenomenon

Non-constant fields: a single pulse

Exactly solvable *inhomogeneous* field configurations⁴:

$$B(x) = B \operatorname{sech}^{2}\left(\frac{x}{\lambda}\right), \qquad E(t) = E \operatorname{sech}^{2}\left(\frac{t}{\tau}\right)$$

Natural dimensionless parameter: $\gamma = \frac{m}{eB\lambda}$ or $\gamma = \frac{m}{eE\tau}$



Connected by simultaneous rotations: $B \mapsto \pm iE$ and $\lambda \mapsto \mp i\tau$

⁴Nikishov, A. I. Nuclear Physics B 21.2 (1970)

Standard approximations

The structure of the exact imaginary part is difficult to probe

Locally constant field approximation (LCFA):

$$\frac{\mathrm{Im}\,\Gamma}{V_3}\bigg|_{\mathrm{LCFA}} \sim \tau \frac{e^2 E^2}{8\pi^3} \bigg(\frac{eE}{m^2}\bigg)^{1/2} \sum_{k=1}^{\infty} \frac{1}{k^{5/2}} \exp\bigg(-\frac{k\pi m^2}{eE}\bigg) \sum_{n=0}^{\infty} c_n \bigg(\frac{eE}{k\pi m^2}\bigg)^n$$

WKB approximation:

$$\frac{\mathrm{Im}\,\Gamma}{V_3}\Big|_{\mathrm{WKB}} = \tau \frac{e^2 E^2}{8\pi^3} \left(\frac{eE}{m^2}\right)^{1/2} (1+\gamma^2)^{5/4} \exp\left(-\frac{\pi m^2}{eE}\frac{2}{\sqrt{1+\gamma^2}+1}\right)$$

It's natural to consider ${\rm Im}\,\Gamma$ at fixed $\gamma=\frac{m}{eE\tau}$ $\$ ('t Hooft coupling)

Pair-production from perturbation theory



New idea: use resurgence to extract this pair-production rate from a weak-field expansion of the *magnetic* effective action, $\Gamma(B, \lambda)$

$$\frac{\Gamma(B,\lambda)}{V_3} \sim \lambda \frac{m^4}{3\pi^2} \sum_{n=0}^{\infty} a_n(\gamma) \left(\frac{eB}{\pi m^2}\right)^{2n+4} \qquad \left(\frac{eB}{m^2} \to 0\right)$$

Borel summation in physics

The Schwinger effect Borel summation in physics Inhomogeneous field: Exact effective a Finite order extrapolation

Summing divergent series

Series in perturbation theory are generically asymptotic

$$f(g) \sim \sum_{n=0}^{\infty} a_n g^n \quad (g \to 0), \qquad a_n \sim n! \quad (n \to \infty)$$

• Zero radius of convergence

Borel (1899): reconstruct functions from their asymptotic data

$$\sum_{n=0}^{\infty} a_n g^n \quad \to \quad \underbrace{\mathcal{B}(s) = \sum_{n=0}^{\infty} \frac{a_n}{n!} s^n}_{\text{Borel transform}} \quad \to \quad \underbrace{\mathcal{S}[f](g) = \int_0^{\infty} \mathrm{d}s \, e^{-s} \mathcal{B}(sg)}_{\text{Borel sum}}$$

- $\cdot \ \mathcal{B}(s)$ has a non-zero radius of convergence \rightarrow singularities
- \cdot Borel singularities \iff non-perturbative physics

If $\mathcal{B}(s)$ has no singularities on the positive real axis, then f(g) is **Borel summable**

 $\mathcal{S}[f](g)=f(g)$

If $\mathcal{B}(s)$ has singularities on the positive real axis, then f(g) is non-Borel summable \rightarrow our resummation is ambiguous



Resurgence

E.g. a single pole: $a_n \sim \sigma A^{-n} n!$

$$\mathcal{B}(s) = rac{o}{1 - (s/A)} \quad o \quad \mathcal{S}[f](g) \sim \operatorname{Re} \mathcal{S}[f] \pm i\sigma \pi e^{-A/g}$$

E.g. a single branch point: $a_n \sim \sigma A^{-n} n! \left(1 + \frac{b_1}{n} + \frac{b_2}{n(n-1)} + \ldots \right)$

$$\mathcal{S}[f](g) \sim \operatorname{Re} \mathcal{S}[f] \pm i\sigma \pi e^{-A/g} \Big(1 + \frac{b_1 g}{g} + \frac{b_2 g^2}{g^2} + \dots \Big)$$

Large-order growth of $g^n \iff$ low-order fluctuations of $e^{-1/g}$ Observables are represented by **resurgent trans-series**

$$f(g) \sim \sum_{n} a_n g^n \implies f(g) = \sum_{nkl} a_{nkl} g^n \left(e^{-A/g} \right)^k \left(\ln\left(-g\right) \right)^l$$

Resurgence: all non-pert. physics is encoded in the $\{a_n\}$

How to fix an ambiguity?

Idea: use physical knowledge of the system to fix the ambiguity



Constant fields:

$$\frac{\Gamma_{\rm EH}(E)}{V_4} = -\frac{e^2 E^2}{8\pi^2} \int_0^\infty \frac{\mathrm{d}s}{s^2} \left(\cot s - \frac{1}{s} + \frac{s}{3}\right) e^{-m^2 s/(eE)} \qquad \text{Borel sum!}$$

$$\frac{\mathrm{Im}\,\Gamma_{\rm EH}(E)}{V_4} = +\frac{e^2 E^2}{8\pi^3} \sum_{k=1}^\infty \frac{1}{k^2} \exp\left(-\frac{k\pi m^2}{eE}\right)$$

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Finite perturbative knowledge

Question: what if we only know the first N perturbative coefficients?

• Ratio tests: $a_n \sim \sigma A^{-n} n! (1 + \ldots) \qquad \checkmark$

• Borel transform:
$$\mathcal{B}_N(s) = \sum_{n=0}^{N-1} \frac{a_n}{n!} s^n$$
 !!! no singularities!
• Borel sum: $\mathcal{S}_N[f](g) \sim \sum_{n=0}^{N-1} a_n g^n$ X no progress

First step: Padé approximation (poles and zeros accumulate to branch points)

$$\mathcal{P}[\mathcal{B}_N](s) = \frac{P_{N/2}(s)}{Q_{N/2}(s)} \qquad \xrightarrow{10}_{-2} \xrightarrow{10}_{-10} \xrightarrow{10}_{-10} \xrightarrow{10}_{-10}$$

Inhomogeneous field: Exact effective action

The Schwinger effect Borel summation in physics Inhomogeneous field: Exact effective action Finite order extrapolation Conclusion

Single peak magnetic field configuration^{5,6}

$$\begin{split} \frac{\Gamma(B,\lambda)}{V_3} &= \lambda \frac{m^4}{3\pi^2} \int_0^\infty \frac{\mathrm{d}s}{e^{\pi m^2 s/(eB)} - 1} \\ &\times \left[\frac{\mathrm{d}z}{\mathrm{d}s} \left(1 - \frac{4z}{3} + \frac{z^2}{5} \,_2 F_1 \big(1, 1, \frac{7}{2}; z \big) \right) - (s \to -s) - \left(\frac{8}{3}s - 2\gamma \left(1 + \frac{\gamma^2}{4}s^2 \right)^{3/2} \mathrm{arcsinh} \left(\frac{\gamma}{2}s \right) \right) \right] \\ &z = -is - \frac{\gamma^2}{4}s^2 \end{split}$$

Weak-field expansion:



⁵Dunne, Gerald, and Theodore M. Hall. Physics Letters B 419.1-4 (1998)

⁶Cangemi, Daniel, Eric D'Hoker, and Gerald Dunne. Physical Review D 52.6 (1995)

Modified Borel transform

Bose-like factor:

$$\frac{1}{e^{\pi m^2 s/(eB)}-1} = \sum_{k=1}^{\infty} e^{-k\pi m^2 s/(eB)} \qquad \text{``sum over instantons''}$$

Euler-Heisenberg can be written in this form

$$\frac{\Gamma_{\rm EH}(B)}{V_4} = \frac{e^2 B^2}{4\pi^4} \sum_{k=1}^{\infty} \frac{1}{k^2} \int_0^\infty \mathrm{d}s \, e^{-k\pi m^2 s/(eB)} \frac{s}{s^2 + 1}$$

"Modified Borel transform" (only one set of singularities)

Inhomogeneous case:

$$\mathcal{B}(s,\gamma) = \frac{\mathrm{d}z}{\mathrm{d}s} \left(1 - \frac{4z}{3} + \frac{z^2}{5} \,_2 F_1\left(1, 1, \frac{7}{2}; z\right) \right) - (s \to -s) - \left(\frac{8}{3}s - 2\gamma \left(1 + \frac{\gamma^2}{4}s^2 \right)^{3/2} \mathrm{arcsinh}\left(\frac{\gamma}{2}s\right) \right)$$

Inhomogeneous B field: new Borel singularities

Three pairs of complex conjugate branch points:



The inhomogeneity causes new singularities to appear!

Darboux's theorem

Sub-leading singularities would give rise to sub-dominant exponential contributions

$$\frac{\operatorname{Im}\Gamma(E,\tau)}{V_3} \sim \sum_{i=1}^{3} \sigma_i(\gamma) \left[\sum_{n=0}^{\infty} \sum_{k=1}^{\infty} c_{nk}^{(i)}(\gamma) \left(\frac{eE}{\pi m^2}\right)^{n+\delta} \left(e^{-\pi m^2 |s_i|/(eE)}\right)^k \right]$$

How to determine $\sigma_i(\gamma)$, $c_{nk}^{(i)}(\gamma)$, and $\delta? \implies$ Darboux's theorem



Exact large-order growth

Expansion about the origin:

$$\mathcal{B}(s,\gamma) = \sum_{n=0}^{\infty} \frac{a_n(\gamma)}{\Gamma(2n+4)\zeta(2n+4)} s^{2n+3}$$

Expansion about each singularity:

$$\mathcal{B}(s,\gamma) = \left(1 - \frac{s}{s_i}\right)^{3/2} \sum_{n=0}^{\infty} b_n^{(i)}(\gamma)(s - s_i)^n + \text{analytic}$$

Large-order growth of coefficients: (non-trivial function of γ !)

$$\frac{a_n(\gamma)}{\Gamma(2n+4)\zeta(2n+4)} \sim 2\sum_{l=0}^{\infty} (-1)^l \binom{2n+\frac{1}{2}-l}{-\frac{5}{2}-l} \left(\frac{b_l^{(1)}(\gamma)}{s_1^{2n+3-l}} + \frac{b_l^{(2)}(\gamma)}{s_2^{2n+3-l}} + \frac{b_3^{(1)}(\gamma)}{s_3^{2n+3-l}}\right)$$

Do these new singularities really contribute?



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Exact large-order growth \rightarrow full trans-series structure of Im $\Gamma(E, \tau)$

$$\frac{\operatorname{Im} \Gamma(E,\tau)}{V_3} \sim \sum_{i=1}^{3} \sigma_i(\gamma) \Biggl[\sum_{n=0}^{\infty} \sum_{k=1}^{\infty} c_{nk}^{(i)}(\gamma) \Biggl(\frac{eE}{\pi m^2} \Biggr)^{n+\delta} \Biggl(e^{-\pi m^2 |s_i|/(eE)} \Biggr)^k \Biggr]$$

$$\sigma_1(\gamma) = \sigma_2(\gamma) = \frac{\tau m^4}{8\sqrt{\pi}} (1+\gamma^2)^{5/4}, \qquad \sigma_3(\gamma) = -\frac{\tau m^4}{4\sqrt{\pi}} \gamma^{5/2}$$

$$c_{nk}^{(j)}(\gamma) = \frac{i^n}{k^{n+5/2}} \frac{\Gamma\left(n + \frac{5}{2}\right)}{\Gamma\left(\frac{5}{2}\right)} \left(\frac{b_n^{(j)}(\gamma)}{b_0^{(j)}(\gamma)}\right)$$
$$\delta = \frac{5}{2}$$

Resurgence confirmed to all orders!

• Perturbative coefficients encode all non-perturbative physics

LCFA and WKB approximations

$$\frac{\operatorname{Im} \Gamma(E,\tau)}{V_3} \sim \sum_{i=1}^3 \sigma_i(\gamma) \Biggl[\sum_{n=0}^\infty \sum_{k=1}^\infty c_{nk}^{(i)}(\gamma) \Biggl(\frac{eE}{\pi m^2} \Biggr)^{n+5/2} \Biggl(e^{-\pi m^2 |s_i|/(eE)} \Biggr)^k \Biggr]$$

 $\underline{\mathsf{LCFA}}: \, \mathsf{set} \, \gamma = 0 \quad \Longrightarrow \quad s_1(0) = 1, \quad s_2(0), s_3(0) \to \infty$

$$\tau \frac{e^2 E^2}{8\pi^3} \left(\frac{eE}{m^2}\right)^{1/2} \sum_{k=1}^{\infty} \frac{1}{k^{5/2}} \exp\left(-\frac{k\pi m^2}{eE}\right) \sum_{n=0}^{\infty} c_{nk}^{(1)}(0) \left(\frac{eE}{k\pi m^2}\right)^n \quad \checkmark$$

<u>WKB</u>: consider only the closest singularity and its leading fluctuation

$$\tau \frac{e^2 E^2}{8\pi^3} \left(\frac{eE}{m^2}\right)^{1/2} (1+\gamma^2)^{5/4} \exp\left(-\frac{\pi m^2}{eE} \frac{2}{\sqrt{1+\gamma^2}+1}\right) \quad \checkmark$$

Full perturbative knowledge \rightarrow full non-perturbative knowledge

Finite order extrapolation

The Schwinger effect Borel summation in physics Inhomogeneous field: Exact effective action Finite order extrapolation Conclusion Assuming no knowledge of the exact effective action, we start with *N* terms in perturbation theory

$$\frac{\Gamma(B,\lambda)}{V_3} \sim \lambda \frac{m^4}{3\pi^2} \sum_{n=0}^{N-1} a_n(\gamma) \left(\frac{eB}{\pi m^2}\right)^{2n+4}$$

With ~ 10 terms, we can deduce the exact leading growth



Singularity structure

Borel transform:
$$\mathcal{B}_N(s,\gamma) = \sum_{n=0}^{N-1} \frac{a_n(\gamma)}{\Gamma(2n+2)} (|s_1|s)^{2n+2}$$

Padé-approximant: $\mathcal{P}[\mathcal{B}_N](s,\gamma) = \frac{P_N(s,\gamma)}{Q_N(s,\gamma)}$ $(\gamma = 1)$



Conformal map: looking under the cut(s)

Conformal map: separate out branch cuts⁷

$$s = \frac{2w}{1 - w^2} \quad \iff \quad w = \frac{s}{\sqrt{1 + s^2} + 1}$$

At large γ we see the new singularities



⁷Costin, Ovidiu, and Gerald V. Dunne. Physics Letters B 808 (2020).

(N = 50)

Extrapolation from weak field to strong field

Weak field:
$$\frac{\Gamma(B,\lambda)}{V_3} \sim \lambda \frac{m^4}{3\pi^2} \sum_{n=0}^{\infty} a_n(\gamma) \left(\frac{eB}{\pi m^2}\right)^{2n+1} \qquad \left(\frac{eB}{m^2} \to 0\right)$$

Strong field: $\frac{\Gamma(B,\lambda)}{V_3} \sim \lambda \frac{m^4}{3\pi^2} \cdot \frac{1}{3} \left(\frac{eB}{m^2}\right)^2 \left[\ln\left(\frac{eB}{m^2}\right) + \ln(2\gamma) - 12\ln A + \frac{2}{3}\right] \qquad \left(\frac{eB}{m^2} \to \infty\right)$

 $2n \pm 4$

Borel sum:
$$\frac{\mathcal{S}[\Gamma_N](B,\lambda)}{V_3} = \lambda \frac{m^4}{3\pi^2} \left(\frac{eB}{\pi m^2}\right)^2 \int_0^\infty \frac{\mathrm{d}s}{s} e^{-\pi m^2 |s_1|s/(eB)} \mathcal{P}[\mathcal{B}_N](s,\gamma)$$



Pair-production rate

Borel sum:
$$\frac{\mathcal{S}_{\theta}[\Gamma_{N}](E,\tau)}{V_{3}} = -\tau \frac{m^{4}}{3\pi^{2}} \left(\frac{eE}{\pi m^{2}}\right)^{2} \int_{0}^{\infty e^{i\theta}} \frac{\mathrm{d}s}{s} e^{-\pi m^{2}|s_{1}|s/(eE)} \mathcal{P}[\mathcal{B}_{N}](-is,\gamma)$$
$$\operatorname{Im} \mathcal{S}[\Gamma_{N}](E,\tau) = \frac{1}{2} \left(\mathcal{S}_{0^{+}}[\Gamma_{N}](E,\tau) - \mathcal{S}_{0^{-}}[\Gamma_{N}](E,\tau)\right)$$



With only 10 terms, we recover the exact pair-production rate with remarkable precision

Conclusion

The Schwinger effect Borel summation in physics Inhomogeneous field: Exact effective action Finite order extrapolation **Conclusion**

Conclusion

- The Schwinger effect is a non-perturbative phenomenon which is difficult to probe for inhomogeneous fields even at one-loop order.
- Resurgence predicts that this information is encoded within the divergence of weak field expansions of the effective action. \checkmark
- Using only a modest amount of perturbative data, we can efficiently extract the strong field behavior and the pair-production rate with a higher precision than both the LCFA and WKB approximation.
- This technique offers a new approach to understand more general field configurations in QED, higher loop contributions, pair-production in dS/AdS backgrounds, etc.

Questions?



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Additional slides

Scattering processes

Nonlinear inverse Compton scattering (NICS)



Nonlinear Breit-Wheeler



LUXE (Laser Und XFEL Experiment) at DESY⁸

Sketch of experimental set-up for NICS



⁸Abramowicz, H., et al. "Letter of intent for the LUXE experiment." arXiv preprint arXiv:1909.00860 (2019).

LUXE (Laser Und XFEL Experiment) at DESY

Sketch of experimental set-up for nonlinear Breit-Wheeler



 $E_e = 17.5 \,\text{GeV}, \quad P_{\gamma} = 30 - 300 \,\text{TW}, \quad I = 10^{21} \,\text{W/cm}^2, \quad 30 \,\text{fs pulses}$

Experimental set-up for NICS



$$E_e = 13 \,\text{GeV}, \quad P_{\gamma} = 10 \,\text{TW}, \quad I = 10^{20} \,\text{W/cm}^2, \quad 30 \,\text{fs pulses}$$

⁹Meuren, Sebastian. "Probing strong-field qed at facet-ii (slac e-320)." Third conference on extremely high intensity laser physics (exhilp). Vol. 7. 2019.