NC STATE UNIVERSITY

Center for Frontiers in Nuclear Science

The role of the chiral anomaly in GPDs

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Based on Andrey Tarasov and Raju Venugopalan
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and in preparation

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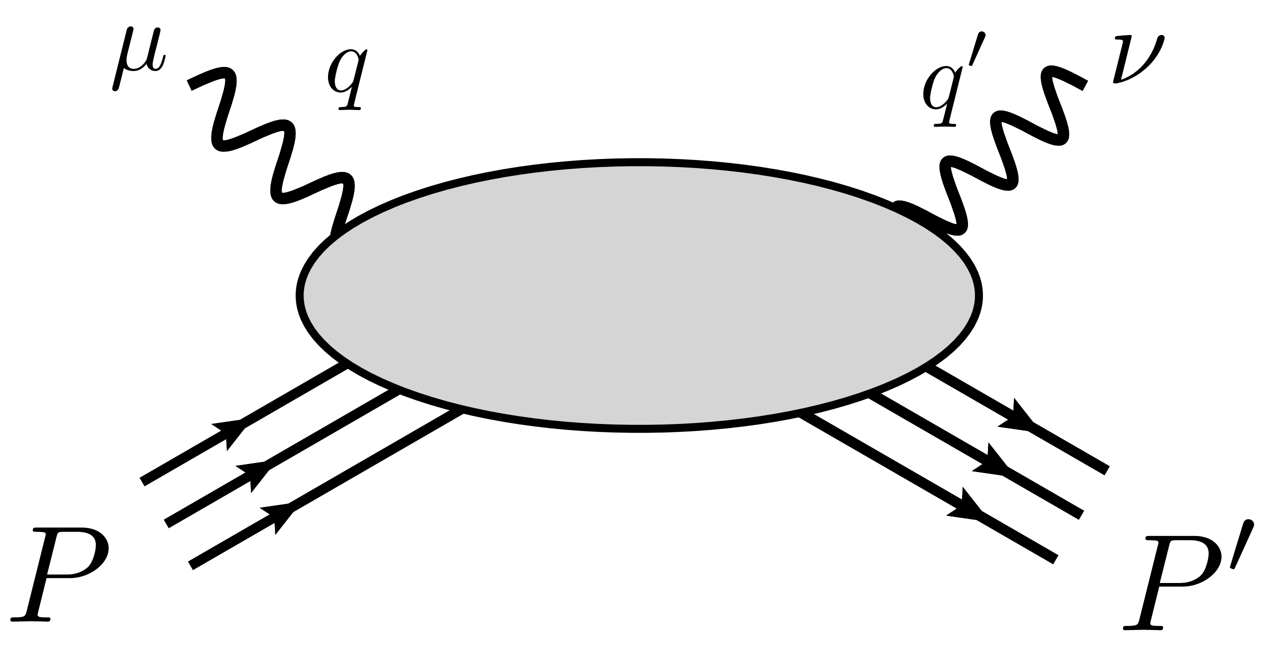
Off-forward Compton amplitude



$$T^{\mu\nu} = i\int d^4z\, e^{\frac{i}{2}(q+q')z}\langle P'|T\{j^\mu(z/2)j^\nu(-z/2)\}|P\rangle$$

Basis of momentum vectors:

$$\bar{P} = \frac{1}{2}(P+P') \qquad \bar{q} = \frac{1}{2}(q+q') \qquad \Delta = P' - P = q - q'$$

Scalar variables:

$$\bar{\omega} = \frac{2\bar{P}\cdot\bar{q}}{\bar{Q}^2} \qquad \vartheta = -\frac{\Delta\cdot\bar{q}}{\bar{Q}^2} \qquad t = \Delta^2 \qquad \bar{Q}^2 = -\bar{q}^2$$

Skewness variable: $\xi = \vartheta/\bar{\omega}$

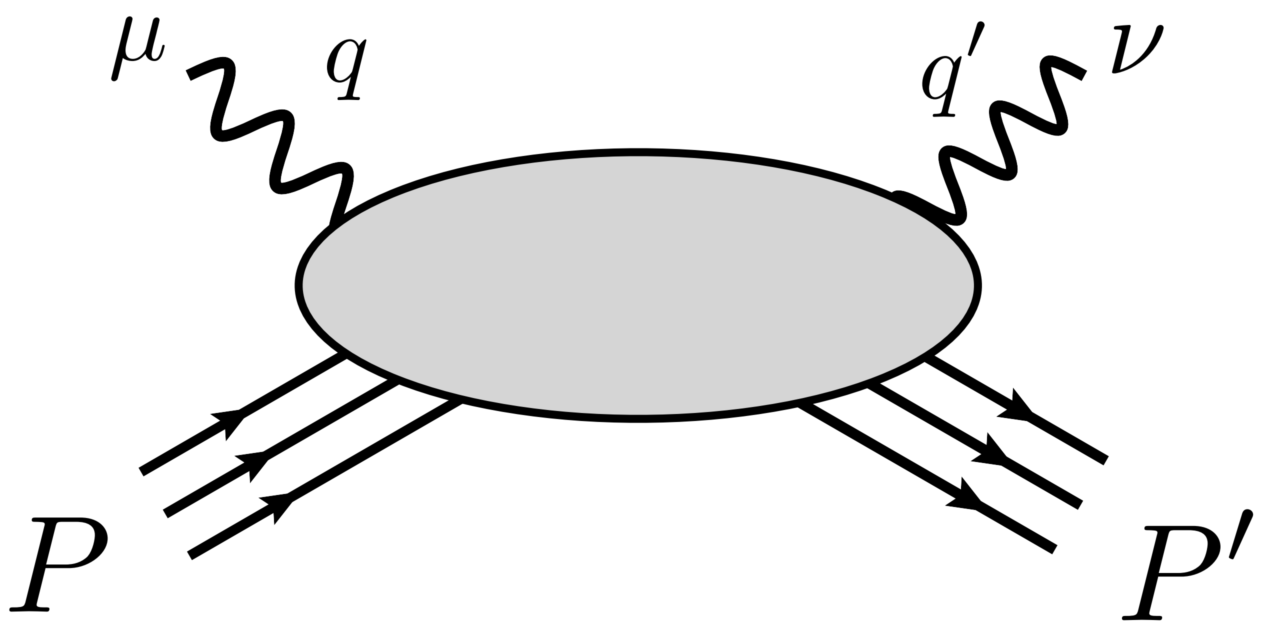
A. Hannaford-Gunn et al. (2022)

General decomposition of the off-forward Compton amplitude contains 18 tensorial structures ⇒ 18 off-forward Compton amplitudes $\mathcal{H}_1$, $\mathcal{E}_1$ etc.:

$$\begin{aligned}
\bar{T}_{\mu\nu} = {} & \frac{1}{2\bar{P}\cdot\bar{q}}\Big[-\Big(h\cdot\bar{q}\mathcal{H}_1 + e\cdot\bar{q}\mathcal{E}_1\Big)g_{\mu\nu} + \frac{1}{\bar{P}\cdot\bar{q}}\Big(h\cdot\bar{q}\mathcal{H}_2 + e\cdot\bar{q}\mathcal{E}_2\Big)\bar{P}_\mu\bar{P}_\nu + \mathcal{H}_3 h_{\{\mu}\bar{P}_{\nu\}}\Big] \\
& + \frac{i}{2\bar{P}\cdot\bar{q}}\epsilon_{\mu\nu\rho\kappa}\bar{q}^\rho\Big(\tilde{h}^\kappa\tilde{\mathcal{H}}_1 + \tilde{e}^\kappa\tilde{\mathcal{E}}_1\Big) + \frac{i}{2(\bar{P}\cdot\bar{q})^2}\epsilon_{\mu\nu\rho\kappa}\bar{q}^\rho\Big[\Big(\bar{P}\cdot\bar{q}\tilde{h}^\kappa - \tilde{h}\cdot\bar{q}\bar{P}^\kappa\Big)\tilde{\mathcal{H}}_2 + \Big(\bar{P}\cdot\bar{q}\tilde{e}^\kappa - \tilde{e}\cdot\bar{q}\bar{P}^\kappa\Big)\tilde{\mathcal{E}}_2\Big] \\
& + \Big(\bar{P}_\mu q'_\nu + \bar{P}_\nu q_\mu\Big)\Big(h\cdot\bar{q}\mathcal{K}_1 + e\cdot\bar{q}\mathcal{K}_2\Big) + \Big(\bar{P}_\mu q'_\nu - \bar{P}_\nu q_\mu\Big)\Big(h\cdot\bar{q}\mathcal{K}_3 + e\cdot\bar{q}\mathcal{K}_4\Big) + q_\mu q'_\nu\Big(h\cdot\bar{q} - e\cdot\bar{q}\Big)\mathcal{K}_5 \\
& + h_{[\mu}\bar{P}_{\nu]}\mathcal{K}_6 + \Big(h_\mu q'_\nu + h_\nu q_\mu\Big)\mathcal{K}_7 + \Big(h_\mu q'_\nu - h_\nu q_\mu\Big)\mathcal{K}_8 + \bar{P}_{\{\mu}\bar{u}(P')i\sigma_{\nu\}\alpha}u(P)\bar{q}^\alpha\mathcal{K}_9
\end{aligned}$$

Perrottet (1973)
Tarrach (1975)
D. Drechsel, G. Knochlein, A. Y. Korchin, A. Metz, and S. Scherer (1998)

Off-forward Compton amplitude



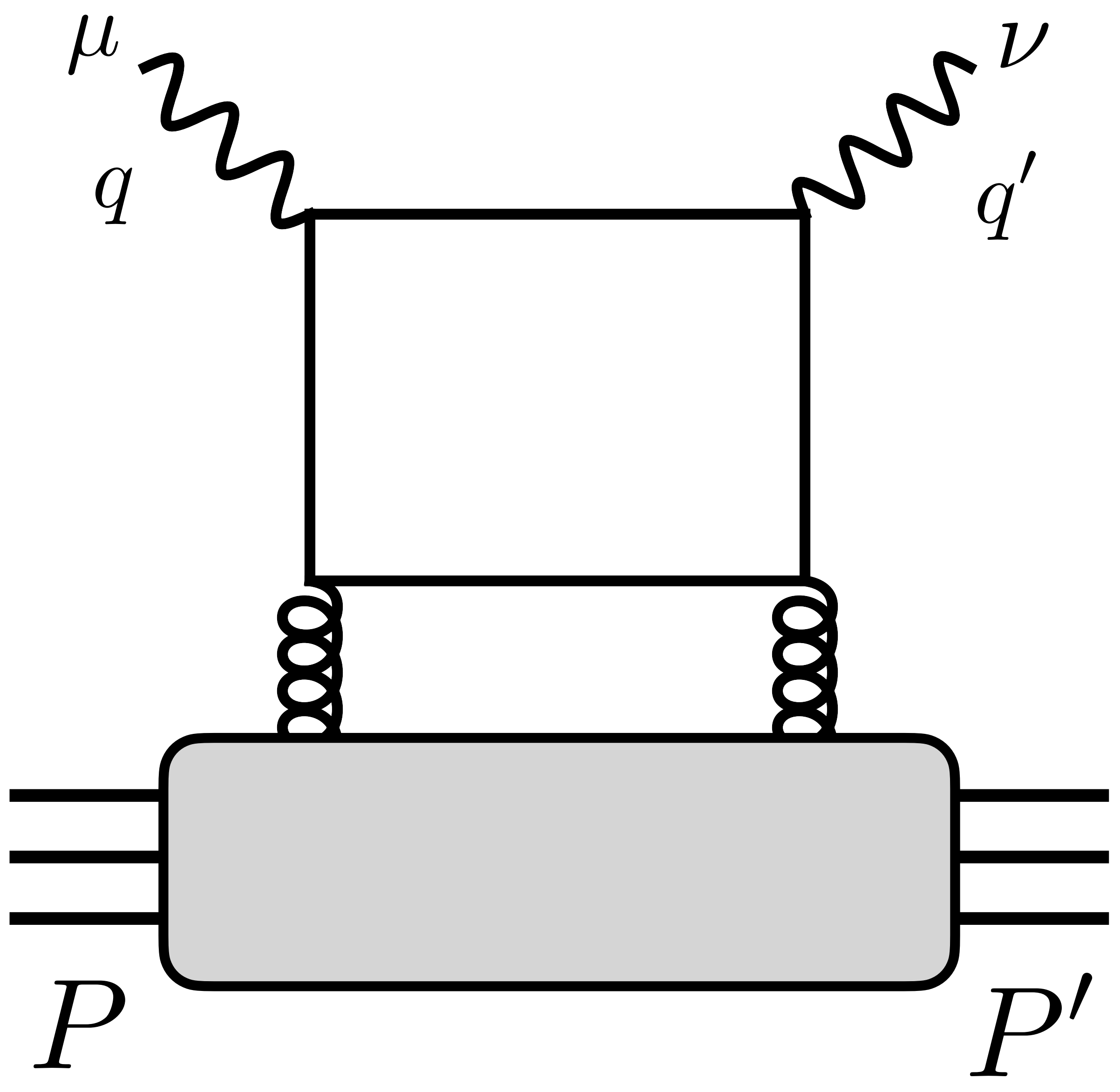
Decomposition depends on the Dirac bilinear:

$$\boxed{h^\mu = \bar{u}'\gamma^\mu u \quad e^\mu = \bar{u}'\frac{i\sigma^{\mu\alpha}\Delta_\alpha}{2M}u \quad \tilde{h}^\mu = \bar{u}'\gamma^\mu\gamma_5 u \quad \tilde{e}^\mu = \frac{\Delta^\mu}{2M}\bar{u}'\gamma_5 u}$$

The amplitudes are related to DIS structure functions:

$$\tilde{\mathcal{H}}_1 \xrightarrow{t\to 0} \tilde{g}_1 \quad \mathrm{Im}\tilde{g}_1 = 2\pi g_1 \quad \tilde{\mathcal{H}}_2 \xrightarrow{t\to 0} \tilde{g}_2 \quad \mathrm{Im}\tilde{g}_2 = 2\pi g_2$$

We are interested in gluonic contribution to the Compton amplitudes. In the leading order it's represented by a quark box diagram with two background gluons.



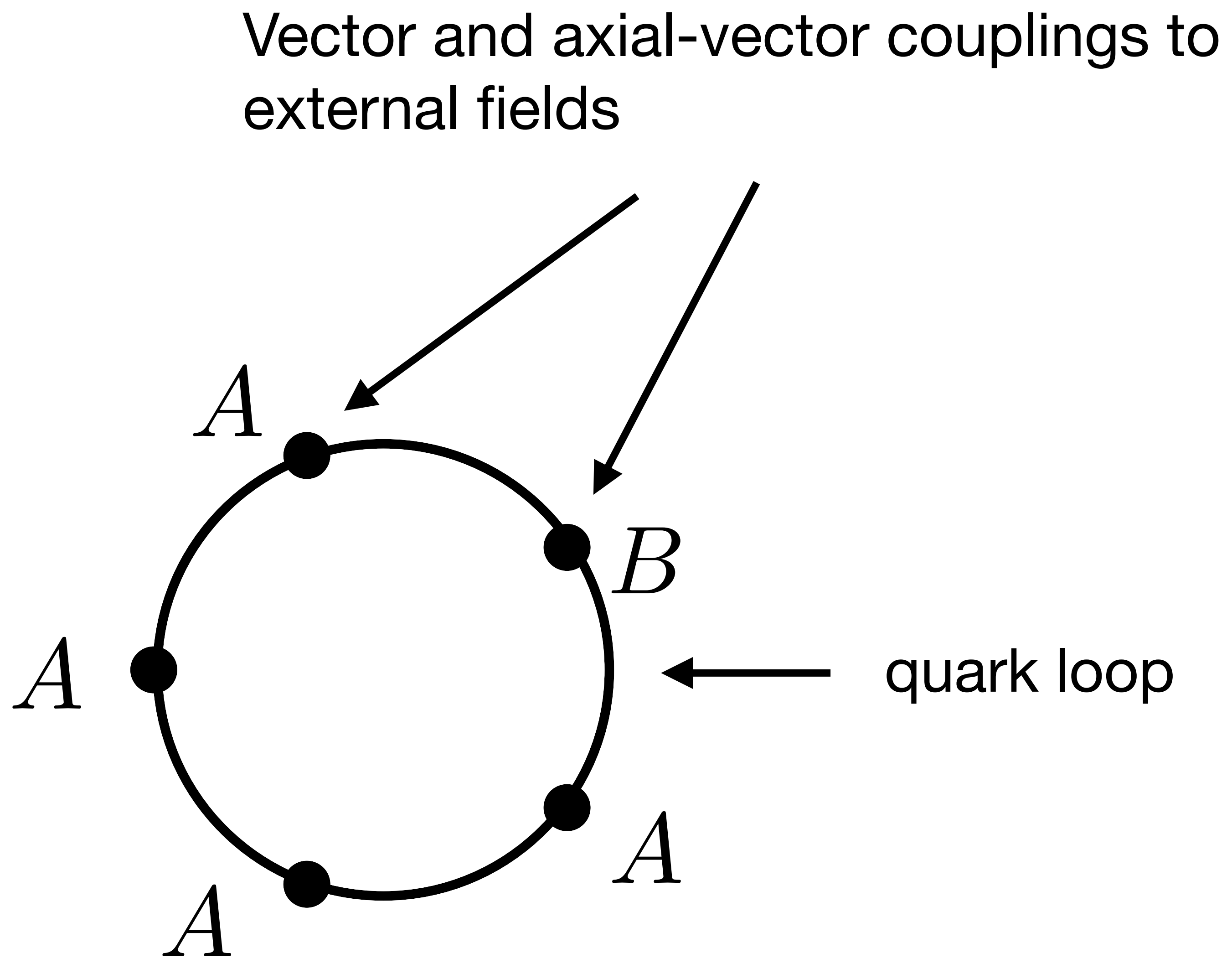
In general, such off-forward calculation in a general kinematics is a formidable tasks. We apply the worldline approach which is a powerful technique for calculation of multi-leg Feynman diagrams

Worldline approach

We start with a general form of the QCD effective action written in the form of a functional integral over quark fields:

$$e^{i\mathcal{W}[A,B]} = \int \mathcal{D}\Psi\mathcal{D}\bar{\Psi}e^{iS[\bar{\Psi},A,B,\Psi]}$$

$$S[\bar{\Psi}, A, B, \Psi] = \int d^4x\bar{\Psi}[i\not{\partial} + \not{A} + \gamma_5\not{B}]\Psi$$


Vector and axial-vector couplings to external fields

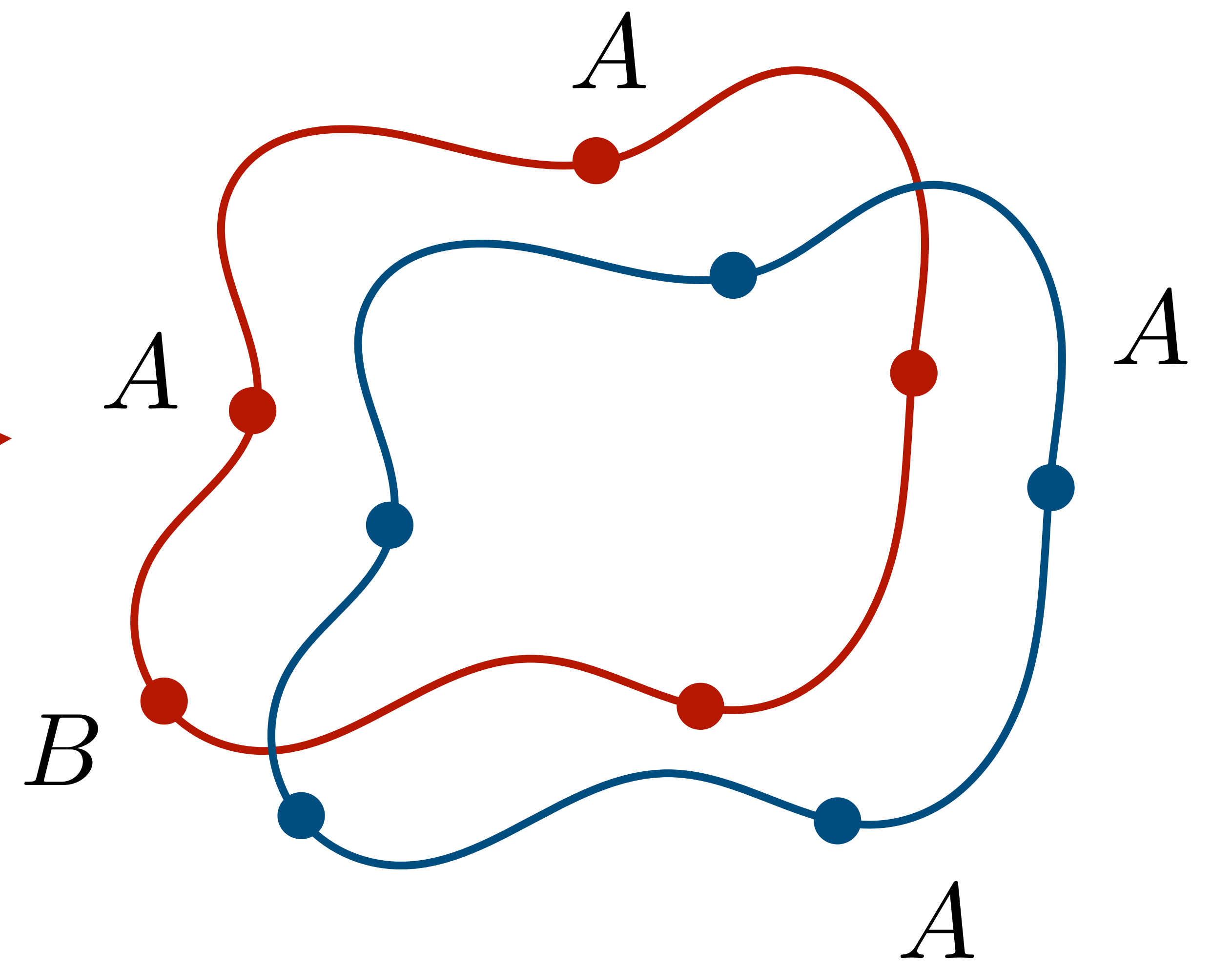
quark loop

In the worldline approach the effective action is rewritten as a quantum mechanical point particle path integrals over classical trajectories $x(\tau)$ and $\psi(\tau)$, where spin degrees of freedom of a particle (quark) are expressed in terms of Grassmann variables ψ.

$$\mathcal{W}[A,B] = -\frac{1}{2}\mathrm{Tr_c}\int_0^\infty \frac{dT}{T}\int_P \mathcal{D}x\int_{AP}\mathcal{D}\psi\; e^{-S_{w.l.}(x,\psi,A,B)}$$

Functional integrals over classical trajectories in two spaces of bosonic and fermionic degrees of freedom

Proper time variable $\tau \in [0,T]$ defines position of a point on the worldline, T is a period (“size”) of the worldline

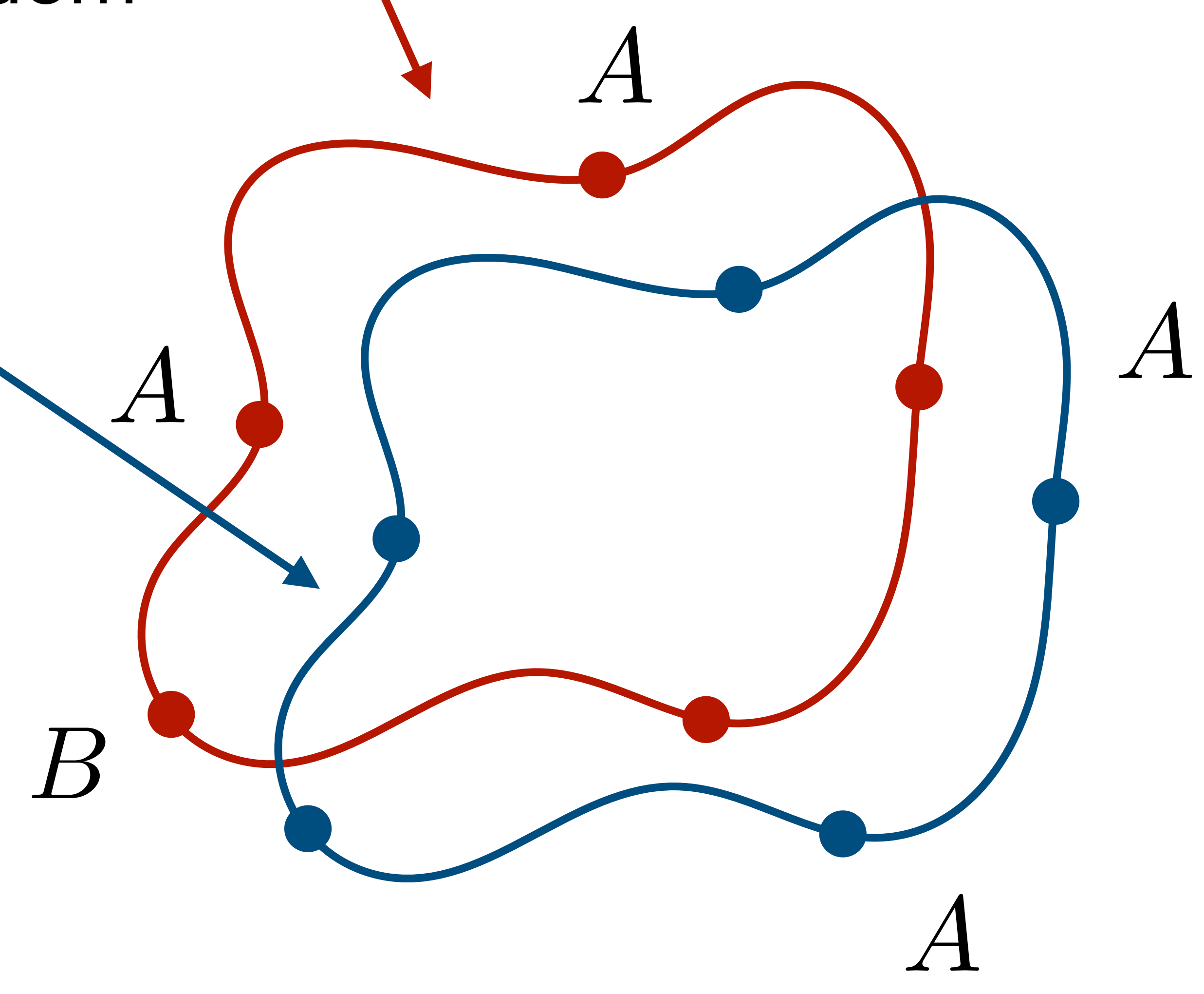


Worldline representation of the effective action

Z. Bern & D.A. Kosower 88
M. J. Strassler 92
E. D'Hoker & D. G. Gagne 96
Schubert 2001

$$\mathcal{W}_{QED}[A] = -\frac{1}{2}\int_0^T \frac{dT}{T}\int_P \mathcal{D}x \int_{AP}\mathcal{D}\psi \exp\Big\{-\int_0^T d\tau\Big(\underbrace{\frac{1}{4}\overset{\text{free prop.}}{\dot{x}^2} + ie\dot{x}^\mu \overset{\text{int. term (gauge phase)}}{A_\mu}}_{\text{scalar QED}} + \underbrace{\frac{1}{2}\psi_\mu\dot{\psi}^\mu - ie\psi^\mu\psi^\nu F_{\mu\nu}}_{\text{fermionic degrees of freedom}}\Big)\Big\}$$

boundary conditions



The approach is explicitly gauge invariant: interaction with the background field via gauge phase (Wilson line) and strength tensor → analogous to the background field method

The functional integrals can be easily evaluated using Green functions (worldline propagators) of the corresponding free field operator:

$$\partial_\tau^2 G_B(\tau,\tau') = 2\delta(\tau-\tau') - \frac{2}{T}$$

$$G_B(\tau,\tau') = |\tau-\tau'| - \frac{(\tau-\tau')^2}{T}$$

$$\partial_\tau G_F(\tau,\tau') = 2\delta(\tau-\tau')$$

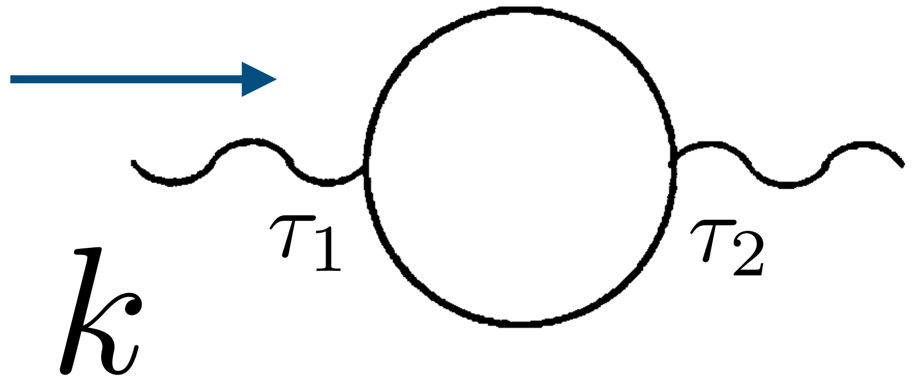
$$G_F(\tau,\tau') = \text{sign}(\tau-\tau')$$

$$\int_P \mathcal{D}y \exp\left\{\frac{1}{2}\int_0^T d\tau d\tau' y^\mu(\tau) G_B^{-1}(\tau,\tau') y_\mu(\tau')\right\} = (4\pi T)^{-D/2}$$

Calculation of the functional integrals is straightforward!

The approach can be efficiently applied to various challenging problems. Example: calculation of the cusp anomalous dimension **Feal, Tarasov, Venugopalan, in preparation**

Example: vacuum polarization tensors



It's easy to write a general expression for a given amplitude

$$\Pi^{\mu\nu}(k_1)(2\pi)^4\delta(k_1+k_2) = -\frac{(ie)^2}{2}\int_0^T \frac{dT}{T}\int_P \mathcal{D}x\int_{AP}\mathcal{D}\psi\,\underbrace{V_1^\mu(k_1)V_2^\mu(k_2)}_{\text{interaction vertexes}}\exp\Big\{\underbrace{-\int_0^T d\tau\left(\frac{1}{4}\dot{x}^2+\frac{1}{2}\psi_\mu\dot{\psi}^\mu\right)}_{\text{free action}}\Big\}$$

Vertexes describe interaction of a worldline with an external current: $V_i^\mu(k_i) \equiv \int_0^T d\tau_i(\dot{x}_i^\mu + 2i\psi_i^\mu k_j\cdot\psi_j)e^{ik_i\cdot x_i}$

The functional integral can be easily evaluated as well:

$$\Pi^{\mu\nu}(k) = 2e^2\int_0^T \frac{dT}{T}(4\pi T)^{-D/2}\int_0^T d\tau_1\int_0^T d\tau_2\Big\{g^{\mu\nu}\ddot{G}_{B12} - k^\mu k^\nu\dot{G}_{B12}^2 - G_{F12}^2(g^{\mu\nu}k^2 - k^\mu k^\nu)\Big\}e^{-k^2G_B(\tau_1,\tau_2)}$$

The really interesting result can be obtained after IBP (integration by part) procedure which eliminated $\ddot{G}_{B12}$ **Strassler (1992)**

$$\Pi^{\mu\nu}(k) = 2e^2\overbrace{\left[\int_0^T \frac{dT}{T}(4\pi T)^{-D/2}\int_0^T d\tau_1\int_0^T d\tau_2(\dot{G}_{B12}^2 - G_{F12}^2)e^{-k^2G_B(\tau_1,\tau_2)}\right]}^{\text{scalar coefficient (scalar integrals over proper-time variables)}}\overbrace{(g^{\mu\nu}k^2 - k^\mu k^\nu)}^{\text{tensorial structure}}$$

After IBP: scalar coefficient multiplied by a tensorial structure
→ Decomposition of the Compton scattering amplitude

The method allows unambiguously define the tensorial structure of the diagram

Spin puzzle and the role of anomaly in polarized DIS

Why calculation in full off-forward kinematics are important? Example: the structure of the $\tilde{\mathcal{H}}_1$ amplitude

$$\bar{T}_{[\mu\nu]} = \frac{i}{2\bar{P}\cdot\bar{q}}\epsilon_{\mu\nu\rho\kappa}\bar{q}^{\rho}\left(\tilde{h}^{\kappa}\tilde{\mathcal{H}}_1 + \tilde{e}^{\kappa}\tilde{\mathcal{E}}_1\right) + \frac{i}{2(\bar{P}\cdot\bar{q})^2}\epsilon_{\mu\nu\rho\kappa}\bar{q}^{\rho}\left[\left(\bar{P}\cdot\bar{q}\tilde{h}^{\kappa} - \tilde{h}\cdot\bar{q}\bar{P}^{\kappa}\right)\tilde{\mathcal{H}}_2 + \left(\bar{P}\cdot\bar{q}\tilde{e}^{\kappa} - \tilde{e}\cdot\bar{q}\bar{P}^{\kappa}\right)\tilde{\mathcal{E}}_2\right] + ..$$

DIS structure functions: $\tilde{\mathcal{H}}_1 \xrightarrow{t\to 0} \tilde{g}_1 \qquad \mathrm{Im}\tilde{g}_1 = 2\pi g_1$

The first moment of g_1 defines quark helicity: $\int_0^1 dx_B\, g_1(x_B, Q^2) = \frac{1}{18}\left(3F + D + 2\,\Sigma(Q^2)\right)\left(1 - \frac{\alpha_s}{\pi} + O(\alpha_s^2)\right) + O\left(\frac{\Lambda^2}{Q^2}\right)$

DIS experiments showed that quarks carry only about 30% of the proton's spin: $\Delta\Sigma \approx 0.32$, which is much smaller than predicted by the quark model $\Delta\Sigma \approx 0.6$ - spin puzzle

$$\frac{1}{2} = \frac{1}{2}\Delta\Sigma + \Delta G + L_q + L_g$$

In the parton model: $S^{\mu}\Delta\Sigma(Q^2) = \frac{1}{M_N}\langle P, S|J_5^{\mu}(0)|P, S\rangle$

isosinglet axial vector current

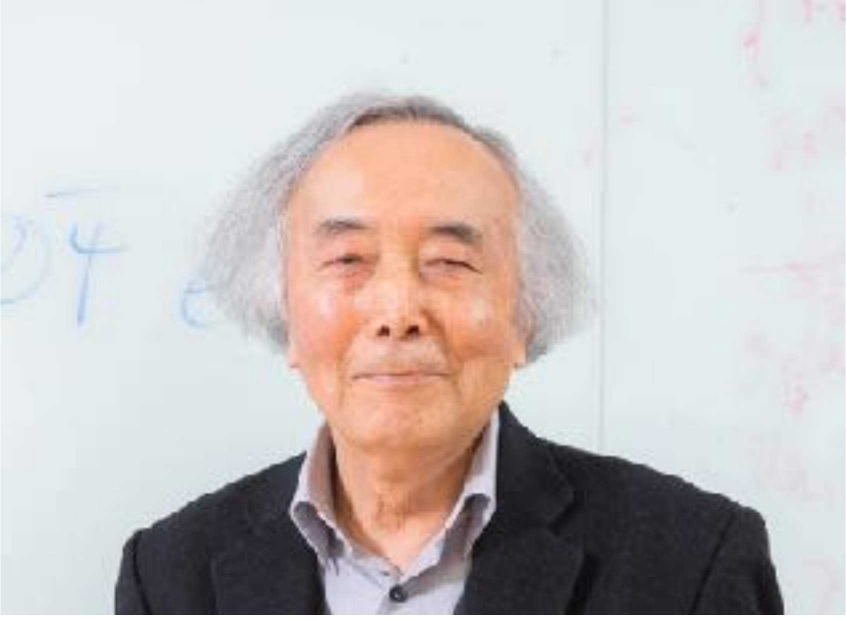
Spin puzzle and the role of anomaly in polarized DIS

$$\frac{1}{2} = \frac{1}{2}\Delta\Sigma + \Delta G + L_q + L_g$$

$$S^\mu \Delta\Sigma(Q^2) = \frac{1}{M_N}\langle P, S | J_5^\mu(0) | P, S\rangle$$

Fundamental property of the current. The anomaly equation

isosinglet axial vector current



$$\partial^\mu J_\mu^5(x) = \frac{n_f \alpha_s}{2\pi} \, \mathrm{Tr}\left(F_{\mu\nu}(x)\tilde{F}^{\mu\nu}(x)\right) \longleftarrow$$

The isosinglet current couples to the topological charge density in the polarized proton!

The anomaly arises from the non-invariance of the path integral measure under chiral (γ_5) rotations. Topological properties of the QCD vacuum! **K. Fujikawa, PRL. 42, 1195 (1979)**

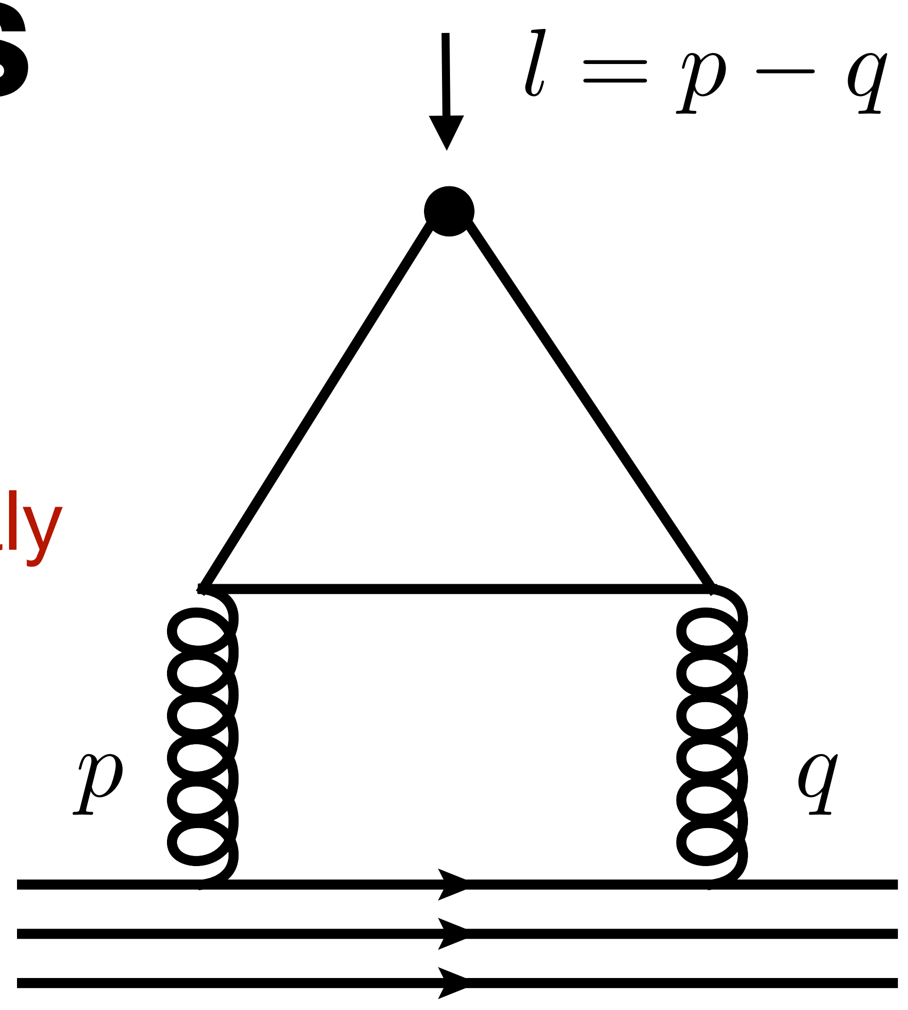
⇩

New tools for an old problem: the small value of $\Delta\Sigma$ can be explained due to interplay between parton dynamics and the topology of the QCD vacuum in the helicity structure of the proton

Hard to see in the standard pQCD calculations! Requires calculations in full kinematics in a general background field

Thinking efficiently about anomalies with worldlines

How does the anomaly manifest itself? To extract the anomaly effects we perform calculation in general kinematics. The exact worldline calculation gives:



exact result! ABJ anomaly

$$\langle P', S | J_5^\mu(0) | P, S\rangle = -i\boxed{\frac{l^\mu}{l^2}}\frac{\alpha_s n_f}{2\pi}\langle P', S|\boxed{\mathrm{Tr}\left(F\tilde{F}\right)}|P, S\rangle$$

infrared pole

topological charge density (pseudoscalar operator)

The worldline approach allows unambiguously define both the kinematic coefficient and the operator structure corresponding to the background gluons

$$\bar{T}_{[\mu\nu]} = \frac{i}{2\bar{P}\cdot\bar{q}}\epsilon_{\mu\nu\rho\kappa}\bar{q}^\rho\left(\tilde{h}^\kappa\tilde{\mathcal{H}}_1 + \tilde{e}^\kappa\tilde{\mathcal{E}}_1\right) + \frac{i}{2(\bar{P}\cdot\bar{q})^2}\epsilon_{\mu\nu\rho\kappa}\bar{q}^\rho\left[\left(\bar{P}\cdot\bar{q}\tilde{h}^\kappa - \tilde{h}\cdot\bar{q}\bar{P}^\kappa\right)\tilde{\mathcal{H}}_2 + \left(\bar{P}\cdot\bar{q}\tilde{e}^\kappa - \tilde{e}\cdot\bar{q}\bar{P}^\kappa\right)\tilde{\mathcal{E}}_2\right] + ..$$

$$\tilde{\mathcal{H}}_1 \xrightarrow{t\to 0} \tilde{g}_1 \longrightarrow \Delta\Sigma$$

the triangle can be related to off-forward amplitudes (first moments)

$$\tilde{e}^\mu = \frac{l^\mu}{2M}\bar{u}'\gamma_5 u$$

pseudoscalar coupling!

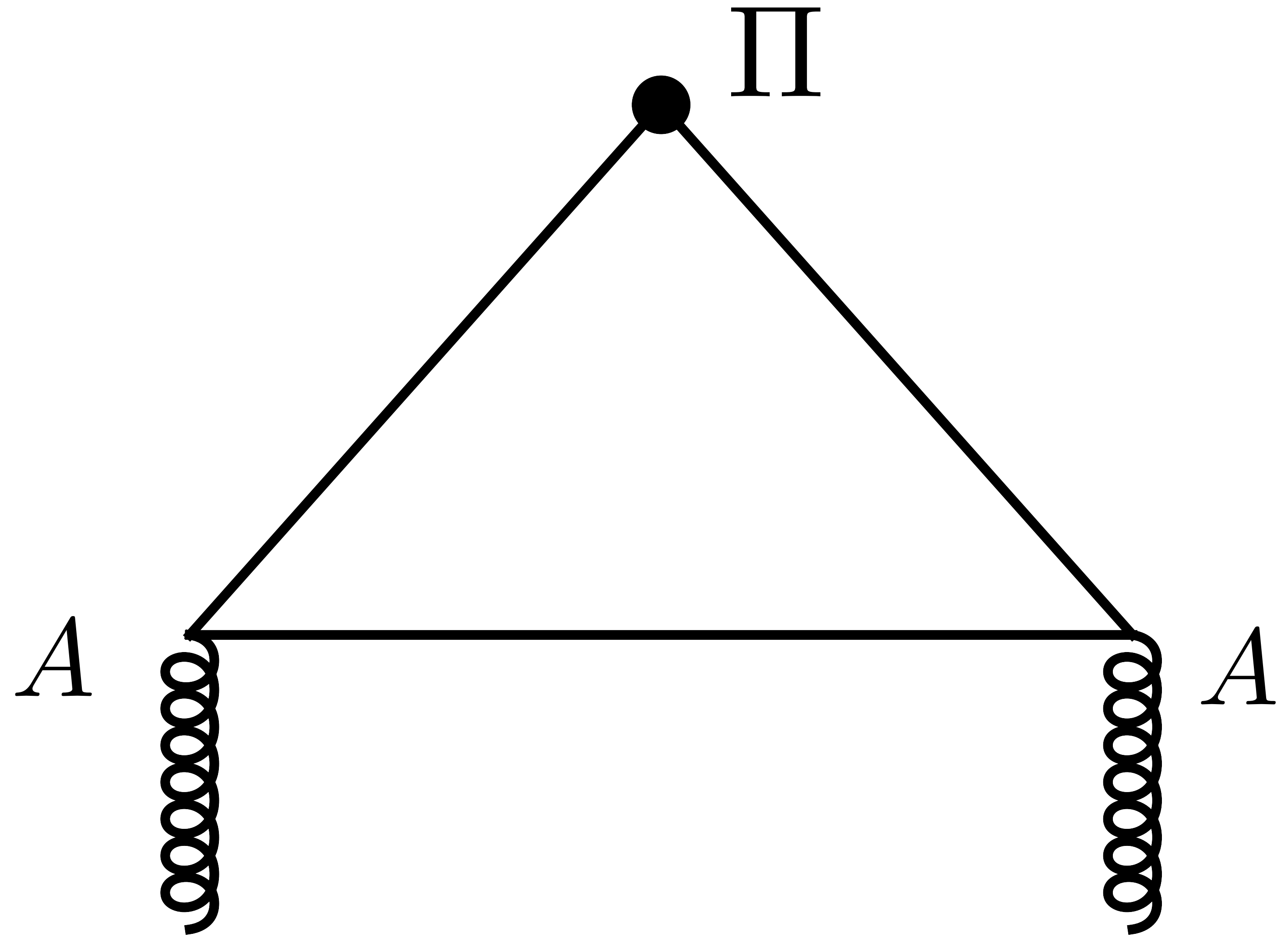
Why poles are important? Mechanism of the pole cancellation allows to relate amplitudes (first moments) between each other and topological properties of the QCD vacuum

R. L. Jaffe, A. Manohar (1990)
Shore, Veneziano (1990)
Tarasov, Venugopalan (2020)
Bhattacharya, Hatta, Vogelsang (2022)

Wess-Zumino-Witten coupling

How does the pole cancel? The crucial role is played by the Wess-Zumino-Witten (WZW) coupling

Using the worldline approach, we calculate the imaginary part of the effective action in the leading order in Π, which generates the isosinglet Wess-Zumino-Witten coupling $\propto \bar{\eta} F \tilde{F}$



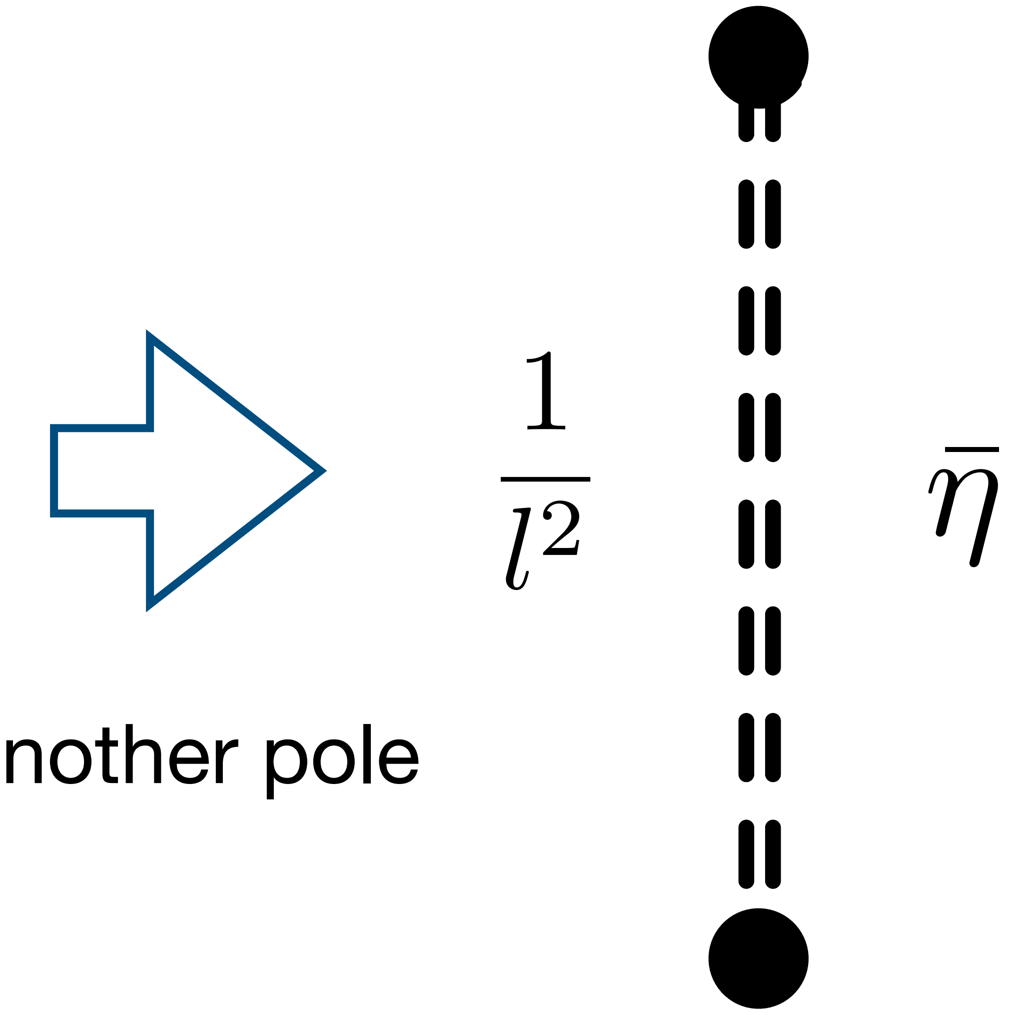
Tarasov, Venugopalan (2022)

$$\mathcal{W}_{\mathcal{I}}[\Pi A^2] = \frac{ig^2\, 2n_f}{16\pi^2} \frac{1}{\Phi} \mathrm{tr}_c \int d^4x\, \Pi(x)\, F_{\mu\nu}(x) \tilde{F}^{\mu\nu}(x)$$

in agreement with the corresponding term in $\mathscr{L}_{\mathrm{WZW}}$ which was derived from chiral perturbation theory **Leutwyler (1996); Herrara-Sikody et al (1997); Leutwyler-Kaiser (2000)**

$$S^{\bar{\eta}}_{\mathrm{WZW}} = -i \frac{\sqrt{2\, n_f}}{F_{\bar{\eta}}} \int d^4x\, \bar{\eta}\, \Omega \qquad \Omega = \frac{\alpha_s}{4\pi} \mathrm{Tr}\left(F \tilde{F}\right)$$

where $\bar{\eta}$ is a massless “primordial” ninth Goldstone boson arising from the spontaneous symmetry breaking of the flavor group $U_L(3) \times U_R(3)$



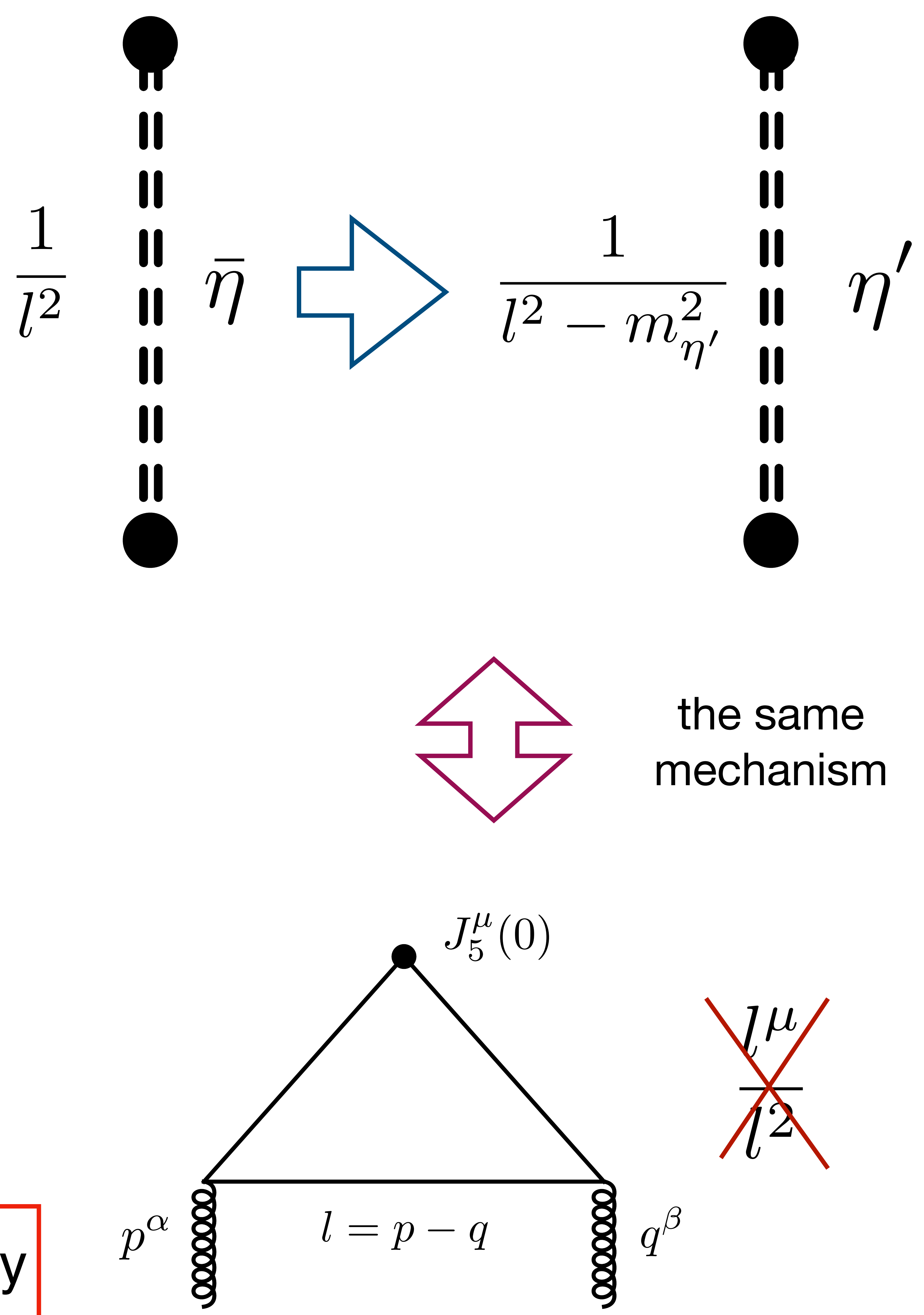
another pole

Anomaly pole and the $U_A(1)$ problem

However $\bar{\eta}$ is not observed. Instead there is a heavy η' ($m_{\eta'} \approx 957 MeV$) - the famous $U_A(1)$ problem.

There is no Goldstone pole just as there is no anomaly pole in the QCD spectrum

We demonstrate that the dynamical interplay between the physics of the anomaly, and that of the isosinglet pseudoscalar $U_A(1)$ sector of QCD resolves both problems simultaneously: the lifting of the $\bar{\eta}$ pole by topological mass generation of the η' and the cancellation of the anomaly pole



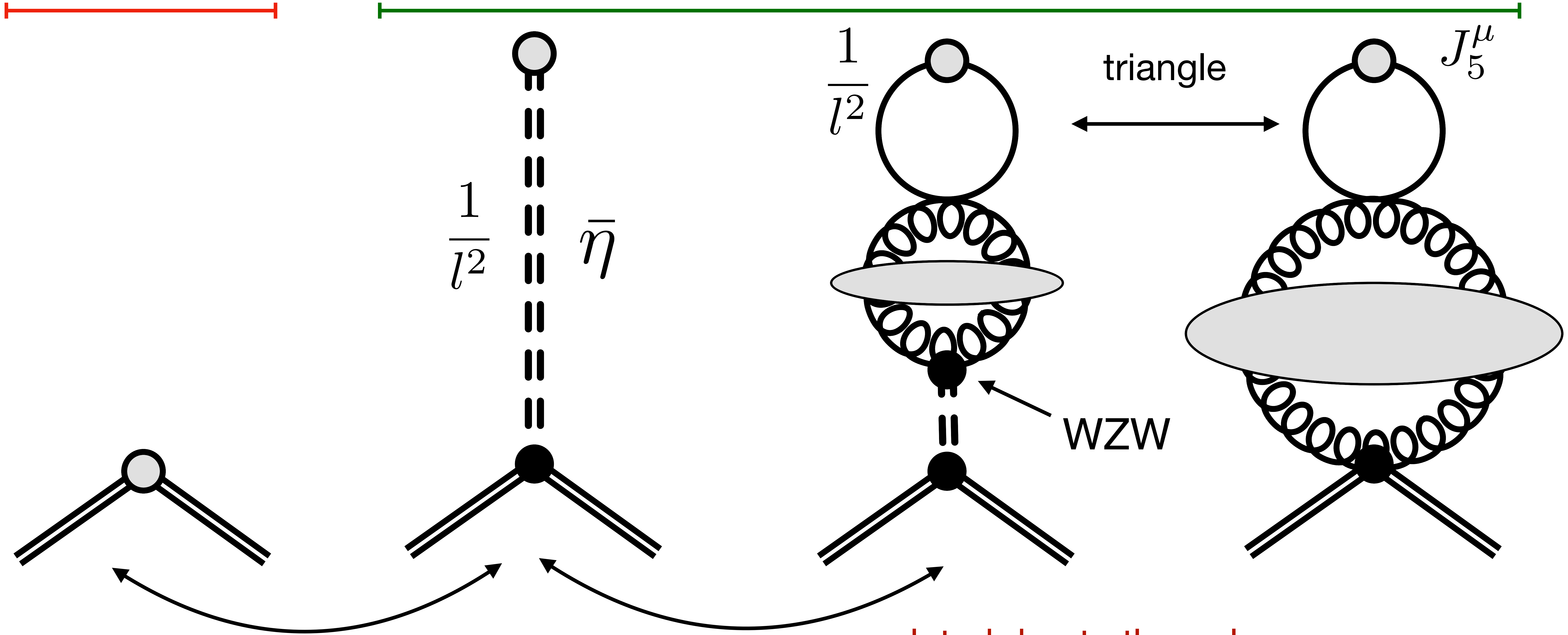
This mechanism relates the helicity structure of the proton to the topology of the QCD vacuum

Pseudovector vs. pseudoscalar coupling

$$\langle P', S | J_5^\mu | P, S \rangle = \bar{u}(P', S) \Big[\gamma^\mu \gamma_5 G_A(l^2) + l^\mu \gamma_5 G_P(l^2) \Big] u(P, S)$$

Direct axial-vector coupling

Pseudoscalar coupling to the polarized proton



This relations will allow us to connect quark helicity Σ to the topological properties of QCD

topological screening

$$\boxed{\Sigma(Q^2) = \sqrt{\frac{2}{3}} \frac{2n_f}{M_N} g_{\eta_0 NN} \sqrt{\chi'(0)}}$$

related due to the anomaly equation

related due to the pole cancellation

We recover a Shore-Veneziano result (derived using anomalous functional chiral Ward identities)

Shore, Veneziano (1992)

Anomaly pole in the box diagram

Why poles are important? Mechanism of the pole cancellation allows to relate amplitudes (first moments) between each other and topological properties of the QCD vacuum

first moments are related (cancellation of the pole + anomaly equation)

$$\bar{T}_{[\mu\nu]} = \frac{i}{2\bar{P}\cdot\bar{q}}\epsilon_{\mu\nu\rho\kappa}\bar{q}^{\rho}\left(\tilde{h}^{\kappa}\tilde{\mathcal{H}}_1 + \tilde{e}^{\kappa}\tilde{\mathcal{E}}_1\right) + \frac{i}{2(\bar{P}\cdot\bar{q})^2}\epsilon_{\mu\nu\rho\kappa}\bar{q}^{\rho}\left[\left(\bar{P}\cdot\bar{q}\tilde{h}^{\kappa} - \tilde{h}\cdot\bar{q}\bar{P}^{\kappa}\right)\tilde{\mathcal{H}}_2 + \left(\bar{P}\cdot\bar{q}\tilde{e}^{\kappa} - \tilde{e}\cdot\bar{q}\bar{P}^{\kappa}\right)\tilde{\mathcal{E}}_2\right] + ..$$

$$\tilde{\mathcal{H}}_1 \xrightarrow{t\to 0} \tilde{g}_1 \longrightarrow \Delta\Sigma \propto \sqrt{\chi'(0)}$$

triangle diagram contribution

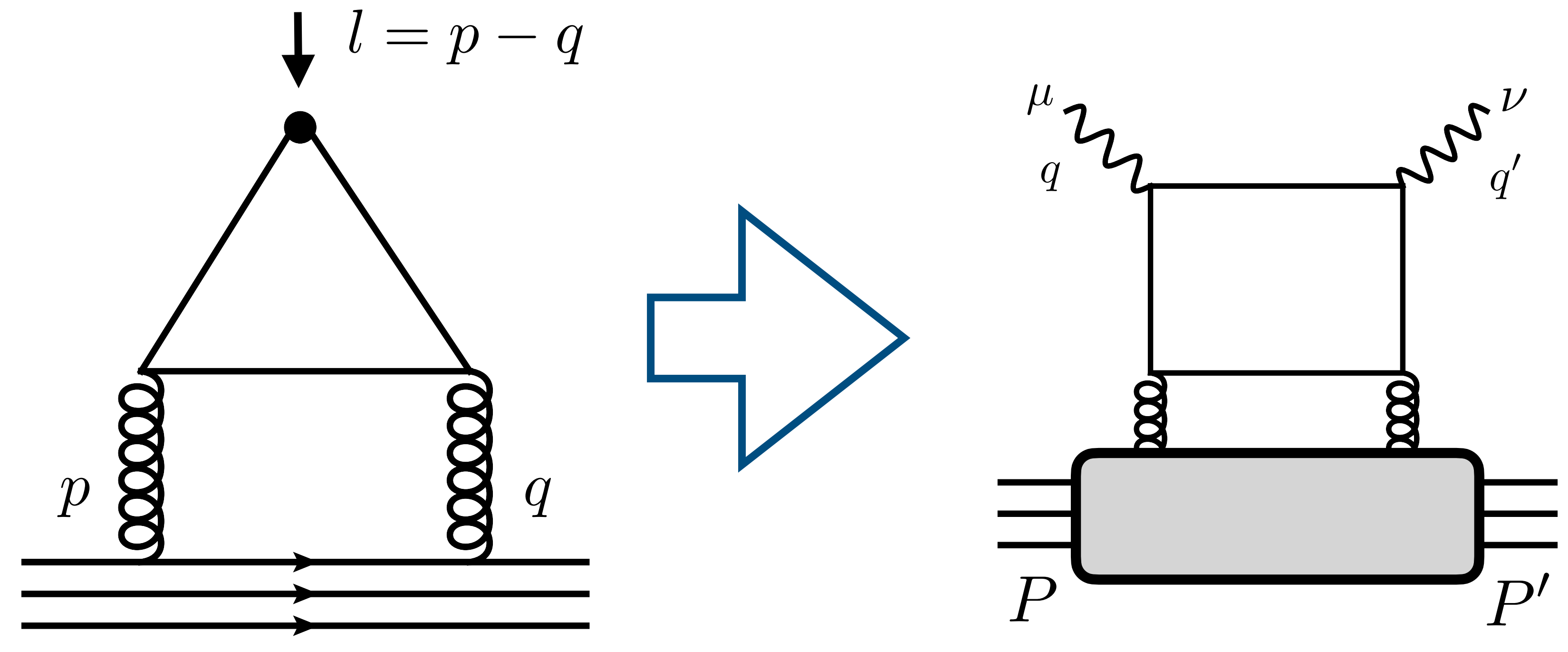
Does this relation go beyond the first moment? → calculation of the off-forward box diagram without kinematical approximations (in the standard calculations $l^2 = 0$)

$$\lim_{l_\mu\to 0} l^{\mu}(\mathrm{Im}\tilde{\mathcal{E}}_1 + \mathrm{Im}\tilde{\mathcal{E}}_2) = \sum_f e_f^2 \frac{\alpha_s}{i\pi M_N}\int_{x_B}^{1}\frac{dx}{x}\left(1-\frac{x_B}{x}\right)\int\frac{d\xi}{2\pi}e^{-i\xi x}\lim_{l_\mu\to 0}\boxed{\frac{l^{\mu}}{l^2}}\langle P', S|\mathrm{Tr_c}\boxed{F_{\alpha\beta}(\xi n)\tilde{F}^{\alpha\beta}(0)}|P, S\rangle + ..$$

infrared pole

non-local operator

Tarasov, Venugopalan (2020)
Bhattacharya, Hatta, Vogelsang (2022)

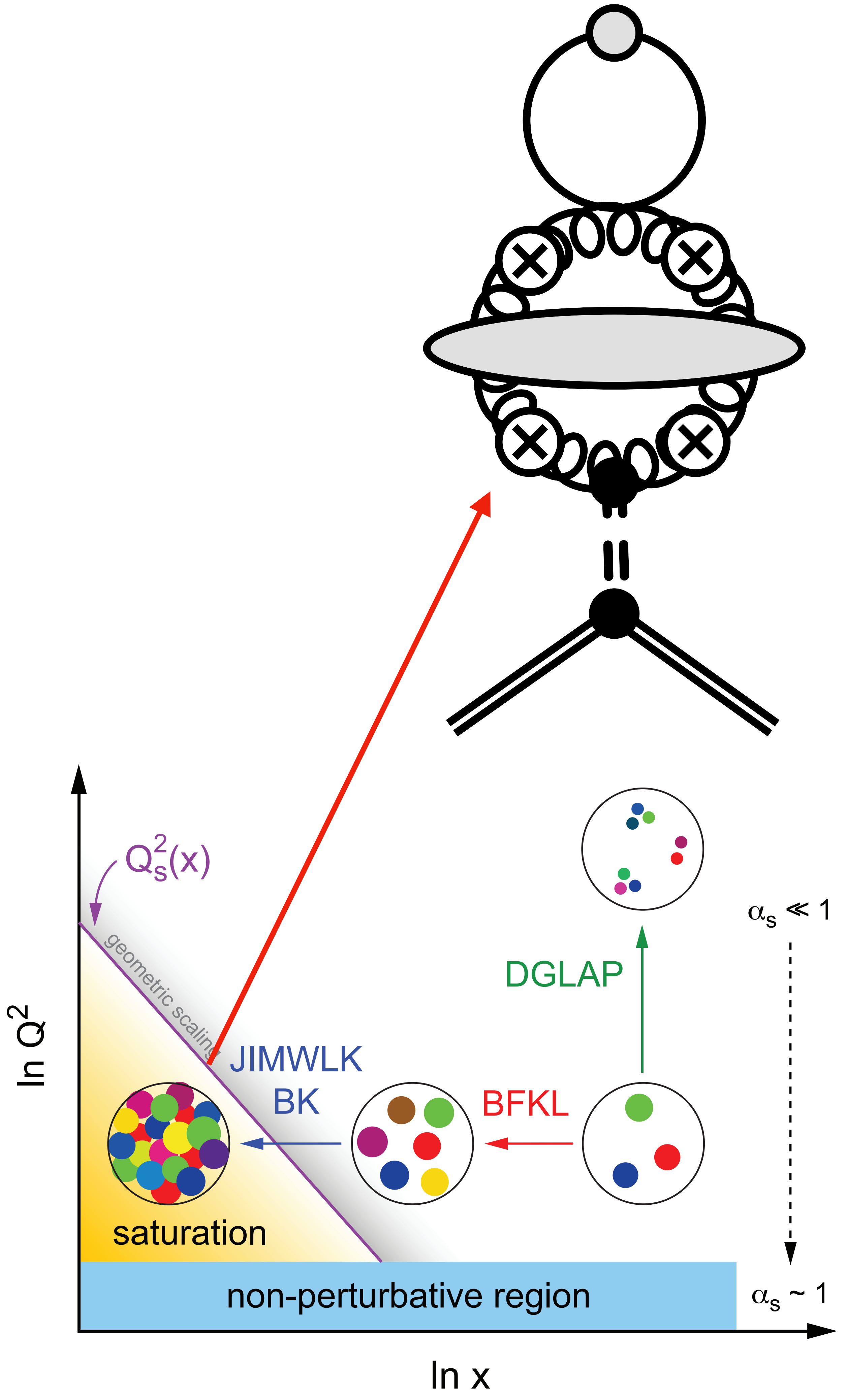


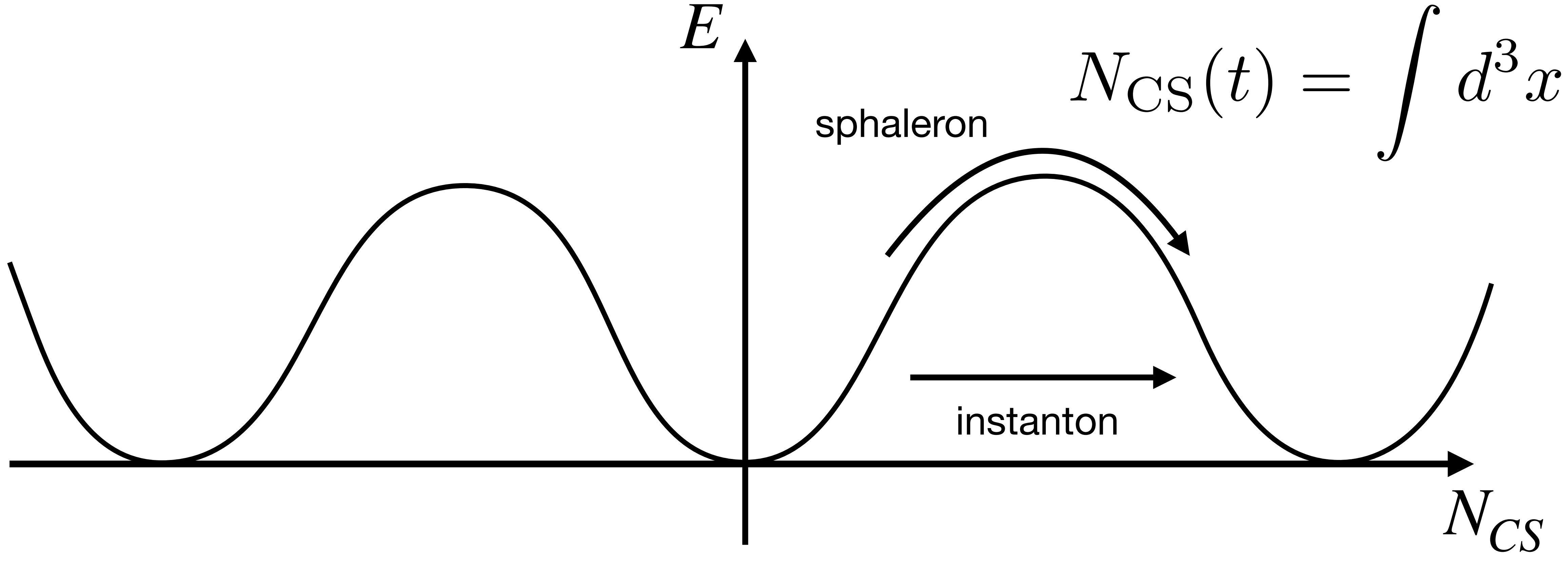
How does the pole cancel? Can we relate amplitudes beyond the first moment (what about symmetric part)? Non-local anomaly equations? What is a logarithmic structure related to the anomaly, interplay between poles and logarithms?

Sphaleron transitions

Get access to topological properties of the QCD vacuum in DIS experiments → $g_1(x_B)$

There are two scales in the problem at small-x_B: $m_{\eta'}^2$ and Q_s^2! Gluon saturation (Q_s^2) can be treated as a perturbation around an instanton solution and induce over the barrier sphaleron-like transitions between different topological sectors of the QCD vacuum, each corresponding to distinct integer valued Chern-Simons number N_{CS}





$$N_{\rm CS}(t) = \int d^3x\, K^0$$

The sphaleron-like transitions induced by interactions with the small-x background fields introduce a “drag force” on $\bar{\eta}$ “axion” propagation proportional to sphaleron transition rate:

“drag force” effect

$$\frac{\partial^2 \eta'}{\partial t^2} = \boxed{-\gamma \frac{\partial \eta'}{\partial t}} - m_{\eta'}^2 \eta' \quad \Longrightarrow \quad g_1^{\rm Regge}(x_B, Q^2) \propto \frac{Q_S^2 m_{\eta'}^2}{F_{\bar{\eta}}^3 M_N} \exp\left(-4\, n_f C\, \frac{Q_S^2}{F_{\bar{\eta}}^2}\right)$$

While at large x_B the gluon field is dominated by the instanton configurations, at small x_B the CGC background ($Q_s^2 > m_{\eta'}^2$) can induce over-the-barrier transitions. We predict over-the-barrier sphaleron transitions between different topological sectors of the QCD vacuum ⇒ can be detected in DIS and DVCS

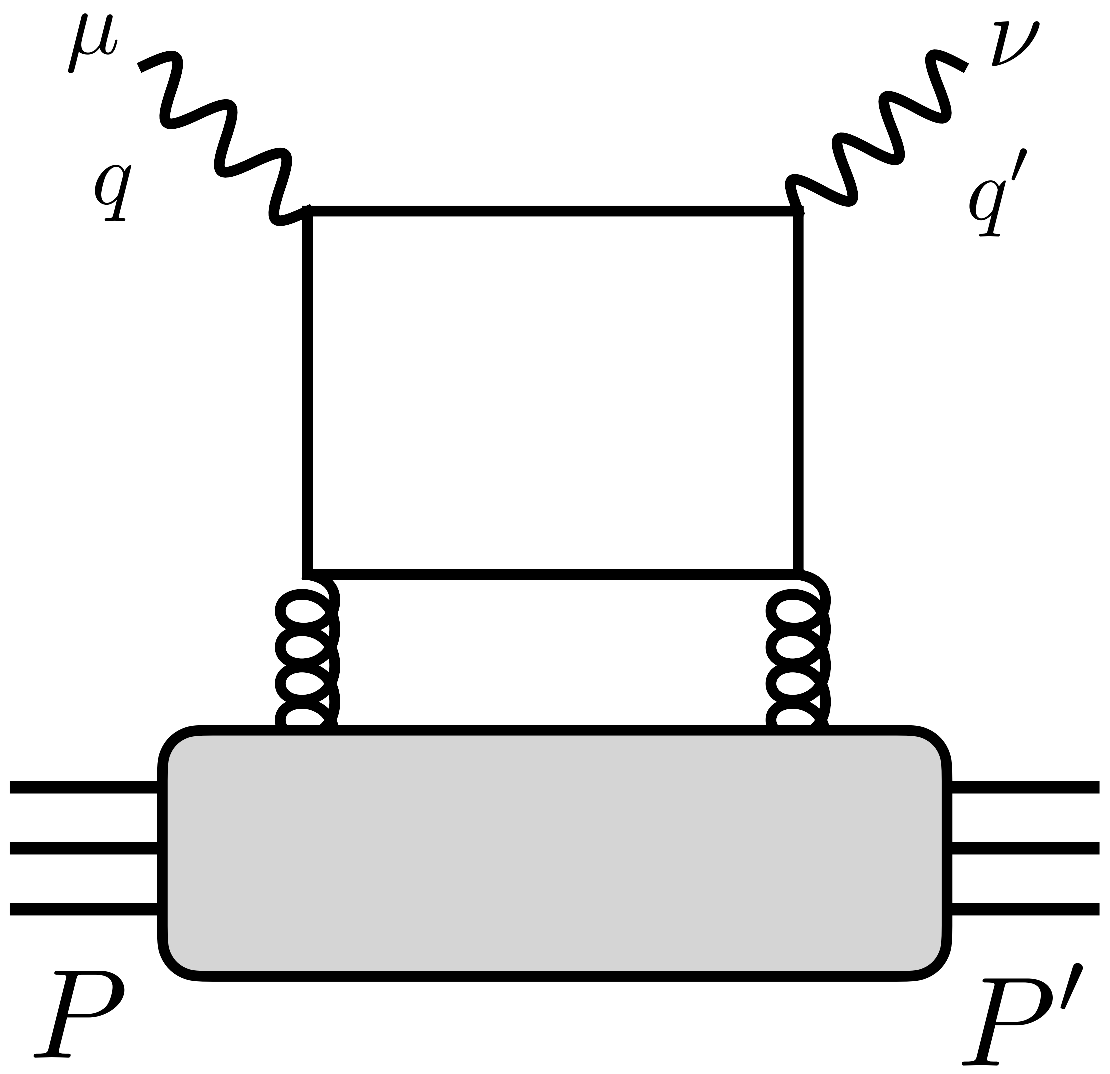
Off-forward Compton amplitudes and the box diagram

These questions motivate further calculation of the off-forward box diagram

$$\bar{T}_{\mu\nu} = \frac{1}{2\bar{P}\cdot\bar{q}}\Big[-\Big(h\cdot\bar{q}\mathcal{H}_1 + e\cdot\bar{q}\mathcal{E}_1\Big)g_{\mu\nu} + \frac{1}{\bar{P}\cdot\bar{q}}\Big(h\cdot\bar{q}\mathcal{H}_2 + e\cdot\bar{q}\mathcal{E}_2\Big)\bar{P}_\mu\bar{P}_\nu + \mathcal{H}_3 h_{\{\mu}\bar{P}_{\nu\}}\Big]$$
$$+\frac{i}{2\bar{P}\cdot\bar{q}}\epsilon_{\mu\nu\rho\kappa}\bar{q}^\rho\Big(\tilde{h}^\kappa\tilde{\mathcal{H}}_1 + \tilde{e}^\kappa\tilde{\mathcal{E}}_1\Big) + \frac{i}{2(\bar{P}\cdot\bar{q})^2}\epsilon_{\mu\nu\rho\kappa}\bar{q}^\rho\Big[\Big(\bar{P}\cdot\bar{q}\tilde{h}^\kappa - \tilde{h}\cdot\bar{q}\bar{P}^\kappa\Big)\tilde{\mathcal{H}}_2 + \Big(\bar{P}\cdot\bar{q}\tilde{e}^\kappa - \tilde{e}\cdot\bar{q}\bar{P}^\kappa\Big)\tilde{\mathcal{E}}_2\Big]$$
$$+\Big(\bar{P}_\mu q'_\nu + \bar{P}_\nu q_\mu\Big)\Big(h\cdot\bar{q}\mathcal{K}_1 + e\cdot\bar{q}\mathcal{K}_2\Big) + \Big(\bar{P}_\mu q'_\nu - \bar{P}_\nu q_\mu\Big)\Big(h\cdot\bar{q}\mathcal{K}_3 + e\cdot\bar{q}\mathcal{K}_4\Big) + q_\mu q'_\nu\Big(h\cdot\bar{q} - e\cdot\bar{q}\Big)\mathcal{K}_5$$
$$+h_{[\mu}\bar{P}_{\nu]}\mathcal{K}_6 + \Big(h_\mu q'_\nu + h_\nu q_\mu\Big)\mathcal{K}_7 + \Big(h_\mu q'_\nu - h_\nu q_\mu\Big)\mathcal{K}_8 + \bar{P}_{\{\mu}\bar{u}(P')i\sigma_{\nu\}\alpha}u(P)\bar{q}^\alpha\mathcal{K}_9$$

How does the box diagram contribute to the off-forward Compton amplitudes in general?

- tensorial structures generated by the box diagram
- pole and non-pole terms
- logarithms
- operator structures corresponding to background gluons



→ Calculation of the box diagram in the most general kinematics

Box diagram in the worldline approach

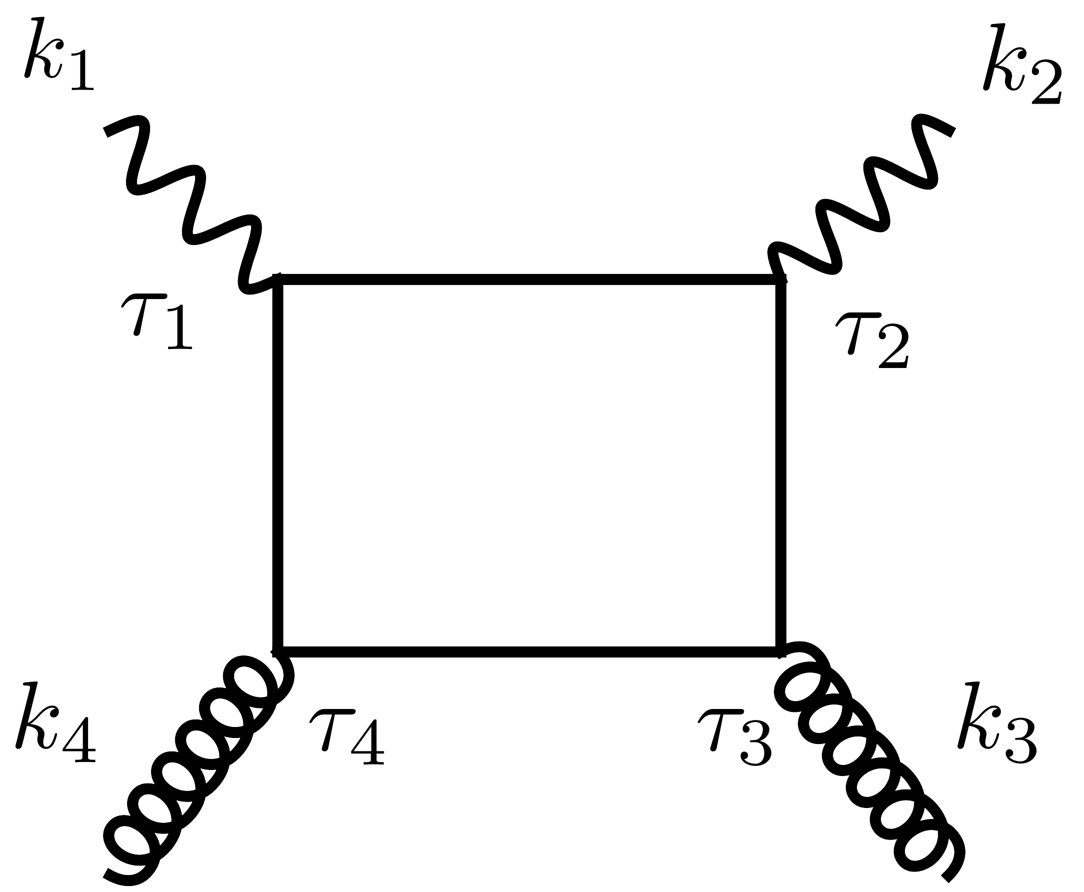
Calculation of the box diagram in the most general kinematics

The worldline approach is a powerful approach which allows to determine both the tensorial structure of the diagram and the corresponding kinematic coefficients. Example: triangle diagram

General expression in the worldline approach:

$$-\frac{e^2g^2}{2}\int_0^T\frac{dT}{T}\int_P\mathcal{D}x\int_{AP}\mathcal{D}\psi\,\underbrace{V_1^\mu(k_1)V_2^\mu(k_2)V_3^\alpha(k_3)V_4^\beta(k_4)}_{\text{interaction vertexes}}$$

$$\times\exp\Big\{\underbrace{-\int_0^T d\tau\Big(\frac{1}{4}\dot{x}^2+\frac{1}{2}\psi_\mu\dot{\psi}^\mu\Big)}_{\text{free action}}\Big\}\underbrace{\epsilon_{1\mu}\epsilon_{2\nu}\mathrm{Tr_c}A_\alpha(k_3)A_\beta(k_4)}_{\text{external fields/pol. vectors}}$$



Functional integrals over trajectories can be easily evaluated:

$$-\frac{e^2g^2}{2}\int_0^\infty\frac{dT}{T}(4\pi T)^{-\frac{D}{2}}\mathrm{Tr_c}\int_0^T\prod_{i=1}^4 d\tau_i P_4(\dot{G}_{Bij},\ddot{G}_{Bij},G_{Fij})\exp\Big[\sum_{i,j=1}^4\frac{1}{2}G_{Bij}k_i\cdot k_j\Big]$$

polynom constructed from worldline propagators

$$P_4(\dot{G}_{Bij},\ddot{G}_{Bij},G_{Fij})=\ddot{G}_{B12}\epsilon_1\cdot\epsilon_2\ddot{G}_{B34}A(k_3)\cdot A(k_4)+\ldots$$

IBP procedure

Previously we saw that to reveal the tensorial structure of the diagram one needs to perform the IBP procedure, which eliminates $\ddot{G}_B$:

$$P_4(\dot{G}_{Bij}, \ddot{G}_{Bij}, G_{Fij}) \to Q_4(\dot{G}_{Bij}, G_{Fij})$$

$$-\frac{e^2 g^2}{2} \int_0^\infty \frac{dT}{T} (4\pi T)^{-\frac{D}{2}} \mathrm{Tr}_c \int_0^T \prod_{i=1}^{4} d\tau_i Q_4(\dot{G}_{Bij}, G_{Fij}) \exp\left[\sum_{i,j=1}^{4} \frac{1}{2} G_{Bij} k_i \cdot k_j\right]$$

The form of the polynomial after IBP procedure is known **Ahmadiniaz, Lopez-Arcos, Lopez-Lopez, Schubert (2020)**

$$Q_4 = Q_4^4 + Q_4^3 + Q_4^2 + Q_4^{22}$$

strength tensors!

↓

$$\begin{aligned} Q_4^4 = & -(\dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B34}\dot{G}_{B41} - G_{F12}G_{F23}G_{F34}G_{F41}) f_{\mu\nu}(k_1) f^{\nu\rho}(k_2) F_{\rho\sigma}(k_3) F^{\sigma\mu}(k_4) \\ & -(\dot{G}_{B23}\dot{G}_{B31}\dot{G}_{B14}\dot{G}_{B42} - G_{F23}G_{F31}G_{F14}G_{F42}) f_{\mu\nu}(k_2) F^{\nu\rho}(k_3) f_{\rho\sigma}(k_1) F^{\sigma\mu}(k_4) \\ & -(\dot{G}_{B31}\dot{G}_{B12}\dot{G}_{B24}\dot{G}_{B43} - G_{F31}G_{F12}G_{F24}G_{F43}) F_{\mu\nu}(k_3) f^{\nu\rho}(k_1) f_{\rho\sigma}(k_2) F^{\sigma\mu}(k_4) \end{aligned}$$

scalar functions

tensorial structure

- The problem simplifies to the calculation of the scalar coefficients and decomposition of the tensorial structure
- The tensorial structure is constructed from gauge covariant objects

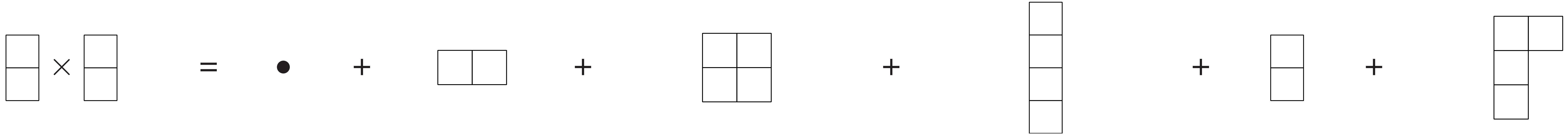
Projectors for the tensorial structure

$$\begin{aligned}Q_4^4 = &-(\dot G_{B12}\dot G_{B23}\dot G_{B34}\dot G_{B41} - G_{F12}G_{F23}G_{F34}G_{F41})f_{\mu\nu}(k_1)f^{\nu\rho}(k_2)F_{\rho\sigma}(k_3)F^{\sigma\mu}(k_4)\\ &-(\dot G_{B23}\dot G_{B31}\dot G_{B14}\dot G_{B42} - G_{F23}G_{F31}G_{F14}G_{F42})f_{\mu\nu}(k_2)F^{\nu\rho}(k_3)f_{\rho\sigma}(k_1)F^{\sigma\mu}(k_4)\\ &-(\dot G_{B31}\dot G_{B12}\dot G_{B24}\dot G_{B43} - G_{F31}G_{F12}G_{F24}G_{F43})\underbrace{F_{\mu\nu}(k_3)f^{\nu\rho}(k_1)f_{\rho\sigma}(k_2)F^{\sigma\mu}(k_4)}_{\text{tensorial structure}}\end{aligned}$$

The tensorial structure contains a direct product of two background gluon strength tensors
→ it needs to be decomposed into irreducible components. There are 6 such components:

$$\mathrm{A}\otimes\mathrm{A} = V_1 \oplus V_2 \oplus V_3 \oplus V_4 \oplus V_5 \oplus V_6$$

e.g. Cvitanovic (2011)



$$F_{\mu_1\nu_1}(k_3)F_{\mu_2\nu_2}(k_4) = g_{\mu_1\sigma_1}g_{\nu_1\rho_1}g_{\mu_2\sigma_2}g_{\nu_2\rho_2}F^{\sigma_1\rho_1}(k_3)F^{\sigma_2\rho_2}(k_4)$$

$$g_{\mu_1\sigma_1}g_{\nu_1\rho_1}g_{\mu_2\sigma_2}g_{\nu_2\rho_2} = \sum_{i=1}^{6} P_i \quad \Rightarrow \quad \boxed{Q_4^4 = \sum_{i=1}^{6} P_i Q_4^4}$$

Each irreducible component should be parametrized in terms of parton distributions

Example: P_4 projector

$$g_{\mu_1\sigma_1}g_{\nu_1\rho_1}g_{\mu_2\sigma_2}g_{\nu_2\rho_2} = \sum_{i=1}^{6} P_i$$

For us particularly interesting is $P_4 = -\frac{1}{4!}\epsilon_{\mu_1\nu_1\mu_2\nu_2}\epsilon_{\sigma_1\rho_1\sigma_2\rho_2}$

$$F_{\mu_1\nu_1}(k_3)F_{\mu_2\nu_2}(k_4) \xrightarrow{P_4} -\frac{1}{12}\epsilon_{\mu_1\nu_1\mu_2\nu_2}F_{\alpha\beta}(k_3)\tilde{F}^{\alpha\beta}(k_4)$$

Applying it to polynomials in the worldline result for the box diagram:

$$P_4 Q_4^{4\mu\nu} = \frac{1}{3}(\dot{G}_{23}\dot{G}_{31}\dot{G}_{14}\dot{G}_{42} - G^F_{23}G^F_{31}G^F_{14}G^F_{42})\epsilon^{\mu\nu\rho\kappa}\bar{q}_\rho\{\Delta_\kappa F_{\alpha\beta}(k_3)\tilde{F}^{\alpha\beta}(k_4)\}$$

The result can be compared with the decomposition of the off-forward Compton amplitude

$$\bar{T}_{\mu\nu} = \frac{i}{2\bar{P}\cdot\bar{q}}\epsilon_{\mu\nu\rho\kappa}\bar{q}^\rho\left(\tilde{h}^\kappa\tilde{\mathcal{H}}_1 + \tilde{e}^\kappa\tilde{\mathcal{E}}_1\right) + \frac{i}{2(\bar{P}\cdot\bar{q})^2}\epsilon_{\mu\nu\rho\kappa}\bar{q}^\rho\left[\left(\bar{P}\cdot\bar{q}\tilde{h}^\kappa - \tilde{h}\cdot\bar{q}\bar{P}^\kappa\right)\tilde{\mathcal{H}}_2 + \left(\bar{P}\cdot\bar{q}\tilde{e}^\kappa - \tilde{e}\cdot\bar{q}\bar{P}^\kappa\right)\tilde{\mathcal{E}}_2\right] + \dots$$

$$\tilde{e}^\mu = \frac{\Delta^\mu}{2M}\bar{u}'\gamma_5 u$$

We find that the P_4 provides contribution to $\tilde{\mathcal{E}}_1$ and $\tilde{\mathcal{E}}_2$

A similar analysis can be performed for other projectors, which provides a whole set of amplitudes with contribution of the box diagram

Example: P₄ projector

The scalar integrals for all six projectors has the same structure!

Applying projector P₄ to the expression for the box diagrams we obtain

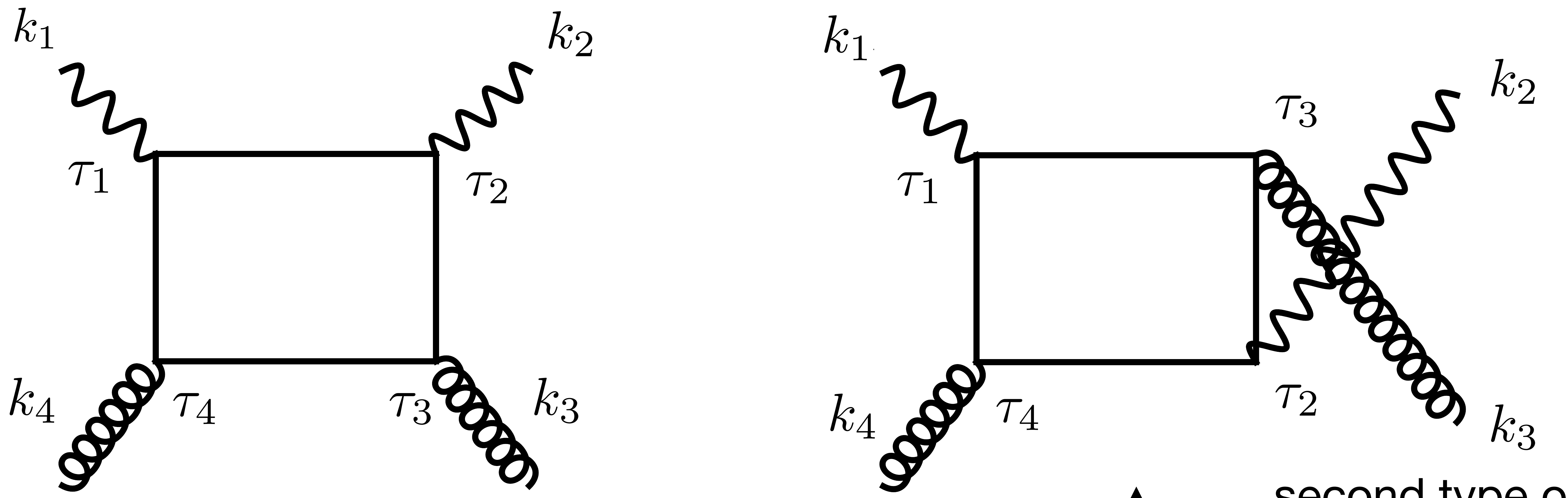
$$\underbrace{-\frac{e^2g^2}{2}\int_0^\infty \frac{dT}{T}(4\pi T)^{-\frac{D}{2}}\int_0^T \prod_{i=1}^{4} d\tau_i \Big[\frac{1}{3}\dot{G}_{23}\dot{G}_{31}\dot{G}_{14}\dot{G}_{42}+\dots\Big]\exp\Big[\sum_{i,j=1}^{4}\frac{1}{2}G_{Bij}k_i\cdot k_j\Big]}_{\text{scalar functions}}\;\underbrace{\epsilon^{\mu\nu\rho\kappa}\bar{q}_\rho\Delta_\kappa \mathrm{Tr}_c F_{\alpha\beta}(k_3)\tilde{F}^{\alpha\beta}(k_4)}_{\text{tensorial structure}}$$

Finally, we need to calculate a scalar integral over proper-time variables

With a certain change of variables the last integral can be rewritten in terms of integration over DGLAP and ERBL regions

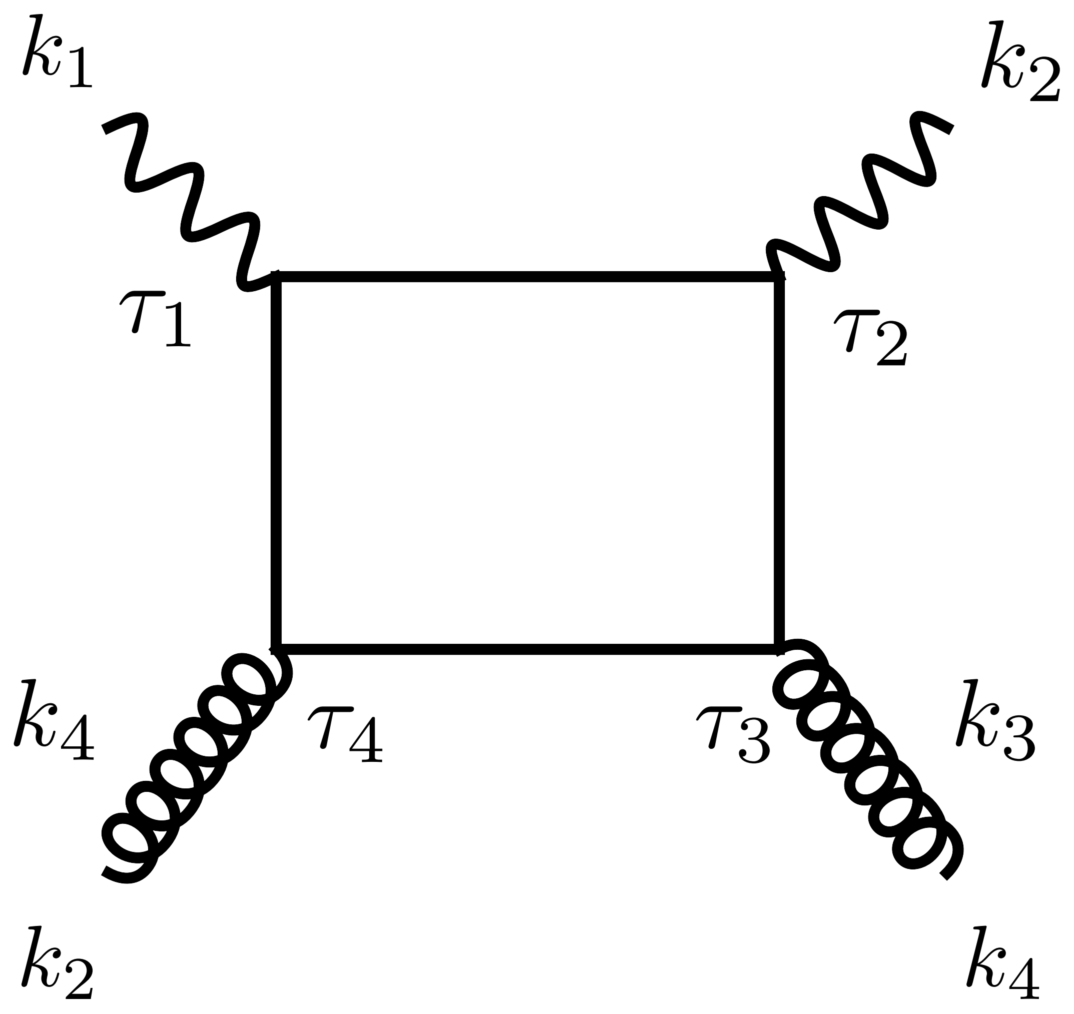
However, to better understand dependence on the kinematical variables we want to perform all integrations

Two types of ordering of the proper-time variables (six contributions in total):



second type of diagrams (with gluons on mass-shell) can be easily calculated

Calculation of the scalar integral



Calculation of the scalar integrals of the first type is more challenging but can be done using differential equations method **Bern, Dixon, Kosower (1993)**

$$\int_0^\infty \frac{dT}{T}(4\pi T)^{-\frac{D}{2}} \int_0^T \prod_{i=1}^4 d\tau_i \Big[\frac{1}{3}\dot{G}_{23}\dot{G}_{31}\dot{G}_{14}\dot{G}_{42} + \dots\Big] \exp\Big[\sum_{i,j=1}^4 \frac{1}{2} G_{Bij} k_i \cdot k_j\Big]$$

After integration over period of the worldline T and a linear change of the proper-time variables ($\tau \to a$) the scalar integral can be rewritten as

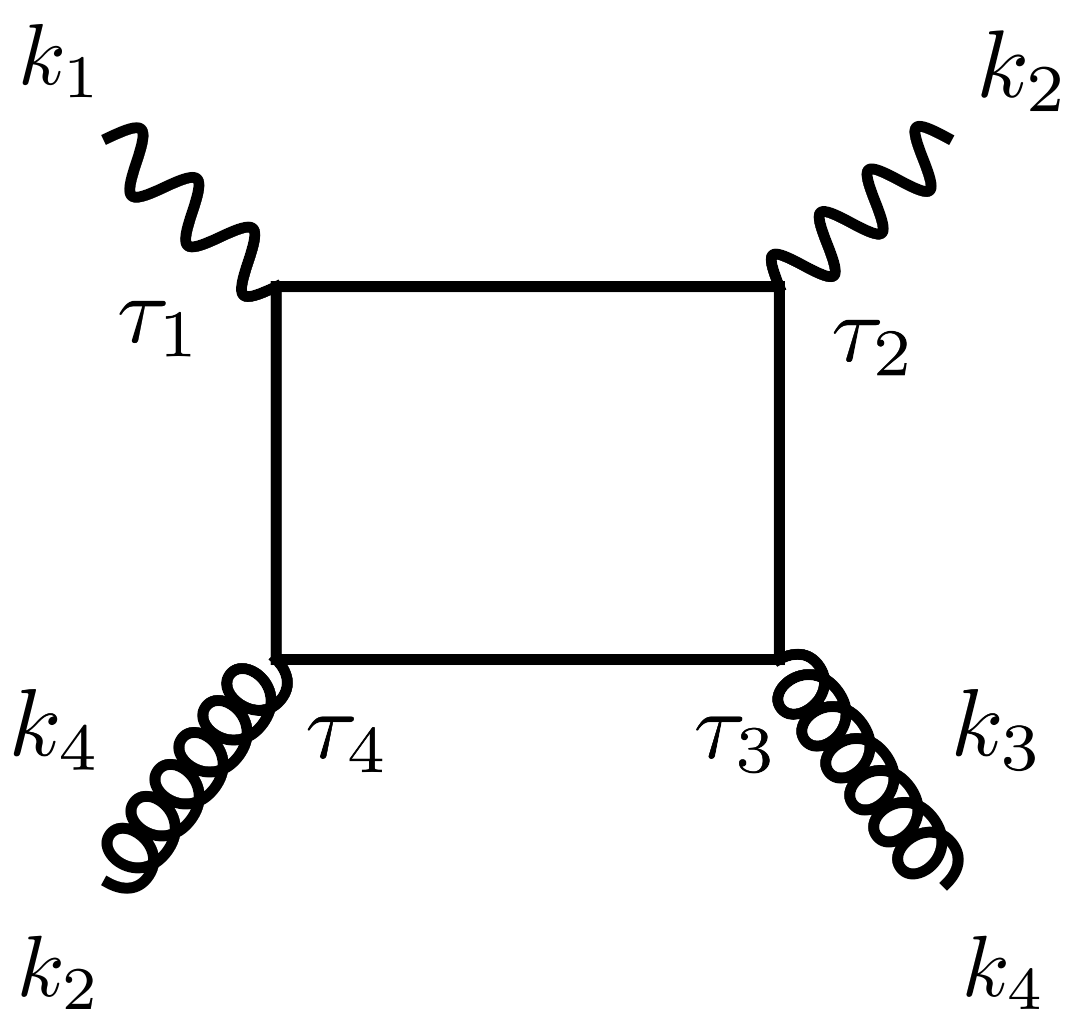
$$I_4[P_m(\{a_i\})] = \Gamma(2+\epsilon)\int_0^1 d^4 a_i \delta(1 - \sum_i a_i) P_4(\{a_i\}) \Big[\sum_{i,j=1}^n S_{ij} a_i a_j - i\epsilon\Big]^{-2-\epsilon}$$

Polynomial of the fourth order defined by the dependence on the worldline propagators, which is generated by a projector Pᵢ

kinematical variables

$S_{12} = S_{21} = -\frac{1}{2}k_1^2$ etc.

Calculation of the scalar integral



The problem is simplified to the calculation of the following scalar integral

$$I_4[P_4(\{a_i\})] = \Gamma(2+\epsilon)\int_0^1 d^4a_i\delta(1-\sum_i a_i)P_4(\{a_i\})\Big[\sum_{i,j=1}^{n} S_{ij}a_ia_j - i\epsilon\Big]^{-2-\epsilon}$$

where P_4 is an arbitrary polynomial of the fourth order which should be represented (for technical reason) as P_3a_k

Using the differentiation technique we obtain the following representation for such integrals:

$$\boxed{I_4[P_3(\{a_i\})a_k] = \alpha_1\alpha_2\alpha_3\alpha_4\frac{\Gamma(2\epsilon-n)}{\Gamma(2\epsilon)}\alpha_kP_3\Big(\alpha_i\Big\{\frac{\partial}{\partial\alpha_i}\Big\}\Big)\hat{I}_4[a_k]}$$

$$S_{ij} = \frac{\rho_{ij}}{\alpha_i\alpha_j}$$

The integral can be obtained by differentiation of the basic integrals $\hat{I}_4[a_k]$

$$\hat{I}_4[a_k] = \Gamma(2+\epsilon)\int_0^1 d^4u\delta(1-u_1-u_2-u_3-u_4)\frac{u_k\big(\sum_{j=1}^4\alpha_ju_j\big)^{-1+2\epsilon}}{\big[\sum_{i,j}\rho_{ij}u_iu_j\big]^{2+\epsilon}}$$

The expression for the basic integrals (up to ϵ^0) can be deduced from literature

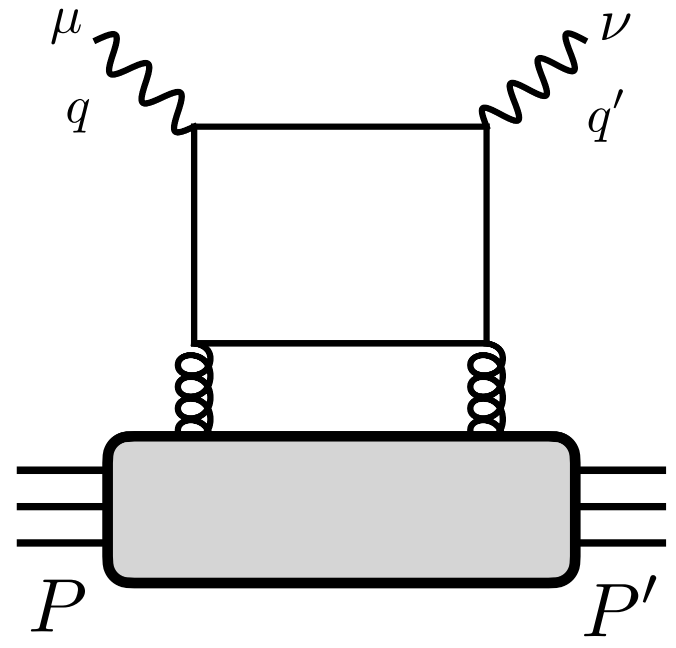
Bern, Dixon, Kosower (1993)
Davydychev, Ussyukina (1991-1994)

No actual calculation of the integrals is required! The integrals can be obtained by differentiation, which can be easily automatized. This solves the problem of calculation of the scalar integrals

Summary

We want to deter the contribution of the box diagram to the off-forward Compton amplitudes

$$\bar{T}_{\mu\nu} = \frac{i}{2\bar{P}\cdot\bar{q}}\epsilon_{\mu\nu\rho\kappa}\bar{q}^{\rho}\left(\tilde{h}^{\kappa}\tilde{\mathcal{H}}_1 + \tilde{e}^{\kappa}\tilde{\mathcal{E}}_1\right) + \frac{i}{2(\bar{P}\cdot\bar{q})^2}\epsilon_{\mu\nu\rho\kappa}\bar{q}^{\rho}\left[\left(\bar{P}\cdot\bar{q}\tilde{h}^{\kappa} - \tilde{h}\cdot\bar{q}\bar{P}^{\kappa}\right)\tilde{\mathcal{H}}_2 + \left(\bar{P}\cdot\bar{q}\tilde{e}^{\kappa} - \tilde{e}\cdot\bar{q}\bar{P}^{\kappa}\right)\tilde{\mathcal{E}}_2\right] + \dots$$



The worldline approach is a powerful approach which allows to determine both the tensorial structure of the diagram and the corresponding kinematic coefficients

The problem can be simplified to decomposition of the tensorial structure and calculation of the kinematic (scalar) coefficient

$$\underbrace{-\frac{e^2g^2}{2}\int_0^{\infty}\frac{dT}{T}(4\pi T)^{-\frac{D}{2}}\int_0^T\prod_{i=1}^{4}d\tau_i\left[\frac{1}{3}\dot{G}_{23}\dot{G}_{31}\dot{G}_{14}\dot{G}_{42} + \dots\right]\exp\left[\sum_{i,j=1}^{4}\frac{1}{2}G_{Bij}k_i\cdot k_j\right]}_{\text{scalar functions}}\ \underbrace{\epsilon^{\mu\nu\rho\kappa}\bar{q}_{\rho}\Delta_{\kappa}\mathrm{Tr}_{\mathrm{c}}F_{\alpha\beta}(k_3)\tilde{F}^{\alpha\beta}(k_4)}_{\text{tensorial structure}}$$

The integrals can be computed using differential equations method, which allows efficient automatization. The calculation is currently in progress

Using projectors onto irreducible components the result can be identified with the corresponding amplitudes

Thank you for your attention!