CFT in Momentum Space, Anomalies, and the Nonlocal Conformal Anomaly Action

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Based on recent works with
S. Lionetti, R. Tommasi, M. Creti', M. Maglio,

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Abstract

Momentum space methods in CFT allow to describe quite efficiently the correlators containing insertions of stress energy tensor (T) and/or axial vector currents, and affected by conformal and chiral anomalies. (TTT, TTJ5, J5JJ,TTTT)

Analysis have been performed up to 4-point functions (4T).

The hierarchy of the conformal Ward identities (CWIs) constraining such correlation functions have been investigated using both free field theory realizations and, nonperturbatively, using their CWIs

By this approach it has also been shown the inconsistency of anomaly induced actions in the Riegert and in the Fradkin-Vilkovisky beyond 3-point functions. Corrections identified for a specific correlator (TTJJ)

We will overview the methodology and the main results in this area, and the central role played by anomaly poles in determining the structure of these interactions.

Yangian Symmetry in momentum space (Maglio, CC)

- *JHEP* 09 (2019) 107 e-Print: 1903.05047

On Some Hypergeometric Solutions of the Conformal Ward Identities of Scalar 4-point Functions in Momentum Space

QUADRATIC CORRECTIONS TO GRAVITY ASSOCIATED WITH THE TRACE ANOMALY

$$\mathcal{S}_2 = \int d^dx \left((R_{\mu\nu\rho\sigma})^2 + a(R_{\mu\nu})^2 + bR^2\right),$$

$$\mathcal{S}_2^{(2)} = \frac{1}{4}\int d^dx\sqrt{g}\left((a+4)h^{\mu\nu}\Box^2 h_{\mu\nu} + (b-1)h\Box^2 h\right),$$

LOVELOCK

$$E_n = \frac{1}{2^{d/2}}\,\delta^{\nu_1\nu_2\ldots\nu_d}_{\mu_1\mu_2\ldots\mu_d}\,R^{\mu_1\mu_2}{}_{\nu_1\nu_2}R^{\mu_3\mu_4}{}_{\nu_3\nu_4}\ldots R^{\mu_{d-1}\mu_d}{}_{\nu_{d-1}\nu_d},$$

$$S = \int d^dx\sqrt{g}\sum_{n=0}^{[d/2]}\alpha_{2n}E_{2n}.$$

0/0 LIMIT

$$\mathcal{S}_{EGB} = \mathcal{S}_{EH} + \mathcal{S}_{GB}(d) \qquad \mathcal{S}_{GB}(d) = \frac{\alpha}{\epsilon}V_E(d)$$

$$\begin{aligned} E_0 &= 1, \\ E_2 &= R, \\ E_4 &= R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2, \end{aligned}$$

TOPOLOGICAL TERMS CAN BE RENDERED DYNAMICS AT D=4, D=6,
VIA A 0/0 PROCEDURE.

He result is a form od dilaton gravity, that can be rendered nonlocal by removing the dilaton.

$$g_{\mu\nu} = e^{2\phi(x)}\bar{g}_{\mu\nu}$$

PoS, CC 2023

$$\sqrt{g}E = \sqrt{\bar{g}}e^{(d-4)\phi}\left\{\bar{E} + (d-3)\bar{\nabla}_\mu \bar{J}^\mu(\bar{g},\phi) + (d-3)(d-4)\bar{K}(\bar{g},\phi)\right\},$$

$$\bar{K}(\bar{g},\phi) = 4\bar{R}^{\mu\nu}\bar{\nabla}_\mu\phi\bar{\nabla}_\nu\phi - 2\bar{R}\bar{\nabla}_\lambda\phi\bar{\nabla}^\lambda\phi + 4(d-2)\bar{\Box}\phi\bar{\nabla}_\lambda\phi\bar{\nabla}^\lambda\phi + (d-1)(d-2)(\bar{\nabla}_\lambda\phi\bar{\nabla}^\lambda\phi)^2$$

$$\begin{aligned}\hat{V}'_E(g,\phi) &\equiv \lim_{\epsilon\to 0}\left(\frac{1}{\epsilon}\left(V_E(g,d) - V_E(\bar{g},d)\right)\right) \\ &= \int d^4x\sqrt{g}\Big[\phi_4 E - (4G^{\mu\nu}(\bar{\nabla}_\mu\phi\bar{\nabla}_\nu\phi) + 2(\nabla_\lambda\phi\nabla^\lambda\phi)^2 + 4\Box\phi\nabla_\lambda\phi\nabla^\lambda\phi)\Big],\end{aligned}$$

Few phenomenological facts about anomalies

Anomalies are quantum violation of classical conservation laws.

For instance, **for chiral anomalies**, they are related to the presence of chiral interactions **that need to be canceled** in the case of chiral gauge theories such as the Standard Model, but they are perfectly fine for currents associated with global symmetries.

A candidate for dark matter, the axion, comes from the spontaneous breaking of a global U(1) (PQ) symmetry at a large scale, **with a physical Nambu Goldstone mode, whose potential is slightly tilted** (a vacuum misalignment) at the QCD confinement phase transition scale, through instanton effects.

Anomalies can be characterised both by topological and non-topological contributions.
For example, **a chiral anomaly is topological** (Pontryagin density) a conformal anomaly is related both to topological (Euler Poincare' density) and to non topological terms (Weyl tensor squared)

The most important dynamical character of the anomaly, from the point of view of an anomaly amplitude, appears in momentum space and is associated with **anomaly poles**.
Anomaly cancelation can be interpreted as cancelation of anomaly poles of a certain interaction.
The interaction mediated by the anomaly pole is, obviously, nonlocal in cooordinate space

$$J_5^\lambda = \bar{\psi}\gamma^\lambda\gamma_5\psi \qquad J^\mu = \bar{\psi}\gamma^\mu\psi$$

Axial-vector and Vector currents
Conserved at classical level

$$\partial_\lambda J_5^\lambda = 0 \qquad \partial_\mu J^\mu = 0 \qquad \partial_\lambda \langle J_5^\lambda \rangle \neq 0$$

At quantum level these conservation equations are violated.
A certain gauge global or local symmetry is violated at quantum level
in the presence of chiral fermions.

Various methods of computations of such anomalies. Quantum averages are computed
in the presence of background fields (gravitational and/or external gauge fields)

CHIRAL ANOMALY

$$\partial_\lambda \langle J_5^\lambda \rangle = a_n F\tilde{F}$$

Similar situation for other symmetries. Diffeomorphism invariance of a certain classical
action requires that the stress energy tensor is covariantly conserved.

$$T^{\mu\nu} \equiv \frac{2}{\sqrt{g}}\frac{\delta S}{\delta g_{\mu\nu}}$$

$$\nabla_\mu T^{\mu\nu} = 0$$

$$g_{\mu\nu}T^{\mu\nu} = 0. \qquad g_{\mu\nu}\langle T^{\mu\nu}\rangle \neq 0.$$

This is a requirement that should
be respected all the time.
But if the action has a conformal symmetry,
then its trace should vanish at classical level.

There can be a Trace anomaly $\quad g_{\mu\nu}\langle T^{\mu\nu}\rangle = \beta(g)FF$

In the presence of a classical external gravitational field, the anomaly functionals include other contributions

$$\nabla_\mu \langle J_5^\mu \rangle = a_1\, \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} + a_2\, \varepsilon^{\mu\nu\rho\sigma} R^{\alpha\beta}{}_{\mu\nu} R_{\alpha\beta\rho\sigma}.$$

$$C^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma} = R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - 2 R^{\mu\nu} R_{\mu\nu} + \frac{1}{3} R^2,$$

$$E_4 \equiv E = R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - 4 R^{\mu\nu} R_{\mu\nu} + R^2.$$

$$g_{\mu\nu} \langle T^{\mu\nu} \rangle = b_1\, E_4 + b_2\, C^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma} + b_3\, \nabla^2 R + b_4\, F^{\mu\nu} F_{\mu\nu},$$

Euler Poincare’ and
Weyl Tensor

These constraints generate, by a functional expansion of the quantum averages wrt the external fileds,
An infinite set of anomalous Ward identities which are hierarchical.

In the case of a trace anomaly, when the classical conformal symmetry of the action is violated at quantum level, we derive an infinite set of conformal Ward identities (CWIs) that constrain these correlators.

Notice that the one can formulated the breaking of conformal symmetry as a violation of Weyl invariance of a certain action. In each free falling frame, one reobtains, one recovers the ordinary conformal WIs of flat space

Axion E&M

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{1}{4}\widetilde{g}\,\varphi\,F_{\mu\nu}\widetilde{F}^{\mu\nu},$$

$$\Box\varphi - \frac{\widetilde{g}}{4}F^{\mu\nu}\widetilde{F}_{\mu\nu} = 0.$$

Modified Maxwell's equations

$$\Box\varphi = -\widetilde{g}\,\boldsymbol{E}\cdot\boldsymbol{B}.$$

$$\Box(\boldsymbol{E} - \frac{1}{2}\widetilde{g}\varphi\boldsymbol{B}) = -\frac{1}{2}\widetilde{g}\varphi\,\Box\boldsymbol{B},$$

$$\Box(\boldsymbol{B} + \frac{1}{2}\widetilde{g}\varphi\boldsymbol{E}) = \frac{1}{2}\widetilde{g}\varphi\,\Box\boldsymbol{E}.$$

$$\cdot\,\boldsymbol{B} = 0,$$

$$\begin{cases} \frac{\partial \boldsymbol{B}}{\partial t} + \nabla\times\boldsymbol{E} = 0, \\ \Box\varphi = -\widetilde{g}\,\boldsymbol{E}\cdot\boldsymbol{B}, \\ \nabla\cdot\boldsymbol{E} = \widetilde{g}\,\nabla\varphi\cdot\boldsymbol{B}, \\ \nabla\times\boldsymbol{B} - \frac{\partial \boldsymbol{E}}{\partial t} = -\widetilde{g}\boldsymbol{B}\frac{\partial\varphi}{\partial t} + \widetilde{g}\boldsymbol{E}\times\nabla\varphi. \end{cases}$$

Similar effects are possible in gravity (Faraday rotations on GWs). (Creti, Tommasi, CC). gravitomagnetism

$$\Box(\boldsymbol{E} - \frac{1}{2}\widetilde{g}\varphi\boldsymbol{B}) = -\frac{1}{2}\widetilde{g}\varphi\,\Box\boldsymbol{B},$$
$$\Box(\boldsymbol{B} + \frac{1}{2}\widetilde{g}\varphi\boldsymbol{E}) = \frac{1}{2}\widetilde{g}\varphi\,\Box\boldsymbol{E}.$$

$$\boldsymbol{D} \equiv \boldsymbol{E} - \frac{1}{2}\widetilde{g}\varphi\boldsymbol{B}$$

$$\boldsymbol{H} \equiv \boldsymbol{B} + \frac{1}{2}\widetilde{g}\varphi\boldsymbol{E}.$$

Rotation of the polarization plane

$$\theta = \frac{1}{2}\widetilde{g}\Delta\varphi,$$

$$\Delta\boldsymbol{E} \equiv \boldsymbol{E}(L) - \boldsymbol{E}(0) = \frac{1}{2}\widetilde{g}\Delta\varphi\boldsymbol{H}(0).$$

$$\Delta\varphi \equiv \varphi(L) - \varphi(0).$$

$$g_{\mu\nu}\langle T^{\mu\nu}\rangle = \mathcal{A} = b_1\, E_4 + b_2\, C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} + b_3\, \nabla^2 R + b_4\, F^{\mu\nu}F_{\mu\nu} + f_1\, \varepsilon^{\mu\nu\rho\sigma} R_{\alpha\beta\mu\nu}R^{\alpha\beta}{}_{\rho\sigma} + f_2\, \varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma},$$

Parity even and parity odd terms in the trace anomaly conjectures since the 70's (Deser Isham Duff)
Source of parity violation generate an issue with unitarity in free field theory realizations (e.g. the Standard Model) due to the fact that the anomaly coefficients are imaginary.

Several subtle issues where identified since the 80's (M. Duff, P. van Nieuwenhuizen) concerning topological contributions to the anomaly in free field theories.

Equivalent field redefinitions at classical level generate different anomalies.

Important issues that need to be explored in the context of free CFT realizations in different dimensions.

$$g_{\mu\nu}\langle T^{\mu\nu}\rangle = (f_1\, \varepsilon^{\mu\nu\rho\sigma} R_{\alpha\beta\mu\nu}R^{\alpha\beta}{}_{\rho\sigma} + f_2\, \varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma},$$

Source of CP violaton in the Early universe

$$F\tilde{F} = kE\cdot B$$

E and B even under C, but E odd and B even under P > CP violation

The Standard Model, even with complex phases in the CKM matrix, is unable to generate significant sources of CP violation.

However, the perturbative realizations of correlators such as T5 JJ, seem to indicate that they are zero In free field theory (Armillis, Delle Rose, CC) , (Bastianelli, Chiese) , (Abdallah, Franchino-Vinas, Frob)

Issues with unitarity due to complex f's : They are non zero (L. Bonora et Al)

CWIs seem to indicate that they can be nonzero (Lionetti, maglio, CC) (more later)
It is not clear whether such CFT's, even for real f's are consistent.

QUESTION:
What happens when conformal symmetry is broken by an anomaly ?

Momentum space techniques
are the most effcient way to investigate correlation functions affected by an anomaly.

We will show that chiral and anomaly interactions can be completely determined by CFT + anomaly poles

Conformal field theory in momentum space and anomaly actions in gravity: The analysis of three- and four-point function
• *Phys.Rept.* 952 (2022) e-Print: 2005.06873 (Maglio, CC)

The conformal anomaly action to fourth order (4T) in *d*=4 in momentum space
•*Eur.Phys.J.C* 81 (2021) 8, 740 e-Print: 2103.13957 (Maglio, Theofilopoulos, CC)

Four-point functions of gravitons and conserved currents of CFT in momentum space: testing the nonlocal action with the TTJJ
•*Eur.Phys.J.C* 83 (2023) 5, 427 e-Print: 2212.12779
(Maglio, Tommasi, CC)

CFT Correlators and CP-Violating Trace Anomalies 2307.03038 [hep-th]
(Lionetti, Maglio, CC)

Topological corrections and conformal backreaction in the Einstein Gauss–Bonnet/Weyl theories of gravity at *D*=4
•*Eur.Phys.J.C* 82 (2022) 12, 1121 e-Print: 2203.04213
(Maglio, Theofilopoulos,CC)

Parity-odd 3-point functions from CFT in momentum space and the chiral anomaly, Lionetti, Maglio, CC)
•*Eur.Phys.J.C* 83 (2023) 6, 502 e-Print: 2303.10710

Einstein Gauss-Bonnet theories as ordinary, Wess-Zumino conformal anomaly actions
•*Phys.Lett.B* 828 (2022) 137020 e-Print: 2201.07515 (Maglio, CC)

Work in preparation with M. Creti, R. Tommasi, D. Melle (4T) and Lionetti, Maglio (J5TT)

$$\Delta^{\lambda\mu\nu}_{\mathbf{AVV}} = \Delta^{\lambda\mu\nu} = i^3 \int \frac{d^4q}{(2\pi)^4} \frac{Tr\left[\gamma^\mu(\not{q}+m)\gamma^\lambda\gamma^5(\not{q}-\not{k}+m)\gamma^\nu(\not{q}-\not{k}_1+m)\right]}{(q^2-m^2)[(q-k_1)^2-m^2][(q-k)^2-m^2]} + \text{exch.}$$

Feynman expansion

$$\Delta^{\lambda\mu\nu}_{\mathbf{AAA}} = \Delta^{\lambda\mu\nu}_3 = i^3 \int \frac{d^4q}{(2\pi)^4} \frac{Tr\left[\gamma^\mu\gamma^5(\not{q}+m)\gamma^\lambda\gamma^5(\not{q}-\not{k}+m)\gamma^\nu\gamma^5(\not{q}-\not{k}_1+m)\right]}{(q^2-m^2)[(q-k_1)^2-m^2][(q-k)^2-m^2]} + \text{exch.}$$

$$k_{1\mu}\Delta^{\lambda\mu\nu}(k_1,k_2) = a_1\epsilon^{\lambda\nu\alpha\beta}k_1^\alpha k_2^\beta$$
$$k_{2\nu}\Delta^{\lambda\mu\nu}(k_1,k_2) = a_2\epsilon^{\lambda\mu\alpha\beta}k_2^\alpha k_1^\beta$$
$$k_\lambda\Delta^{\lambda\mu\nu}(k_1,k_2) = a_3\epsilon^{\mu\nu\alpha\beta}k_1^\alpha k_2^\beta,$$

$$a_1 = -\frac{i}{8\pi^2} \qquad a_2 = -\frac{i}{8\pi^2} \qquad a_3 = -\frac{i}{4\pi^2}.$$

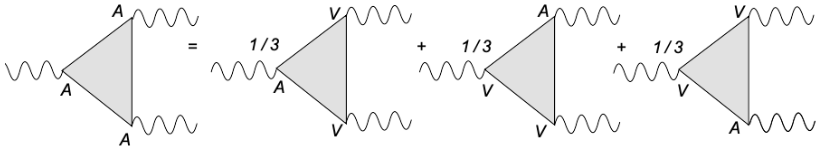
$$(a = \alpha(k_1+k_2)+\beta(k_1-k_2))$$

$$\Delta^{\lambda\mu\nu}(\beta,k_1,k_2) = \Delta^{\lambda\mu\nu}(k_1,k_2) - \frac{i}{4\pi^2}\beta\epsilon^{\lambda\mu\nu\sigma}\left(k_{1\sigma}-k_{2\sigma}\right).$$

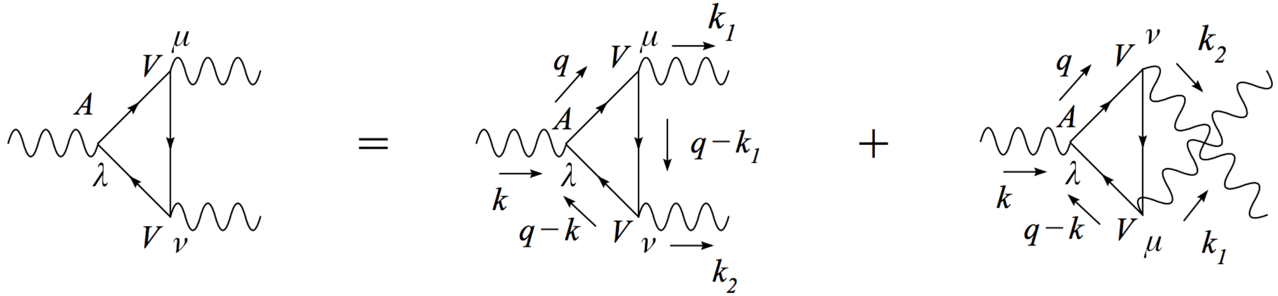
$$\begin{aligned} k_{1\mu}\Delta^{\lambda\mu\nu}(a,k_1,k_2) &= 0, \\ k_{2\nu}\Delta^{\lambda\mu\nu}(a,k_1,k_2) &= 0, \\ k_\lambda\Delta^{\lambda\mu\nu}(a,k_1,k_2) &= -\frac{i}{2\pi^2}\varepsilon^{\mu\nu\alpha\beta}k_1^\alpha k_2^\beta \end{aligned}$$

The external Wis determine the diagram. No need for Renormalization.

Signatures of Chiral and Conformal anomalies



AVV diagram



$$\Delta_0^{\lambda\mu\nu} = A_1(k_1,k_2)\varepsilon[k_1,\mu,\nu,\lambda] + A_2(k_1,k_2)\varepsilon[k_2,\mu,\nu,\lambda] + A_3(k_1,k_2)\varepsilon[k_1,k_2,\mu,\lambda]k_1{}^{\nu}$$
$$+ \; A_4(k_1,k_2)\varepsilon[k_1,k_2,\mu,\lambda]k_2^{\nu} + A_5(k_1,k_2)\varepsilon[k_1,k_2,\nu,\lambda]k_1^{\mu} + A_6(k_1,k_2)\varepsilon[k_1,k_2,\nu,\lambda]k_2^{\mu}.$$

If we change the parameterization of the loop momentum, A1 and A2 change.

$$A_3(k_1,k_2) = -A_6(k_2,k_1) = -16\pi^2 I_{11}(k_1,k_2),$$
$$A_4(k_1,k_2) = -A_5(k_2,k_1) = 16\pi^2\left[I_{20}(k_1,k_2) - I_{10}(k_1,k_2)\right],$$

Some are finite by power counting.
A1 and A2 are not

where the general massive I_{st} integral is defined by

$$I_{st}(k_1,k_2)=\int_0^1 dw\int_0^{1-w} dz w^s z^t \left[z(1-z)k_1^2+w(1-w)k_2^2+2wz(k_1k_2)-m^2\right]^{-1},$$

Impose vector Ward identities

Then A1 ans A2 are fixed without any renormalization

$$\begin{aligned} A_1(k_1,k_2) &= k_1\cdot k_2\, A_3(k_1,k_2)+k_2^2\, A_4(k_1,k_2), \\ A_2(k_1,k_2) &= k_1^2\, A_5(k_1,k_2)+k_1\cdot k_2\, A_6(k_1,k_2), \end{aligned}$$

No renormalization: Chiral anomalies are topological, similarly to the Euler density in the conformal anomaly

$$\begin{aligned} \frac{\partial}{\partial x^\mu}T^{\lambda\mu\nu}_{\mathbf{AVV}}(x,y,z) &= ia_1(\beta)\epsilon^{\lambda\nu\alpha\beta}\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial y^\beta}\left(\delta^4(x-z)\delta^4(y-z)\right), \\ \frac{\partial}{\partial y^\nu}T^{\lambda\mu\nu}_{\mathbf{AVV}}(x,y,z) &= ia_2(\beta)\epsilon^{\lambda\mu\alpha\beta}\frac{\partial}{\partial y^\alpha}\frac{\partial}{\partial x^\beta}\left(\delta^4(x-z)\delta^4(y-z)\right), \\ \frac{\partial}{\partial z^\lambda}T^{\lambda\mu\nu}_{\mathbf{AVV}}(x,y,z) &= ia_3(\beta)\epsilon^{\mu\nu\alpha\beta}\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial y^\beta}\left(\delta^4(x-z)\delta^4(y-z)\right), \end{aligned}$$

Anomalies come from ultralocal terms

$$A_1(s, s_1, s_2) = -\frac{i}{4\pi^2} + \frac{i}{8\pi^2\sigma}\left\{\Phi(s_1, s_2)\frac{s_1 s_2 (s_2 - s_1)}{s} + s_1 (s_2 - s_{12}) \log\left[\frac{s_1}{s}\right] - s_2 (s_1 - s_{12}) \log\left[\frac{s_2}{s}\right]\right\},$$

.

$$\Phi(x, y) = \frac{1}{\lambda}\left\{2[Li_2(-\rho x) + Li_2(-\rho y)] + \ln\frac{y}{x}\ln\frac{1+\rho y}{1+\rho x} + \ln(\rho x)\ln(\rho y) + \frac{\pi^2}{3}\right\},$$

Davydychev

$$\lambda(x, y) = \sqrt{\Delta}, \qquad \Delta = (1 - x - y)^2 - 4xy,$$

s1 and s2 are vector currents virtualities

$$\rho(x, y) = 2(1 - x - y + \lambda)^{-1}, \qquad x = \frac{s_1}{s}, \qquad y = \frac{s_2}{s}.$$

This is not the only parameterization. A second one is the **longitudinal/transverse (LT) decomposition**

$$W^{\lambda\mu\nu} = \frac{1}{8\pi^2}\left[W^{L\,\lambda\mu\nu} - W^{T\,\lambda\mu\nu}\right],$$

$$W^{L\,\lambda\mu\nu} = w_L\, k^{\lambda}\varepsilon[\mu,\nu,k_1,k_2]$$

De Rafael et al

developed in the study of g-2 of the muon

It corrects an erro r in the book by Kerson Huang on particle theory

Only the L part contributes to the Ward Identity

$$\begin{aligned} {W^T}_{\lambda\mu\nu}(k_1,k_2) \;&=\; w_T^{(+)}\left(k^2,k_1^2,k_2^2\right)\, t^{(+)}_{\lambda\mu\nu}(k_1,k_2) + w_T^{(-)}\left(k^2,k_1^2,k_2^2\right)\, t^{(-)}_{\lambda\mu\nu}(k_1,k_2) \\ &+ \widetilde{w}_T^{(-)}\left(k^2,k_1^2,k_2^2\right)\, \widetilde{t}^{(-)}_{\lambda\mu\nu}(k_1,k_2), \end{aligned}$$

$$\begin{aligned} t^{(+)}_{\lambda\mu\nu}(k_1,k_2) \;&=\; k_{1\nu}\,\varepsilon[\mu,\lambda,k_1,k_2] - k_{2\mu}\,\varepsilon[\nu,\lambda,k_1,k_2] - (k_1\cdot k_2)\,\varepsilon[\mu,\nu,\lambda,(k_1-k_2)] \\ &\quad + \frac{k_1^2+k_2^2-k^2}{k^2}\, k_\lambda\, \varepsilon[\mu,\nu,k_1,k_2]\,, \\ t^{(-)}_{\lambda\mu\nu}(k_1,k_2) \;&=\; \left[(k_1-k_2)_\lambda - \frac{k_1^2-k_2^2}{k^2}\, k_\lambda\right] \varepsilon[\mu,\nu,k_1,k_2] \\ \widetilde{t}^{(-)}_{\lambda\mu\nu}(k_1,k_2) \;&=\; k_{1\nu}\,\varepsilon[\mu,\lambda,k_1,k_2] + k_{2\mu}\,\varepsilon[\nu,\lambda,k_1,k_2] - (k_1\cdot k_2)\,\varepsilon[\mu,\nu,\lambda,k]. \end{aligned}$$

Tensor structures involved
In the LT parameterization

Topology comes through the Schouten relations

Armlllis, Delle Rose, CC

$$A_3(k_1,k_2) = \frac{1}{8\pi^2}\left[w_L - \tilde{w}_T^{(-)} - \frac{k^2}{(k_1+k_2)^2} w_T^{(+)} - 2\,\frac{k_1\cdot k_2 - k_2^2}{k^2} w_T^{(-)}\right],$$

$$A_4(k_1,k_2) = \frac{1}{8\pi^2}\left[w_L + 2\,\frac{k_1\cdot k_2}{k^2} w_T^{(+)} - 2\,\frac{k_1\cdot k_2 + k_2^2}{k^2} w_T^{(-)}\right],$$

$$A_5(k_1,k_2) = -A_4(k_2,k_1), \qquad A_6(k_1,k_2) = -A_3(k_2,k_1),$$

$$w_L(k^2,\, k_1^2,\, k_2^2) = \frac{8\pi^2}{k^2}\left[A_1 - A_2\right],$$

$$w_L(k^2,\, k_1^2,\, k_2^2) = \frac{8\pi^2}{k^2}\left[(A_3 - A_6) k_1\cdot k_2 + A_4\, k_2^2 - A_5\, k_1^2\right],$$

$$w_T^{(+)}(k^2,\, k_1^2,\, k_2^2) = -4\pi^2\,(A_3 - A_4 + A_5 - A_6),$$

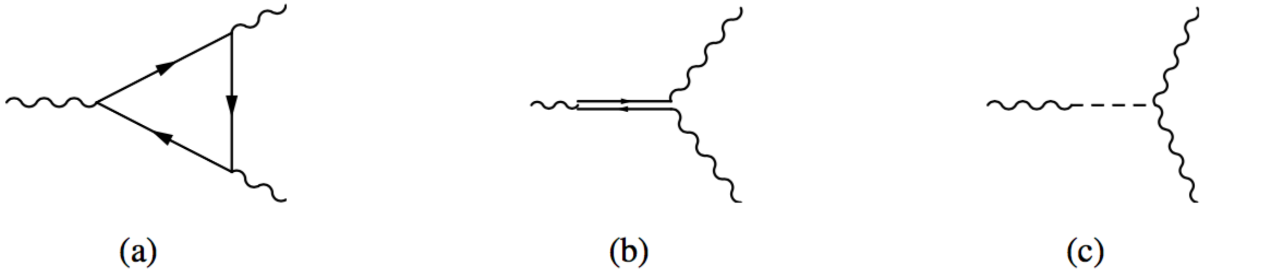
$$w_T^{(-)}(k^2,\, k_1^2,\, k_2^2) = 4\pi^2\,(A_4 + A_5),$$

$$\tilde{w}_T^{(-)}(k^2,\, k_1^2,\, k_2^2) = -4\pi^2\,(A_3 + A_4 + A_5 + A_6),$$

Notice that if you change the parameterization of The momentum in the loop, A1 and A2 will shift by the same amount, but WL will not change.

Notice the presence of a single pole in the Longitudinal component of the AVV diagram.

$$
\begin{aligned}
w_L(s_1, s_2, s) &= -\frac{4i}{s} \\
w_T^{(+)}(s_1, s_2, s) &= i\frac{s}{\sigma} + \frac{i}{2\sigma^2}\Big[(s_{12} + s_2)(3s_1^2 + s_1(6s_{12} + s_2) + 2s_{12}^2)\log\frac{s_1}{s} \\
&+ (s_{12} + s_1)(3s_2^2 + s_2(6s_{12} + s_1) + 2s_{12}^2)\log\frac{s_2}{s} \\
&+ s(2s_{12}(s_1 + s_2) + s_1 s_2(s_1 + s_2 + 6s_{12}))\Phi(s_1, s_2)\Big]
\end{aligned}
$$



The triangle diagram in the fermion case (a), the collinear fermion configuration responsible for the anomaly (b) and a diagrammatic representation of the exchange via an intermediate state (dashed line) (c).

The signature of the chiral anomaly is in the the generation of 1 pole in the axial vector channel

We consider the standard QED lagrangian

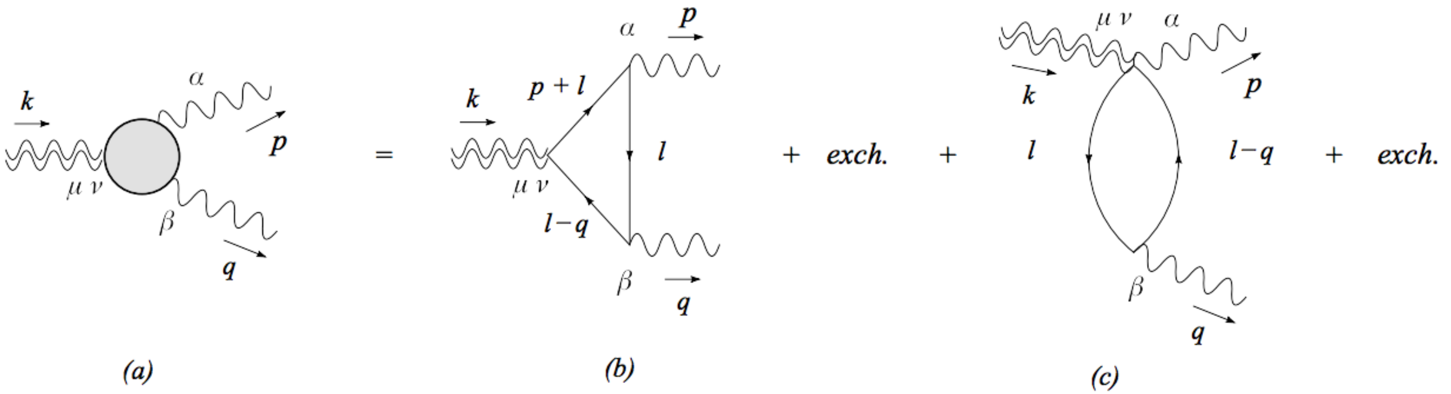
$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\,\bar{\psi}\gamma^{\mu}(\partial_{\mu} - i\,e\,A_{\mu})\psi - m\bar{\psi}\psi,$$

$$T^{\mu\nu} \equiv T^{\mu\nu}_{f} + T^{\mu\nu}_{fp} + T^{\mu\nu}_{ph}$$

$$T^{\mu\nu}_{f} = -i\bar{\psi}\gamma^{(\mu}\overleftrightarrow{\partial}^{\nu)}\psi + g^{\mu\nu}(i\bar{\psi}\gamma^{\lambda}\overleftrightarrow{\partial}_{\lambda}\psi - m\bar{\psi}\psi),$$

$$T^{\mu\nu}_{fp} = -\,eJ^{(\mu}A^{\nu)} + eg^{\mu\nu}J^{\lambda}A_{\lambda}\,,$$

$$T^{\mu\nu}_{ph} = F^{\mu\lambda}F^{\nu}{}_{\lambda} - \frac{1}{4}g^{\mu\nu}F^{\lambda\rho}F_{\lambda\rho},$$

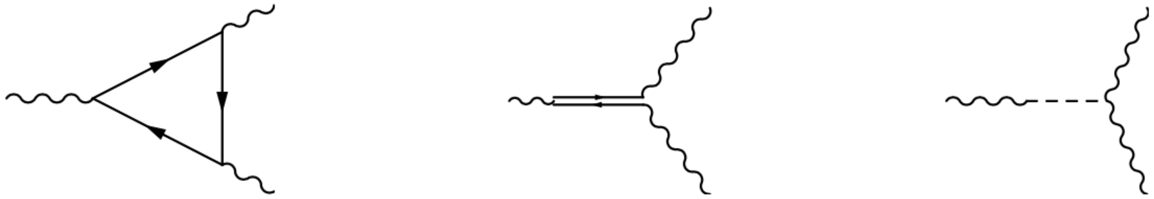


$$\begin{aligned}\langle T^{\mu\nu}_{p}(z)\rangle_{A} \;&\equiv\; \int D\psi D\bar{\psi}\; T^{\mu\nu}_{p}(z)\; e^{i\int d^4x\mathcal{L} + \int J\cdot A(x)d^4x} \\ &=\; \langle T^{\mu\nu}_{p}\, e^{i\int d^4x\, J\cdot A(x)}\rangle\end{aligned}$$

i	$t_i^{\mu\nu\alpha\beta}(p,q)$
1	$\left(k^2 g^{\mu\nu} - k^\mu k^\nu\right) u^{\alpha\beta}(p.q)$
2	$\left(k^2 g^{\mu\nu} - k^\mu k^\nu\right) w^{\alpha\beta}(p.q)$
3	$\left(p^2 g^{\mu\nu} - 4p^\mu p^\nu\right) u^{\alpha\beta}(p.q)$
4	$\left(p^2 g^{\mu\nu} - 4p^\mu p^\nu\right) w^{\alpha\beta}(p.q)$
5	$\left(q^2 g^{\mu\nu} - 4q^\mu q^\nu\right) u^{\alpha\beta}(p.q)$
6	$\left(q^2 g^{\mu\nu} - 4q^\mu q^\nu\right) w^{\alpha\beta}(p.q)$
7	$\left[p\cdot q\, g^{\mu\nu} - 2(q^\mu p^\nu + p^\mu q^\nu)\right] u^{\alpha\beta}(p.q)$
8	$\left[p\cdot q\, g^{\mu\nu} - 2(q^\mu p^\nu + p^\mu q^\nu)\right] w^{\alpha\beta}(p.q)$
9	$\left(p\cdot q\, p^\alpha - p^2 q^\alpha\right)\left[p^\beta\left(q^\mu p^\nu + p^\mu q^\nu\right) - p\cdot q\left(g^{\beta\nu}p^\mu + g^{\beta\mu}p^\nu\right)\right]$
10	$\left(p\cdot q\, q^\beta - q^2 p^\beta\right)\left[q^\alpha\left(q^\mu p^\nu + p^\mu q^\nu\right) - p\cdot q\left(g^{\alpha\nu}q^\mu + g^{\alpha\mu}q^\nu\right)\right]$
11	$\left(p\cdot q\, p^\alpha - p^2 q^\alpha\right)\left[2\, q^\beta q^\mu q^\nu - q^2(g^{\beta\nu}q^\mu + g^{\beta\mu}q^\nu)\right]$
12	$\left(p\cdot q\, q^\beta - q^2 p^\beta\right)\left[2\, p^\alpha p^\mu p^\nu - p^2(g^{\alpha\nu}p^\mu + g^{\alpha\mu}p^\nu)\right]$
13	$(p^\mu q^\nu + p^\nu q^\mu)g^{\alpha\beta} + p\cdot q\,(g^{\alpha\nu}g^{\beta\mu} + g^{\alpha\mu}g^{\beta\nu}) - g^{\mu\nu}u^{\alpha\beta}$ $-(g^{\beta\nu}p^\mu + g^{\beta\mu}p^\nu)q^\alpha - (g^{\alpha\nu}q^\mu + g^{\alpha\mu}q^\nu)p^\beta$

$$\underline{\underline{\mathbf{F_1(s;\, s_1,\, s_2,\, 0)}}} \;=\; -\frac{e^2}{18\pi^2 s},$$

$$\Gamma^{\mu\nu\alpha\beta}(p,q) = \sum_{i=1}^{13} F_i(s; s_1, s_2, m^2)\; t_i^{\mu\nu\alpha\beta}(p,q)\,,$$



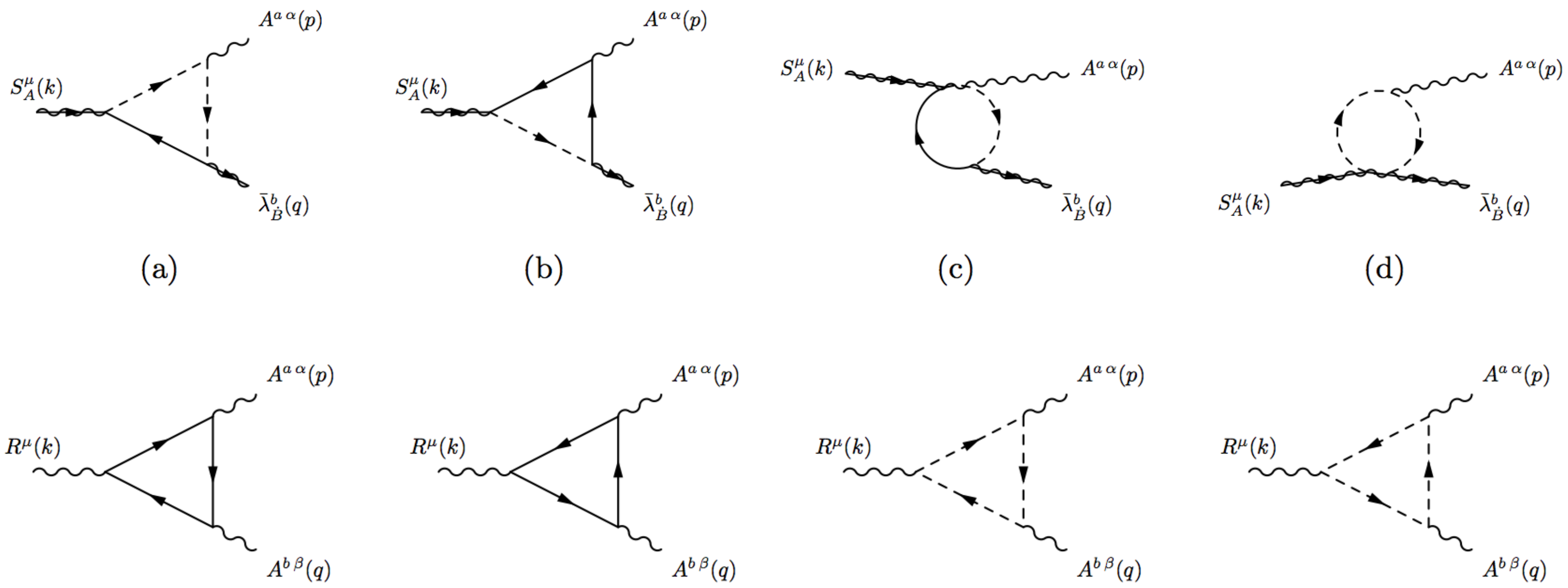
(a) (b) (c)

$$
\begin{aligned}
R^\mu &= \bar\lambda^a\bar\sigma^\mu\lambda^a + \frac{1}{3}\left(-\bar\chi_i\bar\sigma^\mu\chi_i + 2i\phi_i^\dagger\mathcal{D}^\mu_{ij}\phi_j - 2i(\mathcal{D}^\mu_{ij}\phi_j)^\dagger\phi_i\right) , \\
S^\mu_A &= i(\sigma^{\nu\rho}\sigma^\mu\bar\lambda^a)_A F^a_{\nu\rho} - \sqrt{2}(\sigma_\nu\bar\sigma^\mu\chi_i)_A(\mathcal{D}^\nu_{ij}\phi_j)^\dagger - i\sqrt{2}(\sigma^\mu\bar\chi_i)\mathcal{W}_i^\dagger(\phi^\dagger) \\
&- ig(\phi_i^\dagger T^a_{ij}\phi_j)(\sigma^\mu\bar\lambda^a)_A + S^\mu_{I\,A} , \\
T^{\mu\nu} &= -F^{a\,\mu\rho}F^{a\,\nu}{}_\rho + \frac{i}{4}\left[\bar\lambda^a\bar\sigma^\mu(\delta^{ac}\overrightarrow{\partial^\nu} - g\,t^{abc}A^{b\,\nu})\lambda^c + \bar\lambda^a\bar\sigma^\mu(-\delta^{ac}\overleftarrow{\partial^\nu} - g\,t^{abc}A^{b\,\nu})\lambda^c + (\mu\leftrightarrow\nu)\right] \\
&+ (\mathcal{D}^\mu_{ij}\phi_j)^\dagger(\mathcal{D}^\nu_{ik}\phi_k) + (\mathcal{D}^\nu_{ij}\phi_j)^\dagger(\mathcal{D}^\mu_{ik}\phi_k) + \frac{i}{4}\Big[\bar\chi_i\bar\sigma^\mu(\delta_{ij}\overrightarrow{\partial^\nu} + igT^a_{ij}A^{a\,\nu})\chi_j \\
&+ \bar\chi_i\bar\sigma^\mu(-\delta_{ij}\overleftarrow{\partial^\nu} + igT^a_{ij}A^{a\,\nu})\chi_j + (\mu\leftrightarrow\nu)\Big] - \eta^{\mu\nu}\mathcal{L} + T^{\mu\nu}_I ,
\end{aligned}
$$

$$
\begin{aligned}
\partial_\mu R^\mu &= \frac{g^2}{16\pi^2}\left(T(A) - \frac{1}{3}T(R)\right) F^{a\,\mu\nu}\tilde F^a_{\mu\nu} , \\
\bar\sigma_\mu S^\mu_A &= -i\frac{3\,g^2}{8\pi^2}\left(T(A) - \frac{1}{3}T(R)\right)\left(\bar\lambda^a\bar\sigma^{\mu\nu}\right)_A F^a_{\mu\nu} , \\
\eta_{\mu\nu}T^{\mu\nu} &= -\frac{3\,g^2}{32\pi^2}\left(T(A) - \frac{1}{3}T(R)\right) F^{a\,\mu\nu}F^a_{\mu\nu} .
\end{aligned}
$$

Similar pattern
in a superconformal
theory

Chiral and trace anomalies are related to anomaly poles.

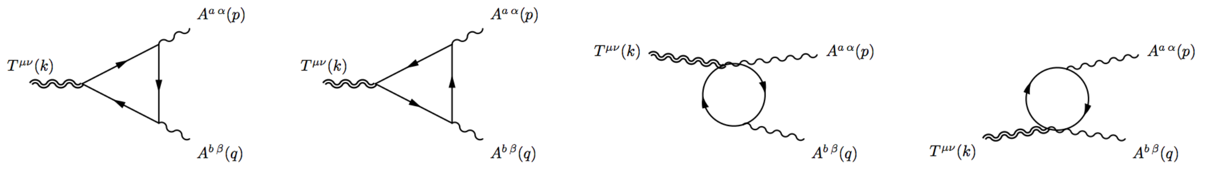


Similar pattern as for the TJJ correlator, just more complex.
One single form factor generates the anomaly

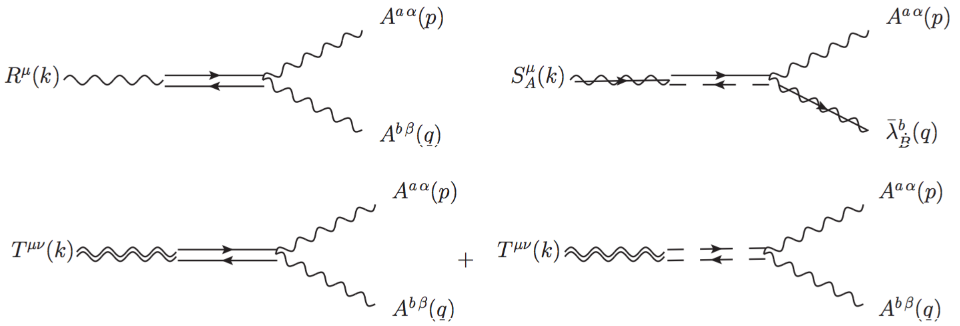
If we move away from the conformal limit
And give the fields a mass “m” then
The anomaly form factor is more
complicated

$$\Gamma^{\mu\alpha\beta}_{(R)}(p,q) \;=\; i\frac{g^2\,T(R)}{12\pi^2}\,\Phi_1(k^2,m^2)\,\frac{k^\mu}{k^2}\varepsilon[p,q,\alpha,\beta]\,,$$

$$\Gamma^{\mu\alpha}_{(S)}(p,q) \;=\; i\frac{g^2 T(R)}{6\pi^2\,k^2}\,\Phi_1(k^2,m^2)\,s_1^{\mu\alpha}$$



(a) (b) (c) (d)



$$S_{\text{axion}} \;=\; -\frac{g^2}{4\pi^2}\left(T(A)-\frac{T(R)}{3}\right)\int d^4z\,d^4x\,\partial^\mu B_\mu(z)\,\frac{1}{\Box_{zx}}\,\frac{1}{4}F_{\alpha\beta}(x)\tilde F^{\alpha\beta}(x) \qquad ($$

$$S_{\text{dilatino}} \;=\; \frac{g^2}{2\pi^2}\left(T(A)-\frac{T(R)}{3}\right)\int d^4z\,d^4x\left[\partial_\nu\Psi_\mu(z)\sigma^{\mu\nu}\sigma^\rho\frac{\overleftarrow{\partial}_\rho}{\Box_{zx}}\,\bar\sigma^{\alpha\beta}\bar\lambda(x)\frac{1}{2}F_{\alpha\beta}(x)+h.c.\right] \qquad ($$

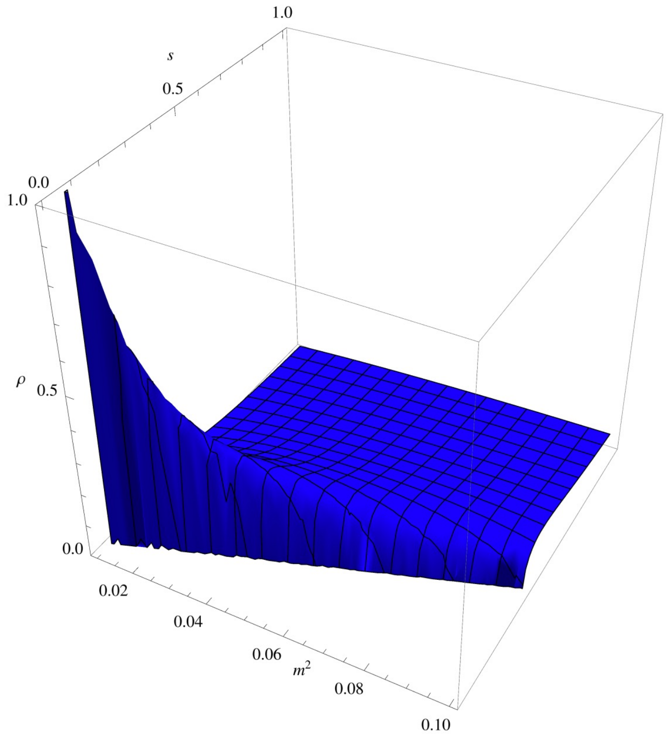
$$S_{\text{dilaton}} \;=\; -\frac{g^2}{8\pi^2}\left(T(A)-\frac{T(R)}{3}\right)\int d^4z\,d^4x\,\left(\Box h(z)-\partial^\mu\partial^\nu h_{\mu\nu}(z)\right)\,\frac{1}{\Box_{zx}}\,\frac{1}{4}F_{\alpha\beta}(x)F^{\alpha\beta}(x). \qquad ($$

$$\frac{1}{\pi}\int_0^{\infty} \rho(s,m^2)ds = f,$$

SUM RULE

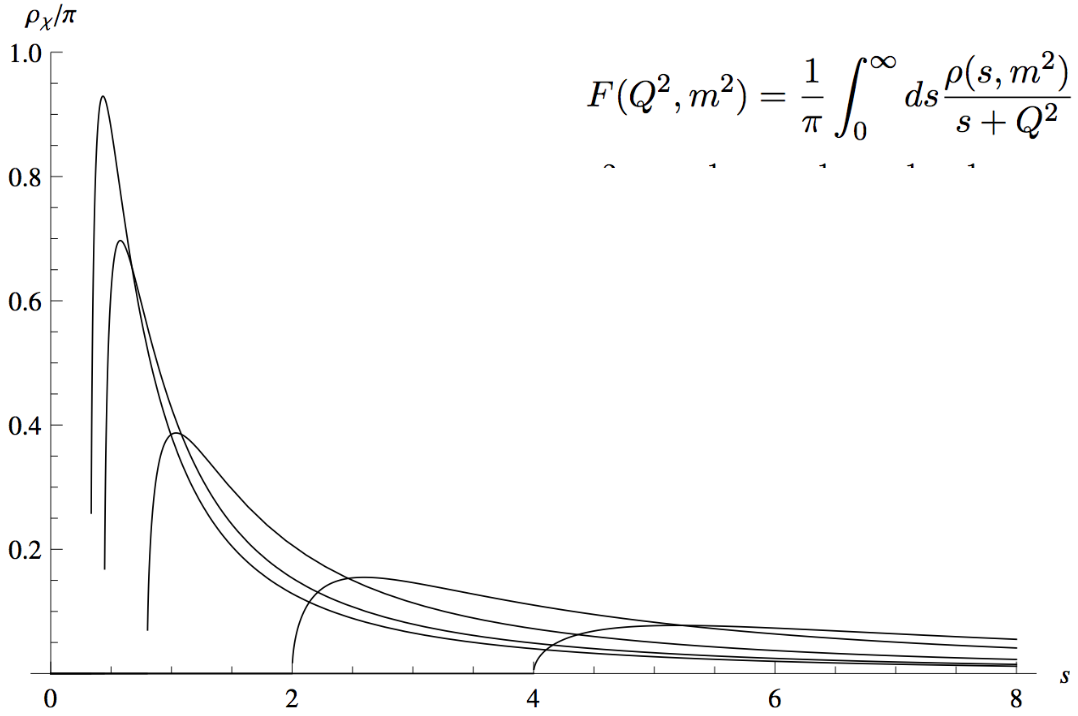
with the constant f independent of any mass (or other) parameter which characterizes the thresholds or the strengths of the resonant states eventually present in the integration region ($s > 0$).

$$m_n^2 = \frac{4m^2}{n}.$$

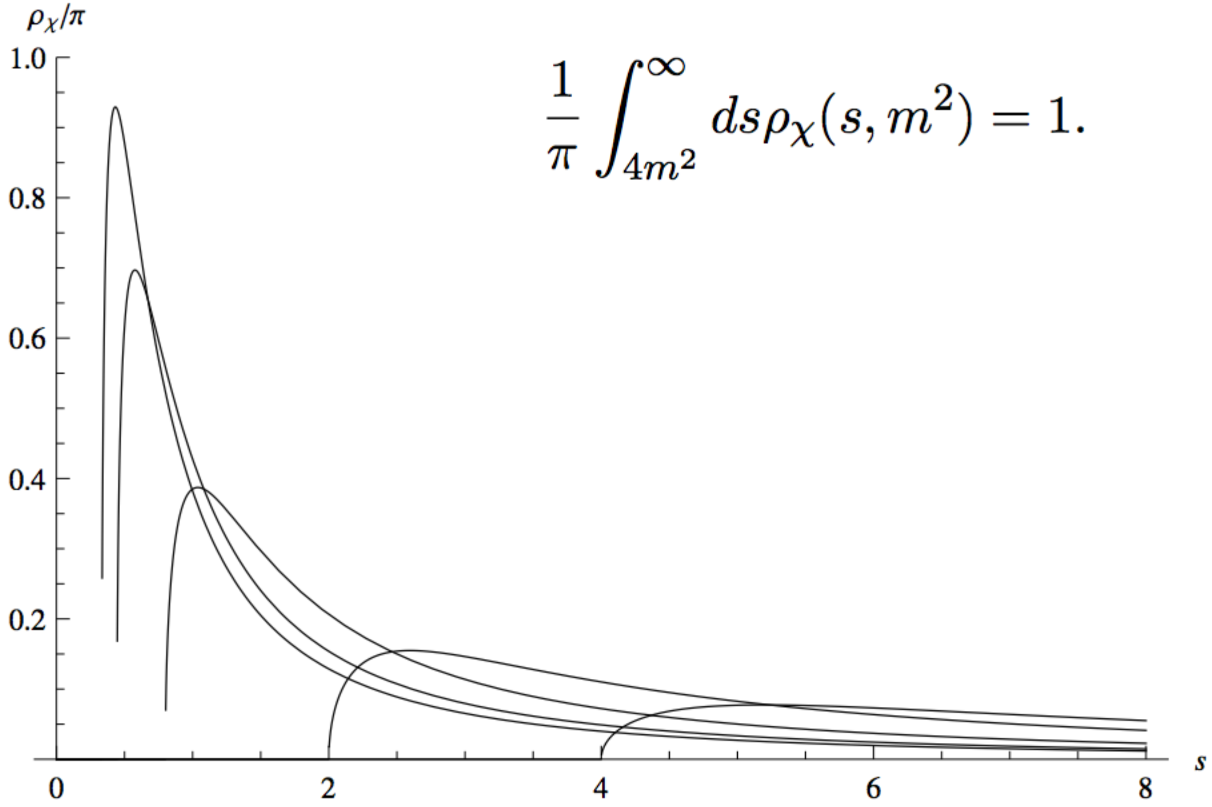


(a)

Figure 7: 3-D Plot of the spectral density ρ_χ in the variables s and m^2.



$$F(Q^2,m^2) = \frac{1}{\pi}\int_0^{\infty} ds \frac{\rho(s,m^2)}{s+Q^2},$$



$$\frac{1}{\pi}\int_{4m^2}^{\infty} ds\,\rho_\chi(s,m^2) = 1.$$

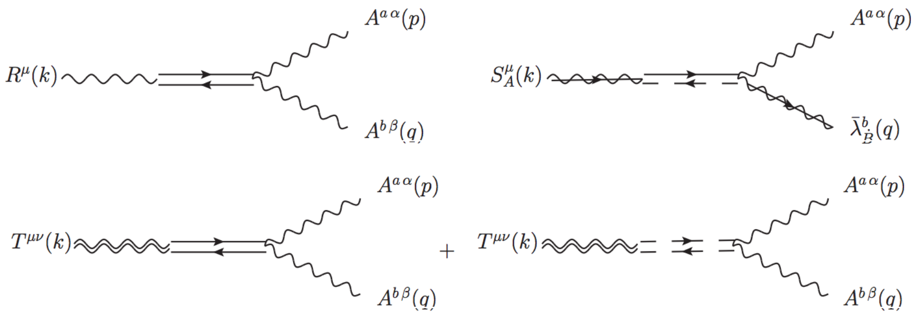
EXACT SUM RULE

The form factor that
carries the chiral
And conformal anomaly
away from the critical point c
shows a branch cut.

The spectral dentity exhibits a pole
As m→0

$$\lim_{m\to 0}\rho_\chi(s,m^2) = \lim_{m\to 0}\frac{2\pi m^2}{s^2}\log\left(\frac{1+\sqrt{\tau(s,m^2)}}{1-\sqrt{\tau(s,m^2)}}\right)\theta(s-4m^2) = \pi\delta(s)$$

Delle Rose, CC



THESE ANALYSIS
ARE PURELY
PERTURBATIVE.

[Trace Anomaly, Massless Scalars and the Gravitational Coupling of QCD](#).
Armillis, Delle Rose, CC
#Published in: *Phys.Rev.D* 82 (2010) 064023, e-Print: [1005.4173](#) [hep-ph]

$$\begin{aligned} T_{\mu\nu} &= -g_{\mu\nu}\mathcal{L}_{QCD} - F^a_{\mu\rho}F^{a\rho}_{\nu} - \frac{1}{\xi}g_{\mu\nu}\partial^\rho(A^a_\rho\partial^\sigma A^a_\sigma) + \frac{1}{\xi}(A^a_\nu\partial_\mu(\partial^\sigma A^a_\sigma) + A^a_\mu\partial_\nu(\partial^\sigma A^a_\sigma)) \\ &+ \frac{i}{4}\Big[\overline{\psi}\gamma_\mu(\overrightarrow{\partial}_\nu - igT^aA^a_\nu)\psi - \overline{\psi}(\overleftarrow{\partial}_\nu + igT^aA^a_\nu)\gamma_\mu\psi + \overline{\psi}\gamma_\nu(\overrightarrow{\partial}_\mu - igT^aA^a_\mu)\psi \\ &- \overline{\psi}(\overleftarrow{\partial}_\mu + igT^aA^a_\mu)\gamma_\nu\psi\Big] + \partial_\mu\overline{\omega}^a(\partial_\nu\omega^a - gf^{abc}A^c_\nu\omega^b) + \partial_\nu\overline{\omega}^a(\partial_\mu\omega^a - gf^{abc}A^c_\mu\omega^b), \end{aligned}$$

$$\begin{aligned} T^{g.f.}_{\mu\nu} &= \frac{1}{\xi}\left[A^a_\nu\partial_\mu(\partial\cdot A^a) + A^a_\mu\partial_\nu(\partial\cdot A^a)\right] - \frac{1}{\xi}g_{\mu\nu}\left[-\frac{1}{2}(\partial\cdot A)^2 + \partial^\rho(A^a_\rho\partial\cdot A^a)\right], \\ T^{gh}_{\mu\nu} &= \partial_\mu\bar{\omega}^a D^{ab}_\nu\omega^b + \partial_\nu\bar{\omega}^a D^{ab}_\mu\omega^b - g_{\mu\nu}\partial^\rho\bar{\omega}^a D^{ab}_\rho\omega^b. \end{aligned}$$

$$\mathcal{L}_{int} = -\frac{1}{2}\kappa\, h^{\mu\nu}T_{\mu\nu}.$$

$$\begin{aligned} \partial^\mu T_{\mu\nu} &= -\frac{\delta S}{\delta\psi}\partial_\nu\psi - \partial_\nu\bar{\psi}\frac{\delta S}{\delta\bar{\psi}} + \frac{1}{2}\partial^\mu\left(\frac{\delta S}{\delta\psi}\sigma_{\mu\nu}\psi - \bar{\psi}\sigma_{\mu\nu}\frac{\delta S}{\delta\bar{\psi}}\right) - \partial_\nu A^a_\mu\frac{\delta S}{\delta A^a_\mu} \\ &+ \partial_\mu\left(A^a_\nu\frac{\delta S}{\delta A^a_\mu}\right) - \frac{\delta S}{\delta\omega^a}\partial_\nu\omega^a - \partial_\nu\bar{\omega}^a\frac{\delta S}{\delta\bar{\omega}^a} \end{aligned}$$

Functional derivation of WIs

$$Z[J,\eta,\bar\eta,\chi,\bar\chi,h]=\int \mathcal{D}A\,\mathcal{D}\psi\,\mathcal{D}\bar\psi\,\mathcal{D}\omega\,\mathcal{D}\bar\omega\,\exp\left\{i\int d^4x\,(\mathcal{L}+J_\mu A^\mu + \bar\eta\psi+\bar\psi\eta+\bar\chi\omega+\bar\omega\chi+h_{\mu\nu}T^{\mu\nu})\right\},$$

$$\exp i\,W[J,\eta,\bar\eta,\chi,\bar\chi,h]=\frac{Z[J,\eta,\bar\eta,\chi,\bar\chi,h]}{Z[0,0,0,0,0,0]}$$

$$\Gamma[A_c,\bar\psi_c,\psi_c,\bar\omega_c,\omega_c,h]=W[J,\eta,\bar\eta,\chi,\bar\chi,h]-\int d^4x\,\left(J_\mu A^\mu_c+\bar\eta\psi_c+\bar\psi_c\eta+\bar\chi\omega_c+\bar\omega_c\chi\right).$$

Semiclassical Formulation via the effective action

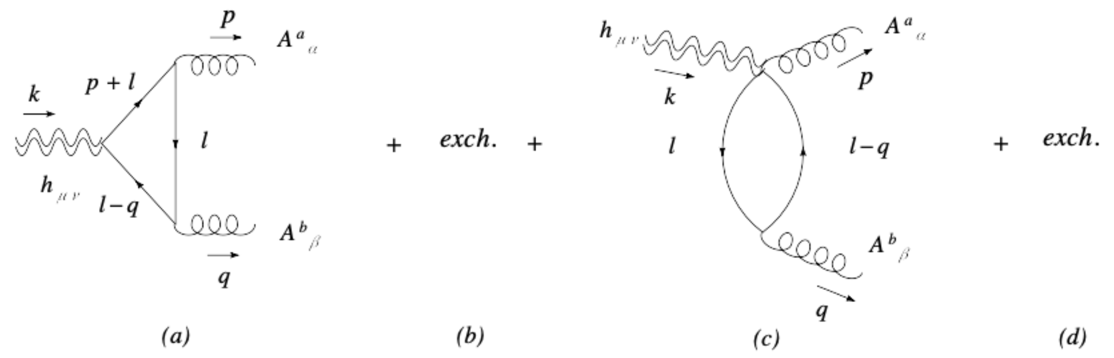
$$\begin{aligned}\partial_\mu\frac{\delta\Gamma}{\delta h_{\mu\nu}} &= -\frac{\delta\Gamma}{\delta\psi_c}\partial^\nu\psi_c-\partial^\nu\bar\psi_c\frac{\delta\Gamma}{\delta\bar\psi_c}+\frac{1}{2}\partial_\mu\left(\frac{\delta\Gamma}{\delta\psi_c}\sigma^{\mu\nu}\psi_c-\bar\psi_c\sigma^{\mu\nu}\frac{\delta\Gamma}{\delta\bar\psi_c}\right)-\partial^\nu A^\mu_c\frac{\delta\Gamma}{\delta A^\mu_c}+\partial^\mu\left(A^\nu_c\frac{\delta\Gamma}{\delta A^\mu_c}\right)\\ &- \frac{\delta\Gamma}{\delta\omega_c}\partial^\nu\omega_c-\partial^\nu\bar\omega_c\frac{\delta\Gamma}{\delta\bar\omega_c}, \qquad (36)\end{aligned}$$

$$\begin{aligned}\partial^\mu\langle T_{\mu\nu}(x)A^a_\alpha(x_1)A^b_\beta(x_2)\rangle_{trunc} &= -\partial_\nu\delta^4(x_1-x)D^{-1}_{\alpha\beta}(x_2,x)-\partial_\nu\delta^4(x_2-x)D^{-1}_{\alpha\beta}(x_1,x)\\ &+ \partial^\mu\left(g_{\alpha\nu}\delta^4(x_1-x)D^{-1}_{\beta\mu}(x_2,x)+g_{\beta\nu}\delta^4(x_2-x)D^{-1}_{\alpha\mu}(x_1,x)\right)\end{aligned}$$

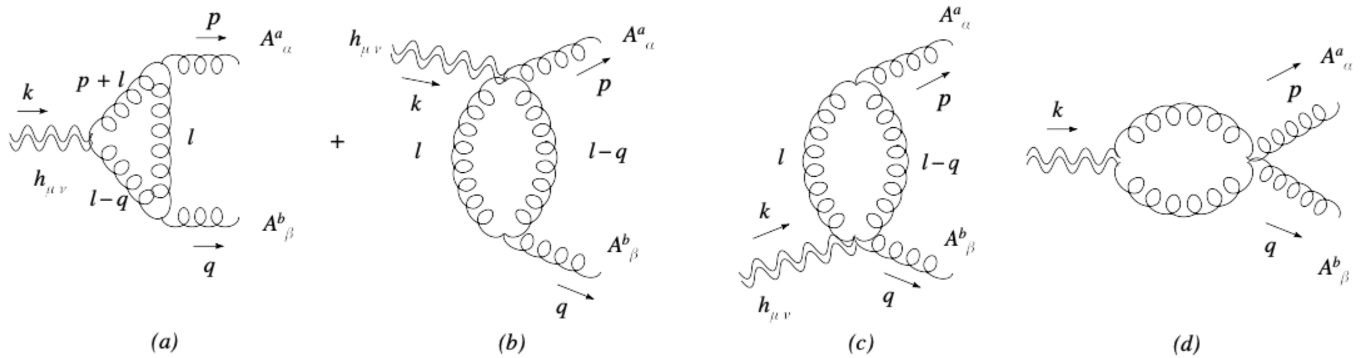
$$k^\mu\langle T_{\mu\nu}(k)A_\alpha(p)A_\beta(q)\rangle_{trunc}=q_\mu D^{-1}_{\alpha\mu}(p)g_{\beta\nu}+p_\mu D^{-1}_{\beta\mu}(q)g_{\alpha\nu}-q_\nu D^{-1}_{\alpha\beta}(p)-p_\nu D^{-1}_{\alpha\beta}(q)\,.$$

WI in momentum space

$$\langle T^\mu_\mu(0)A_\alpha(p)A_\beta(-p)\rangle_{trunc}=\left(2-d+p\cdot\frac{\partial}{\partial p}\right)D^{-1}_{\alpha\beta}(p)$$



contributions



$$\Gamma^{\mu\nu\alpha\beta}(p,q) = \Gamma_q^{\mu\nu\alpha\beta}(p,q) + \Gamma_g^{\mu\nu\alpha\beta}(p,q),$$

$$\begin{aligned}
\phi_1^{\mu\nu\alpha\beta}(p,q) &= (s\, g^{\mu\nu} - k^\mu k^\nu)\, u^{\alpha\beta}(p,q), \\
\phi_2^{\mu\nu\alpha\beta}(p,q) &= -2\, u^{\alpha\beta}(p,q)\left[s\, g^{\mu\nu} + 2(p^\mu\, p^\nu + q^\mu\, q^\nu) - 4\,(p^\mu\, q^\nu + q^\mu\, p^\nu)\right], \\
\phi_3^{\mu\nu\alpha\beta}(p,q) &= (p^\mu q^\nu + p^\nu q^\mu) g^{\alpha\beta} + \frac{s}{2}\left(g^{\alpha\nu} g^{\beta\mu} + g^{\alpha\mu} g^{\beta\nu}\right) \\
&\quad - g^{\mu\nu}\left(\frac{s}{2} g^{\alpha\beta} - q^\alpha p^\beta\right) - \left(g^{\beta\nu} p^\mu + g^{\beta\mu} p^\nu\right) q^\alpha - (g^{\alpha\nu} q^\mu + g^{\alpha\mu} q^\nu) p^\beta,
\end{aligned}$$

Fourier transform of second variation of FF

$$u^{\alpha\beta}(p,q) \equiv (p\cdot q)\, g^{\alpha\beta} - q^\alpha\, p^\beta\,,$$

$$\Phi_{1q}(s,\,0,\,0,\,m^2) = -\frac{g^2}{36\pi^2 s} + \frac{g^2 m^2}{6\pi^2 s^2} - \frac{g^2\, m^2}{6\pi^2 s}\mathcal{C}_0(s,0,0,m^2)\left[\frac{1}{2} - \frac{2m^2}{s}\right],$$

$$\Phi_{2q}(s,\,0,\,0,\,m^2) = -\frac{g^2}{288\pi^2 s} - \frac{g^2 m^2}{24\pi^2 s^2} - \frac{g^2\, m^2}{8\pi^2 s^2}\mathcal{D}(s,0,0,m^2) - \frac{g^2\, m^2}{12\pi^2 s}\mathcal{C}_0(s,0,0,m^2)\left[\frac{1}{2} + \frac{m^2}{s}\right],$$

$$\Phi_{3q}(s,\,0,\,0,\,m^2) = \frac{11 g^2}{288\pi^2} + \frac{g^2 m^2}{8\pi^2 s} + g^2 \mathcal{C}_0(s,0,0,m^2)\left[\frac{m^4}{4\pi^2 s} + \frac{m^2}{8\pi^2}\right] + \frac{5\, g^2\, m^2}{24\pi^2 s}\mathcal{D}(s,0,0,m^2) + \frac{g^2}{24\pi^2}\mathcal{B}_0^{\overline{MS}}(s,m^2),$$

Pole in the quark sector

Similar pole in the gluon sector

$$\Phi_1(s,0,0) = -\frac{g^2}{72\pi^2\, s}(2n_f - 11C_A) + \frac{g^2}{6\pi^2}\sum_{i=1}^{n_f} m_i^2 \left\{\frac{1}{s^2} - \frac{1}{2s}\mathcal{C}_0(s,0,0,m_i^2)\left[1 - \frac{4m_i^2}{s}\right]\right\}, \quad (92)$$

$$\Phi_2(s,0,0) = -\frac{g^2}{288\pi^2\, s}(n_f - C_A) - \frac{g^2}{24\pi^2}\sum_{i=1}^{n_f} m_i^2 \left\{\frac{1}{s^2} + \frac{3}{s^2}\mathcal{D}(s,0,0,m_i^2) + \frac{1}{s}\mathcal{C}_0(s,0,0,m_i^2)\left[1 + \frac{2m_i^2}{s}\right]\right\}, \quad (93)$$

$$\Phi_3(s,0,0) = \frac{g^2}{288\pi^2}(11n_f - 65C_A) - \frac{g^2\, C_A}{8\pi^2}\left[\frac{11}{6}\mathcal{B}_0^{\overline{MS}}(s,0) - \mathcal{B}_0^{\overline{MS}}(0,0) + s\,\mathcal{C}_0(s,0,0,0)\right] + \frac{g^2}{8\pi^2}\sum_{i=1}^{n_f}\left\{\frac{1}{3}\mathcal{B}_0^{\overline{MS}}(s,m_i^2) + m_i^2\left[\frac{1}{s} + \frac{5}{3s}\mathcal{D}(s,0,0,m_i^2) + \mathcal{C}_0(s,0,0,m_i^2)\left[1 + \frac{2m_i^2}{s}\right]\right]\right\},$$

Notice that the residui, combined, equal the beta function

$$= cF_{\alpha\beta}F^{\alpha\beta} = -\frac{e^2}{24\pi^2}F_{\alpha\beta}F^{\alpha\beta}.$$

$$
\begin{aligned}
S_{pole} &= -\frac{c}{6}\int d^4x\, d^4y\, R^{(1)}(x)\,\Box^{-1}(x,y)\,F^a_{\alpha\beta}\,F^{a\,\alpha\beta} \\
&= \frac{1}{3}\,\frac{g^3}{16\pi^2}\left(-\frac{11}{3}\,C_A + \frac{2}{3}\,n_f\right)\int d^4x\, d^4y\, R^{(1)}(x)\,\Box^{-1}(x,y)\,F_{\alpha\beta}F^{\alpha\beta}
\end{aligned}
$$

$$
R^{(1)}_x \equiv \partial^x_\mu\,\partial^x_\nu\,h^{\mu\nu} - \Box\,h, \qquad h = \eta_{\mu\nu}\,h^{\mu\nu}
$$

$$
c = -2\,\frac{\beta(g)}{g}.
$$

$$
S_{anom}[g,A] = \frac{1}{8}\int d^4x\sqrt{-g}\int d^4x'\sqrt{-g'}\left(E - \frac{2}{3}\Box R\right)_x \Delta_4^{-1}(x,x')\left[2b\,F + b'\left(E - \frac{2}{3}\Box R\right) + 2\,c\,F_{\mu\nu}F^{\mu\nu}\right]_{x'}
$$

$$
F = C_{\lambda\mu\nu\rho}C^{\lambda\mu\nu\rho} = R_{\lambda\mu\nu\rho}R^{\lambda\mu\nu\rho} - 2R_{\mu\nu}R^{\mu\nu} + \frac{R^2}{3}
$$

$$
E = {}^*R_{\lambda\mu\nu\rho}\,{}^*R^{\lambda\mu\nu\rho} = R_{\lambda\mu\nu\rho}R^{\lambda\mu\nu\rho} - 4R_{\mu\nu}R^{\mu\nu} + R^2
$$

$$
\Delta_4 \equiv \nabla_\mu\left(\nabla^\mu\nabla^\nu + 2R^{\mu\nu} - \frac{2}{3}Rg^{\mu\nu}\right)\nabla_\nu = \Box^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu + \frac{1}{3}(\nabla^\mu R)\nabla_\mu - \frac{2}{3}R\Box\,.
$$

$4d$ Einstein Gauss-Bonnet Gravity without a Dilaton [1]

[a]Claudio Corianò, [a][b]Mario Creti, [a] Stefano Lionetti, [c]Matteo Maria Maglio,
[a]Riccardo Tommasi

PoS Corfu 2022

$$\Delta_4 \equiv \nabla_\mu \left(\nabla^\mu \nabla^\nu + 2R^{\mu\nu} - \frac{2}{3} R g^{\mu\nu} \right) \nabla_\nu = \Box^2 + 2R^{\mu\nu} \nabla_\mu \nabla_\nu + \frac{1}{3} (\nabla^\mu R) \nabla_\mu - \frac{2}{3} R \Box \, .$$

Paneitz operator

$$\sqrt{-g}\, \Delta_4 \chi_0 = \sqrt{-\bar{g}}\, \bar{\Delta}_4 \chi_0,$$

Weyl invariant if acting on conformal scalars (ie fields of vanishing scaling dimensions)

CFT in coordinate space (scalar primary operators) in d=4

$$\langle \mathcal{O}_i(x_i)\,\mathcal{O}_j(x_j)\rangle = \frac{C_{ij}}{|x_i - x_j|^{\Delta_i+\Delta_j}}.$$

$$\langle \mathcal{O}_1(x_1)\,\mathcal{O}_2(x_2)\,\mathcal{O}_3(x_3)\rangle = \frac{C_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}\;x_{23}^{\Delta_2+\Delta_3-\Delta-1}\;x_{13}^{\Delta_3+\Delta_1-\Delta_2}}.$$

dilatation

$$\left[\sum_{j=1}^{n}\Delta_j + \sum_{j=1}^{n} x_j^{\alpha}\frac{\partial}{\partial x_j^{\alpha}}\right]\langle \mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\rangle =$$

Special conformal

$$\sum_{j=1}^{n}\left(2\Delta_j\, x_j^{\kappa} + 2x_j^{\kappa}\, x_j^{\alpha}\frac{\partial}{\partial x_{j\alpha}} - x_j^2\frac{\partial}{\partial x_{j\kappa}}\right)\langle \mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\rangle = 0,$$

No anomalies yet and no spin !

The rederivation of these expressions in momentum space is in

Solving the Conformal Constraints for Scalar Operators in Momentum Space and the Evaluation of Feynman's Master Integrals

•*JHEP* 07 (2013) 011 e-Print: 1304.6944 (Delle Rose, Mottola, Serino, CC)

Recovering the correlators from momentum space may not be easy as in coordinate space, But once you do it, you can connect with Amplitudes and test the expressions against Free field theory realizations

$$\left[-\sum_{r=1}^{n-1}\left(p_{r\,\mu}\frac{\partial}{\partial p_{r\,\mu}}+d\right)+\sum_{r=1}^{n}\eta_r\right]\langle \mathcal{O}_1^{i_1}(p_1)\dots\mathcal{O}_r^{i_r}(p_r)\dots\mathcal{O}_n^{i_n}(p_n)\rangle=0\,,$$

$$\sum_{r=1}^{n-1}\left(p_{r\,\mu}\frac{\partial^2}{\partial p_r^\nu\partial p_{r\,\nu}}-2\,p_{r\,\nu}\frac{\partial^2}{\partial p_r^\mu\partial p_{r\,\nu}}+2(\eta_r-d)\frac{\partial}{\partial p_r^\mu}+2(\Sigma_{\mu\nu}^{(r)})^{i_r}_{\ j_r}\frac{\partial}{\partial p_{r\,\nu}}\right)$$
$$\times\langle \mathcal{O}_1^{i_1}(p_1)\dots\mathcal{O}_r^{j_r}(p_r)\dots\mathcal{O}_n^{i_n}(p_n)\rangle=0\,,$$

Transform the eqs to momentum space and solve them. They can be mapped to generalized hypergeometric functions

No anomalies and no spin

$$G_{123}(p_1^2,p_2^2,p_3^2)=(p_3^2)^{-d+\frac{1}{2}(\eta_1+\eta_2+\eta_3)}\,\Phi(x,y)\qquad\text{with}\qquad x=\frac{p_1^2}{p_3^2}\,,\quad y=\frac{p_2^2}{p_3^2}\,,$$

ansatz

$$\begin{aligned}G_{123}(p_1^2,p_2^2,p_3^2)\;=\;&\frac{c_{123}\,\pi^d\,4^{d-\frac{1}{2}(\eta_1+\eta_2+\eta_3)}\,(p_3^2)^{-d+\frac{1}{2}(\eta_1+\eta_2+\eta_3)}}{\Gamma\left(\frac{\eta_1}{2}+\frac{\eta_2}{2}-\frac{\eta_3}{2}\right)\Gamma\left(\frac{\eta_1}{2}-\frac{\eta_2}{2}+\frac{\eta_3}{2}\right)\Gamma\left(-\frac{\eta_1}{2}+\frac{\eta_2}{2}+\frac{\eta_3}{2}\right)\Gamma\left(-\frac{d}{2}+\frac{\eta_1}{2}+\frac{\eta_2}{2}+\frac{\eta_3}{2}\right)}\Bigg\{\\
&\Gamma\left(\eta_1-\frac{d}{2}\right)\Gamma\left(\eta_2-\frac{d}{2}\right)\Gamma\left(d-\frac{\eta_1}{2}-\frac{\eta_2}{2}-\frac{\eta_3}{2}\right)\Gamma\left(\frac{d}{2}-\frac{\eta_1}{2}-\frac{\eta_2}{2}+\frac{\eta_3}{2}\right)\\
&\times F_4\left(\frac{d}{2}-\frac{\eta_1+\eta_2-\eta_3}{2},d-\frac{\eta_1+\eta_2+\eta_3}{2};\frac{d}{2}-\eta_1+1,\frac{d}{2}-\eta_2+1;x,y\right)\\
&+\Gamma\left(\frac{d}{2}-\eta_1\right)\Gamma\left(\eta_2-\frac{d}{2}\right)\Gamma\left(\frac{\eta_1}{2}-\frac{\eta_2}{2}+\frac{\eta_3}{2}\right)\Gamma\left(\frac{d}{2}+\frac{\eta_1}{2}-\frac{\eta_2}{2}-\frac{\eta_3}{2}\right)\\
&\times x^{\eta_1-\frac{d}{2}}\,F_4\left(\frac{d}{2}-\frac{\eta_2+\eta_3-\eta_1}{2},\frac{\eta_1+\eta_3-\eta_2}{2};-\frac{d}{2}+\eta_1+1,\frac{d}{2}-\eta_2+1;x,y\right)\\
&+\Gamma\left(\eta_1-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}-\eta_2\right)\Gamma\left(-\frac{\eta_1}{2}+\frac{\eta_2}{2}+\frac{\eta_3}{2}\right)\Gamma\left(\frac{d}{2}-\frac{\eta_1}{2}+\frac{\eta_2}{2}-\frac{\eta_3}{2}\right)\\
&\times y^{\eta_2-\frac{d}{2}}\,F_4\left(\frac{d}{2}-\frac{\eta_1+\eta_3-\eta_2}{2},\frac{\eta_2+\eta_3-\eta_1}{2};\frac{d}{2}-\eta_1+1,-\frac{d}{2}+\eta_2+1;x,y\right)\\
&+\Gamma\left(\frac{d}{2}-\eta_1\right)\Gamma\left(\frac{d}{2}-\eta_2\right)\Gamma\left(\frac{\eta_1}{2}+\frac{\eta_2}{2}-\frac{\eta_3}{2}\right)\Gamma\left(-\frac{d}{2}+\frac{\eta_1}{2}+\frac{\eta_2}{2}+\frac{\eta_3}{2}\right)\\
&\times x^{\eta_1-\frac{d}{2}}y^{\eta_2-\frac{d}{2}}\,F_4\left(-\frac{d}{2}+\frac{\eta_1+\eta_2+\eta_3}{2},\frac{\eta_1+\eta_2-\eta_3}{2};-\frac{d}{2}+\eta_1+1,-\frac{d}{2}+\eta_2+1;x,y\right)\Bigg\}\,.\end{aligned}$$

Delle Rose, Serino, Mototla, CC

General solution in terms of a single constant C123

$$F_4(\alpha,\beta;\gamma,\gamma';x,y)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_{i+j}\,(\beta)_{i+j}}{(\gamma)_i\,(\gamma')_j}\frac{x^i}{i!}\frac{y^j}{j!}$$

4 point functions

Maglio, C. C.

JHEP 09 (2019) 107 e-Print: 1903.05047

$$\langle O(p_1)\, O(p_2)\, O(p_3)\, O(\bar{p}_4)\rangle = \Phi(p_1, p_2, p_3, p_4, s, t).$$

$$\left[\sum_{i=1}^{4} \Delta_i - 3d - \sum_{i=1}^{3} p_i^\mu \frac{\partial}{\partial p_i^\mu}\right] \langle O(p_1)\, O(p_2)\, O(p_3)\, O(\bar{p}_4)\rangle = 0$$

$$\left[(\Delta_t - 3d) - \sum_{i=1}^{4} p_i \frac{\partial}{\partial p_i} - s\frac{\partial}{\partial s} - t\frac{\partial}{\partial t}\right] \Phi(p_1, p_2, p_3, p_4, s, t) = 0,$$

$$\sum_{i=1}^{3} \left[2(\Delta_i - d)\frac{\partial}{\partial p_{i\,\kappa}} - 2p_i^\alpha \frac{\partial^2}{\partial p_i^\alpha \partial p_i^\kappa} + p_i^\kappa \frac{\partial^2}{\partial p_i^\alpha \partial p_{i\,\alpha}}\right] \langle O(p_1)\, O(p_2)\, O(p_3)\, O(\bar{p}_4)\rangle = 0.$$

$$\sum_{i=1}^{3} p_i^\kappa\, C_i = 0,$$

$$\begin{aligned} C_2 = \Bigg\{ & \frac{\partial^2}{\partial p_2^2} + \frac{(d - 2\Delta_2 + 1)}{p_2}\frac{\partial}{\partial p_2} - \frac{\partial^2}{\partial p_4^2} - \frac{(d - 2\Delta_4 + 1)}{p_4}\frac{\partial}{\partial p_4} \\ & + \frac{1}{s}\frac{\partial}{\partial s}\left(p_1 \frac{\partial}{\partial p_1} + p_2 \frac{\partial}{\partial p_2} - p_3 \frac{\partial}{\partial p_3} - p_4 \frac{\partial}{\partial p_4}\right) + \frac{(\Delta_3 + \Delta_4 - \Delta_1 - \Delta_2)}{s}\frac{\partial}{\partial s} \\ & + \frac{1}{t}\frac{\partial}{\partial t}\left(p_2 \frac{\partial}{\partial p_2} + p_3 \frac{\partial}{\partial p_3} - p_1 \frac{\partial}{\partial p_1} - p_4 \frac{\partial}{\partial p_4}\right) + \frac{(\Delta_1 + \Delta_4 - \Delta_2 - \Delta_3)}{t}\frac{\partial}{\partial t} \\ & + \frac{(p_2^2 - p_4^2)}{st}\frac{\partial^2}{\partial s \partial t} \Bigg\} \Phi(p_1, p_2, p_3, p_4, s, t) = 0 \end{aligned}$$

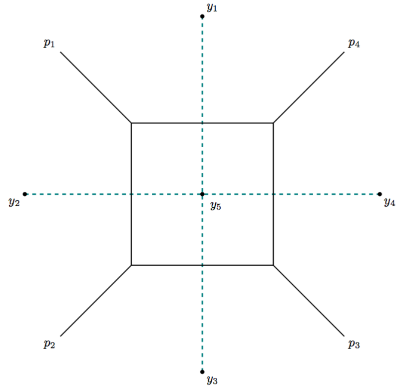
One specific structure. There are 3 of them.

For some special choices of the scaling dimensions
We can solve the equations

YANGIAN SYMMETRY: 4 point functions can also be fixed in the scalar case, probably
The reconstruction can be also performed in the tensor case

DUAL CONFORMAL/CONFORMAL

$$\Phi_{Box}(p_1,p_2,p_3,p_4) = \int \frac{d^d k}{k^2(k+p_1)^2(k+p_1+p_2)^2(k+p_1+p_2+p_3)^2}$$



and apply the redefinition in terms of momentum variables y_i

$$k = y_{51}, \qquad p_1 = y_{12}, \qquad p_2 = y_{23}, \qquad p_3 = y_{34}$$

with $y_{ij} = y_i - y_j$, thereby rewriting the integral in the form

$$\Phi_{Box}(y_1,y_2,y_3,y_4) = \int \frac{d^d y_5}{y_{15}^2 y_{25}^2 y_{35}^2 y_{45}^2}$$

Maglio, CC

$$\langle O(p_1)O(p_2)O(p_3)O(p_4)\rangle =$$
$$= \sum_{a,b} c(a,b)\Bigg[(s^2\,t^2)^{\Delta-\frac{3}{4}d} \left(\frac{p_1^2 p_3^2}{s^2t^2}\right)^a \left(\frac{p_2^2 p_4^2}{s^2t^2}\right)^b F_4\left(\alpha(a,b),\beta(a,b),\gamma(a),\gamma'(b),\frac{p_1^2p_3^2}{s^2t^2},\frac{p_2^2p_4^2}{s^2t^2}\right)$$
$$+ (s^2\,u^2)^{\Delta-\frac{3}{4}d} \left(\frac{p_2^2 p_3^2}{s^2u^2}\right)^a \left(\frac{p_1^2 p_4^2}{s^2u^2}\right)^b F_4\left(\alpha(a,b),\beta(a,b),\gamma(a),\gamma'(b),\frac{p_2^2p_3^2}{s^2u^2},\frac{p_1^2p_4^2}{s^2u^2}\right)$$
$$+ (t^2\,u^2)^{\Delta-\frac{3}{4}d} \left(\frac{p_1^2 p_2^2}{t^2u^2}\right)^a \left(\frac{p_3^2 p_4^2}{t^2u^2}\right)^b F_4\left(\alpha(a,b),\beta(a,b),\gamma(a),\gamma'(b),\frac{p_1^2p_2^2}{t^2u^2},\frac{p_3^2p_4^2}{t^2u^2}\right)\Bigg]$$

$$\mathrm{K}_i = \frac{\partial^2}{\partial p_i^2} + \frac{(d-2\Delta_i+1)}{p_i}\frac{\partial}{\partial p_i}, \qquad i = 1,\dots,4\,,$$
$$\mathrm{K}_{ij} = \mathrm{K}_i - \mathrm{K}_j\ .$$

$$\alpha(a,b) = \frac{3}{4}d - \Delta + a + b, \qquad \beta(a,b) = \frac{3}{4}d - \Delta + a + b,$$
$$\gamma(a) = \frac{d}{2} - \Delta + 1 + 2a, \qquad \gamma'(b) = \frac{d}{2} - \Delta + 1 + 2b.$$

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The inclusion of anomalies

Tensor correlators in coordinate space had been studied long ago by Osborn and Petkou.

The important step in these analsys was the inclusion of the anomaly contribution in coordinate space

Implications of conformal invariance in field theories for general dimensions

- *Annals Phys.* 231 (1994) 311-362 e-Print: hep-th/9307010

Conserved currents and the energy momentum tensor in conformally invariant theories for general dimensions

- *Nucl.Phys.B* 483 (1997) 431-474 e-Print: hep-th/9605009 (Erdmenger Osborn)

Not so practical beyond 3 point functions

Anomalies come from the coalescence of all the external coordinates.

$$\partial_\nu \langle T^{\mu\nu}(x_1)T^{\rho\sigma}(x_2)T^{\alpha\beta}(x_3)\rangle = \Big[\langle T^{\rho\sigma}(x_1)T^{\alpha\beta}(x_3)\rangle\partial^\mu\delta(x_1,x_2) + \langle T^{\alpha\beta}(x_1)T^{\rho\sigma}(x_2)\rangle\partial^\mu\delta(x_1,x_3)\Big]$$
$$- \Big[\delta^{\mu\rho}\langle T^{\nu\sigma}(x_1)T^{\alpha\beta}(x_3)\rangle + \delta^{\mu\sigma}\langle T^{\nu\rho}(x_1)T^{\alpha\beta}(x_3)\rangle\Big]\partial_\nu\delta(x_1,x_2)$$
$$- \Big[\delta^{\mu\alpha}\langle T^{\nu\beta}(x_1)T^{\rho\sigma}(x_2)\rangle + \delta^{\mu\beta}\langle T^{\nu\alpha}(x_1)T^{\rho\sigma}(x_2)\rangle\Big]\partial_\nu\delta(x_1,x_3)\,.$$

TENSOR CORRELATORS

Conservation WI

$$p_{1\nu_1}\,\langle T^{\mu_1\nu_1}(p_1)\,T^{\mu_2\nu_2}(p_2)\,T^{\mu_3\nu_3}(p_3)\rangle = -p_2^{\mu_1}\,\langle T^{\mu_2\nu_2}(p_1+p_2)T^{\mu_3\nu_3}(p_3)\rangle - p_3^{\mu_1}\,\langle T^{\mu_2\nu_2}(p_2)T^{\mu_3\nu_3}(p_1+p_3)\rangle$$
$$+ p_{2\alpha}\left[\delta^{\mu_1\nu_2}\,\langle T^{\mu_2\alpha}(p_1+p_2)T^{\mu_3\nu_3}(p_3)\rangle + \delta^{\mu_1\mu_2}\,\langle T^{\nu_2\alpha}(p_1+p_2)T^{\mu_3\nu_3}(p_3)\rangle\right]$$
$$+ p_{3\alpha}\left[\delta^{\mu_1\nu_3}\,\langle T^{\mu_3\alpha}(p_1+p_3)T^{\mu_2\nu_2}(p_2)\rangle + \delta^{\mu_1\mu_3}\,\langle T^{\nu_3\alpha}(p_1+p_3)T^{\mu_2\nu_2}(p_2)\rangle\right].$$

while naive scale invariance gives the traceless condition

$$g_{\mu\nu}\,\langle T^{\mu\nu}\rangle = 0.$$

$$\beta_a(S) = -\frac{3\pi^2}{720}\,, \qquad \beta_b(S) = \frac{\pi^2}{720}\,,$$
$$\beta_a(F) = -\frac{9\pi^2}{360}\,, \qquad \beta_b(F) = \frac{11\pi^2}{720}$$
$$\beta_a(G) = -\frac{18\pi^2}{360}\,, \qquad \beta_b(G) = \frac{31\pi^2}{360}$$

special CWI's take the form

$$0 = K^\kappa\,\langle T^{\mu_1\nu_1}(x_1)T^{\mu_2\nu_2}(x_2)T^{\mu_3\nu_3}(x_3)\rangle = \sum_{i=1}^{3} K^\kappa_{i\,scalar}(x_i)\,\langle T^{\mu_1\nu_1}(x_1)T^{\mu_2\nu_2}(x_2)T^{\mu_3\nu_3}(x_3)\rangle$$
$$+ 2\left(\delta^{\mu_1\kappa}x_{1\rho} - \delta^\kappa_\rho x_1^{\mu_1}\right)\langle T^{\rho\nu_1}(x_1)T^{\mu_2\nu_2}(x_2)T^{\mu_3\nu_3}(x_3)\rangle + 2\left(\delta^{\nu_1\kappa}x_{1\rho} - \delta^\kappa_\rho x_1^{\nu_1}\right)\langle T^{\mu_1\rho}(x_1)T^{\mu_2\nu_2}(x_2)T^{\mu_3\nu_3}(x_3)\rangle$$
$$+ 2\left(\delta^{\mu_2\kappa}x_{2\rho} - \delta^\kappa_\rho x_2^{\mu_2}\right)\langle T^{\mu_1\nu_1}(x_1)T^{\rho\nu_2}(x_2)T^{\mu_3\nu_3}(x_3)\rangle + 2\left(\delta^{\nu_2\kappa}x_{2\rho} - \delta^\kappa_\rho x_2^{\nu_2}\right)\langle T^{\mu_1\nu_1}(x_1)T^{\mu_2\rho}(x_2)T^{\mu_3\nu_3}(x_3)\rangle$$
$$+ 2\left(\delta^{\mu_3\kappa}x_{3\rho} - \delta^\kappa_\rho x_3^{\mu_3}\right)\langle T^{\mu_1\nu_1}(x_1)T^{\mu_2\nu_2}(x_2)T^{\rho\nu_3}(x_3)\rangle + 2\left(\delta^{\nu_3\kappa}x_{3\rho} - \delta^\kappa_\rho x_3^{\nu_3}\right)\langle T^{\mu_1\nu_1}(x_1)T^{\mu_2\nu_2}(x_2)T^{\mu_3\rho}(x_3)\rangle$$

Aomalous conformal WI

$$C^2 = R_{abcd}R^{abcd} - \frac{4}{d-2}R_{ab}R^{ab} + \frac{2}{(d-2)(d-1)}R^2, \qquad E = R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2$$

$$\begin{aligned} g_{\mu_1\nu_1} \langle T^{\mu_1\nu_1}(p_1)T^{\mu_2\nu_2}(p_2)T^{\mu_3\nu_3}(p_3)\rangle \\ = 4\,\mathcal{A}^{\mu_2\nu_2\mu_3\nu_3}(p_2,p_3) - 2\,\langle T^{\mu_2\nu_2}(p_1+p_2)T^{\mu_3\nu_3}(p_3)\rangle - 2\,\langle T^{\mu_2\nu_2}(p_2)T^{\mu_3\nu_3}(p_1+p_3)\rangle \end{aligned}$$

$$\begin{aligned} = 4\left[\beta_a\left[C^2\right]^{\mu_2\nu_2\mu_3\nu_3}(p_2,p_3) + \beta_b\left[E\right]^{\mu_2\nu_2\mu_3\nu_3}(p_2,p_3)\right] \\ - 2\,\langle T^{\mu_2\nu_2}(p_1+p_2)T^{\mu_3\nu_3}(p_3)\rangle - 2\,\langle T^{\mu_2\nu_2}(p_2)T^{\mu_3\nu_3}(p_1+p_3)\rangle\,. \end{aligned}$$

Are affected by the anomaly

$$\sum_{j=1}^{2}\left[2(\Delta_j-d)\frac{\partial}{\partial p_j^{\kappa}}-2p_j^{\alpha}\frac{\partial}{\partial p_j^{\alpha}}\frac{\partial}{\partial p_j^{\kappa}}+(p_j)_{\kappa}\frac{\partial}{\partial p_j^{\alpha}}\frac{\partial}{\partial p_{j\alpha}}\right]\langle T^{\mu_1\nu_1}(p_1)\,T^{\mu_2\nu_2}(p_2)\,T^{\mu_3\nu_3}(\bar{p}_3)\rangle$$
$$+\,2\left(\delta^{\kappa(\mu_1}\frac{\partial}{\partial p_1^{\alpha_1}}-\delta^{\kappa}_{\alpha_1}\delta^{\lambda(\mu_1}\frac{\partial}{\partial p_1^{\lambda}}\right)\langle T^{\nu_1)\alpha_1}(p_1)\,T^{\mu_2\nu_2}(p_2)\,T^{\mu_3\nu_3}(\bar{p}_3)\rangle$$
$$+\,2\left(\delta^{\kappa(\mu_2}\frac{\partial}{\partial p_2^{\alpha_2}}-\delta^{\kappa}_{\alpha_2}\delta^{\lambda(\mu_2}\frac{\partial}{\partial p_2^{\lambda}}\right)\langle T^{\nu_2)\alpha_2}(p_2)\,T^{\mu_3\nu_3}(\bar{p}_3)\,T^{\mu_1\nu_1}(p_1)\rangle=0.$$

projectors

Reconstruction in the BMS approach

$$\pi^{\mu}_{\alpha}=\delta^{\mu}_{\alpha}-\frac{p^{\mu}p_{\alpha}}{p^2},\qquad \tilde{\pi}^{\mu}_{\alpha}=\frac{1}{d-1}\pi^{\mu}_{\alpha}$$
$$\Pi^{\mu\nu}_{\alpha\beta}=\frac{1}{2}\left(\pi^{\mu}_{\alpha}\pi^{\nu}_{\beta}+\pi^{\mu}_{\beta}\pi^{\nu}_{\alpha}\right)-\frac{1}{d-1}\pi^{\mu\nu}\pi_{\alpha\beta},$$
$$\mathcal{I}^{\mu\nu}_{\alpha}=\frac{1}{p^2}\left[2p^{(\mu}\delta^{\nu)}_{\alpha}-\frac{p_{\alpha}}{d-1}(\delta^{\mu\nu}+(d-2)\frac{p^{\mu}p^{\nu}}{p^2})\right]$$
$$\mathcal{I}^{\mu\nu}_{\alpha\beta}=\mathcal{I}^{\mu\nu}_{\alpha}p_{\beta}=\frac{p_{\beta}}{p^2}\left(p^{\mu}\delta^{\nu}_{\alpha}+p^{\nu}\delta^{\mu}_{\alpha}\right)-\frac{p_{\alpha}p_{\beta}}{p^2}\left(\delta^{\mu\nu}+(d-2)\frac{p^{\mu}p^{\nu}}{p^2}\right)$$
$$\mathcal{L}^{\mu\nu}_{\alpha\beta}=\frac{1}{2}\left(\mathcal{I}^{\mu\nu}_{\alpha\beta}+\mathcal{I}^{\mu\nu}_{\beta\alpha}\right)\qquad \tau^{\mu\nu}_{\alpha\beta}=\tilde{\pi}^{\mu\nu}\delta_{\alpha\beta}$$

$$T^{\mu\nu}=t^{\mu\nu}+t^{\mu\nu}_{loc}$$

transverse traceless sector

$$\langle t^{\mu_1\nu_1}(p_1)t^{\mu_2\nu_2}(p_2)t^{\mu_3\nu_3}(p_3)\rangle=\Pi_1{}^{\mu_1\nu_1}_{\alpha_1\beta_1}\Pi_2{}^{\mu_2\nu_2}_{\alpha_2\beta_2}\Pi_3{}^{\mu_3\nu_3}_{\alpha_3\beta_3}\,\langle T^{\alpha_1\beta_1}(p_1)T^{\alpha_2\beta_2}(p_2)T^{\alpha_3\beta_3}(p_3)\rangle$$

the intermediate steps are rather technical (ee BMS, "Implications of conformal symmetry in momentum space")

$$
\begin{aligned}
&K_{13}A_1 = 0 && K_{23}A_1 = 0 \\
&K_{13}A_2 = 8A_1 && K_{23}A_2 = 8A_1 \\
&K_{13}A_2(p_1 \leftrightarrow p_3) = -8A_1 && K_{23}A_2(p_1 \leftrightarrow p_3) = 0 \\
&K_{13}A_2(p_2 \leftrightarrow p_3) = 0 && K_{23}A_2(p_2 \leftrightarrow p_3) = -8A_1 \\
&K_{13}A_3 = 2A_2 && K_{23}A_3 = 2A_2 \\
&K_{13}A_3(p_1 \leftrightarrow p_3) = -2A_2(p_1 \leftrightarrow p_3) && K_{23}A_3(p_1 \leftrightarrow p_3) = 0 \\
&K_{13}A_3(p_2 \leftrightarrow p_3) = 0 && K_{23}A_3(p_2 \leftrightarrow p_3) = -2A_2(p_2 \leftrightarrow p_3) \\
&K_{13}A_4 = -4A_2(p_2 \leftrightarrow p_3) && K_{23}A_4 = -4A_2(p_1 \leftrightarrow p_3) \\
&K_{13}A_4(p_1 \leftrightarrow p_3) = 4A_2(p_2 \leftrightarrow p_3) && K_{23}A_4(p_1 \leftrightarrow p_3) = 4A_2(p_2 \leftrightarrow p_3) - 4A_2 \\
&K_{13}A_4(p_2 \leftrightarrow p_3) = 4A_2(p_1 \leftrightarrow p_3) - 4A_2 && K_{23}A_4(p_2 \leftrightarrow p_3) = 4A_2(p_1 \leftrightarrow p_3) \\
&K_{13}A_5 = 2\left[A_4 - A_4(p_1 \leftrightarrow p_3)\right] && K_{23}A_5 = 2\left[A_4 - A_4(p_2 \leftrightarrow p_3)\right]
\end{aligned}
$$

primary WI's

and secondary WI's which connect 3- and 2-point functions

The primary can be solved in temrs of 3K integrals and define a generalised hypergeometric system of Appell type for F4.

The generality of the BMS soluton, needs to be investigated from free field theory
in order to explore whether the structure of the anomalous correlator is the one predicted from free field theory
The free field theory approach is exactly equivalent to the general solutions since at d=4. TTT has 3 constants of Integration and there are 3 free field theories available for its representations.

Notice that the scaling dimension of T is fixed, equal to d.
THings would be different for arbitrary scalar operators

Drastic simplifications of the 3K Bessel functions integrals, in terms of standard perturbative master integrals

Phys.Lett.B 781 (2018) Maglio, CC

Renormalization, Conformal Ward Identities and the Origin of a Conformal Anomaly Pole

Conformal field theory in momentum space and anomaly actions in gravity: The analysis of three- and four-point function

Maglio, CC, Phys. Reports 2022

4 point functions

Eur.Phys.J.C 80 (2020) 6, 540
• e-Print: 1912.01907

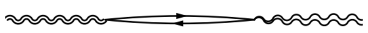
Maglio, Theofilopoulos, CC

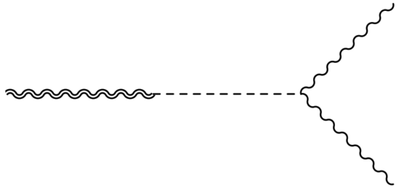
The TJJ pole?

Also in this case the free feld theory can be **Compared with the general BMS one**

$$
\begin{aligned}
0 &= K_{13}A_1 \\
0 &= K_{13}A_2 + 2A_1 \\
0 &= K_{13}A_3 - 4A_1 \\
0 &= K_{13}A_3(p_2 \leftrightarrow p_3) \\
0 &= K_{13}A_4 - 2A_3(p_2 \leftrightarrow p_3)
\end{aligned}
\qquad
\begin{aligned}
0 &= K_{23}A_1 \\
0 &= K_{23}A_2 \\
0 &= K_{23}A_3 - 4A_1 \\
0 &= K_{23}A_3(p_2 \leftrightarrow p_3) + 4A_1 \\
0 &= K_{23}A_4 + 2A_3 - 2A_3(p_2 \leftrightarrow p_3).
\end{aligned}
$$

$$
\begin{aligned}
A_1 &= 4(F_7 - F_3 - F_5) - 2p_2^2 F_9 - 2p_3^2 F_{10} \\
A_2 &= 2(p_1^2 - p_2^2 - p_3^2)(F_7 - F_5 - F_3) - 4p_2^2 p_3^2 (F_6 - F_8 + F_4) - 2F_{13} \\
A_3 &= p_3^2(p_1^2 - p_2^2 - p_3^2)F_{10} - 2p_2^2\, p_3^2 F_{12} - 2F_{13} \\
A_3(p_2 \leftrightarrow p_3) &= p_2^2(p_1^2 - p_2^2 - p_3^2)F_9 - 2p_2^2 p_3^2 F_{11} - 2F_{13} \\
A_4 &= (p_1^2 - p_2^2 - p_3^2)F_{13},
\end{aligned}
$$





$$
\mathcal{S}_A \sim \beta(e) \int d^4x\, d^4y\, R^{(1)}(x) \left(\frac{1}{\square}\right)(x,y) F^{\mu\nu} F_{\mu\nu}(y),
$$

$$
F_1 = \frac{(d-4)}{p_1^2(d-1)} \left[F_{13} - p_2^2\, F_3 - p_3^2\, F_5 - p_2 \cdot p_3\, F_7\right]
$$

F1 has a pole coming from the trace WI, while F2 does not

$$
F_2 = \frac{(d-4)}{p_1^2(d-1)} \left[p_2^2\, F_4 + p_3^2\, F_6 + p_2 \cdot p_3\, F_8\right].
$$

Anomalies and renormalization are connected.
The result is a true nonlocal interaction

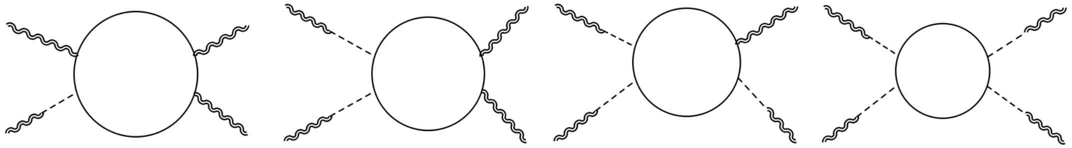
$$\langle T^{\mu_1\nu_1}(p_1)T^{\mu_2\nu_2}(p_2)T^{\mu_3\nu_3}(p_3)T^{\mu_4\nu_4}(\bar{p}_4)\rangle_{poles} =$$

$$= \frac{\pi^{\mu_1\nu_1}(p_1)}{3}\,\langle T(p_1)T^{\mu_2\nu_2}(p_2)T^{\mu_3\nu_3}(p_3)T^{\mu_4\nu_4}(\bar{p}_4)\rangle_{anomaly} + (perm.)$$

$$- \frac{\pi^{\mu_1\nu_1}(p_1)}{3}\frac{\pi^{\mu_2\nu_2}(p_2)}{3}\,\langle T(p_1)T(p_2)T^{\mu_3\nu_3}(p_3)T^{\mu_4\nu_4}(\bar{p}_4)\rangle_{anomaly} + (perm.)$$

$$+ \frac{\pi^{\mu_1\nu_1}(p_1)}{3}\frac{\pi^{\mu_2\nu_2}(p_2)}{3}\frac{\pi^{\mu_3\nu_3}(p_2)}{3}\,\langle T(p_1)T(p_2)T(p_3)T^{\mu_4\nu_4}(\bar{p}_4)\rangle_{anomaly} + (perm.)$$

$$- \frac{\pi^{\mu_1\nu_1}(p_1)}{3}\frac{\pi^{\mu_2\nu_2}(p_2)}{3}\frac{\pi^{\mu_3\nu_3}(p_3)}{3}\frac{\pi^{\mu_4\nu_4}(p_4)}{3}\,\langle T(p_1)T(p_2)T(p_3)T(\bar{p}_4)\rangle_{anomaly}\,.$$



Maglio, Theofilopoulos, CC

Figure 3 The Weyl-variant contributions from $\mathcal{S}_A$ to the renormalized vertex for the 4T with the corresponding bilinear mixings in $d = 4$

$$\mathcal{S}_A = \int d^4x_1 d^4x_2 \langle T\cdot h(x_1)T\cdot h(x_2)\rangle + \int d^4x_1 d^4x_2 d^4x_3 \langle T\cdot h(x_1)T\cdot h(x_2)T\cdot h(x_3)\rangle_{pole}$$

$$+ \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \left(\langle T\cdot h(x_1)T\cdot h(x_2)T\cdot h(x_3)T\cdot h(x_4)\rangle_{pole} + \right.$$

$$\left. + \langle T\cdot h(x_1)T\cdot h(x_2)T\cdot h(x_3)T\cdot h(x_4)\rangle_{0T}\right),$$

MISSING TERMS IN THE NONLOCAL ANOMALY-INDUCED ACTION

Four-Point Functions of Gravitons and Conserved Currents of CFT in Momentum Space: Testing the Nonlocal Action with the TTJJ

MAGLIO, TOMMASI, CC

EPJC 2023

$$\mathcal{S}^{(2)}_{anom} = -\frac{\beta_C}{6}\int d^4x \int d^4x' \left\{ (\sqrt{-g}\,F^2)^{(1)}_x \left(\frac{1}{\Box_0}\right)_{xx'} R^{(1)}_{x'} + F^2_x \left(\frac{1}{\Box_0}\right)_{xx'} R^{(2)}_{x'} \right.$$
$$+ \int d^4x'' \Bigg[F^2_x \left(\frac{1}{\Box_0}\right)_{xx'} (\Box_1)_{x'} \left(\frac{1}{\Box_0}\right)_{x'x''} R^{(1)}_{x''} - \frac{1}{6} F^2_x \left(\frac{1}{\Box_0}\right)_{xx'} R^{(1)}_{x'} \left(\frac{1}{\Box_0}\right)_{x'x''} R^{(1)}_{x''}$$
$$\left. + \frac{1}{3} R^{(1)}_x \left(\frac{1}{\Box_0}\right)_{xx'} F^2_{x'} \left(\frac{1}{\Box_0}\right)_{x'x''} R^{(1)}_{x''} \Bigg] \right\}.$$

(Maglio, Theofilopoulos, CC, arXiv:2103.13957 , **EPJ C**)

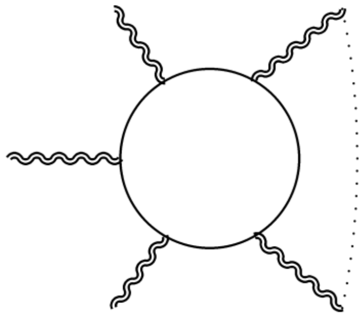
The action describes the Conformal Backreaction

Maglio, Theofilopoulos, CC

$$\mathcal{S}(g) = \mathcal{S}(\bar{g}) + \sum_{n=1}^{\infty} \frac{1}{2^n n!} \int d^d x_1 \ldots d^d x_n \sqrt{g_1} \ldots \sqrt{g_n} \, \langle T^{\mu_1 \nu_1} \ldots T^{\mu_n \nu_n} \rangle_{\bar{g}} \delta g_{\mu_1 \nu_1}(x_1) \ldots \delta g_{\mu_n \nu_n}(x_n).$$

Diagrammatically, for a scalar theory in a flat background, it takes the form

$$\mathcal{S}(g) = \sum_n$$



(n-point)

$$\mathcal{S}_A \sim \int d^4x \, d^4y R^{(1)}(x) \left(\frac{1}{\Box}\right)(x,y) \left(b' \, E_4^{(2)}(y) + b \, (C^2)^{(2)}(y)\right),$$

$$\frac{1}{p^2} \hat{\pi}^{\mu\nu} \leftrightarrow R^{(1)} \frac{1}{\Box}$$

$$\mathcal{Z}_B(g) = \mathcal{N}\int D\Phi e^{-S_0(g,\chi)},$$

$$e^{-\mathcal{S}_B(g)} = \mathcal{Z}_B(g) \leftrightarrow \mathcal{S}_B(g) = -\log \mathcal{Z}_B(g).$$

$$\mathcal{Z}_R(g) = \mathcal{N}\int D\Phi e^{-S_0(g,\Phi)+b'\frac{1}{\epsilon}V_E(g)+b\frac{1}{\epsilon}V_{C^2}(g)}.$$

$$V_{C^2}(d) \equiv \mu^{\varepsilon}\int d^dx\,\sqrt{-g}\,C^2$$

$$V_E(d) \equiv \mu^{\varepsilon}\int d^dx\,\sqrt{-g}\,E,$$

$$g_{\mu\nu} = e^{2\phi(x)}\bar{g}_{\mu\nu} \qquad \bar{g}_{\mu\nu} = e^{-2\phi}g_{\mu\nu}.$$

$$V_E = 4\pi\chi(\mathcal{M}).$$

$$V'_E = \frac{\partial V_E(d)}{\partial d}\,|_{d=4} \qquad V'_{C^2} = \frac{\partial V_{C^2}(d)}{\partial d}\,|_{d=4}$$

Dilaton explicit in the effective action

$$\mathcal{S}_R \equiv \mathcal{S}_R(4) = \mathcal{S}_f(4) + V'_E(4) + V'_{C^2}(4).$$

RECENT Proposal

Evading Lovelock's theorem by a renormalization of the coupling (Glavan and Lin; R. Mann et al)

$$\mathcal{S}_{EGB} = \mathcal{S}_{EH} + \mathcal{S}_{GB}(d) \qquad \mathcal{S}_{GB}(d) = \frac{\alpha}{\epsilon} V_E(d)$$

$$\frac{1}{\kappa}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda_0 g_{\mu\nu}\right) + \alpha(V_E(d))_{\mu\nu} = 0,$$

$$\mathcal{S}_{GB}^{(WZ)} = \alpha \int d^4x \sqrt{\bar{g}} \Big[\phi \bar{E} - \Big(4\bar{G}^{\mu\nu}(\bar{\nabla}_\mu \phi \bar{\nabla}_\nu \phi) + 2(\bar{\nabla}_\lambda \phi \bar{\nabla}^\lambda \phi)^2 + 4\bar{\Box}\phi \bar{\nabla}_\lambda \phi \bar{\nabla}^\lambda \phi\Big)\Big],$$

This action is quartic

Can we remove the dilaton? •e-Print: 2201.07515 Maglio CC

In DR this is possible by a finite renormalization of the E counterterm

$$S_{EGB} = S_{EH} + \alpha V_E,$$

$$E_{ext} = E + \frac{\epsilon}{2(d-1)^2} R^2,$$

Several papers, R. Mann et al, Glavan and Lin PRL

$$\delta_\sigma \mathcal{S} = \frac{1}{(4\pi)^2} \int d^4x \sqrt{g}\, \sigma \left(b_1 C^{(4)}_{\mu\nu\rho\sigma} C^{(4)\mu\nu\rho\sigma} + b_2 E_4 + b_3 \Box R \right)$$

$$\mathcal{S}^{(3)}_{\rm anom} = \frac{1}{9} \int d^4x \int d^4x' \int d^4x'' \Big\{ (\partial_\mu R^{(1)})_x \left(\frac{1}{\overline{\Box}}\right)_{xx'} \left(R^{(1)\mu\nu} - \frac{1}{3}\eta^{\mu\nu} R^{(1)}\right)_{x'} \left(\frac{1}{\overline{\Box}}\right)_{x'x''} (\partial_\nu R^{(1)})_{x''} \Big\}$$
$$- \frac{1}{6} \int d^4x \int d^4x' \left(E^{(2)} \right)_x \left(\frac{1}{\overline{\Box}}\right)_{xx'} R^{(1)}_{x'} + \frac{1}{18} \int d^4x\, R^{(1)} \left(2\, R^{(2)} + (\sqrt{-g})^{(1)} R^{(1)} \right) \qquad (9.$$

$$\mathcal{S}^{NL}_{\rm anom}[g] = \frac{1}{4} \int d^4x \sqrt{-g_x} \left(E - \frac{2}{3} \Box R \right)_x \int d^4x' \sqrt{-g_{x'}}\, D_4(x,x') \left[\frac{b'}{2} \left(E - \frac{2}{3} \Box R \right) + b\, C^2 \right]_{x'}$$

$$\Delta_4 \equiv \Box^2 + 2 R^{\mu\nu} \nabla_\mu \nabla_\nu - \frac{2}{3} R \Box + \frac{1}{3} (\nabla^\mu R) \nabla_\mu$$

Topological Corrections and Conformal Backreaction in the Einstein Gauss-Bonnet/Weyl Theories of Gravity at $D = 4$

Maglio, CC, Theofilopoulos,

Conclusions

The breaking of conformal symmetry is associated to the the propagation of massless effective states in the effective action.

For chiral anomalies, the interactions can be reconstructed by a combination of the Anomaly pole + CWIs. We have shown it in the case of the AVV, for the J5TT (work in preparation)

For parity breaking trace/conformal anomalies, we have also shown that
the reconstruction can also be based entirely on the selection of an anomaly pole to solve the CWIs.

We have used the TTJJ correlator to show that the anomaly induced actions either in the Riegert form or in the Fradkin-Vilkovisky form miss crucial Weyl invariant terms in order to be consistent with the CWIs
and identified such terms

Applicatons

Condensed Matter theory: application of this class of nonlocal actions in the context of topological Materials (via Luttinger formula)

THANKS FOR THE INVITATION and for YOUR ATTENTION