# Observables for scattering on targets with arbitrary spin

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# Matrix elements for composite particles with arbitrary spin

- Decompose matrix element in independent non-perturbative objects while maintaining manifest Lorentz invariance

$$\langle d' | J^\mu | d \rangle = -\left( \left\{ G_1 \left[ \epsilon^{'*} \cdot \epsilon \right] - G_3 \frac{\left( \epsilon'^* \cdot q \right) \left( \epsilon \cdot q \right)}{2m_d^2} \right\} 2P^\mu + G_M \left[ \epsilon^\mu \left( \epsilon'^* \cdot q \right) - \epsilon'^{*\mu} (\epsilon \cdot q) \right] \right)$$

- Spin-$j$ fields embedded in objects with $> 2j+1$ components
  $\rightarrow$ 4-vector, Rarita-Schwinger, Fierz-Pauli, ...
  - Need for constraints
  - Kinematical singularities

- Use $(2j+1)$-component (chiral) spinors [$(j,0)$ & $(0,j)$ irreps.]
  [Joos; Barut-Muzinich-Williams; Weinberg 63+]

# Advantages of chiral spinor construction

- Leads to **systematic approach** for any spin $j$
- "Basic" algebraic construction su(2) $\rightarrow$ su(N) $\rightarrow$ sl(2,$\mathbb{C}$)
- Covariant "multipole" basis emerges $\rightarrow$ physical interpretation
- Parity conserving interactions $\rightarrow$ generalized Dirac algebra
- Easy to implement different types of spin
  $\rightarrow$ canonical, helicity, light-front
- Exact degrees of freedom, no need for constraints

# Lorentz group basics

- Algebra for Generators of the Lorentz group

$$[\mathbb{J}_l, \mathbb{J}_m] = i\epsilon_{lmn}\mathbb{J}_n \ , \quad [\mathbb{J}_l, \mathbb{K}_m] = i\epsilon_{lmn}\mathbb{K}_n \ , \quad [\mathbb{K}_l, \mathbb{K}_m] = -i\epsilon_{lmn}\mathbb{J}_n$$

- Two independent su(2) subalgebras → irreps $(j_A, j_B)$

$$\mathbb{A}_m = \frac{1}{2}(\mathbb{J}_m + i\mathbb{K}_m) \quad , \quad \mathbb{B}_m = \frac{1}{2}(\mathbb{J}_m - i\mathbb{K}_m)$$

$$[\mathbb{A}_l, \mathbb{A}_m] = i\epsilon_{lmn}\mathbb{A}_n \ , \quad [\mathbb{B}_l, \mathbb{B}_m] = i\epsilon_{lmn}\mathbb{B}_n \ , \quad [\mathbb{A}_l, \mathbb{B}_m] = 0$$

- Simplest irreps that contain spin-$j$ → ($2j + 1$ components)
  - Right-handed $(j, 0)$: $\mathbb{K}_m \to -i\mathbb{J}_m$
  - Left-handed $(0, j)$: $\mathbb{K}_m \to +i\mathbb{J}_m$

# Weinberg's causal chiral fields (massive)

- Lorentz invariant S-matrix using a Hamiltonian density built up from causal fields

$$U_{[\Lambda,a]}\psi_{\sigma(x)}U^{-1}_{[\Lambda,a]} = \sum_{\sigma'}\left(D^{(j)}_{[\Lambda^{-1}]}\right)_{\sigma\sigma'}\psi_{\sigma'(\Lambda x+a)}$$

- No EoM for chiral fields (only obey KG eq.)

- Spinors appearing in the fields (not invariants, depend on choice boost)

Canonical $\rightarrow$ $D^{(j)}_{[L(p)]} = e^{-\eta\hat{p}\cdot J^{(j)}}$

$$\overline{D}^{(j)}_{[L(p)]} = e^{+\eta\hat{p}\cdot J^{(j)}}$$

# Propagator and spinors: $t$–tensors

- Propagator numerator

$$\Pi^{(j)}_{\sigma\sigma'}(p) = m^{2j} D^{(j)}_{\sigma\sigma'}[L_p] \left( D^{(j)}_{\sigma'\sigma''}[L_p] \right)^{\dagger} = m^{2j} \left( e^{-2\eta \hat{\boldsymbol{p}}\cdot \boldsymbol{J}^{(j)}} \right)_{\sigma\sigma'}$$

$$\overline{\Pi}^{(j)}_{\sigma\sigma'}(p) = m^{2j} \overline{D}^{(j)}_{\sigma\sigma'}[L_p] \left( \overline{D}^{(j)}_{\sigma'\sigma''}[L_p] \right)^{\dagger} = m^{2j} \left( e^{2\eta \hat{\boldsymbol{p}}\cdot \boldsymbol{J}^{(j)}} \right)_{\sigma\sigma'}$$

- Introduction of 2j-rank $t$–tensors

$$\Pi^{(j)}_{\sigma\sigma'}(p) = t^{\mu_1\mu_2\ldots\mu_{2j}}_{\sigma\sigma'} p_{\mu_1} p_{\mu_2} \ldots p_{\mu_{2j}}$$

totally symmetric

$$\overline{\Pi}^{(j)}_{\sigma\sigma'}(p) = \bar{t}^{\mu_1\mu_2\ldots\mu_{2j}}_{\sigma\sigma'} p_{\mu_1} p_{\mu_2} \ldots p_{\mu_{2j}}$$

covariantly traceless

$$g_{\mu_k\mu_l} t^{\mu_1\ldots\mu_k\ldots\mu_l\ldots\mu_{2j}}_{\sigma\sigma'} = 0$$

- Central role of $t$–tensors

$$D^{(j)}_{[L(p)]} = t^{\mu_1\mu_2\ldots\mu_{2j}} \tilde{p}_{\mu_1} \tilde{p}_{\mu_2} \ldots \tilde{p}_{\mu_{2j}}$$

used to construct boosts/spinors

$$\overline{D}^{(j)}_{[L(p)]} = \bar{t}^{\mu_1\mu_2\ldots\mu_{2j}} \tilde{p}_{\mu_1} \tilde{p}_{\mu_2} \ldots \tilde{p}_{\mu_{2j}}$$

($\tilde{p}^{\mu}$ **not** 4-vectors)
**Same** for any spin!

Canonical:

$$\tilde{p}^{\mu}_{\mathrm{C}} = \sqrt{\frac{1}{2m(m+p^0)}}(p^0 + m, \boldsymbol{p})$$

For helicity and LF spinors similar expression (but $\mathbb{C}$–numbers)

# $t$–tensors

- Generalization of $\sigma^\mu = (1, \boldsymbol{\sigma}) \quad \bar{\sigma}^\mu = (1, -\boldsymbol{\sigma}^\mu)$ to arbitrary spin

- Intertwining map:

$$\begin{array}{cc} (j,0)\otimes(0,j) & [\text{rank–2 in sl}(2,\mathbb{C})] \\ & \updownarrow \\ (j,j) & [\text{rank–}2j \text{ symm. traceless in so}(3,1)] \end{array}$$

- Recursion relation between different spins (CG)

$$\boldsymbol{t}_{\sigma\dot{\tau}}^{\mu_1\mu_2\ldots\mu_{2j}} = \langle j\sigma | j-\frac{1}{2}\sigma_1 \frac{1}{2}\sigma_2\rangle \langle j\dot{\tau}|j-\frac{1}{2}\dot{\tau}_1 \frac{1}{2}\dot{\tau}_2\rangle\, \boldsymbol{t}_{\sigma_1\dot{\tau}_1}^{\mu_1\mu_2\ldots\mu_{2j-1}}\, \boldsymbol{t}_{\sigma_2\dot{\tau}_2}^{\mu_{2j}}$$

- Contain a basis of su(N): use to expand $\langle\lambda'|\hat{O}|\lambda\rangle$.

# Bi-spinors $(j,0)\oplus(0,j)$

- For Parity conserving interactions the direct sum of both chiral representations is used, **like the spin 1/2 case**
- Boosts and bispinor (Weyl rep.)

$$u^{(j)}_{(p,s)} = \mathcal{D}^{(j)}_{[L_p]} \overset{\circ}{u}{}^{(j)}_s = \begin{pmatrix} D^{(j)}_{[L_p]} & 0 \\ 0 & \overline{D}^{(j)}_{[L_p]} \end{pmatrix} \overset{\circ}{u}{}^{(j)}_s = \begin{pmatrix} \Pi^{(j)}(\tilde{p}) & 0 \\ 0 & \overline{\Pi}^{(j)}(\tilde{p}) \end{pmatrix} \overset{\circ}{u}{}^{(j)}_s$$

$$\overset{\circ}{u}{}^{(j)}_s = \begin{pmatrix} \overset{\circ}{\phi}{}^{(j)}_s \\ \overset{\circ}{\phi}{}^{(j)}_s \end{pmatrix} \quad , \qquad \overset{\circ}{\phi}{}^{(j)}_s = m^j \begin{pmatrix} \vdots \\ 1 \\ \vdots \end{pmatrix} \qquad \text{(1 in the s–th position)}$$

- Adjoint bispinor $(\mathcal{D}^{\dagger}_{[\Lambda]} = \beta \mathcal{D}^{-1}_{[\Lambda]} \beta)$

$$\bar{u}^{(j)}_{(p,s)} = \overset{\circ}{u}{}^{(j)\dagger}_s \begin{pmatrix} 0 & \Pi^{(j)}(\tilde{p}) \\ \overline{\Pi}^{(j)}(\tilde{p}) & 0 \end{pmatrix} \quad ; \quad \beta = \begin{pmatrix} 0 & 1^{(j)} \\ 1^{(j)} & 0 \end{pmatrix}$$

# Dirac Eq. & Gamma matrices

- The bispinor satisfy the Dirac eq.

$$\left(\gamma^{\mu_1\cdots\mu_{2j}} p_{\mu_1}\cdots p_{\mu_{2j}} - m^{2j}\right) u^{(j)}(p,s) = 0$$

$$\bar{u}^{(j)}(p,s)\left(\gamma^{\mu_1\cdots\mu_{2j}} p_{\mu_1}\cdots p_{\mu_{2j}} - m^{2j}\right) = 0$$

- Gamma matrices (chiral rep.)

$$\gamma^{\mu_1\cdots\mu_{2j}} = \begin{pmatrix} 0 & t^{\mu_1\cdots\mu_{2j}} \\ \bar{t}^{\mu_1\cdots\mu_{2j}} & 0 \end{pmatrix} \; ; \quad \beta = \gamma^{0\cdots 0} = \begin{pmatrix} 0 & 1^{(j)} \\ 1^{(j)} & 0 \end{pmatrix} \; ; \quad \gamma_5 = \begin{pmatrix} -1^{(j)} & 0 \\ 0 & 1^{(j)} \end{pmatrix}$$

# Algorithm for construction of t-tensors

$$\Pi^{(j)}_{\sigma\sigma'}(p) = m^{2j}\left(e^{-2\eta\hat{p}\cdot\boldsymbol{J}^{(j)}}\right)_{\sigma\sigma'} = t^{\mu_1\mu_2\ldots\mu_{2j}}_{\sigma\sigma'}\, p_{\mu_1} p_{\mu_2}\ldots p_{\mu_{2j}}$$

- The 0-th degree polynomial in the $J$'s is always $t^{0\ldots0} = 1$
- The linear polynomials are the rotation group generators $\quad t^{0\ldots i\ldots 0} = \frac{2}{2j}J_i = \frac{1}{j}J_i$
- From pairwise symmetrizations of the rotation generators

$$t^{0\ldots m\ldots 0\ldots n\ldots 0} = t^{mn0\ldots0} = \frac{1}{\frac{(2j)!}{2!(2j-2)!}}\left(\{J_m, J_n\} - \frac{1}{3}\delta_{mn}\sum_{r=1}^{3}\{J_r, J_r\}\right) + \frac{1}{3}t^{0\ldots0}\delta_{mn}$$

$$= \frac{j}{(2j-1)}\left(\left\{t^{m0\ldots0}, t^{n0\ldots0}\right\} - \frac{1}{j}\delta_{mn}t^{0\ldots0}\right)$$

# Algorithm for construction of t-tensors

- Continues for higher orders [Jordan algebra, **not** universal]
  - Matrices have more and more off-diagonal elements

$$t^{lmn0\ldots0} = t^{0\ldots0l0\ldots0m0\ldots0n0\ldots0} = \frac{j}{(2j-2)}\frac{1}{3}\Big( \left\{t^{l0\ldots0}, t^{mn0\ldots0}\right\} + \left\{t^{m0\ldots0}, t^{nl0\ldots0}\right\} + \left\{t^{n0\ldots0}, t^{lm0\ldots0}\right\} - \frac{2}{j}\left\{\delta_{lm}t^{n0\ldots0} + \delta_{ln}t^{m0\ldots0} + \delta_{mn}t^{l0\ldots0}\right\}\Big)$$

- Stops after $2j$ steps (eigenvalue eq.) $(J-s)(J-s-1)\ldots(J+s) = 0$

- $t$-tensors contain a basis for su(N=2j+1)
  (Universal Enveloping Algebra)

# Algorithm for construction of t-tensors (II)

- Use recursion relation

$$t^{\mu_1\mu_2\cdots\mu_{2j}}_{\sigma\dot{\tau}} = \langle j\sigma | j-\frac{1}{2}\sigma_1 \frac{1}{2}\sigma_2\rangle \, \langle j\dot{\tau}| j-\frac{1}{2}\dot{\tau}_1 \frac{1}{2}\dot{\tau}_2\rangle \; t^{\mu_1\mu_2\cdots\mu_{2j-1}}_{\sigma_1\dot{\tau}_1} \; t^{\mu_{2j}}_{\sigma_2\dot{\tau}_2}$$

- Efficient in +-RL Lorentz coordinates
  - Pauli matrices have only 1 non-zero element (=2)
  - $t^{\mu_1\mu_2\cdots\mu_{2j}}$ elements in that basis have only 1 non-zero matrix element
    → position follows from +-RL counting
    → value from CG recursion
- Appropriate for efficient numerical implementation

# $t^\mu$–tensor for Spin 1/2

0–th order terms in $J_i^{(1/2)}$: $t^0 = 1$

Linear terms in $J_i^{(1/2)}$: $t^i = \frac{1}{1/2} J_i^{(1/2)} = \sigma_i$
(Pauli matrices)

$$J_1^{(1/2)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , J_2^{(1/2)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , J_3^{(1/2)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Quadratic terms in $J_i^{(1/2)}$

$$(J^{(1/2)} - \tfrac{1}{2}1)(J^{(1/2)} + \tfrac{1}{2}1) = 0 \implies (J^{(1/2)})^2 = c_0 1 + c_2 J^{(1/2)}$$

# $t^{\mu\nu}$–tensor for Spin 1

0–th order terms in $J_i^{(1)}$: $t^{00} = 1$

Linear terms in $J_i^{(1)}$: $t^{0i} = J_i^{(1)}$

$$t^{01} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \;,\; t^{02} = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \;,\; t^{03} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Quadratic terms in $J_i^{(1)}$: $t^{ij} = \{J_i^{(1)}, J_j^{(1)}\} - 1\delta_{ij}$

$$t^{11} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \;,\; t^{22} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \;,\; t^{33} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$t^{12} = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \;,\; t^{13} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \;,\; t^{23} = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

Cubic terms in $J_i^{(1)}$: $(J^{(1)} - 1)(J^{(1)})(J^{(1)} + 1) = 0 \implies (J^{(1)})^3 = c_0 1 + c_2 J^{(1)} + c_3 (J^{(1)})^2$

# Algebra of $t$–tensors

## Reduction for Cubic Monomials

- Central role of the covariant $t$–tensors
  $\to$ spinors, boosts, propagators, gamma matrices
- Bilinear calculus involve products with alternating "barring" pattern: $t\bar{t}t\cdots$
- Matrices in $t$–tensors: $su(2j+1)$ basis $\to$ Products can be **linearized**
- Cubic products are reduced with an Invariant Tensor

$$t^{\mu_1\cdots\mu_{2j}}\,\bar{t}^{\rho_1\cdots\rho_{2j}}\,t^{\sigma_1\cdots\sigma_{2j}} = \frac{1}{[(2j)!]^2}\underset{\{\rho_1\ldots\rho_{2j}\}}{\mathcal{S}}\;\underset{\{\sigma_1\ldots\sigma_{2j}\}}{\mathcal{S}}\left(\prod_{l=1}^{2j}\mathcal{C}^{\mu_l\rho_l\sigma_l\alpha_l}\right)t_{\alpha_1\cdots\alpha_{2j}}$$

$$\bar{t}^{\mu_1\cdots\mu_{2j}}\,t^{\rho_1\cdots\rho_{2j}}\,\bar{t}^{\sigma_1\cdots\sigma_{2j}} = \frac{1}{[(2j)!]^2}\underset{\{\rho_1\ldots\rho_{2j}\}}{\mathcal{S}}\;\underset{\{\sigma_1\ldots\sigma_{2j}\}}{\mathcal{S}}\left(\prod_{l=1}^{2j}\bar{\mathcal{C}}^{\mu_l\rho_l\sigma_l\alpha_l}\right)\bar{t}_{\alpha_1\cdots\alpha_{2j}}$$

$$\mathcal{C}^{\mu\rho\alpha\beta} = g^{\mu\rho}g^{\alpha\beta} - g^{\mu\alpha}g^{\rho\beta} + g^{\mu\beta}g^{\rho\alpha} + i\epsilon^{\mu\rho\alpha\beta} = \left(\bar{\mathcal{C}}^{\mu\rho\alpha\beta}\right)^*$$ (Lorentz Invariants)

- **Trade matrix multiplication by number multiplication**

# Algebra of $t$–tensors

## Reduction for Quadratic Monomials

- Central role of the covariant $t$–tensors
  $\rightarrow$ spinors, boosts, propagators, gamma matrices

- Since, $t^{0\cdots 0} = \bar{t}^{\,0\cdots 0} = 1 \quad \rightarrow \quad t^{\mu_1\cdots\mu_{2j}}\,\bar{t}^{\nu_1\cdots\nu_{2j}} = t^{\mu_1\cdots\mu_{2j}}\,\bar{t}^{\nu_1\cdots\nu_{2j}}\left(t^{\rho_1\cdots\rho_{2j}}\,\eta_{\rho_1}\cdots\eta_{\rho_{2j}}\right)$

$$t^{\mu_1\cdots\mu_{2j}}\,\bar{t}^{\rho_1\cdots\rho_{2j}} = \frac{1}{(2j)!}\underset{\{\rho_1\ldots\rho_{2j}\}}{\mathcal{S}}\left(\prod_{l=1}^{2j}\mathcal{C}^{\mu_l\rho_l\sigma_l\alpha_l}\,\eta_{\sigma_l}\right)t_{\alpha_1\cdots\alpha_{2j}}$$

$\eta^{\mu} = (1, 0, 0, 0)$

$\mathcal{C}^{\mu\rho\sigma\alpha}\,\eta_{\sigma} = g^{\mu\rho}\,\eta^{\alpha} - g^{\rho\alpha}\,\eta^{\mu} + g^{\mu\alpha}\,\eta^{\rho} + i\,\epsilon^{\mu\rho\sigma\alpha}\,\eta_{\sigma}$ (Rotational Invariant)

- General result $\left(\mathcal{Q}^{\mu\rho\alpha}_{\text{red}} = -\mathcal{Q}^{\rho\mu\alpha}_{\text{red}} \equiv \mathcal{C}^{\mu\rho\sigma\alpha}\,\eta_{\sigma} - g^{\mu\rho}\,\eta^{\alpha}\right)$

$$t^{\mu_1\cdots\mu_{2j}}\,\bar{t}^{\rho_1\cdots\rho_{2j}} = \sum_{m=0}^{2j}\frac{1}{(2j)!}\underset{\{\rho_1\ldots\rho_{2j}\}}{\mathcal{S}}\left[\sum_{n=1}^{B^{2j}_{m}}\left(\prod_{l\in\pi_{m,n}}\mathcal{Q}^{\mu_l\rho_l\alpha_l}_{\text{red}}\;\prod_{k\in\pi^{\complement}_{m,n}}g^{\mu_k\rho_k}\,\eta^{\alpha_k}\right)\right]t_{\alpha_1\cdots\alpha_{2j}}$$

each $0 \leq m \leq 2j$ corresponds to a Lorentz rank-$m$ **bitensor** $[(m, 0) \oplus (0, m)]$.

## Covariant $sl(2,C)$ Multipole expansion from quadratic monomials

- $m=0$ → Identity $\quad \prod_{r=1}^{0} \mathcal{Q}_{\text{red}}^{\mu_r\rho_r\alpha_r} \left(\prod_{s=1}^{2j} \eta^{\alpha_s}\right) t_{\alpha_1\cdots\alpha_{2j}} = t_{0\cdots 0} = 1$

- $m=1$ → generators of Lorentz transf. $[(1,0)\oplus(0,1)]$
  $\left(i\left[\mathbb{M}^{\mu\rho},\mathbb{M}^{\nu\lambda}\right] = g^{\rho\lambda}\mathbb{M}^{\mu\nu} - g^{\mu\nu}\mathbb{M}^{\rho\lambda} + g^{\rho\nu}\mathbb{M}^{\mu\lambda} - g^{\mu\lambda}\mathbb{M}^{\rho\nu}\right)$

$$\prod_{r=1}^{1} \mathcal{Q}_{\text{red}}^{\mu_r\rho_r\alpha_r} \left(\prod_{s=2}^{2j} \eta^{\alpha_s}\right) (j) t_{\alpha_1\cdots\alpha_{2j}} = \mathcal{Q}_{\text{red}}^{\mu\rho\alpha}(j) t_{\alpha 0\cdots 0} = -i\mathbb{M}^{\mu\rho}$$

- For general $m$ $[(m,0)\oplus(0,m)]$

$$\prod_{r=1}^{m} \mathcal{Q}_{\text{red}}^{\mu_r\rho_r\alpha_r}(j) t_{\alpha_1\cdots\alpha_m 0\cdots 0} = \frac{(-i)^m}{m!} \underset{\{\mu_1\rho_1,\cdots,\mu_m\rho_m\}}{\mathcal{S}} \prod_{r=1}^{m} \mathbb{M}^{\mu_r\rho_r} - \text{(Lower Multipoles)}$$

  Decompose operators with **physical interpretation** for each term
  → mono-, di-, quadrupole, ...

See also [Cotogno, Lorcé, Lowdon, Morales PRD 2020)]

# Algebra of $\gamma$-tensors

## Generalized Dirac basis (Weyl rep)

- $2j$-rank symmetric tensors: $\gamma^{\mu_1\cdots\mu_{2j}} = \begin{pmatrix} 0 & t^{\mu_1\cdots\mu_{2j}} \\ \bar{t}^{\mu_1\cdots\mu_{2j}} & 0 \end{pmatrix}$
  (2j+1) independent matrices
- $2j$-rank symmetric psuedo-tensors: $\gamma^{\mu_1\cdots\mu_{2j}}\gamma_5 = \begin{pmatrix} 0 & t^{\mu_1\cdots\mu_{2j}} \\ -\bar{t}^{\mu_1\cdots\mu_{2j}} & 0 \end{pmatrix}$
  (2j+1) independent matrices
- $4j$-rank bi-tensors: $\gamma^{\mu_1\cdots\mu_{2j}}\gamma^{\rho_1\cdots\rho_{2j}} = \begin{pmatrix} t^{\mu_1\cdots\mu_{2j}}\bar{t}^{\rho_1\cdots\rho_{2j}} & 0 \\ 0 & \bar{t}^{\mu\cdots}t^{\rho_1\cdots\rho_{2j}} \end{pmatrix}$
  2(2j+1) independent matrices
  $\rightarrow$ including $1$, $\gamma_5$

$$\gamma^{\mu_1\cdots\mu_{2j}}\gamma^{\rho_1\cdots\rho_{2j}} = \sum_{m=0}^{2j}\frac{1}{(2j)!}\underset{\{\rho_1\cdots\rho_{2j}\}}{\mathcal{S}}\sum_{n=1}^{B_m^{2j}}\Big(\mathrm{Re}\left\{\prod_{l\in\pi_{m,n}}\mathcal{Q}_{\mathrm{red}}^{\mu_l\rho_l\alpha_l}\right\}\gamma_{\alpha_1\cdots\alpha_{2j}}\gamma^{0\cdots0}$$

$$+i\,\mathrm{Im}\left\{\prod_{l\in\pi_{m,n}}\mathcal{Q}_{\mathrm{red}}^{\mu_l\rho_l\alpha_l}\right\}\gamma_{\alpha_1\cdots\alpha_{2j}}\gamma_5\gamma^{0\cdots0}\Big)\prod_{k\in\pi_{m,n}^{\complement}}g^{\mu_k\rho_k}\eta^{\alpha_k}$$

# Dirac Bilinear Calculus Generalization

## Generalized Bilinears

- Chains of $t\bar{t}t\ldots$ contracted with $\tilde{p}^\mu$ and external 4-vectors $(P, \Delta, n)$

$$\bar{u}^{(j)}_{(p_f,s_f)}\Gamma u^{(j)}_{(p_i,s_i)} = \mathring{u}^{(j)\dagger}_{s_f} \begin{pmatrix} 0 & t^{\beta_1\cdots}\tilde{p}^f_{\beta_1\cdots} \\ \bar{t}^{\beta_1\cdots}(\tilde{p}^f_{\beta_1\cdots})^* & 0 \end{pmatrix} \Gamma \begin{pmatrix} t^{\alpha_1\cdots}\tilde{p}^i_{\beta_1\cdots} & 0 \\ 0 & \bar{t}^{\bar{\alpha}_1\cdots}(\tilde{p}^i_{\alpha_1\cdots})^* \end{pmatrix} \mathring{u}^{(j)}_{s_i}$$

Canonical: $\tilde{p}^\mu_{\mathrm{C}} = \sqrt{\frac{1}{2m(m+p^0)}}(p^0 + m, \boldsymbol{p}\ )$

- How to implement helicity / LF spinors?
  - Use their **complex** $\tilde{p}^\mu$ $\rightarrow$ complicates reduction
  - **Keep** $\tilde{p}^\mu_C$ but **change** $\mathring{u}^{(j)}$ [Melosh rotation] $\rightarrow$ efficient
- $2j$-rank Tensor bilinear $\quad \widetilde{P} = \frac{1}{2}(\tilde{p}_f + \tilde{p}_i), \quad \widetilde{\Delta} = \tilde{p}_f - \tilde{p}_i$

$$\bar{u}_f\, \gamma^{\mu_1\cdots\mu_{2j}}\, u_f = m^{2j}\prod_{l=1}^{2j}\left[2\left(\widetilde{P}^{\mu_l}\widetilde{P}^{\tau_l} - \frac{1}{4}\widetilde{\Delta}^{\mu_l}\widetilde{\Delta}^{\tau_l}\right) - \left(\widetilde{P}^2 - \frac{1}{4}\tilde{\Delta}^2\right)g^{\mu_l\tau_l} + i\varepsilon^{\mu_l\tau_l\widetilde{P}\widetilde{\Delta}}\right]\langle\lambda_f|t_{\tau_1\cdots\tau_{2j}}|\lambda_i\rangle$$

$$+m^{2j}\prod_{l=1}^{2j}\left[2\left(\widetilde{P}^{\mu_l}\widetilde{P}^{\tau_l} - \frac{1}{4}\widetilde{\Delta}^{\mu_l}\widetilde{\Delta}^{\tau_l}\right) - \left(\widetilde{P}^2 - \frac{1}{4}\tilde{\Delta}^2\right)g^{\mu_l\tau_l} + i\varepsilon^{\mu_l\tau_l\widetilde{P}\widetilde{\Delta}}\right]^*\langle\lambda_f|\bar{t}_{\tau_1\cdots\tau_{2j}}|\lambda_i\rangle$$

# On-Shell Identities

## Generalized Gordon Identities

- Use Dirac Equation $\left(\gamma^{\mu_1\ldots\mu_{2j}} p_{\mu_1}\ldots p_{\mu_{2j}} - m^{2j}\right) u_p^s = 0$

$$\bar{u}_{p'}^{s'}\left(\Gamma\right) u_p^s = \frac{1}{2\bar{m}^{2j}} u_{p'}^{s'} \left( \left\{ \not{P}^{(j)}, \Gamma \right\} + \frac{1}{2}\left[ \not{\Delta}^{(j)}, \Gamma \right] \right) u_p^s$$

$$0 = \bar{u}_{p'}^{s'} \left( \frac{1}{2}\left\{ \not{\Delta}^{(j)}, \Gamma \right\} + \left[ \not{P}^{(j)}, \Gamma \right] \right) u_p^s$$

$$P_{\mu_1\ldots\mu_{2j}} = \frac{1}{2}\left(p'_{\mu_1}\ldots p'_{\mu_{2j}} + p_{\mu_1}\ldots p_{\mu_{2j}}\right)$$

$$\Delta_{\mu_1\ldots\mu_{2j}} = p'_{\mu_1}\ldots p'_{\mu_{2j}} - p_{\mu_1}\ldots p_{\mu_{2j}}$$

$$P^{\mu_1\ldots\mu_{2j}}\Delta_{\mu_1\ldots\mu_{2j}} = 0$$

$$P^{\mu_1\ldots\mu_{2j}}_{(p',p)} = +P^{\mu_1\ldots\mu_{2j}}_{(p,p')}$$

$$\Delta^{\mu_1\ldots\mu_{2j}}\left(p',p\right) = -\Delta^{\mu_1\ldots\mu_{2j}}\left(p,p'\right)$$

- Useful to reduce independent Dirac structures
- Rewrite independent terms to those with sl(2,C) multipoles appearing

# Applications

- Using basis of bilinears and Gordon identities we can identify minimal set of independent bilinears
- These will form the basis in decompositions of matrix elements of QCD operators (currents/correlators)
- Each basis element comes with FF/distribution
- Has multipole interpretation, construction is **identical** for all spin cases
- **Unified framework** to discuss spin in hadronic physics
- Intuition from spin-1/2 carries over
- Extensions possible to transition matrix elements

# Spin 1 Example: EM Current

Using spinor representation: $\langle p', s' \left| j^\mu(0) \right| p, s \rangle = \overset{\circ}{\phi}{}^{(1)}_{s'} \Gamma^\mu_{(p',p)} \overset{\circ}{\phi}{}^{(1)}_{s}$

$P = \frac{1}{2}(p' + p)$

$\Delta = p' - p \quad (\Delta^2 = -Q^2)$

$n_t^\nu = (1, 0, 0, 0)$

$$m^2 \Gamma^\mu_{(p',p)} = 2P^\mu \left[ P^2 1 G_C\left(Q^2\right) - \Delta^\rho \Delta^\sigma \left( t_{\rho\sigma} - \frac{1}{3} g_{\rho\sigma} 1 \right) G_Q\left(Q^2\right) \right] - i\epsilon^{\mu\rho\sigma\lambda} \left[ \Delta_\rho P_\sigma \left( t_{\lambda\nu} - \frac{1}{3} g_{\lambda\nu} 1 \right) n_t^\nu G_M\left(Q^2\right) \right]$$

Using polarization vectors: $\langle p', s' \left| j^\mu(0) \right| p, s \rangle = \varepsilon^{*\,\alpha}_{s'}(p') \, \Gamma^\mu_{\alpha\beta}(P, \Delta) \varepsilon^\beta_s(p)$

[Wang & Lorcé (2022)]

$$\Gamma^{\mu\alpha\beta} = 2P^\mu \left( \Pi^{\alpha\beta} G_C\left(Q^2\right) - \frac{\Delta^\rho \Delta^\sigma \left(\Sigma_{\rho\sigma}\right)^{\alpha\beta}}{2m^2} \frac{P^2}{m^2} G_Q\left(Q^2\right) \right) - i\epsilon^{\mu\rho\sigma\lambda} \left( \frac{\Delta_\rho P_\sigma \left(\Sigma_\lambda\right)^{\alpha\beta}}{\sqrt{P^2}} G_M\left(Q^2\right) \right)$$

# Summary

- Construction allows for efficient and manifestly covariant calculations
- Central role of the covariant $t$–tensors
  $\longrightarrow$ spinors, boosts, propagators, gamma matrices
- Very simple "basis ingredient"
  $\longrightarrow$ reps of generators of rotations
- Covariant $sl(2,\mathbb{C})$–multipole basis for operators
  $\longrightarrow$ transparent interpretation
- Unique framework for **any spin**
  $\longrightarrow$ intuition from spin 1/2
- Avoids calculations with (Dirac) matrices
  $\longrightarrow$ everything reduces to number multiplication ($\mathcal{C}^{\mu\rho\sigma\alpha}$, $\mathcal{Q}^{\mu\rho\alpha}$)