

Glueon distributions in exclusive Compton Scattering from low to moderate x

Renaud Boussarie

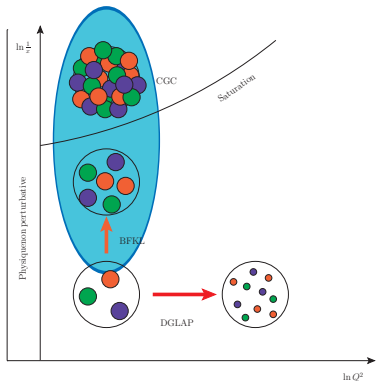
RBRC workshop on GPDs



In collaboration with Y. Mehtar-Tani

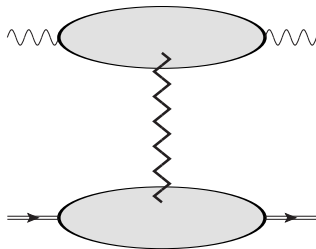
QCD at small $x_{Bj} \sim Q^2/s$

Regge limit: $Q^2 \ll s$



The Pomeron

Regge theory: for asymptotic values of s , an **effective particle with the quantum numbers of the vacuum** is exchanged

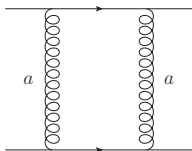


Positive C -parity: **Pomeron** exchange, negative C -parity: **Odderon** exchange

- How can we understand the Pomeron and the Odderon in perturbative QCD?
- How does it couple to hadrons?

Naive perturbative Pomeron and Odderon

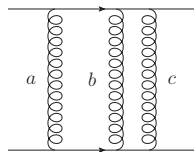
Naive perturbative description of the target hadron



Two gluons on a color singlet state

$$\text{tr}(t^a t^a)$$

Leading Pomeron



Three gluons on a color singlet state

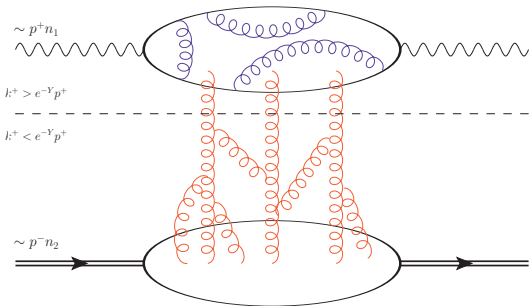
$$\text{tr}(t^a t^b t^c) = \frac{1}{4}(d^{abc} + i f^{abc})$$

d^{abc} : subleading Pomeron

f^{abc} : leading Odderon

Most general description: semi-classical small x frameworks

Rapidity separation

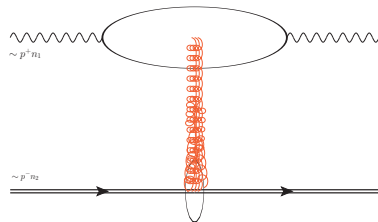
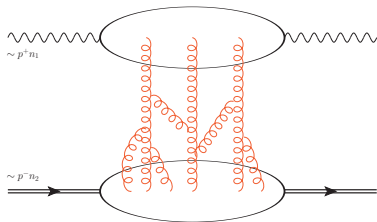


Let us split the gluonic field between "fast" and "slow" gluons

$$\begin{aligned}
 \mathcal{A}^{\mu a}(k^+, k^-, k) &= A^{\mu a}(|k^+| > e^{-Y} p^+, k^-, k) \\
 &+ A_{cl}^{\mu a}(|k^+| < e^{-Y} p^+, k^-, k)
 \end{aligned}$$

$$e^{-Y} \ll 1$$

Large longitudinal boost to the projectile frame



$$A_{\text{cl}}^+(x^+, x^-, \mathbf{x})$$

$$A_{\text{cl}}^-(x^+, x^-, \mathbf{x})$$

$$A_{\text{cl}}^i(x^+, x^-, \mathbf{x})$$

$$\Lambda \sim \sqrt{\frac{s}{m_t^2}}$$

$$\frac{1}{\Lambda} A_{\text{cl}}^+(\Lambda x^+, \frac{x^-}{\Lambda}, \mathbf{x})$$

$$\Lambda A_{\text{cl}}^-(\Lambda x^+, \frac{x^-}{\Lambda}, \mathbf{x})$$

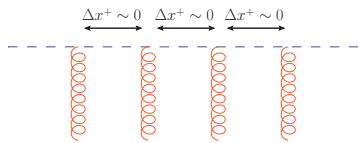
$$A_{\text{cl}}^i(\Lambda x^+, \frac{x^-}{\Lambda}, \mathbf{x})$$

$$A_{\text{cl}}^\mu(x) \rightarrow A_{\text{cl}}^-(x) n_2^\mu = \delta(x^+) \mathbf{A}(x) n_2^\mu + O(\sqrt{\frac{m_t^2}{s}})$$

Shock wave approximation

Effective Feynman rules in the slow background field

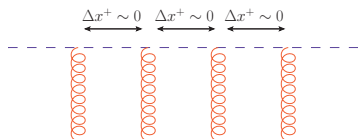
Effective fermion propagator in the external classical field



- $A_{cl}^i = 0, A_{cl}^+ = 0$: the Dirac structure **factorizes**
- A_{cl} does not depend on x^- : **conservation** of + momentum
- A_{cl} is **peaked** around $x^+ = 0$:
 - Most external propagators get **factorized out**
 - Gaussians $\sim \delta$ functions: **conservation of transverse position**
 - Possibility to **extend Wilson lines** to infinity $[x^+, y^+]_x = [\infty^+, -\infty^+]_x \equiv U_x$

Effective Feynman rules in the slow background field

The interactions with the background field can be **exponentiated**

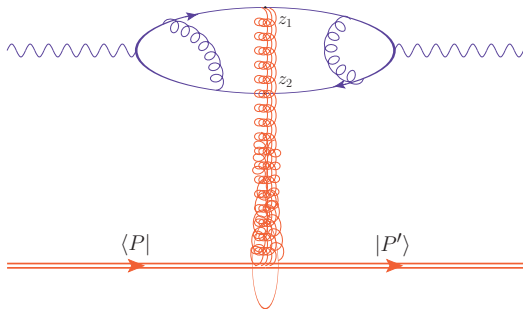


$$D_F(x_2, x_0)|_{x_2^+ > 0, x_0^+ < 0} = \int d^D x_1 \delta(x_1^+) D_0(x_2, x_1) \gamma^+ U_{x_1} D_0(x_1, x_0)$$

Each fast parton is dressed by an infinite Wilson line

$$U_x \equiv \mathcal{P} \exp \left[ig \int_{-\infty}^{+\infty} dx \cdot A_{cl}(x) \right]$$

Factorized picture



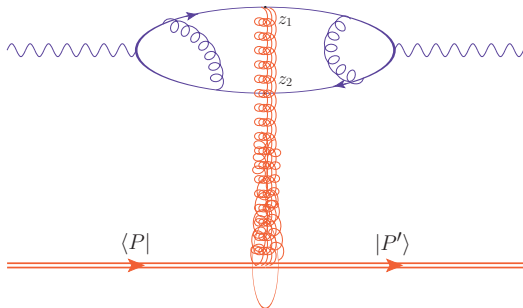
Factorized amplitude

$$\mathcal{S} = \int dx_1 dx_2 \Phi^Y(x_1, x_2) \langle P' | [\text{Tr}(U_{x_1}^Y U_{x_2}^{Y\dagger}) - N_c] | P \rangle$$

Written similarly for any number of Wilson lines in **any color representation!**

Y independence: **B-JIMWLK**, **BK** equations. Resums **logarithms of s**

Factorized picture



Factorized amplitude

$$\mathcal{A}^{Y_c} = \int d^{D-2} \vec{z}_1 d^{D-2} \vec{z}_2 \Phi^{Y_c}(\vec{z}_1, \vec{z}_2) \langle P' | [\text{Tr}(U_{\vec{z}_1}^{Y_c} U_{\vec{z}_2}^{Y_c \dagger}) - N_c] | P \rangle$$

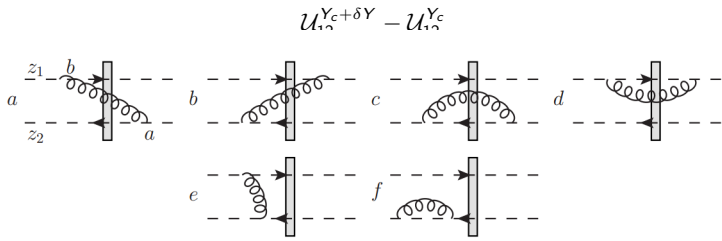
Dipole operator $U_{ij}^{Y_c} = \frac{1}{N_c} \text{Tr}(U_{\vec{z}_i}^{Y_c} U_{\vec{z}_j}^{Y_c \dagger}) - 1$

Written similarly for any number of Wilson lines in any color representation!

Y_c independence: B-JIMWLK hierarchy of equations

[Balitsky, Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov, Kovner]

Evolution for the dipole operator



B-JIMWLK hierarchy of equations

[Balitsky, Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov, Kovner]

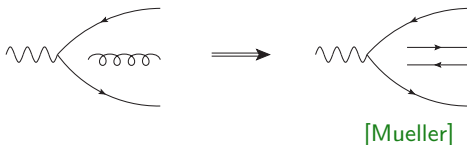
$$\frac{\partial U_{12}^{Y_c}}{\partial Y_c} = \frac{\alpha_s N_c}{2\pi^2} \int d\vec{z}_3 \frac{\vec{z}_{12}^2}{\vec{z}_{13}^2 \vec{z}_{23}^2} \left[U_{13}^{Y_c} + U_{32}^{Y_c} - U_{12}^{Y_c} + U_{13}^{Y_c} U_{32}^{Y_c} \right]$$

$$\frac{\partial U_{13}^{Y_c} U_{32}^{Y_c}}{\partial Y_c} = \dots$$

Evolves a **dipole** into a **double dipole**

The BK equation

Mean field approximation, or 't Hooft planar limit $N_c \rightarrow \infty$ in the dipole B-JIMWLK equation



⇒ **BK equation** [Balitsky, 1995] [Kovchegov, 1999]

$$\frac{\partial \langle u_{12}^{Y_c} \rangle}{\partial Y_c} = \frac{\alpha_s N_c}{2\pi^2} \int d\vec{z}_3 \frac{\vec{z}_{12}^2}{\vec{z}_{13}^2 \vec{z}_{23}^2} \left[\langle u_{13}^{Y_c} \rangle + \langle u_{32}^{Y_c} \rangle - \langle u_{12}^{Y_c} \rangle + \langle u_{13}^{Y_c} \rangle \langle u_{32}^{Y_c} \rangle \right]$$

BFKL/BKP part
Triple pomeron vertex

Non-linear term : one type of **saturation**

Non-perturbative elements are **compatible with CGC-type models**

The seemingly incompatible nature of the distributions

Two different kinds of gluon distributions

Moderate x distributions

Low x distributions

TMD, PDF...

Dipole scattering amplitude

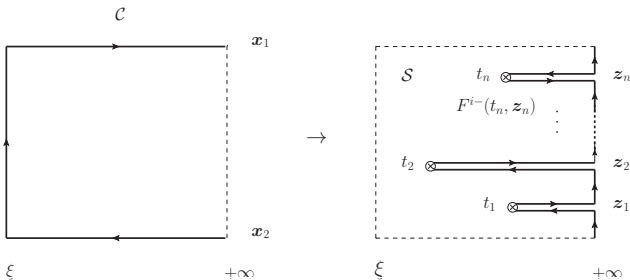
$$\langle P | F^{-i} W F^{-j} W | P \rangle$$

$$\langle P | \text{tr}(U_1 U_2^\dagger) | P \rangle$$

The Wilson line \leftrightarrow parton distribution equivalence

Most general equivalence: use the **Non-Abelian Stokes theorem**

[RB, Mehtar-Tani]



$$\mathcal{P} \exp \left[\oint_C dx_\mu A^\mu(x) \right] = \mathcal{P} \exp \left[\int_S d\sigma_{\mu\nu} WF^{\mu\nu} W^\dagger \right]$$

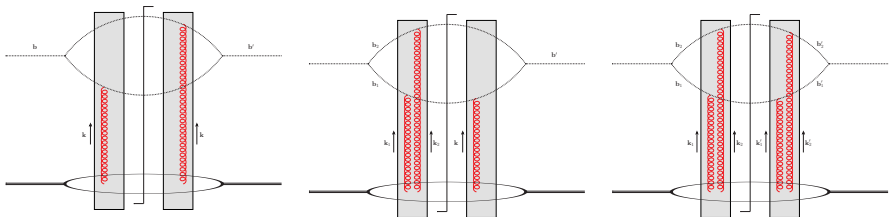
$$U_{x_{1\perp}} U_{x_{2\perp}}^\dagger = [\hat{x}_{1\perp}, \hat{x}_{2\perp}]$$

Inclusive low x cross section

Inclusive low x cross section = TMD cross section

[Altioluk, RB, Kotko], [Altioluk, RB]

Generalizes [Dominguez, Marquet, Xiao, Yuan]

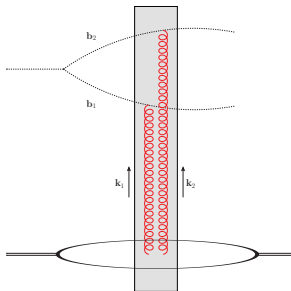


$$\begin{aligned}
 \sigma &= \mathcal{H}_2^{ij}(k) \otimes f_2^{ij}(x=0, k) \\
 &+ \mathcal{H}_3^{ijk}(k, k_1) \otimes f_3^{ijk}(x=0, x_1=0, k, k_1) \\
 &+ \mathcal{H}_4^{ijkl}(k, k_1, k_1') \otimes f_4^{ijkl}(x=0, x_1=0, x_1'=0, k, k_1, k_1')
 \end{aligned}$$

All distributions are evaluated in the **strict $x = 0$ limit**

Exclusive low x cross section

Exclusive low x amplitude = GTMD amplitude
[Altinoluk, RB]

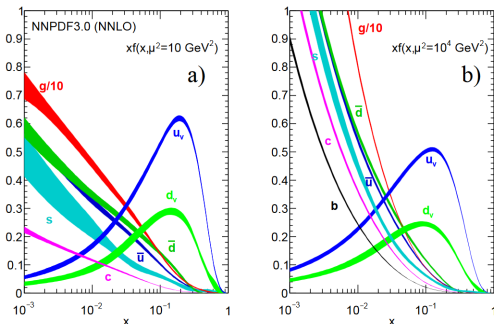


$$\mathcal{H}^{ij}(\mathbf{k}_1, \mathbf{k}_2) \otimes f^{ij}(x=0, \xi=0; \mathbf{k}, \Delta)$$

Every exclusive low x process probes
a **Wigner distribution!**

All distributions are evaluated in the **strict $x = 0$ limit**

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[NNLO NNPDF3.0 global analysis, taken from PDG2018]

Instabilities in the collinear corner of the phase space

All distributions are evaluated in the **strict $x = 0$ limit**

Hard part \mathcal{H} and gluon distribution f for an inclusive observable:

Bjorken limit

$$s \sim Q^2$$

$$\int dx f(x) \mathcal{H}(x)$$

Leading twist of the CGC

$$s \gg Q^2, Q^2 \rightarrow \infty$$

$$f(0) \int dx \mathcal{H}(x)$$

Strong **mismatch beyond LL**: the PDF is not a constant in $x \simeq 0$.

Too late to restore a dependence on x via evolution: x is already integrated over

Summary so far

Distributions involved in pQCD observables

Overarching scheme?

$$f(x_1 \dots x_n; k_{\perp 1} \dots k_{\perp n})$$

Bjorken limit

$$s \sim Q^2$$

$$f(x; 0_{\perp}) + O(Q^{-2})$$

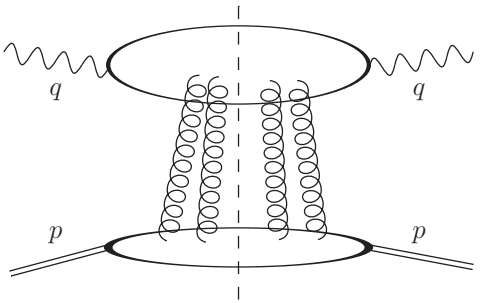
Regge limit

$$s \gg Q^2$$

$$f(0 \dots 0, k_{\perp 1} \dots k_{\perp n}) + O(x_{Bj})$$

Look for an interpolating scheme for simple observables

An interpolating scheme for processes which factorize collinearly in the Bjorken limit



Bjorken limit

$$s \sim Q^2$$

$$f(x, k_{\perp} = 0) + O(Q^{-2})$$

Regge limit

$$s \gg Q^2$$

$$f(x = 0, k_{\perp}) + O(x_{\text{Bj}})$$

Interpolation?

$$s \gtrsim Q^2$$

$$f(x, k_{\perp}) + O(x_{\text{Bj}} Q^{-2})$$

Basic observation: in both limits, $k^+ \simeq 0$ for t -channel gluons

Factorization in k^+ space is consistent

[Balitsky, Tarasov]

Building a semi-classical picture

Still factorizing gluons depending on k^+ in $A^+ = 0$ gauge

Necessary gluon fields in the [Regge limit](#):

$$A^\mu(x) = A^-(x^+, 0^-, \mathbf{x}) n_2^\mu$$

Necessary gluon fields in the [Bjorken limit](#)?

$$A^\mu(x) = A^-(x^+, x^-, \mathbf{x}) n_2^\mu + A_\perp^\mu(x^+, x^-, \mathbf{x})$$

Building a semi-classical picture

Still factorizing gluons depending on k^+ in $A^+ = 0$ gauge

Necessary gluon fields in the **Regge limit**:

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$$A^\mu(x) = A^-(x^+, \mathbf{x}^-, \mathbf{x}) n_2^\mu + A_\perp^\mu(x^+, \mathbf{x}^-, \mathbf{x})$$

Dependence on x^- : **sub-sub-leading** in twist counting

Building a semi-classical picture

Still factorizing gluons depending on k^+ in $A^+ = 0$ gauge

Necessary gluon fields in the **Regge limit**:

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Necessary gluon fields in the **Bjorken limit**?

$$A^\mu(x) = A^-(x^+, 0^-, \mathbf{x}) n_2^\mu + A_{\perp}^\mu(x^+, 0^-, \mathbf{x})$$

Non-zero A_{\perp} : only **two A^i** contribute to DDVCS

They can be computed using **Ward-Takahashi**: only necessary for consistency checks, **can be dropped**.

Building a semi-classical picture

Still factorizing gluons depending on k^+ in $A^+ = 0$ gauge

Necessary gluon fields in the [Regge limit](#):

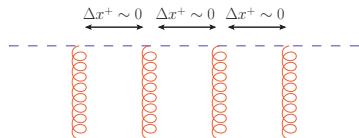
$$A^\mu(x) = A^-(x^+, 0^-, \mathbf{x}) n_2^\mu$$

Necessary gluon fields in the [Bjorken limit](#):

$$A^\mu(x) = A^-(x^+, 0^-, \mathbf{x}) n_2^\mu$$

Effective Feynman rules in the slow background field

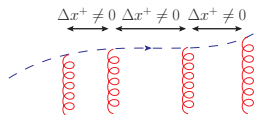
Effective fermion propagator in the external classical field



- $A_{\text{cl}}^i = 0$, $A_{\text{cl}}^+ = 0$: the Dirac structure **factorizes**
- A_{cl} does not depend on x^- : **conservation** of $+$ momentum
- A_{cl} is **peaked** around $x^+ = 0$:
 - Most external propagators get **factorized out**
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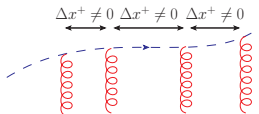
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Effective Feynman rules in the slow background field

Effective fermion propagator in the external classical field

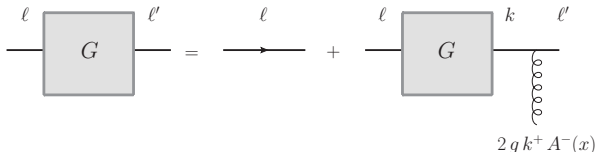


- $A_{\text{cl}}^i = 0$, $A_{\text{cl}}^+ = 0$: the Dirac structure **factorizes**
- A_{cl} does not depend on x^- : **conservation** of $+$ momentum

$$D_F(\ell', \ell) = i \frac{\gamma^+}{2\ell^+} (2\pi)^D \delta^D(\ell' - \ell) + i \frac{\ell' \gamma^+ \ell}{2\ell^+} G_{\text{scal}}(\ell', \ell)$$

Effective Feynman rules in the slow background field

Effective scalar propagator in the external classical field



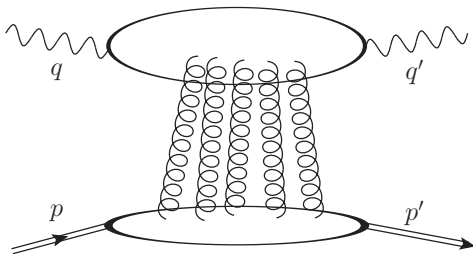
$$G_{\text{scal}}(\ell', \ell) - G_0(\ell')(2\pi)^D \delta^D(\ell' - \ell)$$

$$= 2g \int d^D z \int \frac{d^D k}{(2\pi)^D} e^{i(\ell' - k) \cdot z} G_0(\ell') (k \cdot A)(z) G_{\text{scal}}(k, \ell).$$

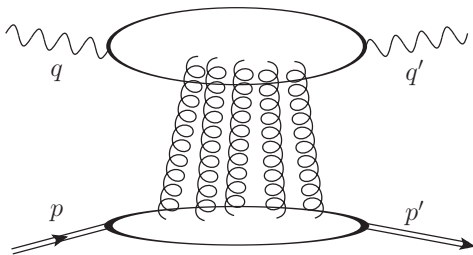
In coordinate space, it satisfies the **Klein-Gordon equation in a potential**

$$[-\square_z + 2igA(z) \cdot \partial_z] G_{\text{scal}}(z, z_0) = \delta^D(z - z_0)$$

Application to the exclusive
 $\gamma^{(*)}(q)P(p) \rightarrow \gamma^{(*)}(q')P(p')$ amplitude



Double, Spacelike, and Timelike exclusive Compton Scattering

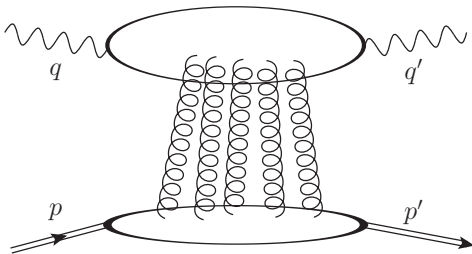


Longitudinal momentum variables:

$$x, \quad \xi \sim \frac{-q^2 + q'^2}{2q \cdot (p + p')}, \quad x_{Bj} = \frac{-q^2 - q'^2}{2q \cdot (p + p')}$$

Can we restore the dependence on all 3 variables in our CGC-like scheme?

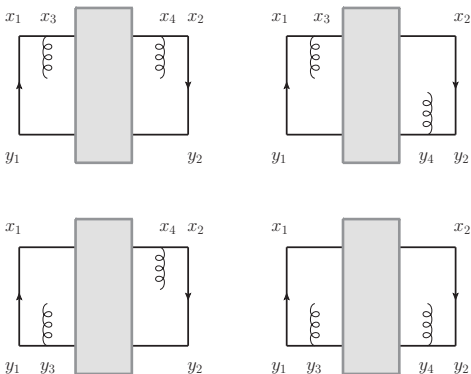
Double, Spacelike, and Timelike exclusive Compton Scattering



$$\mathcal{A} = \frac{e^2}{\mu^{d-2}} \varepsilon_q^\mu \varepsilon_{q'}^{\nu*} \sum_f q_f^2 \int \frac{d^D l}{(2\pi)^D} \int \frac{d^D k}{(2\pi)^D}$$

$$\times \langle p' | \text{tr} [\gamma_\nu D_F(k, l) \gamma_\mu D_F(-q + l, -q' + l + k)] | p \rangle$$

Some operator algebra



$$\begin{aligned}
 & \text{tr } G_{\text{sca1}}^R(x_2, x_1) G_{\text{sca1}}^A(y_1, y_2) \\
 &= 16g^2 \int d^D x_3 \int d^D x_4 \int d^D y_3 \int d^D y_4 \delta(y_3^+ - x_3^+) \delta(x_4^+ - y_4^+) \\
 &\times (\partial_{x_3}^+ G_0^R)(x_3, x_1) (\partial_{x_4}^+ G_0^R)(x_2, x_4) (\partial_{y_3}^+ G_0^A)(y_1, y_3) (\partial_{y_4}^+ G_0^A)(y_4, y_2) \\
 &\times \text{tr} \left\{ [A^-(y_3) - A^-(x_3)] G_{\text{sca1}}^A(y_3, y_4) [A^-(y_4) - A^-(x_4)] G_{\text{sca1}}^R(x_4, x_3) \right\}
 \end{aligned}$$

(First) final result

Fully general result

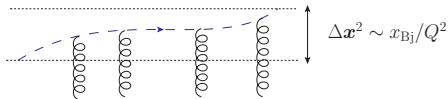
$$\mathcal{A} \propto \mathcal{U}^{ij}(z, \ell_1, \ell_2) \otimes_{z, \ell_1, \ell_2} (\partial^i \Phi)(z, \ell_1) (\partial^j \Phi^*)(z, \ell_2)$$

- Φ : standard wave functions
- \mathcal{U}^{ij} : generalization of the dipole operator

Contains unnecessary subleading powers of x_{Bj}, ξ and Q, Q'

Further simplifications

Partial twist expansion



Typical transverse recoil of a fast parton:
 $\Delta x^2 \sim 1/(q^+(p^- + p'^-)) \sim 1/s$

$1/s$: eikonally suppressed in the **Regge** limit

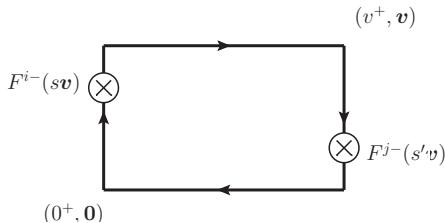
$1/s \sim 1/Q^2$: twist suppressed in in the **Bjorken** limit.

We can **get rid of all corrections from transverse recoils** without loss of accuracy

Further simplifications

Partial twist expansion

$$\frac{\langle p' | \mathcal{U}^{ij}(z, \ell_1, \ell_2) | p \rangle}{\langle p | p \rangle} \simeq -i \frac{(2\pi)^d}{8z\bar{z}(q^+)^2} \int dx \frac{\mathcal{G}^{ij}(x, \ell_2 - \ell_1)}{x - x_{Bj} - \frac{(\frac{\ell_1 + \ell_2}{2})^2}{z\bar{z}q^+(p^- + p'^-)} + i0},$$



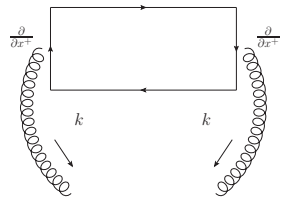
x-dependent unintegrated GPD

$$\mathcal{G}^{ij}(x, \xi, \mathbf{k}, \mathbf{\Delta}) \equiv \frac{2}{p^- + p'^-} \int \frac{dv^+}{2\pi} e^{ix \frac{p^- + p'^-}{2} v^+} \int \frac{d^d \mathbf{v}}{(2\pi)^d} e^{-i(\mathbf{k} \cdot \mathbf{v})} \int_0^1 ds ds' \\ \times \langle p' | \text{tr}_c \{ [v^+, 0^+]_0 F^{i-}(0^+, sv) [0^+, v^+]_v F^{j-}(v^+, s'v) \} | p \rangle$$

The unintegrated PDF

uGPD as a finite Wilson loop

$$\begin{aligned}
 & \int d^2k e^{i(k \cdot r)} r^i r^j \mathcal{G}^{ij}(x, \xi, k, \Delta) \\
 &= \frac{1}{\alpha_s} \int \frac{d^4 v_1 d^4 v_2}{(2\pi)^4} \delta(v_1^-) \delta(v_2^-) e^{-i(k - \frac{\Delta}{2}) \cdot v_1 + i(k + \frac{\Delta}{2}) \cdot v_2} \\
 & \times \frac{\partial}{\partial v_1^+} \frac{\partial}{\partial v_2^+} \frac{\langle p' | \text{tr}[v_1^+, v_2^+]_{v_1} [v_1, v_2]_{v_2^+} [v_2^+, v_1^+]_{v_2} [v_2, v_1]_{v_1^+} | p \rangle}{\langle p | p \rangle}
 \end{aligned}$$



x-dependent unintegrated GPD \Leftrightarrow FT of a finite Wilson loop

(Actual) final result

Final expression for the amplitude

$$\begin{aligned}
 \mathcal{A} &= g^2 \sum_f q_f^2 \int_0^1 \frac{dz}{2\pi} \int \frac{d^d \ell}{(2\pi)^d} \int d^d \mathbf{k} \\
 &\times (\partial^i \Phi)(z, \ell - \mathbf{k}/2) (\partial^j \Phi^*)(z, \ell + \mathbf{k}/2) \\
 &\times \int dx \frac{\mathcal{G}^{ij}(x, \xi, \mathbf{k}, \Delta)}{x - x_{Bj} - \frac{\ell^2}{2z\bar{z}q^+P^-} + i0}
 \end{aligned}$$

Standard wave functions Φ x -dependent unintegrated GPD $\mathcal{G}^{ij}(x, \xi, \mathbf{k}, \Delta)$

Summary

Interpolating scheme for exclusive Compton scattering

Overarching scheme

$$\int d\mathbf{x} \int d^d \mathbf{k} \mathcal{G}^{ij}(\mathbf{x}, \xi, \mathbf{k}, \mathbf{\Delta}) H^{ij}(\mathbf{x}, \xi, \mathbf{k}, \mathbf{\Delta})$$

Bjorken limit?

$$\int d\mathbf{x} H^{ij}(\mathbf{x}, \xi, \mathbf{0}, \mathbf{\Delta}) \times [\int d^d \mathbf{k} \mathcal{G}^{ij}(\mathbf{x}, \xi, \mathbf{k}, \mathbf{\Delta})]$$

Regge limit?

$$\int d^d \mathbf{k} \mathcal{G}^{ij}(\mathbf{0}, \xi, \mathbf{k}, \mathbf{\Delta}) \times [\int d\mathbf{x} H^{ij}(\mathbf{x}, \xi, \mathbf{k}, \mathbf{\Delta})]$$

Did we find the right interpolating scheme?

Bjorken limit and Regge limit

The Bjorken limit

Recovering the Bjorken limit

The Bjorken limit is reached by **neglecting transverse momentum transfert** from the target:

$$|\ell| \sim Q \gg |k|$$

Key observation: \mathcal{G}^{ij} integrates into GPDs

$$\begin{aligned} & \int d^d k (\partial^i \phi)(z, \ell - k/2) (\partial^j \phi^*)(z, \ell + k/2) \mathcal{G}^{ij}(x, k) \\ & \simeq (\partial^i \phi)(z, \ell) (\partial^j \phi^*)(z, \ell) \int d^d k \mathcal{G}^{ij}(x, k) \\ & \simeq (\partial^i \phi)(z, \ell) (\partial^j \phi^*)(z, \ell) G^{ij}(x) \end{aligned}$$

We fully recover the well-known one-loop exclusive Compton scattering amplitudes

The Bjorken limit

Unpolarized contribution

$$\begin{aligned}
& 2\alpha_{\text{em}}\alpha_s \sum_f q_f^2 \int dx \frac{(\varepsilon_q \cdot \varepsilon_{q'}) G(x, \xi, \mathbf{\Delta})}{(x + \xi - i0x_{\text{Bj}})^2 (x - \xi + i0x_{\text{Bj}})^2} \\
& \times \frac{1}{2\xi} \left\{ \left[(x_{\text{Bj}} + \xi)(x^2 - \xi^2 + 4\xi x_{\text{Bj}} + 4\xi x) \ln \left(\frac{x_{\text{Bj}} + x - i0}{x_{\text{Bj}} + \xi - i0} \right) \right. \right. \\
& - \frac{x_{\text{Bj}} + \xi}{2} (x^2 - \xi^2 + 2\xi x_{\text{Bj}} + 2\xi x) \left[\ln^2 \left(\frac{x_{\text{Bj}} + x - i0}{\xi} \right) - \ln^2 \left(\frac{x_{\text{Bj}} + \xi - i0}{\xi} \right) \right] \\
& + \frac{x_{\text{Bj}} - \xi}{2} (x^2 - \xi^2 - 2\xi x_{\text{Bj}} - 2\xi x) \left[\ln^2 \left(\frac{x_{\text{Bj}} + x - i0}{\xi} \right) - \ln^2 \left(\frac{x_{\text{Bj}} - \xi - i0}{\xi} \right) \right] \\
& \left. \left. - (x_{\text{Bj}} - \xi)(x^2 - \xi^2 - 4\xi x_{\text{Bj}} + 4\xi) \ln \left(\frac{x_{\text{Bj}} + x - i0}{x_{\text{Bj}} - \xi - i0} \right) \right] + (x \rightarrow -x) \right\}
\end{aligned}$$

The Bjorken limit

Polarized contribution

$$\begin{aligned}
& 2\alpha_{\text{em}}\alpha_s \sum_f q_f^2 \int dx \frac{\epsilon^{mn} e_h^m e_{h'}^{n*} \tilde{G}(x, \xi, \mathbf{\Delta})}{(x + \xi - i0x_{\text{Bj}})^2 (x - \xi + i0x_{\text{Bj}})^2} \\
& \times \left\{ \left[2(2x^2 + \xi^2) \ln \left(\frac{x_{\text{Bj}} + x - i0}{\xi} \right) \right. \right. \\
& + 3x(x_{\text{Bj}} + \xi) \ln \left(\frac{x_{\text{Bj}} + x - i0}{x_{\text{Bj}} + \xi - i0} \right) + 3x(x_{\text{Bj}} - \xi) \ln \left(\frac{x_{\text{Bj}} + x - i0}{x_{\text{Bj}} - \xi - i0} \right) \\
& - \frac{1}{2}x(x_{\text{Bj}} + \xi) \left[\ln^2 \left(\frac{x_{\text{Bj}} + x - i0}{\xi} \right) - \ln^2 \left(\frac{x_{\text{Bj}} + \xi - i0}{\xi} \right) \right] \\
& - \frac{1}{2}x(x_{\text{Bj}} - \xi) \left[\ln^2 \left(\frac{x_{\text{Bj}} + x - i0}{\xi} \right) - \ln^2 \left(\frac{x_{\text{Bj}} - \xi - i0}{\xi} \right) \right] \\
& \left. \left. - \frac{1}{2}(x^2 + \xi^2) \ln^2 \left(\frac{x_{\text{Bj}} + x - i0}{\xi} \right) \right] - (x \rightarrow -x) \right\}
\end{aligned}$$

The Bjorken limit

Transversity contribution

$$2\alpha_{\text{em}}\alpha_s \sum_f q_f^2 \tau^{mn,ij} \mathbf{e}_h^m \mathbf{e}_{h'}^{n*} \int dx \frac{G_T^{ij}(x, \xi, \mathbf{\Delta})}{(x - \xi + i0_{x_{\text{Bj}}})^2 (x + \xi - i0_{x_{\text{Bj}}})^2} \\ \times \left[(x^2 - \xi^2) + (x_{\text{Bj}}^2 - \xi^2) \ln \frac{(x_{\text{Bj}} - x - i0)(x_{\text{Bj}} + x - i0)}{(x_{\text{Bj}} - \xi - i0)(x_{\text{Bj}} + \xi - i0)} \right]$$

The Regge limit

Recovering the Regge limit? What is x ?

Naive argument

- In the Regge limit, the amplitude is dominated by its **imaginary part**
- Leading order amplitude:

$$\text{Im}\mathcal{A}_{LO} \propto \text{Im} \int dx H^q(x, \xi, t) \frac{1}{x - x_{Bj} + i\epsilon} = -\pi H^q(x_{Bj}, \xi, t)$$

- Hence **take $x = x_{Bj}$**

Problems

- At NLO, the **x cut** is way more complicated
- For DDVCS and for TCS, **s -channel cuts** also contribute to the imaginary part

The Bjorken limit

Recovering the Regge limit

The Regge limit is reached by neglecting x_{Bj} and setting $\frac{\ell^2}{z\bar{z}} \ll q \cdot P$, then taking the x cut:

$$\frac{1}{x - x_{Bj} - \frac{\ell^2}{2z\bar{z}q^+P^-} + i0} \rightarrow \frac{1}{x + i0} \rightarrow -i\pi\delta(x),$$

then taking $x_{Bj}, \xi \ll 1$.

Key observation:

$$\begin{aligned} & \int \frac{d^d \ell_1}{(2\pi)^d} \int \frac{d^d \ell_2}{(2\pi)^d} e^{-i(\ell_1 \cdot r_1) + i(\ell_2 \cdot r_2)} \mathbf{r}_1^i \mathbf{r}_2^j [x G^{ij}(x, \ell_2 - \ell_1)]_{x=0} \\ &= \frac{N_c}{2\pi^2 \alpha_s} \delta^d(\mathbf{r}_1 - \mathbf{r}_2) \int \frac{d^d \mathbf{v}_2}{(2\pi)^d} \text{Re} \frac{\langle P | 1 - \frac{1}{N_c} \text{tr}_c (U_{\mathbf{v}_2 + \mathbf{r}_1} U_{\mathbf{v}_2}^\dagger) | P \rangle}{\langle P | P \rangle} \end{aligned}$$

The Regge limit

Recovering the Regge limit

$$\begin{aligned}
& (\partial^i \Phi)(z, \ell - \frac{k}{2})(\partial^j \Phi^*)(z, \ell + \frac{k}{2}) \otimes_{\ell, k} x G^{ij}(x, \mathbf{k}) \delta(x) \\
& \rightarrow \Psi(z, \mathbf{r}_1) \Psi^*(z, \mathbf{r}_2) \otimes_{\mathbf{r}_1, \mathbf{r}_2} r_1^i r_2^j [x G^{ij}(x, \mathbf{k})]_{x=0} \\
& \rightarrow \Psi(z, \mathbf{r}_1) \Psi^*(z, \mathbf{r}_2) \otimes_{\mathbf{r}_1, \mathbf{r}_2} \delta^d(\mathbf{r}_1 - \mathbf{r}_2) U U \\
& \rightarrow |\Psi(z, \mathbf{r})|^2 \otimes_{\mathbf{r}} D(\mathbf{r})
\end{aligned}$$

We fully recover the small- x description of exclusive Compton scattering e.g. [Hatta, Xiao, Yuan].

Rq: $x = 0$ is the reason why wave functions involve the same dipole size in the wave functions

Non-commutativity of the limits

Summary

Interpolating scheme for exclusive Compton scattering

Overarching scheme

$$\int d\mathbf{x} \int d^d \mathbf{k} \mathcal{G}^{ij}(\mathbf{x}, \xi, \mathbf{k}, \mathbf{\Delta}) H^{ij}(\mathbf{x}, \xi, \mathbf{k}, \mathbf{\Delta})$$

Bjorken limit

$$\int d\mathbf{x} H^{ij}(\mathbf{x}, \xi, \mathbf{0}, \mathbf{\Delta}) \times [\int d^d \mathbf{k} \mathcal{G}^{ij}(\mathbf{x}, \xi, \mathbf{k}, \mathbf{\Delta})]$$

Regge limit

$$\int d^d \mathbf{k} \mathcal{G}^{ij}(\mathbf{0}, \xi, \mathbf{k}, \mathbf{\Delta}) \times [\int d\mathbf{x} H^{ij}(\mathbf{x}, \xi, \mathbf{k}, \mathbf{\Delta})]$$

We found an interpolating scheme

Double limit

Do the two limits commute?

Leading twist limit of the Regge limit

$$\lim_{Q^2+Q'^2 \rightarrow \infty} \mathcal{A}_{\text{Regge}} = g^2 \sum_f q_f^2 \int_0^1 \frac{dz}{2\pi} \int \frac{d^d \ell}{(2\pi)^d} \times (-i\pi) G^{ij}(0, \xi, t) (\partial^i \Phi)(z, \ell) (\partial^j \Phi^*)(z, \ell)$$

Eikonal limit of the Bjorken limit

$$\lim_{x_{\text{Bj}}, \xi \rightarrow 0} \mathcal{A}_{\text{Bjorken}} = g^2 \sum_f q_f^2 \int_0^1 \frac{dz}{2\pi} \int \frac{d^d \ell}{(2\pi)^d} \times \lim_{x_{\text{Bj}}, \xi \rightarrow 0} \int dx \frac{G^{ij}(x, \xi, t) (\partial^i \Phi)(z, \ell) (\partial^j \Phi^*)(z, \ell)}{x - x_{\text{Bj}} - \frac{\ell^2}{2z\bar{z}q^+P^-} + i0}$$

Double limit

Do the two limits commute?

If $G^{ij}(x, \xi, t)$ is a constant at $x = 0$:

$$\begin{aligned} & \int dx \frac{G^{ij}(x, \xi, t)(\partial^i \Phi)(z, \ell)(\partial^j \Phi^*)(z, \ell)}{x - x_{Bj} - \frac{\ell^2}{2z\bar{z}q^+P^-} + i0} \\ & \simeq G^{ij}(0, \xi, t) \int dx \frac{(\partial^i \Phi)(z, \ell)(\partial^j \Phi^*)(z, \ell)}{x - x_{Bj} - \frac{\ell^2}{2z\bar{z}q^+P^-} + i0} \\ & = G^{ij}(0, \xi, t)(\partial^i \Phi)(z, \ell)(\partial^j \Phi^*)(z, \ell) \\ & \quad \times \ln \left(\frac{1 - x_{Bj} - \frac{\ell^2}{z\bar{z}\frac{Q^2+Q'^2}{2}}\xi + i0}{-1 - x_{Bj} - \frac{\ell^2}{z\bar{z}\frac{Q^2+Q'^2}{2}}\xi + i0} \right) \end{aligned}$$

and thus

$$\begin{aligned} & \lim_{x_{Bj}, \xi \rightarrow 0} \int dx \frac{G^{ij}(x, \xi, t)(\partial^i \Phi)(z, \ell)(\partial^j \Phi^*)(z, \ell)}{x - x_{Bj} - \frac{\ell^2}{2z\bar{z}q^+P^-} + i0} \\ & \simeq -i\pi G^{ij}(0, \xi, t)(\partial^i \Phi)(z, \ell)(\partial^j \Phi^*)(z, \ell) \end{aligned}$$

Double limit

Do the two limits commute?

Leading twist limit of the Regge limit

$$\lim_{Q^2+Q'^2 \rightarrow \infty} \mathcal{A}_{\text{Regge}} = g^2 \sum_f q_f^2 \int_0^1 \frac{dz}{2\pi} \int \frac{d^d \ell}{(2\pi)^d} \\ \times (-i\pi) G^{ij}(0, \xi, t) (\partial^i \Phi)(z, \ell) (\partial^j \Phi^*)(z, \ell)$$

Eikonal limit of the Bjorken limit **provided the GPDs are constant at $x = 0$**

$$\lim_{x_{\text{Bj}}, \xi \rightarrow 0} \mathcal{A}_{\text{Bjorken}} = g^2 \sum_f q_f^2 \int_0^1 \frac{dz}{2\pi} \int \frac{d^d \ell}{(2\pi)^d} \\ \times (-i\pi) G^{ij}(0, \xi, t) (\partial^i \Phi)(z, \ell) (\partial^j \Phi^*)(z, \ell)$$

Checked with explicit final expressions for both double limits

Conclusion

Where do we stand?

Bad news

- Semi-classical small x physics has, **at its core**, issues with **collinear logarithms**
- The problem can be traced down **to the very starting point**

Good news

- We now have a **minimal correction** of semi-classical small x which solves the problem **from first principles**
- Wave functions, and thus hard parts, are **not modified by the scheme**
- All we need is the right evolution equation...

BACKUP

The energy denominators

$$\begin{aligned}
 & \text{tr} G_{\text{scal}}^R(x_2, x_1) G_{\text{scal}}^A(y_1, y_2) \\
 &= 16g^2 \int d^D x_3 \int d^D x_4 \int d^D y_3 \int d^D y_4 \delta(y_3^+ - x_3^+) \delta(x_4^+ - y_4^+) \\
 &\times (\partial_{x_3}^+ G_0^R)(x_3, x_1) (\partial_{x_4}^+ G_0^R)(x_2, x_4) (\partial_{y_3}^+ G_0^A)(y_1, y_3) (\partial_{y_4}^+ G_0^A)(y_4, y_2) \\
 &\times \text{tr} \left\{ [A^-(y_3) - A^-(x_3)] G_{\text{scal}}^A(y_3, y_4) [A^-(y_4) - A^-(x_4)] G_{\text{scal}}^R(x_4, x_3) \right\}
 \end{aligned}$$

Can be proven via the **repeated use of Klein-Gordon in a potential**, or by proving the generalization to G_{scal} of the relation

$$\frac{\partial}{\partial x^+} [y^+, x^+]_{x_1} [x^+, z^+]_{x_2} = -ig[y^+, x^+]_{x_1} [A^-(x^+, x_1) - A^-(x^+, x_2)] [x^+, z^+]_{x_2}$$

Structurally ready for a so-called dilute (perturbative) expansion

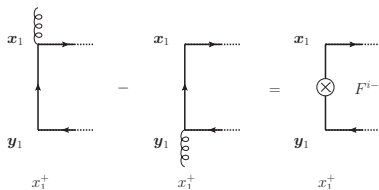
The **free propagators G_0** provide the energy denominators.

Two useful technical details

- The classical field does not depend on x^- so $G_{\text{scal}}(x, x_0)$ only depends on $(x^- - x_0^-)$, not on each separately: we can define

$$G_{\text{scal}}(x, x_0) \equiv \int \frac{dp^+}{2\pi} \frac{e^{-ip^+(x^- - x_0^-)}}{2ip^+} (\mathbf{x} | \mathcal{G}_{p^+}(x^+, x_0^+) | \mathbf{x}_0)$$

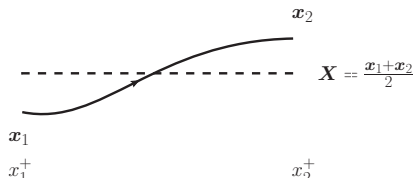
\mathcal{G} satisfies the Schrödinger equation instead of Klein-Gordon



- Since $A^i = 0$, we have

$$A^-(x^+, \mathbf{x}) - A^-(x^+, \mathbf{y}) = -(\mathbf{x}^i - \mathbf{y}^i) \int_0^1 ds F^{i-}(x^+, s\mathbf{x} + (1-s)\mathbf{y})$$

Partial twist expansion



$$(\mathbf{x}_1 | \mathcal{G}_{\rho^+}^R(x_1^+, x_2^+) | \mathbf{x}_2) \simeq \theta(\rho^+) (\mathbf{x}_1 | \mathcal{G}_{\rho^+}^{(0)R}(x_1^+, x_2^+) | \mathbf{x}_2) [x_1^+, x_2^+] \frac{x_1 + x_2}{2}$$

$$(\mathbf{y}_2 | \mathcal{G}_{\rho^+}^A(x_2^+, x_1^+) | \mathbf{y}_1) \simeq \theta(-\rho^+) (\mathbf{y}_2 | \mathcal{G}_{\rho^+}^{(0)A}(x_2^+, x_1^+) | \mathbf{y}_1) [x_2^+, x_1^+] \frac{y_1 + y_2}{2}$$

[Altinoluk, Armesto, Beuf, Martinez, Salgado]

$$F^{i-}(x_1^+, s\mathbf{x}_1 + \bar{s}\mathbf{y}_1) \simeq F^{i-}\left(x_1^+, s\frac{\mathbf{x}_1 + \mathbf{x}_2}{2} + \bar{s}\frac{\mathbf{y}_1 + \mathbf{y}_2}{2}\right)$$