Theory of Resurgence

bridging perturbative and non-perturbative physics

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BNL Colloquium, 10/18/2023

A comprehensive introduction:

GB, Aniceto, Schiappa; A Primer on Resurgent Transseries and Their Asymptotics

Physics Reports 809 (2019), arXiv:1802.10441

Perturbation Theory

Perturbation theory is a ubiquitous (often the only) analytical tool we have in quantum field theory

$$\mathcal{O}(g) \sim c_0 + c_1 g + c_2 g^2 + \ldots$$

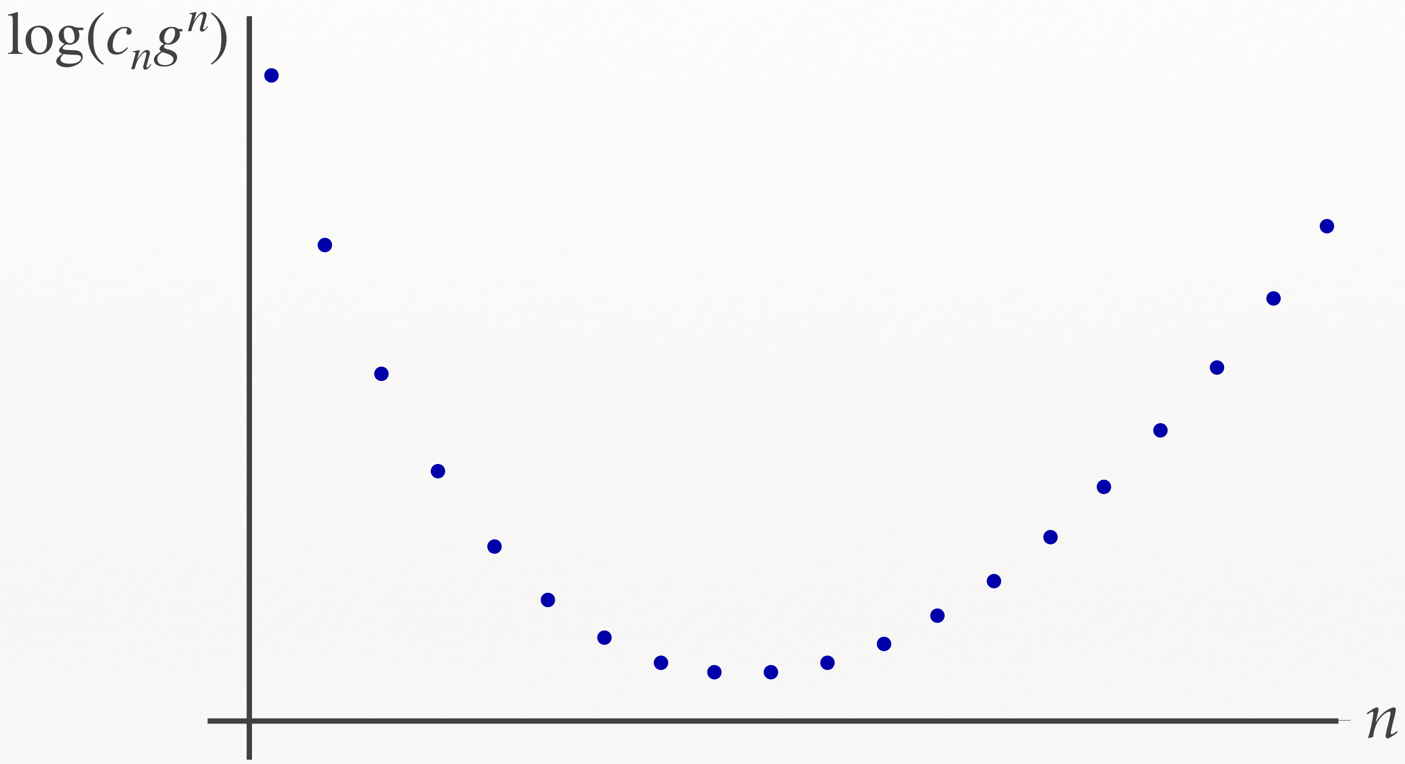
Most of the time the calculation of coefficients are challenging, they have to be regularized, renormalized etc…
But after all the dust settles we still have to deal with another problem:

typically $c_n \sim n!$

Perturbative series has zero radius of convergence [Dyson, '52]
We have to give meaning to the sum above

Perturbation Theory

$$\mathcal{O}(g) \sim \sum_n c_n g^n, \quad c_n \sim n!$$



Partial sums eventually diverge!

Perturbation Theory

"Divergent series are the invention of the devil, and it is shameful to base on them any demonstration whatsoever…"

Niels Abel, 1828

Perturbation Theory

"Divergent series are the invention of the devil, and it is shameful to base on them any demonstration whatsoever… Yet for the most part, the results are valid, it is true, but it is a curious thing. I am looking for the reason, a most interesting problem"

Niels Abel, 1828

Perturbation Theory

(1802-1829)

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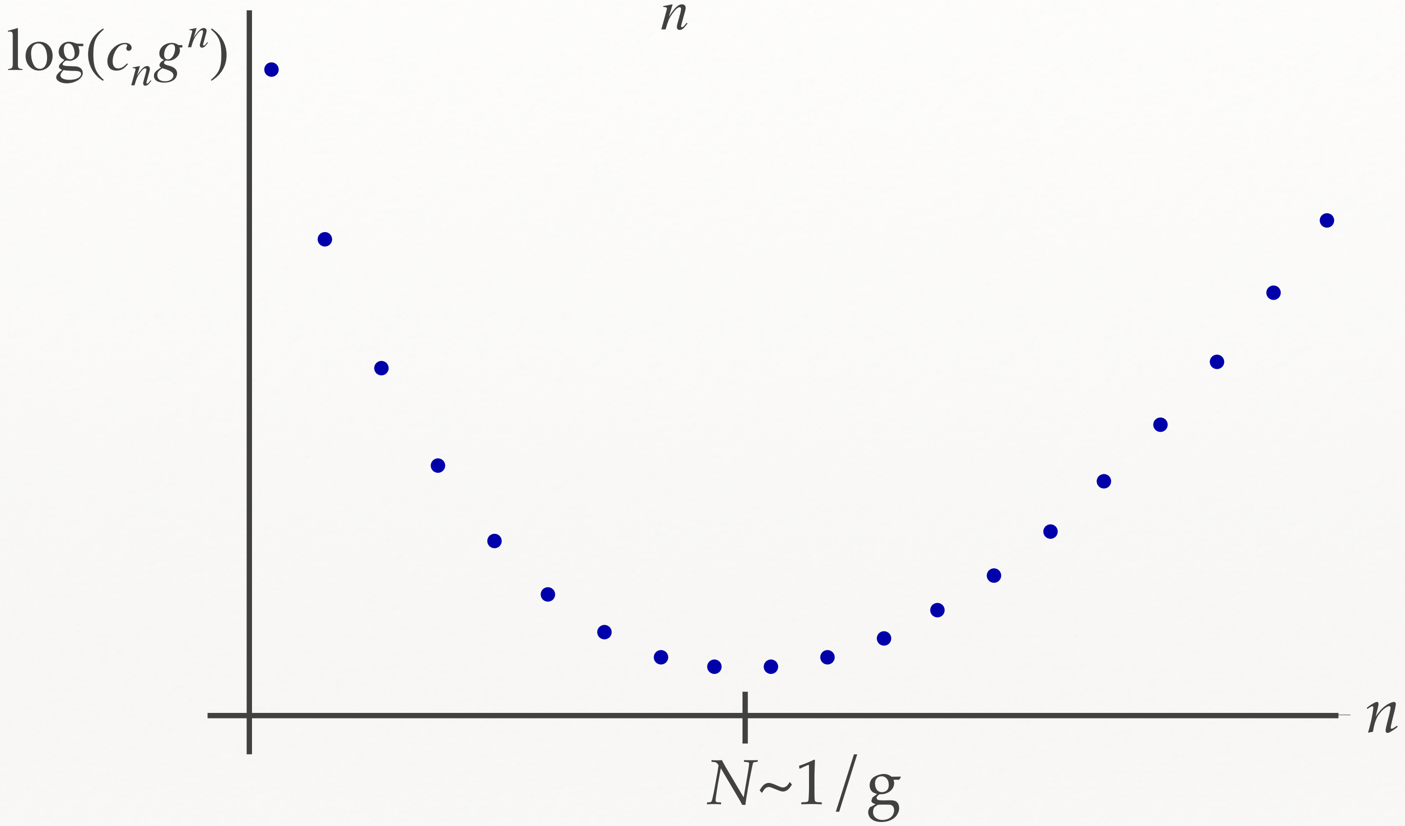
Niels Abel, 1828

Divergent - Asymptotic Series

$$\mathcal{O}(g) \sim \sum_n c_n g^n, \quad c_n \sim n!$$



"optimal truncation": $\mathcal{O}(g) \approx \sum_{n=0}^{N-1} c_n g^n + R_N(g)$ error: $R_N(g) \sim \frac{N!}{g^N} \sim e^{-1/g}$

Divergent -Asymptotic Series

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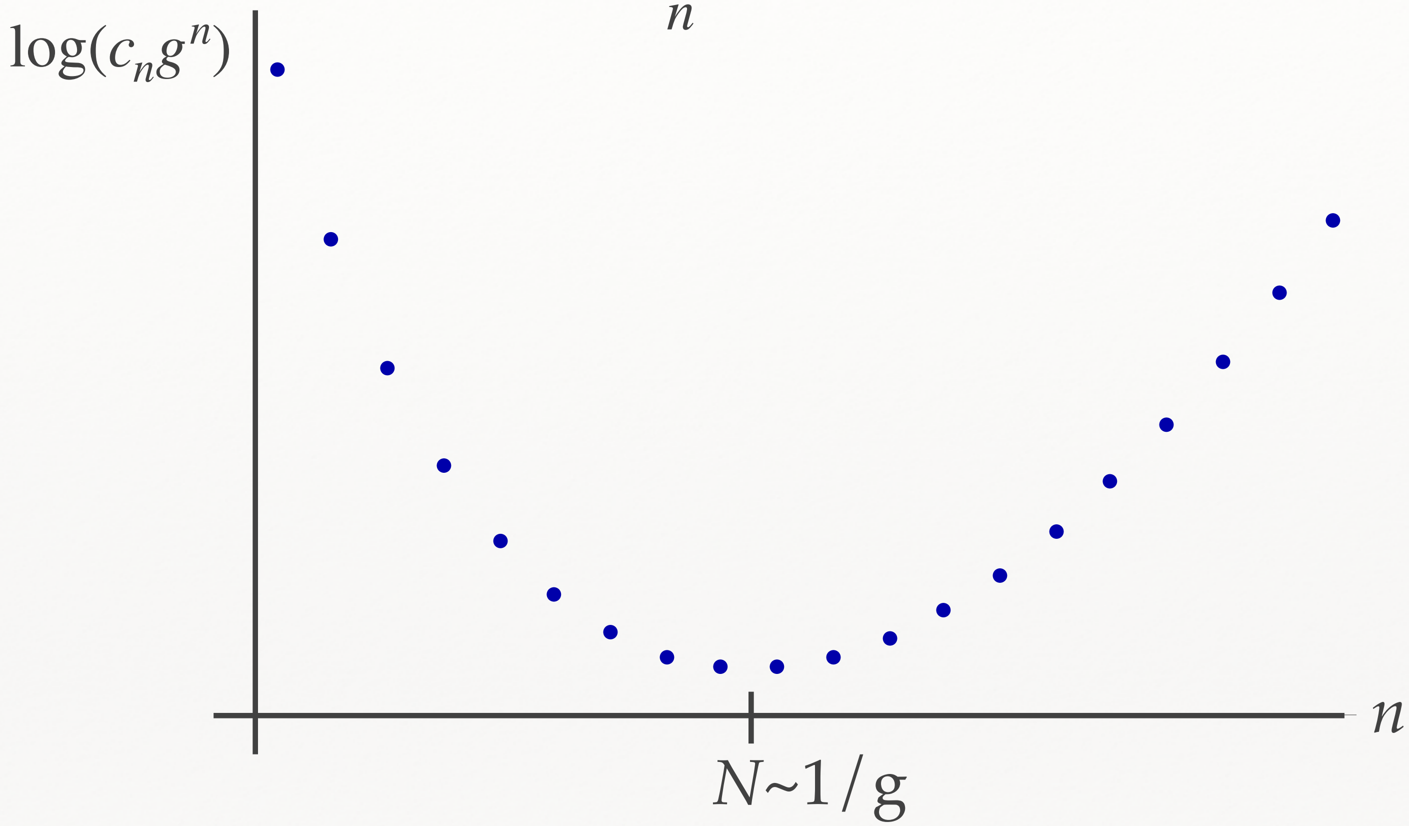

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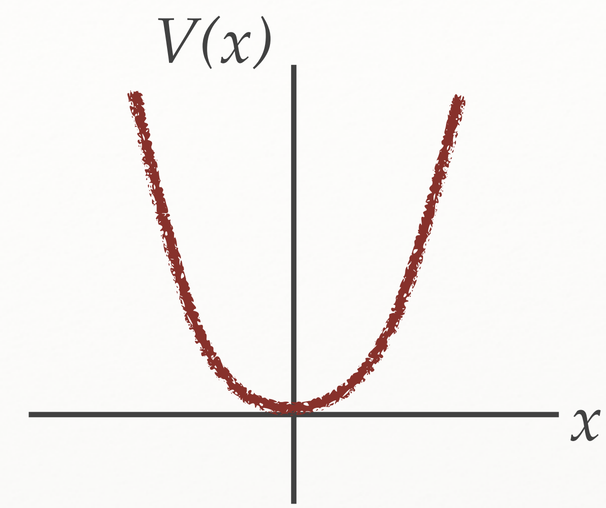
Non-perturbative physics? 🤔

Quartic Oscillator

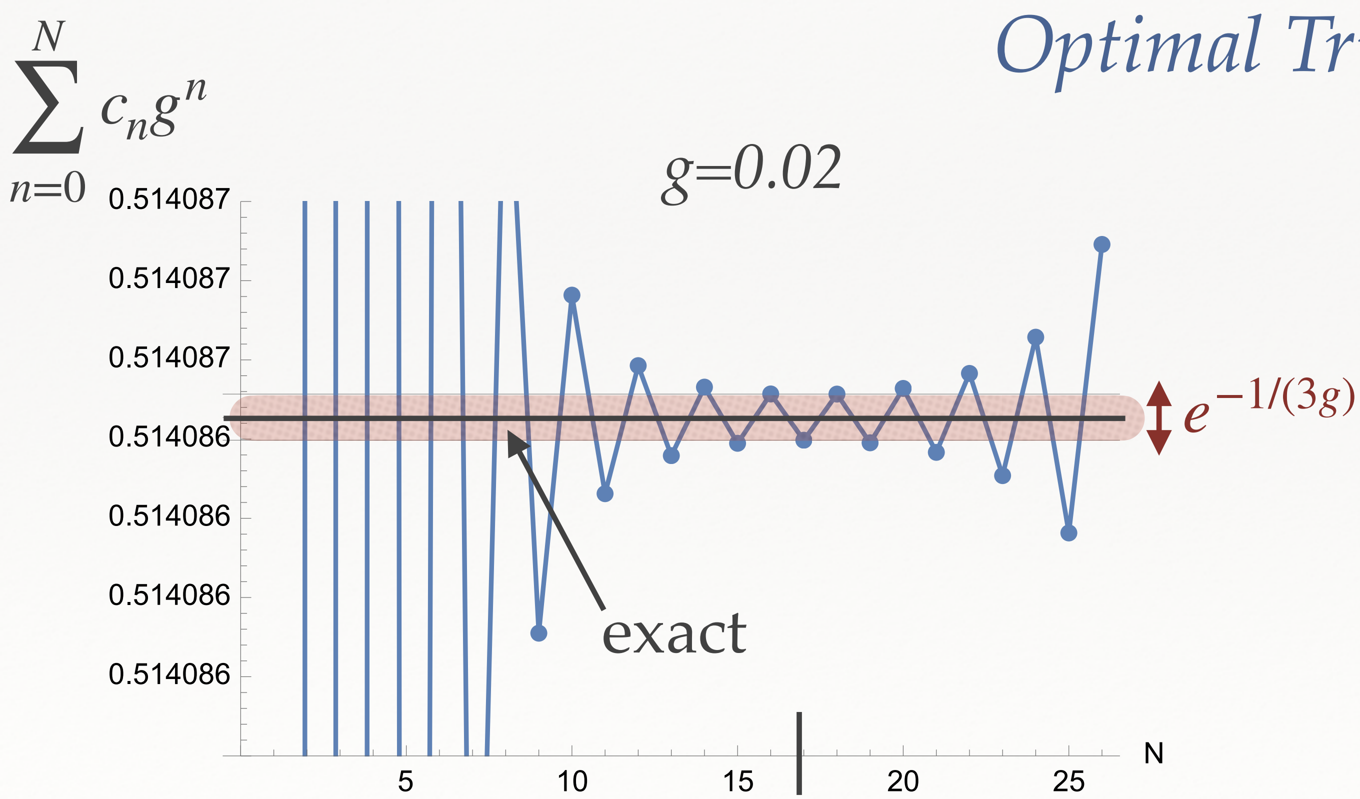
$$H = \frac{1}{2}p^2 + \frac{1}{2}x^2 + gx^4$$

E(g)= harmonic oscillator + corrections(g)

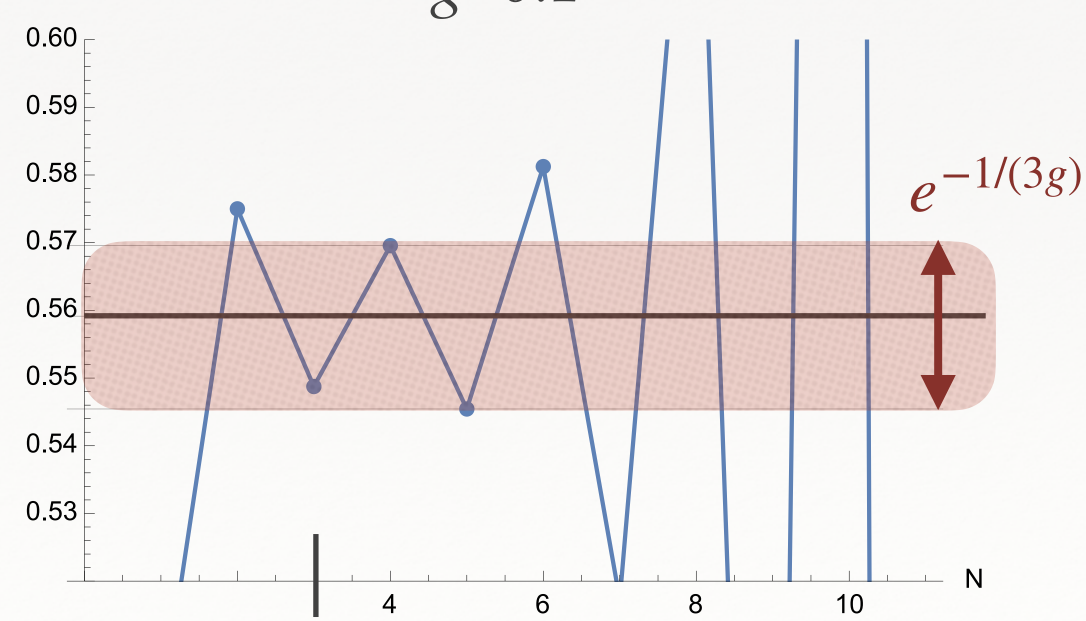
$$E_{gr}(g) = \frac{1}{2} + \frac{3}{4}g - \frac{21}{8}g^2 + \frac{333}{16}g^3 - \frac{30885}{128}g^4 + \ldots \qquad c_n \sim 3^n n!$$



Optimal Truncation



1/(3g)~17



1/(3g)~3

Divergent -Asymptotic Series

The way that perturbation series diverges contains non-perturbative information

One can systematically go beyond optimal truncation via the theory of resurgence

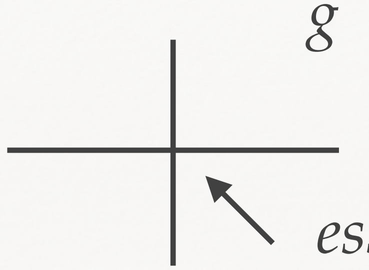
It relates perturbative and non-perturbative physics

Let's first introduce the basic tools

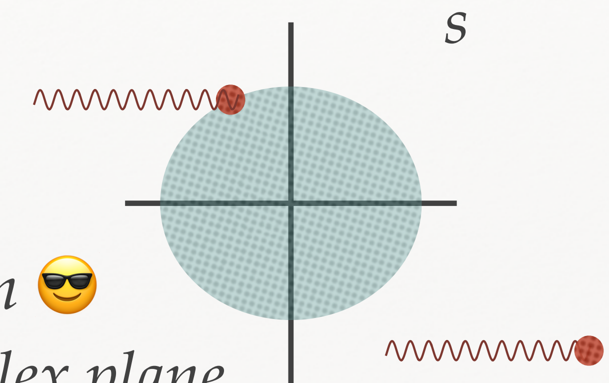
Borel Resummation

We can convert the problem of summing a divergent series into a problem of complex analysis of analytic functions

$\mathcal{O}(g) \sim \sum_n c_n g^n$: *divergent* → *Borel Transform* → $\hat{\mathcal{O}}(s) = \sum_n \frac{c_n}{n!} s^n$: *convergent*



essential singularity at origin 😱

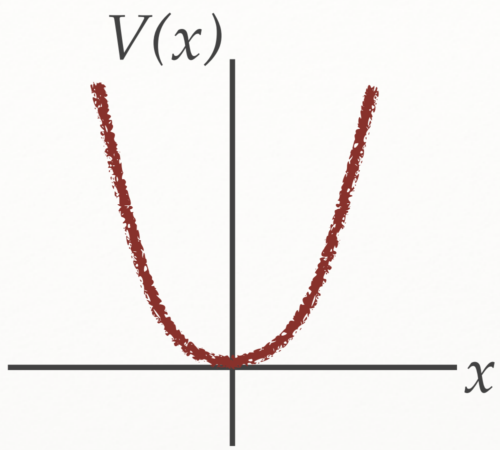


analytic at origin 😎
singularities in complex plane

Borel resummation: $\mathcal{B}\mathcal{O}(g) = \frac{1}{g}\int_0^\infty ds e^{-s/g} \hat{\mathcal{O}}(s)$

Quartic Oscillator

$$E_{gr}(g) = \frac{1}{2} + \frac{3}{4}g - \frac{21}{8}g^2 + \frac{333}{16}g^3 - \frac{30885}{128}g^4 + \ldots$$



$$\mathcal{B}E_{gr}(g) = \frac{1}{g}\int_0^\infty ds e^{-s/g}\hat{E}_{gr}(s)$$

can be resummed by conventional methods (Pade etc..)

no singularities along s>0

Optimal Truncation (g=0.1)

3 terms: $\dfrac{E_{exact} - E_{optimal}}{E_{exact}} = 0.018$

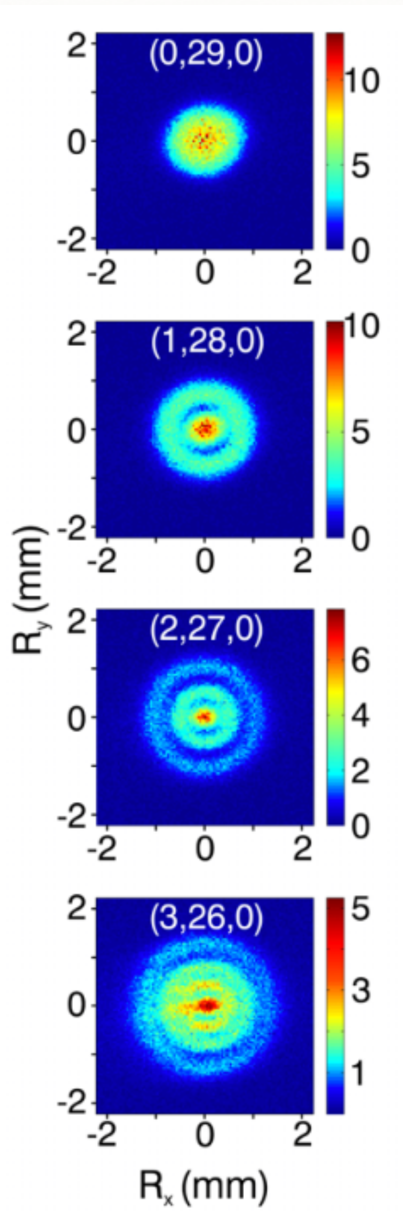
Adding more terms makes it worse…

Borel Resummation (g=0.1)

3 terms: $\dfrac{E_{exact} - E_{Borel-Pade}}{E_{exact}} \approx 6 \times 10^{-4}$

10 terms: $\dfrac{E_{exact} - E_{Borel-Pade}}{E_{exact}} \approx 2 \times 10^{-8}$

Stark Effect



[A. S. Stodolna et al, PRL 110, 213001 (2013)]

Ground state energy of hydrogen atom in constant electric field E

$$\frac{E_{gr}}{E_h} \approx -\frac{1}{2} - 2.25\left(\frac{E}{\mathscr{E}_c}\right)^2 - 55.54\left(\frac{E}{\mathscr{E}_c}\right)^4 + \ldots$$

$$E_h = \frac{e^4 m_e}{\hbar^2} = 27.2\text{eV} \qquad \mathscr{E}_c = \frac{e^5 m_e^2}{\hbar^4} = 51V/\text{Å}$$

N	$E^{(N)}$
0	−0.5
2	−2.25
4	−55.546 875
6	−4 907.771 484 375
8	−794 236.926 452 636 718
10	−194 531 960.466 499 329
12	−66 263 036 523.689 170 9
14	−29 924 943 988 411.939 5
16	−17 346 970 495 631 198.5

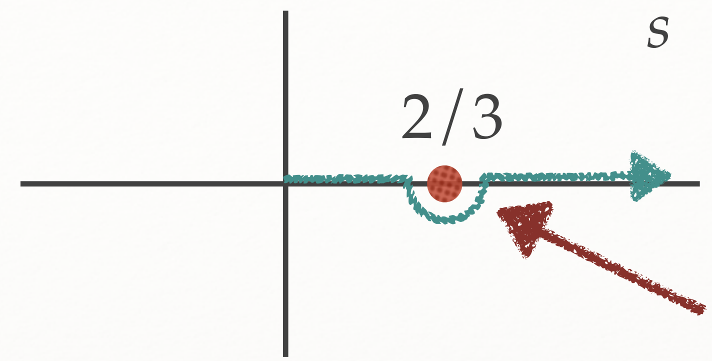
[Silverstone, '78]

$$c_n \sim \frac{(2n)!}{(2/3)^n}$$

Factorial growth
Non-alternating

Stark Effect

$$\hat{E}_{gr}(s) = \sum_{n=0}^{\infty} \frac{E_n}{(2n)!} s^{2n}$$



$$\mathscr{B}E_{gr}(E) = \frac{1}{E} \int_0^{\infty} ds e^{-s\mathscr{E}_c/E} \hat{E}_{gr}(s)$$

singularity
along the integration contour! 😱 *?*

no need to panic… $\mathrm{Im}E_{gr} \sim e^{-\frac{2}{3}\frac{\mathscr{E}_c}{E}}$ *Tunneling ionization rate!*

Non-perturbative physics

Divergence of perturbation series

$$c_n \sim \frac{(2n)!}{(2/3)^{2n}}$$

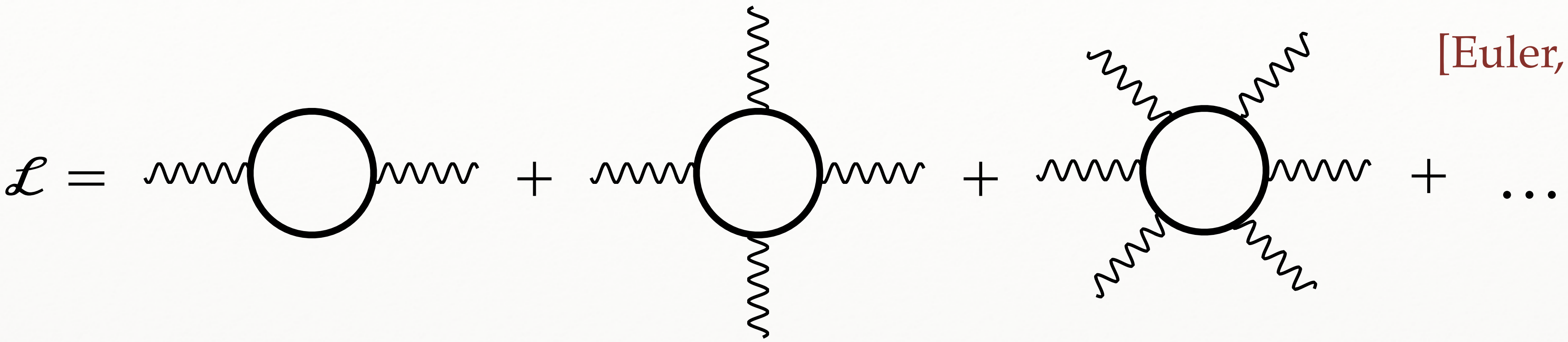
Singularities in Borel plane

$$s_{sing.} = \frac{2}{3}$$

QFT example: Euler-Heisenberg

Effective action for QED in a constant electromagnetic background

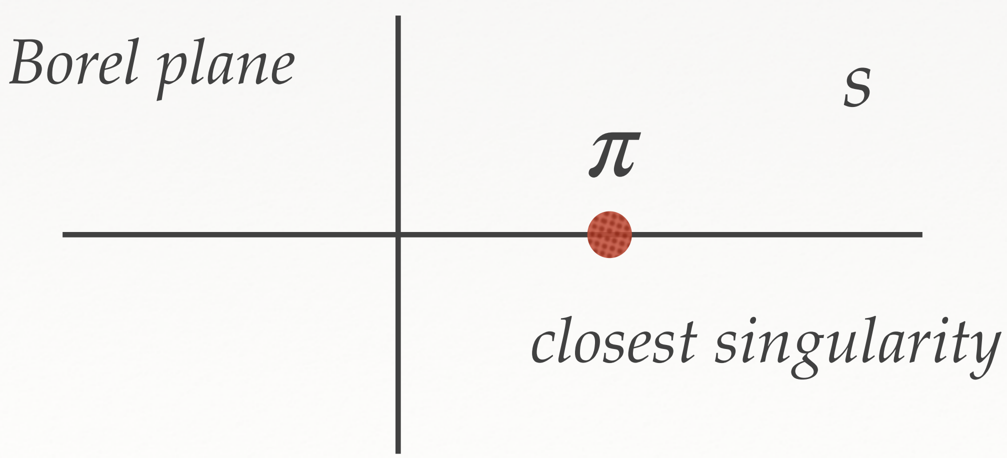
[Euler, Heisenberg '36]

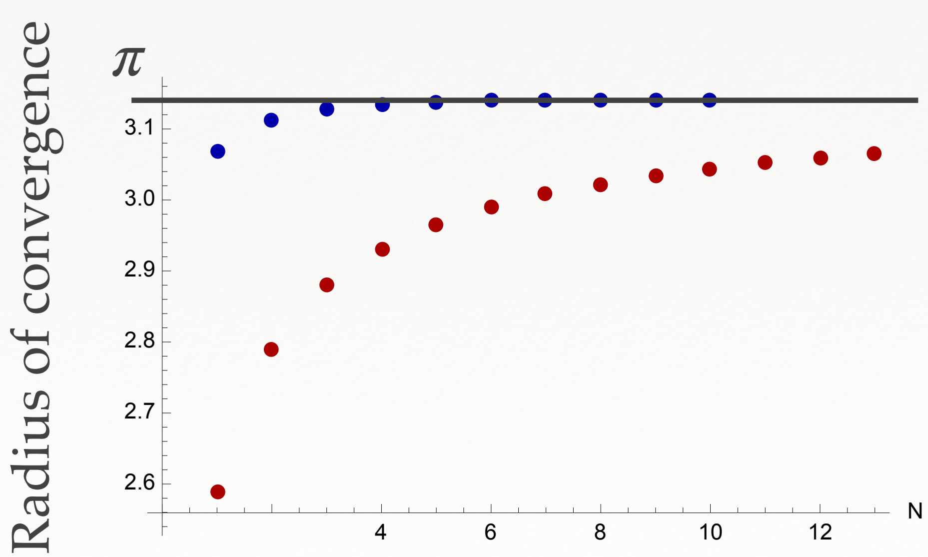


for B=0

$$\mathscr{L} = \frac{1}{2}\vec{E}^2 + \frac{e^4\hbar^2}{360\pi^2 c^6 m_e^4}\vec{E}^4 + \frac{e^6\hbar^4}{630\pi^2 c^{12} m_e^8}\vec{E}^6 + \ldots = \sum_{n=0}^{\infty} c_n \left(\frac{E}{\mathscr{E}_c}\right)^{2n}$$

$$\mathscr{E}_c = \frac{m_e^2 c^3}{\hbar e}$$

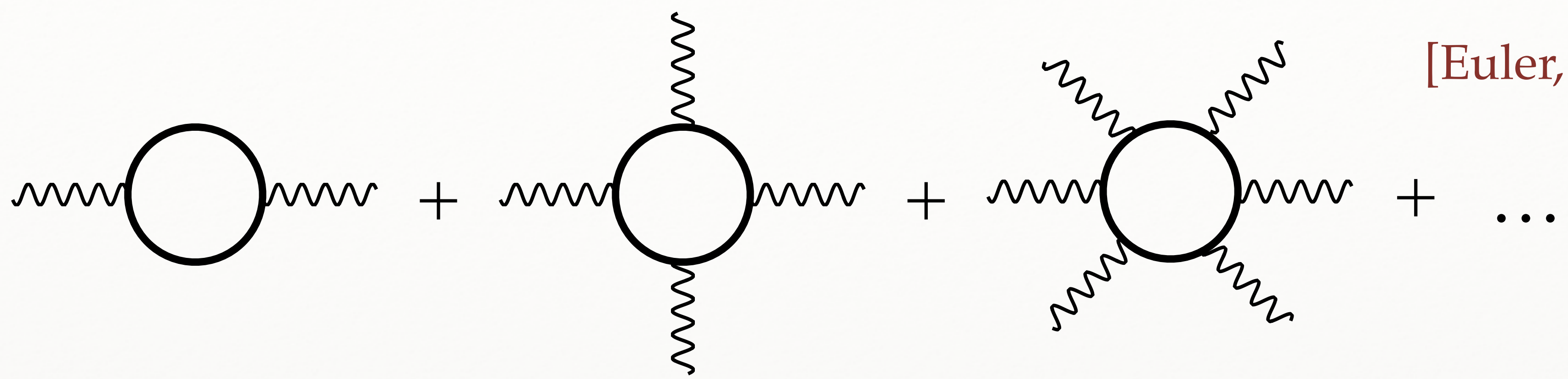




Euler-Heisenberg

Effective action for QED in a constant electromagnetic background

[Euler, Heisenberg '36]

$\mathcal{L} =$ 

for B=0 $\quad \mathscr{L} = \frac{1}{2}\vec{E}^2 + \frac{e^4\hbar^2}{360\pi^2 c^6 m_e^4}\vec{E}^4 + \frac{e^6\hbar^4}{630\pi^2 c^{12} m_e^8}\vec{E}^6 + \ldots = \sum_{n=0}^{\infty} c_n \left(\frac{E}{\mathscr{E}_c}\right)^{2n}$

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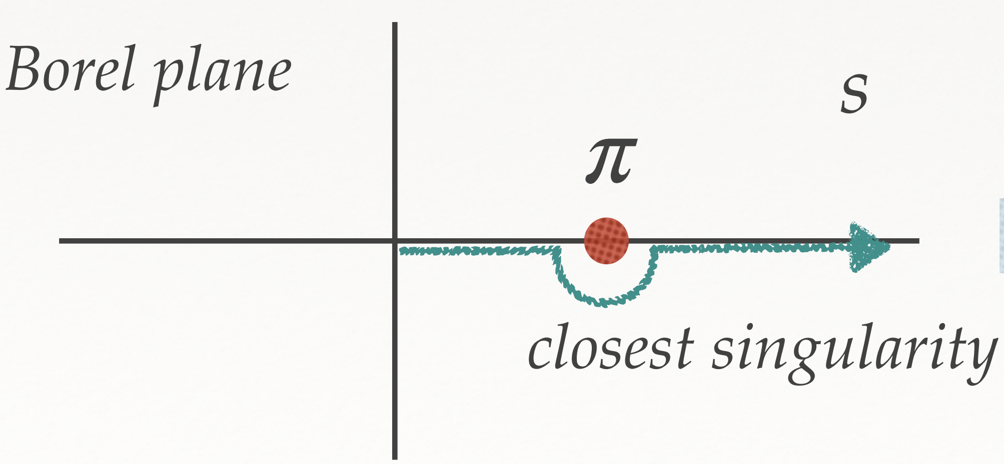


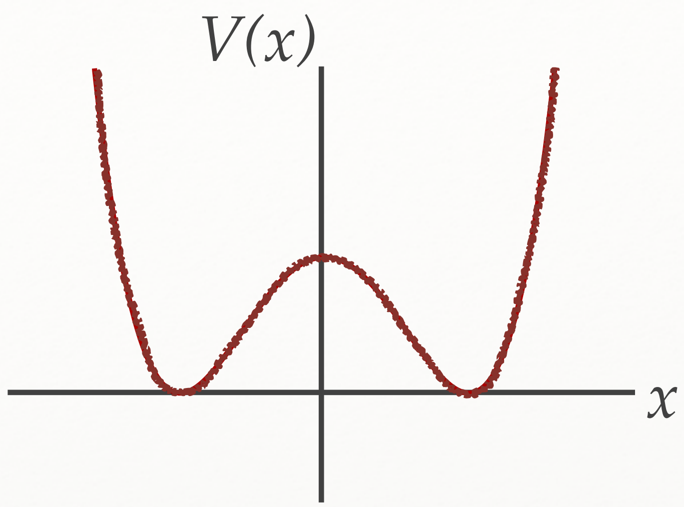
Schwinger pair production rate

$$\mathrm{Im}\mathscr{L} \sim e^{-\pi\frac{\mathscr{E}_c}{E}}$$

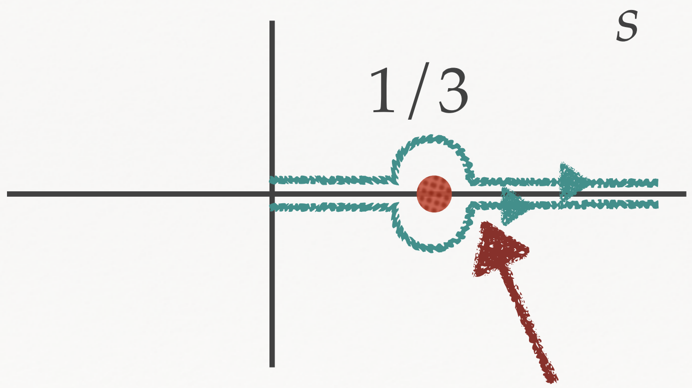
$$\mathscr{E}_c = \frac{m_e^2 c^3}{\hbar e} = 1.32 \times 10^{18} V/m$$

Double well

$$H = \frac{1}{2}p^2 + \frac{1}{2}x^2(1-\sqrt{g}x)^2$$



$$E_{gr}(g) = \frac{1}{2} - g - \frac{9}{2}g^2 - \frac{89}{2}g^3 - \frac{5013}{8}g^4 + \dots$$



singularity
along the integration contour! 😱

$$\mathrm{Im}E_{gr} \sim \pm\, e^{-\frac{1}{3g}} \quad ?$$

No instability: E_{gr} must be real
Choice of the contour is ambiguous

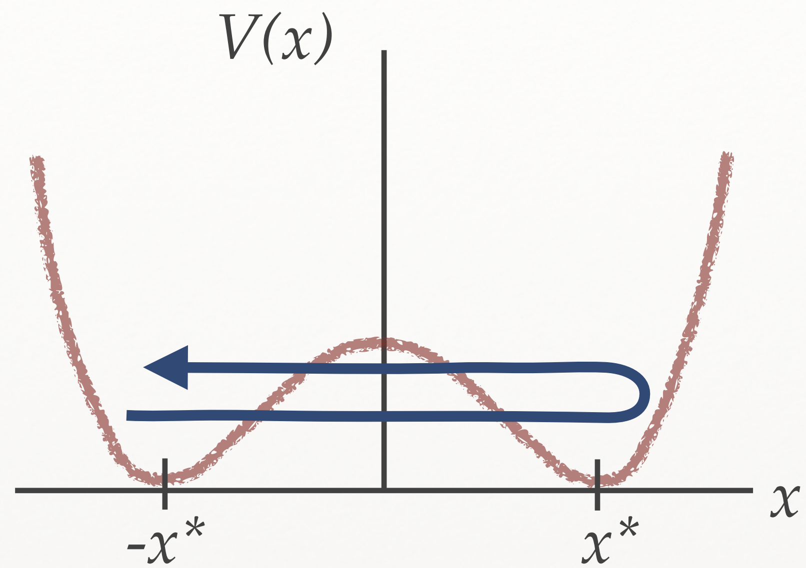
Double Well: Instantons to the Rescue

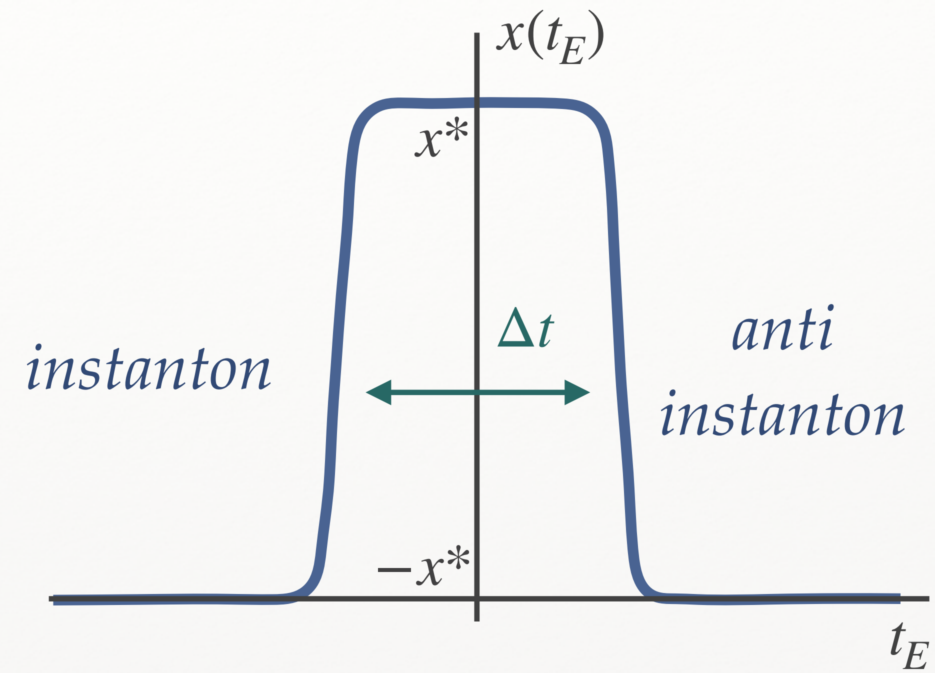
Non-perturbative contributions to the path integral

Instanton action: $S_I = \frac{1}{6}$

Double tunneling $\sim e^{-\frac{2S_I}{g}}$

``instanton-anti-instanton"





Path integral sums over all separations (quasi-zero mode)

Quasi-zero mode integral is ill defined for $g>0$

Evaluate at $g<0$, analytically continue to $g>0$ [Bogomolnyi, Zinn-Justin '80s]

The result is 2-fold ambiguous: $\mathrm{Im}E_{I\bar{I}} \sim \mp\, e^{-\frac{2S_I}{g}}$

Double Well: Instantons to the Rescue

$$\mathrm{Im}E_{pert.} + \mathrm{Im}E_{I\bar{I}} = 0 \quad \text{up to} \quad \mathcal{O}(e^{-4S_I/g})$$

Borel resummation of perturbation series (2-fold ambigious)

Non-perturbative instanton gas (2-fold ambigious)

Resurgence:

Perturbative + non-perturbative fluctuations can be meaningfully resummed

There are *quantitative* relations between perturbative and non-perturbative sectors

Supernumerary rainbows

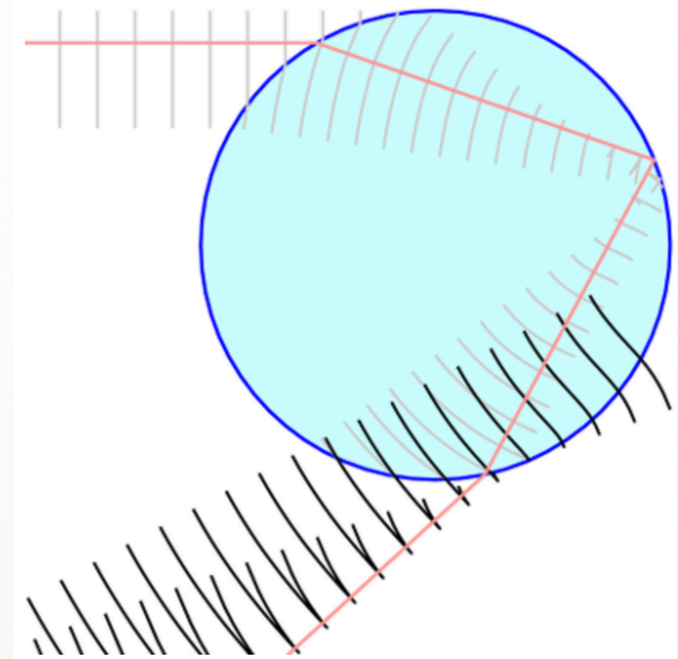
$$\mathrm{Im}E_{pert.} + \mathrm{Im}E_{I\bar{I}} = 0 \quad \text{up to} \quad \mathcal{O}(e^{-4S_I/g})$$

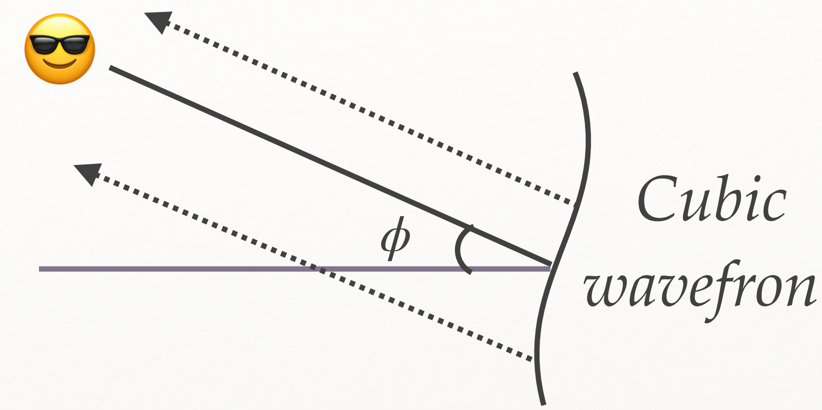
Why does this happen?

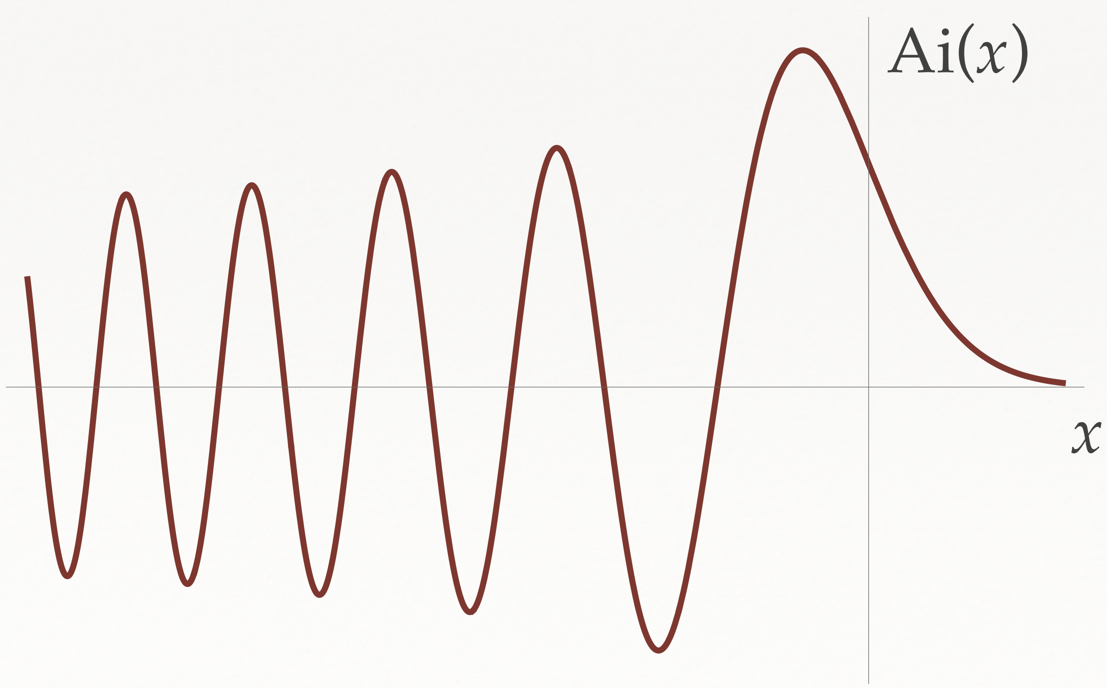
[image: J. Bahrdt wiktionary.org]

Supernumerary rainbows

[image: B. Casselman, ams.org, The Mathematics of the Rainbow, Part II]







$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} dt \cos\left(\frac{t^3}{3} + xt\right)$$

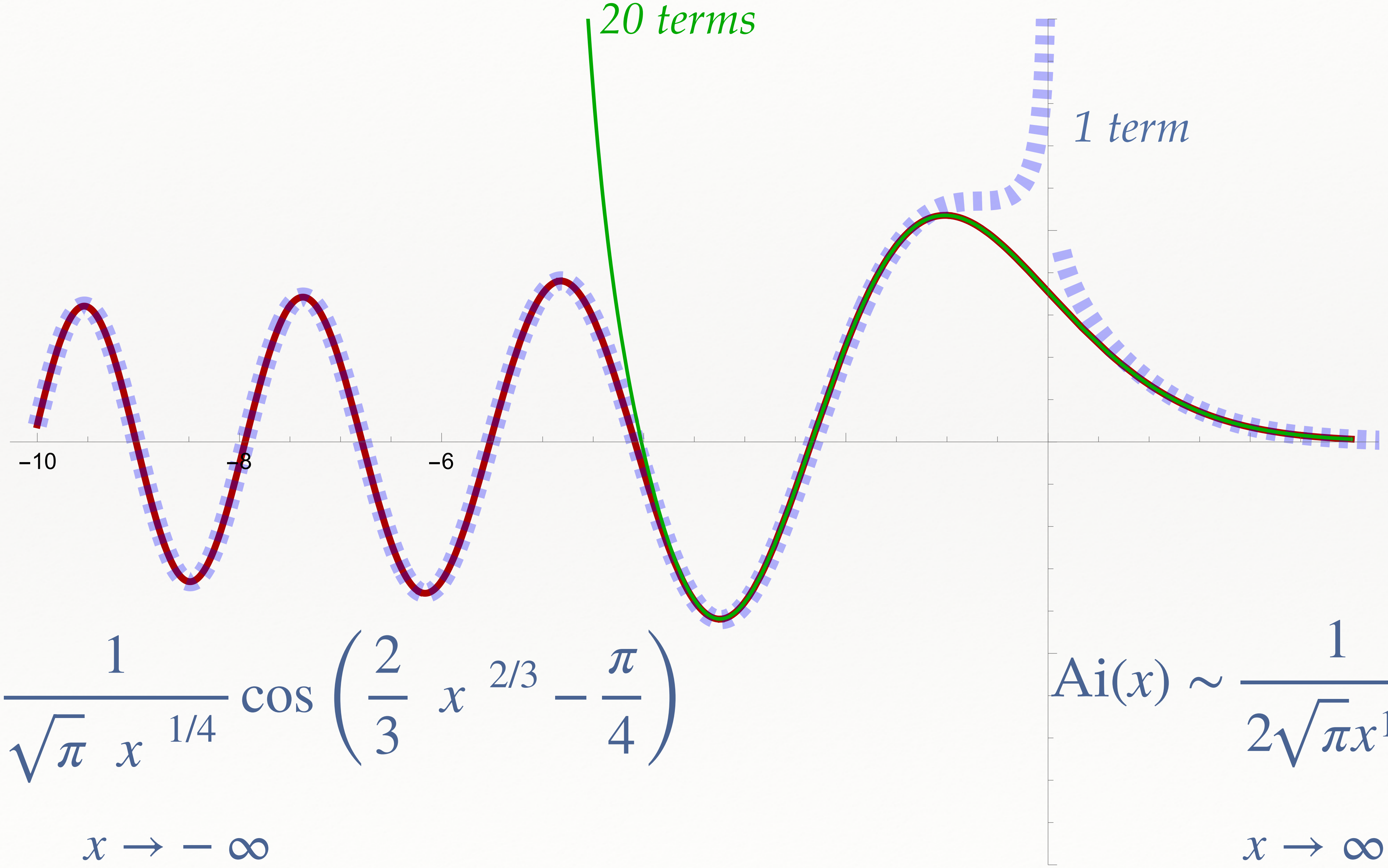
Airy equation

Airy : convergent expansion around $x=0$

[On the intensity of light in the neighbourhood of a caustic, 1838]

Stokes: asymptotic expansion $x \to \infty$

[On the numerical calculation of a class of definite integrals and infinite series, 1850]



$$\mathrm{Ai}(x) \sim \frac{1}{\sqrt{\pi}\ x^{1/4}} \cos\left(\frac{2}{3}\ x^{2/3} - \frac{\pi}{4}\right)$$

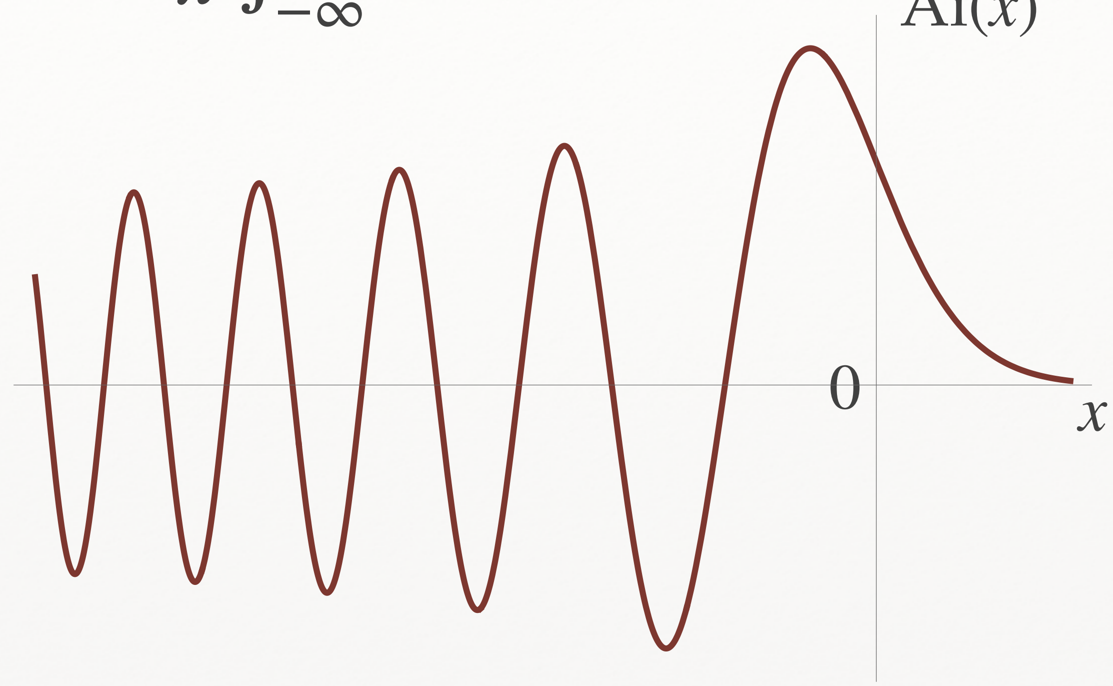
$x \to -\infty$

$$\mathrm{Ai}(x) \sim \frac{1}{2\sqrt{\pi} x^{1/4}} e^{-\frac{2}{3}x^{2/3}}$$

$x \to \infty$

Stokes' puzzle

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} dt e^{i\left(\frac{t^3}{3} + xt\right)}$$



$$\cos\left(\frac{2}{3}\ x^{\ 2/3} - \frac{\pi}{4}\right)$$

$$e^{-\frac{2}{3}x^{2/3}}$$

two exponents

one exponent

Stokes was puzzled by the fact that the same function is represented by two exponentials on one side and one exponential on the other side

Stokes' puzzle

When the cat's away the mice may play. You are the cat and I am the poor little mouse. I have been doing what I guess you won't let me do when we are married, sitting up till 3 o'clock in the morning fighting hard against a mathematical difficulty. Some years ago I attacked an integral of Airy's, and after a severe trial reduced it to a readily calculable form. But there was one difficulty about it which, though I tried till I almost made myself ill, I could not get over, and at last I had to give it up and profess myself unable to master it*. I took it up again a few days ago, and after a two or three days' fight, the last of which I sat up till 3, I at last mastered it. I don't say you won't let me work at such things, but you will keep me to more regular hours. A little out of the way now and then does not signify, but there should not be too much of it. It is not the mere sitting up but the hard thinking combined with it......

PEMBROKE COLLEGE, CAMBRIDGE,
March 28, 1857.

Stokes' letter to his fiancee

VI. *On the Discontinuity of Arbitrary Constants which appear in Divergent Developments. By* G. G. STOKES, M.A., D.C.L., Sec. R.S., *Fellow of Pembroke College, and Lucasian Professor of Mathematics in the University of Cambridge.*

[Read *May* 11, 1857.]

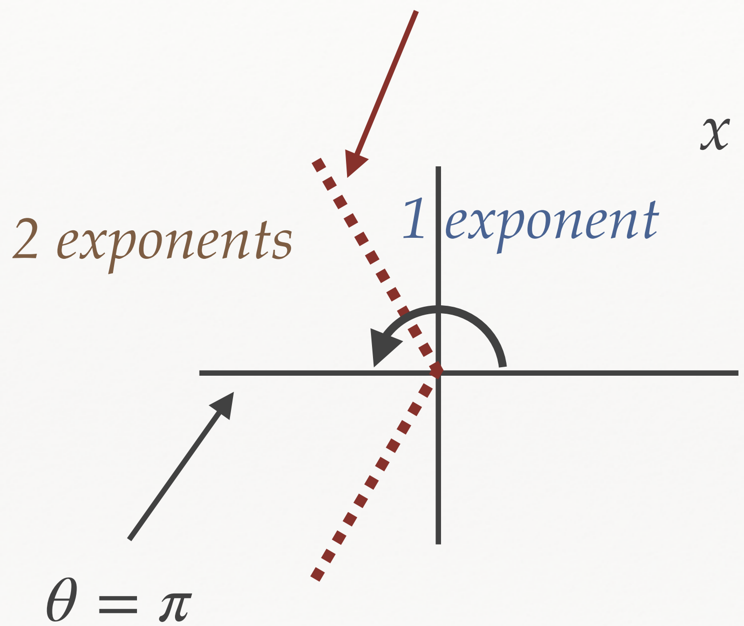
IN a paper "On the Numerical Calculation of a class of Definite Integrals and Infinite Series," printed in the ninth volume of the *Transactions* of this Society, I succeeded in developing the integral $\int_0^\infty \cos\frac{\pi}{2}(w^3 - mw)\,dw$ in a form which admits of extremely easy numerical calculation when m is large, whether positive or negative, or even moderately large.

Stokes phenomenon

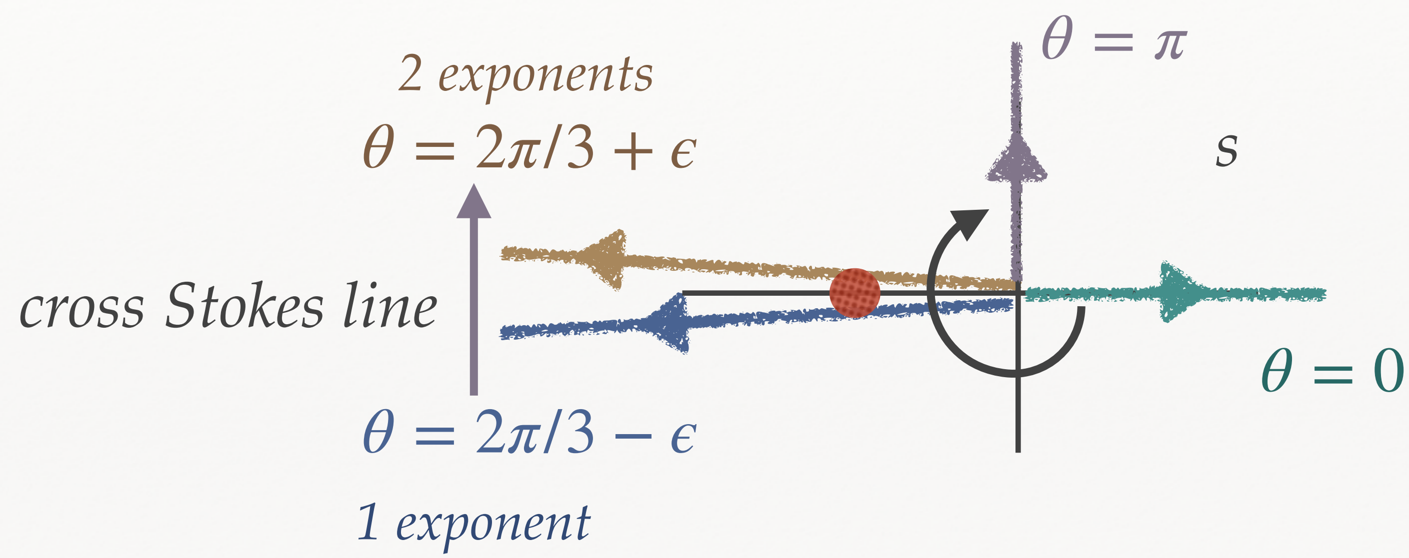
The ``coupling constant'' in Stokes' asymptotic expansion is $x^{-3/2}$

The direction of Borel integration is determined by $\theta = \arg x$

Stokes line: The new exponent is born when it is crossed
(exponentially small compared to the other one)



Both exponents have equal magnitude



As θ passes through the critical value, the inferior term enters as it were into a mist, is hidden for a little from view, and comes out with its coefficient changed. The range during which the inferior term remains in a mist decreases indefinitely as the modulus r increases indefinitely.

[On the discontinuity of arbitrary constants that appear as multipliers of semi-convergent series, 1902]

Resurgence

$$\mathrm{Im}E_{pert.} + \mathrm{Im}E_{I\bar{I}} + \mathrm{Im}E_{I\bar{I}I\bar{I}} + \ldots = 0$$

These ambiguities occur because $g>0$ is a Stokes line!

For an un-ambiguous, well defined expansion we have to incorporate the exponentially small terms from the start

$$\mathcal{O}(g) = \sum_{n} c_n g^n + \sum_{n,k,l} c_n^{(k,l)} (e^{-a/g})^k (\log g)^l$$

perturbative *non-perturbative*

"Trans-series"

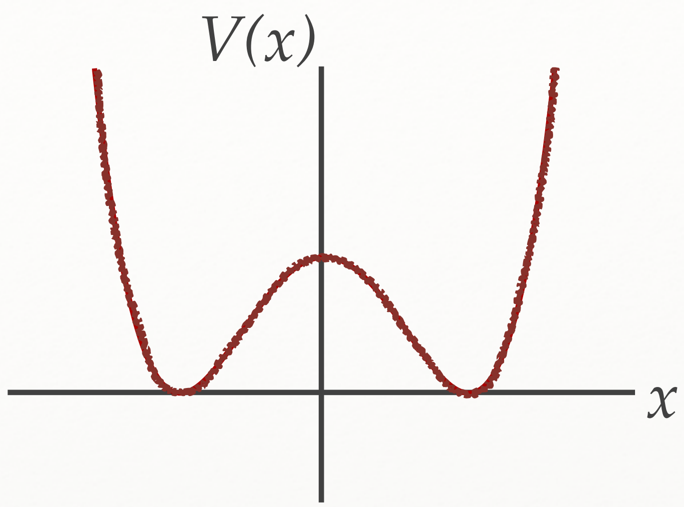
[J. Écalle, '80s]

Resurgence is a framework which consistently keeps track of all the Stokes phenomena to *all orders*

quantifies the relations between c_n and $c_n^{(k,l)}$

Resurgence

$$E_{gr}(g) = \frac{1}{2} - g - \frac{9}{2}g^2 - \frac{89}{2}g^3 - \frac{5013}{8}g^4 + \dots$$



Perturbative expansion

$$c_n \sim 3^n \left(1 - \frac{53}{6} \cdot \frac{1}{3} \cdot \frac{1}{n} - \frac{1277}{72} \cdot \frac{1}{3^2} \cdot \frac{1}{n(n-1)} + \dots\right)$$

Fluctuations around instanton-anti-instanton

$$\mathrm{Im}E \sim e^{-\frac{1}{3g}} \left(1 - \frac{53}{6}g - \frac{1277}{72}g^2 + \dots\right)$$

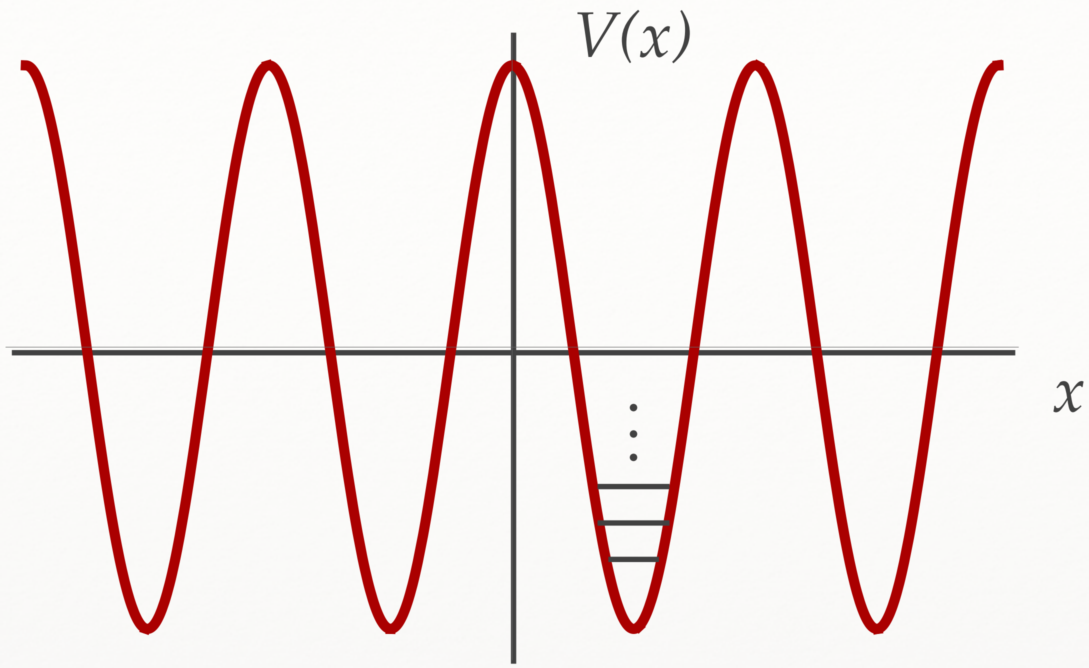
Resurgence

"…resurgent functions display at each of their singular points a behaviour closely related to their behaviour at the origin. Loosely speaking, these functions resurrect, or surge up - in a slightly different guise, as it were - at their singularities"

Écalle, '80s

Periodic potential (Mathieu)

$$H = \frac{1}{2}p^2 + \cos x$$



Perturbative expansion ($\hbar N \ll 1$):

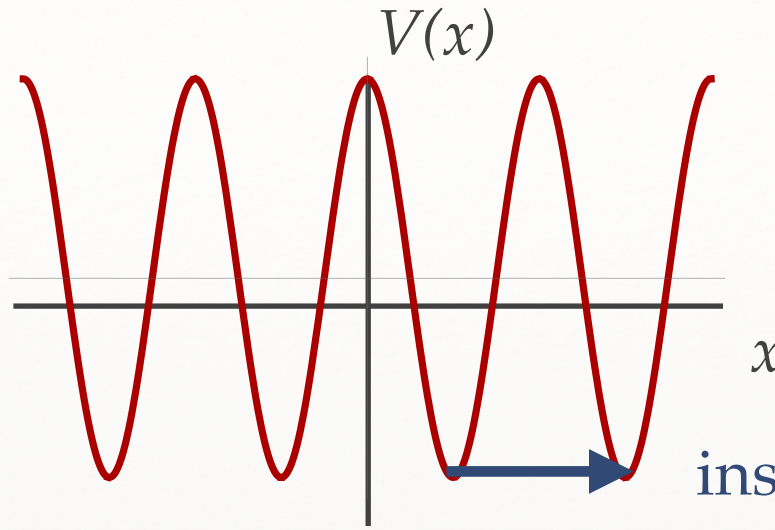
N: level number

harmonic oscillator + corrections

$$E_N(\hbar) \sim -1 + \hbar\left[N + \frac{1}{2}\right] - \frac{\hbar^2}{16}\left[\left(N + \frac{1}{2}\right)^2 + \frac{1}{4}\right] - \frac{\hbar^3}{16^2}\left[\left(N + \frac{1}{2}\right)^3 + \frac{3}{4}\left(N + \frac{1}{2}\right)\right] - \ldots$$

Non-perturbative sector

$E_N(\hbar)$ has a resurgent trans-series expansion for $\hbar N \ll 1$



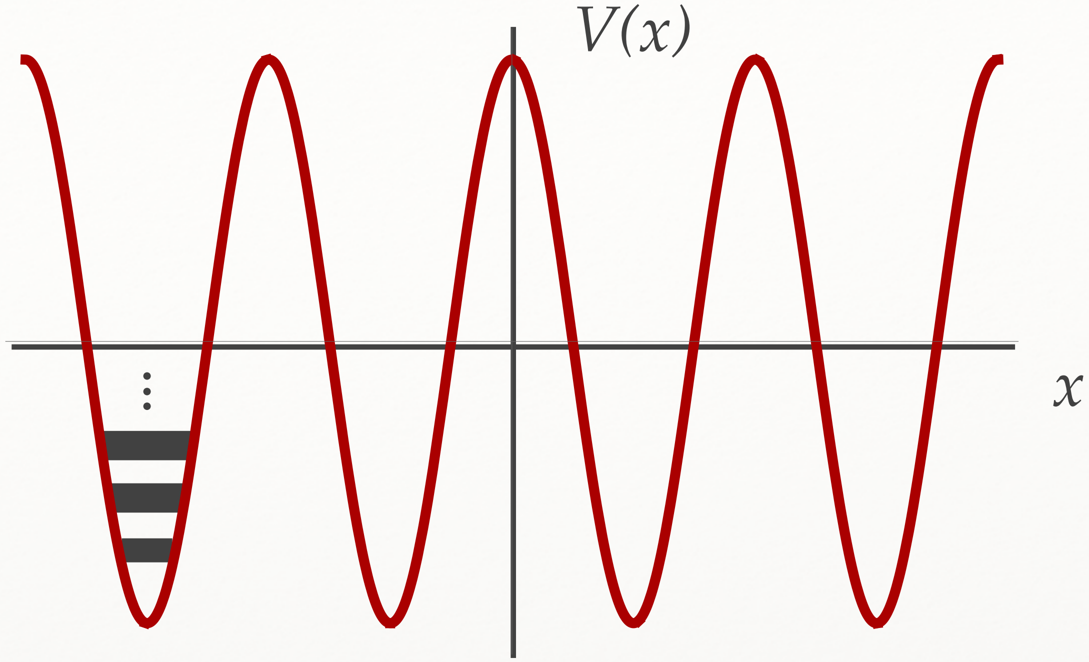
Large order growth (ground state)

$$c_n(0) \sim \frac{n!}{16^n}\left(1 - \frac{5}{2}\cdot\frac{1}{n} - \frac{13}{8}\cdot\frac{1}{n(n-1)} - \ldots\right)$$

Fluctuations around instanton - anti-instanton

$$\operatorname{Im} E_0(\hbar) \sim \pi\,\exp\left[-\frac{8}{\hbar}\right]\left(1 - \frac{5}{2}\cdot\left(\frac{\hbar}{16}\right)^2 - \frac{13}{8}\cdot\left(\frac{\hbar}{16}\right)^4 - \ldots\right)$$

Non-perturbative sector



The physical spectrum has *exponentially small bands* for $N\hbar \ll 1$

Perturbative expansion: center of the band

$$E_N^{pt}(\hbar) \sim -1 + \hbar\left[N + \frac{1}{2}\right] - \frac{\hbar^2}{16}\left[\left(N + \frac{1}{2}\right)^2 + \frac{1}{4}\right] - \frac{\hbar^3}{16^2}\left[\left(N + \frac{1}{2}\right)^3 + \frac{3}{4}\left(N + \frac{1}{2}\right)\right] - \dots$$

Non-perturbative expansion: width of the band

$$\Delta E_N^{\text{band}} \sim \sqrt{\frac{2}{\pi}}\frac{2^{4(N+1)}}{N!}\left(\frac{2}{\hbar}\right)^{N-1/2} \exp\left[-\frac{8}{\hbar}\right]\left\{1 - \frac{\hbar}{32}\left[3\left(N+\frac{1}{2}\right)^2 + 4\left(N + \frac{1}{2}\right) + \frac{3}{4}\right] + O(\hbar^2)\right\}$$

Non-perturbative sector

Perturbative expansion: center of the band

$K = N + 1/2$

$$E_N^{pt}(\hbar) \sim -1 + \hbar K - \frac{\hbar^2}{16}\left(K^2 + \frac{1}{4}\right) - \frac{\hbar^3}{16^2}\left(K^3 + \frac{3}{4}K\right) - \frac{\hbar^4}{16^3}\left(\frac{5K^4}{2} + \frac{17K^2}{4} + \frac{9}{32}\right)$$

$$-\frac{\hbar^5}{16^4}\left(\frac{33K^5}{4} + \frac{205K^3}{8} + \frac{405K}{64}\right) - \ldots$$

Non-perturbative expansion: width of the band

$$\Delta E_N^{\text{band}} \sim \frac{\partial E_N^{pt}}{\partial N} \exp\left[-\frac{8}{\hbar}\left(1 + \frac{\hbar}{16^2}\left(3K^2 + \frac{3}{4}\right) - \frac{\hbar^2}{16^3}\left(5K^3 + \frac{17K}{4}\right) - \frac{\hbar^3}{16^4}\left(\frac{55K^4}{4} + \frac{205K^2}{8} + \frac{135}{64}\right)\right)\right]$$

density of states

the coefficients look suspiciously similar… 🤔

$P\!=\!NP$

Perturbative expansion

$$\hat{E}^{pt}_N(\hbar) \sim K - \frac{\hbar}{16}\left(K^2 + \frac{1}{4}\right) - \frac{\hbar^2}{16^2}\left(K^3 + \frac{3}{4}K\right) - \ldots$$

Non-perturbative expansion

$$\Delta E^{\text{band}}_N \sim \frac{\partial E^{pt}_N}{\partial N} \exp\left[-\frac{1}{2}A_N(\hbar)\right]$$

[Zinn-Justin, Jentschura, '04]

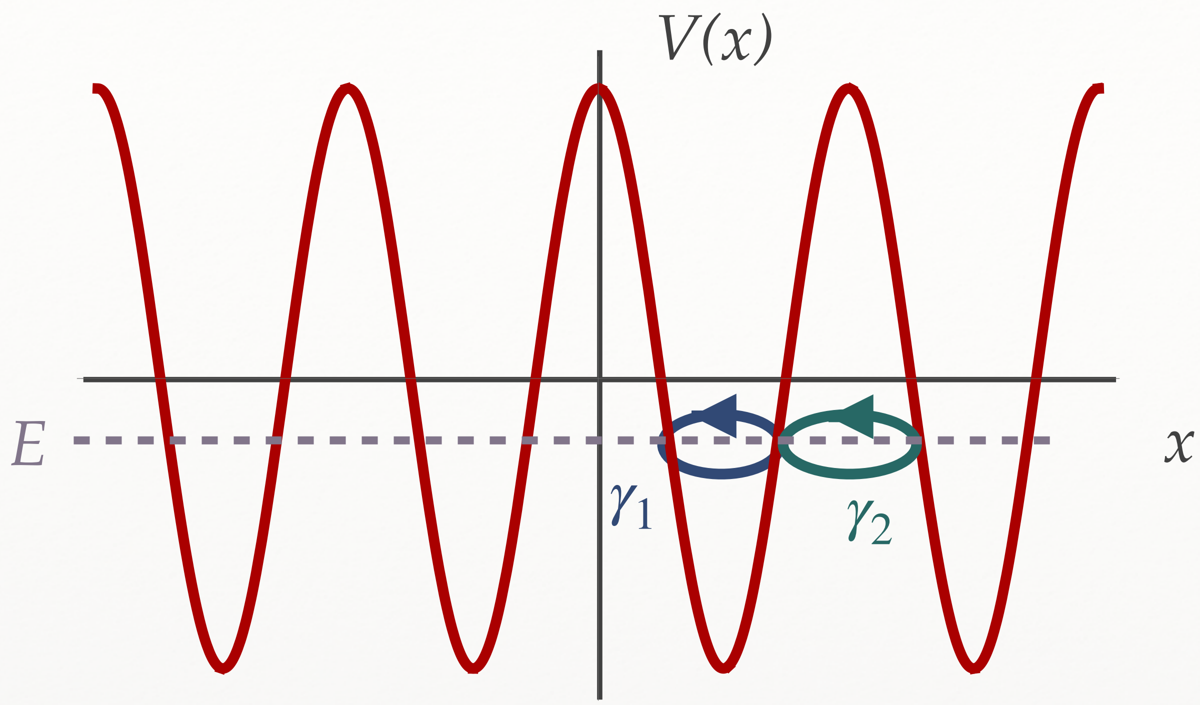
$$\frac{\partial \hat{E}^{pt}_N}{\partial N} = -\frac{\hbar}{16}\left(2K + \hbar\frac{\partial A_N}{\partial \hbar}\right)$$

[Hoe, D'etat et al. '81, Alvarez, Casares '00, Dunne, Ünsal, '14,…]

Low order terms in perturbative series ⟷ Low order terms in fluctuations around instantons

All non-perturbative data is encoded in perturbative expansion!

Geometric origin of P=NP



WKB actions:

$$P(E,\hbar) \sim \sqrt{2(E-V)} - \frac{\hbar^2}{2^6}\frac{\sqrt{2}(V')^2}{(E-V)^{5/2}} - \dots$$

perturbative

$$S_1(E;\hbar) = \oint_{\gamma_1} P(E,\hbar)$$

non-perturbative (tunneling)

$$S_2(E;\hbar) = \oint_{\gamma_2} P(E,\hbar)$$

$$\left(S_1(E;\hbar) - \hbar\frac{S_1(E;\hbar)}{\partial\hbar}\right)\frac{\partial S_2(E;\hbar)}{\partial E} - \left(S_2(E;\hbar) - \hbar\frac{S_2(E;\hbar)}{\partial\hbar}\right)\frac{\partial S_1(E;\hbar)}{\partial E} = iS_I$$

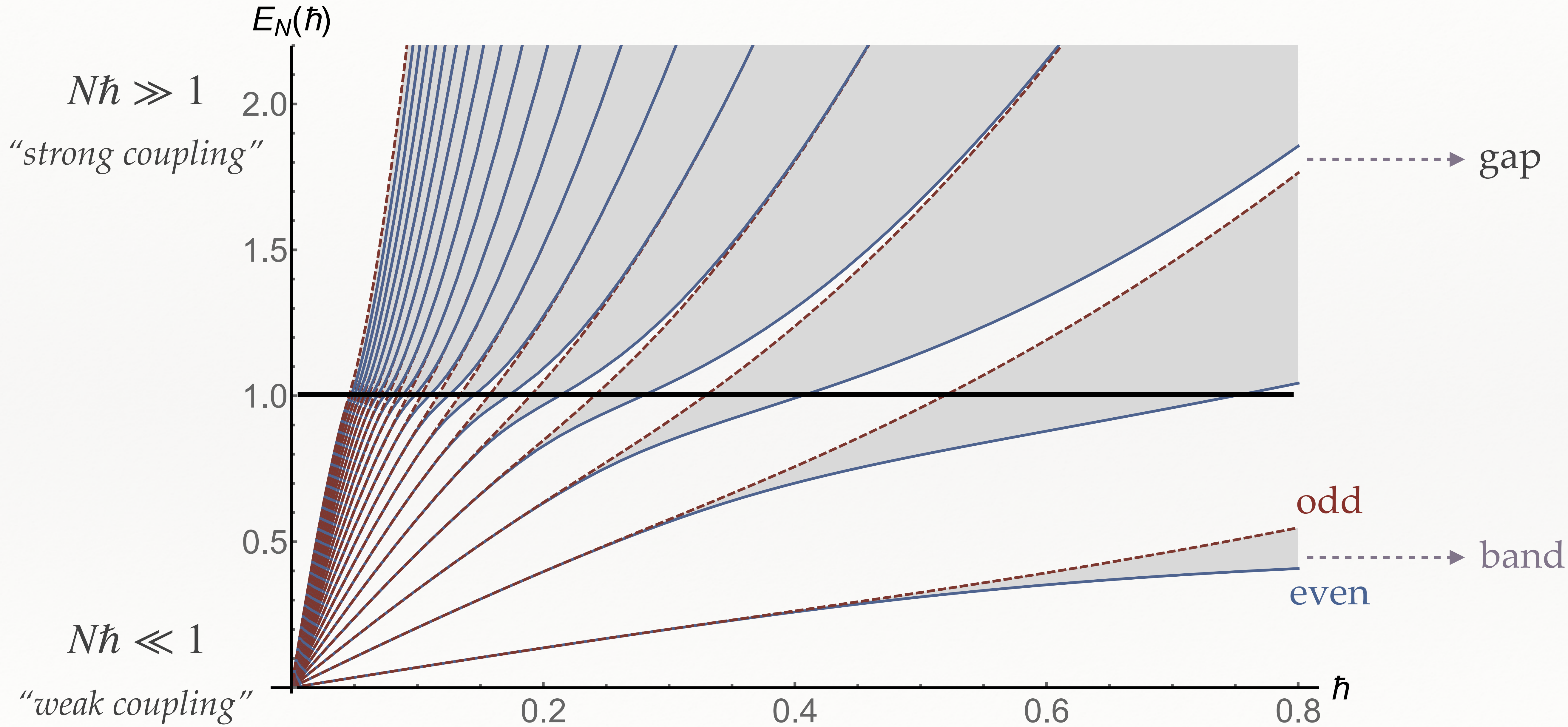
[GB, Dunne '15]

related to SUSY gauge theories, topological string theory,….

[Codesido,Marino, Schiappa '18] *holomorphic anomaly*,
[Gorsky, Milkehin, '14] *Whitham hierarchy*

Connecting weak and strong coupling

P=*NP* relation holds *everywhere* in the spectrum!



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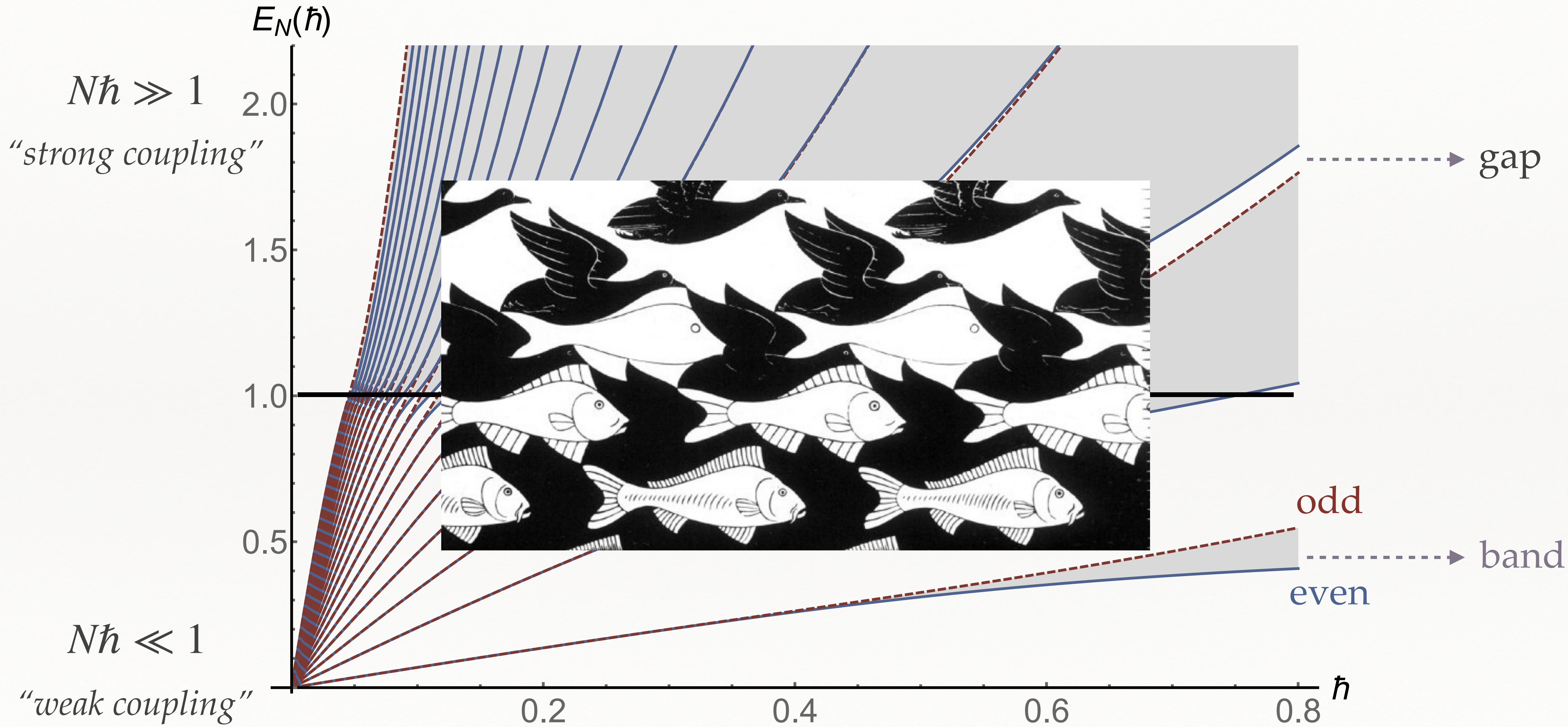
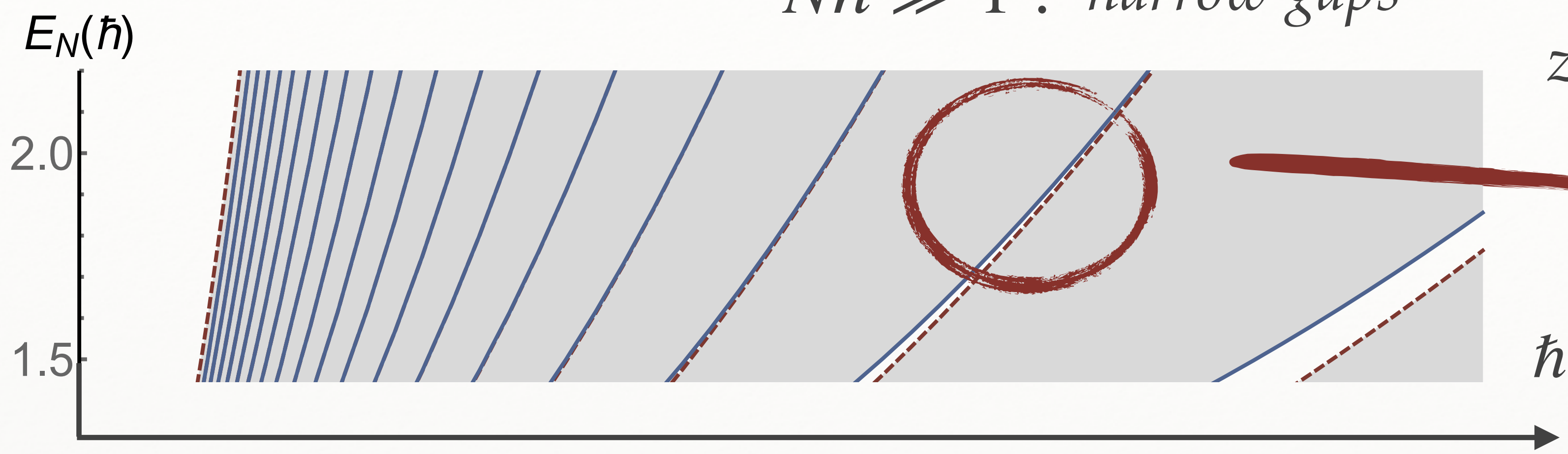
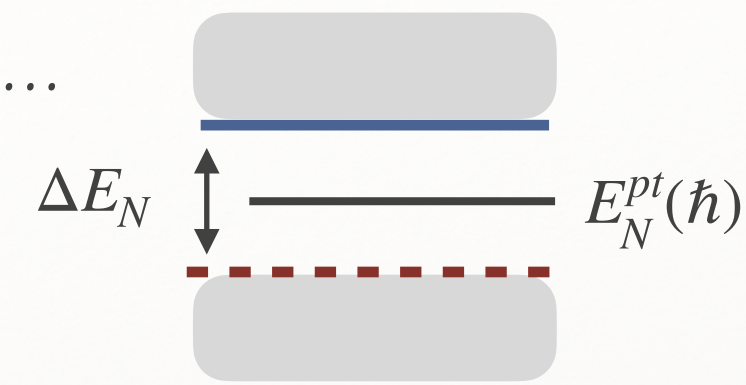


Transmutation of trans-series

$N\hbar \gg 1$: *narrow gaps*



zoom in…



$E_N^{pt}(\hbar)$: *center of gap, "perturbative"*

ΔE_N : *gap width, "1-instanton"*

$$E(\hbar) \sim \frac{\hbar^2}{2}\left(N^2 + \frac{1}{8(N^2-1)}\left(\frac{1}{\hbar}\right)^4 + \frac{5N^2+7}{512(N^2-1)^3(N^2-4)}\left(\frac{1}{\hbar}\right)^8 + \ldots\right) \pm \frac{1}{2^{N-2}\Gamma^2(N)\hbar^{2N-2}}\left(1+\mathcal{O}(\hbar^{-4})\right) + \ldots$$

related by P =NP!

[GB et al, in progress]

New result in a very old problem!

[Mathieu, 1868]

Full structure of the trans-series in an open problem…

Implications for $\mathcal{N}=2$ SUSY theory ,2d CFTs, conformal blocks…

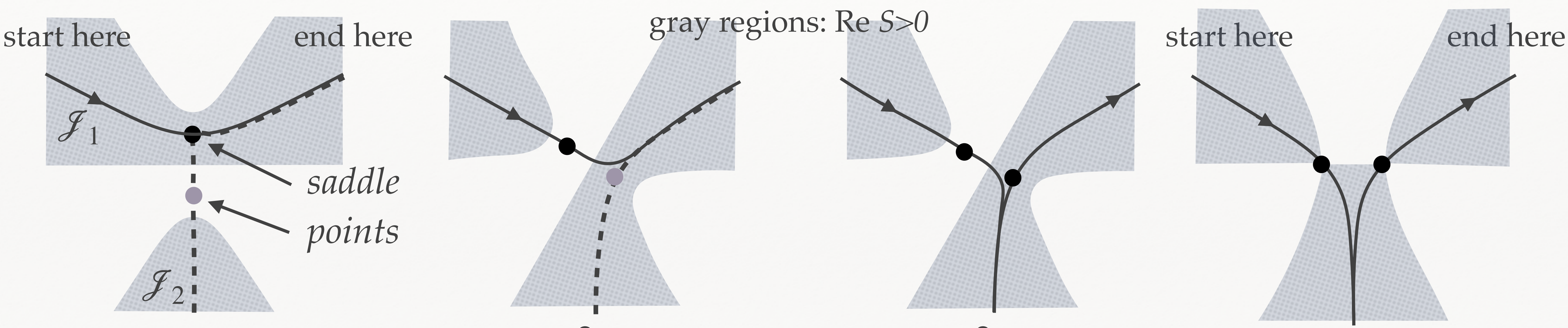
Path integral perspective

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} dt e^{-S(t)} = \frac{1}{\pi}\int_{\mathscr{C}} dt e^{-S(t)}$$

$$S(t) = -i\left(\frac{t^3}{3} + xt\right)$$

$$\theta = \arg x$$

$\mathscr{J}_1, \mathscr{J}_2$: steepest descent contours= ``Lefschetz thimbles”



$\theta = 0$

$\theta = \frac{2\pi}{3} - \epsilon$ → $\theta = \frac{2\pi}{3} + \epsilon$

$\theta = \pi$

Stokes phenomenon

$\mathscr{C} = \mathscr{J}_1$

one thimble

one exponent

→

$\mathscr{C} = \mathscr{J}_1 + \mathscr{J}_2$

two thimbles

two exponents

Path integral perspective

$$\mathcal{O}(g) = \sum_n c_n g^n + \sum_{n,k,l} c_n^{(k,l)} (e^{-a/g})^k (\log g)^l$$

perturbative *non-perturbative*

multi instanton actions

$$Z = \int \mathcal{D}\phi e^{-\frac{1}{g}S[\phi]} = \sum_{i=\text{saddles}} e^{-\frac{1}{g}S[\phi_i]} \int_{\mathcal{J}_i} \mathcal{D}\phi e^{-\frac{1}{g}\left(S[\phi] - S[\phi_i]\right)}$$

fluctuations around saddles
= *path integral over thimble* $\mathcal{J}_i$

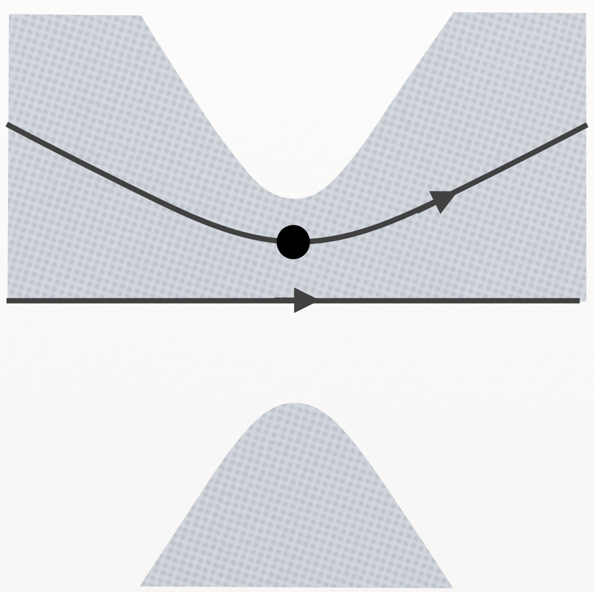
``Exact semi-classics"

Analytical continuation of path integrals

[Pham, Fedoryuk, Witten, Kontsevich,…]

Beyond semi-classics

Even when there is no small parameter in the theory, we can numerically compute the path integral by Monte-Carlo methods



$$\int_{\mathbb{R}} e^{i\left(\frac{t^3}{3}+xt\right)} = \int_{\mathscr{C}} e^{i\left(\frac{t^3}{3}+xt\right)}$$

highly oscillatory integrand

"Sign problem"

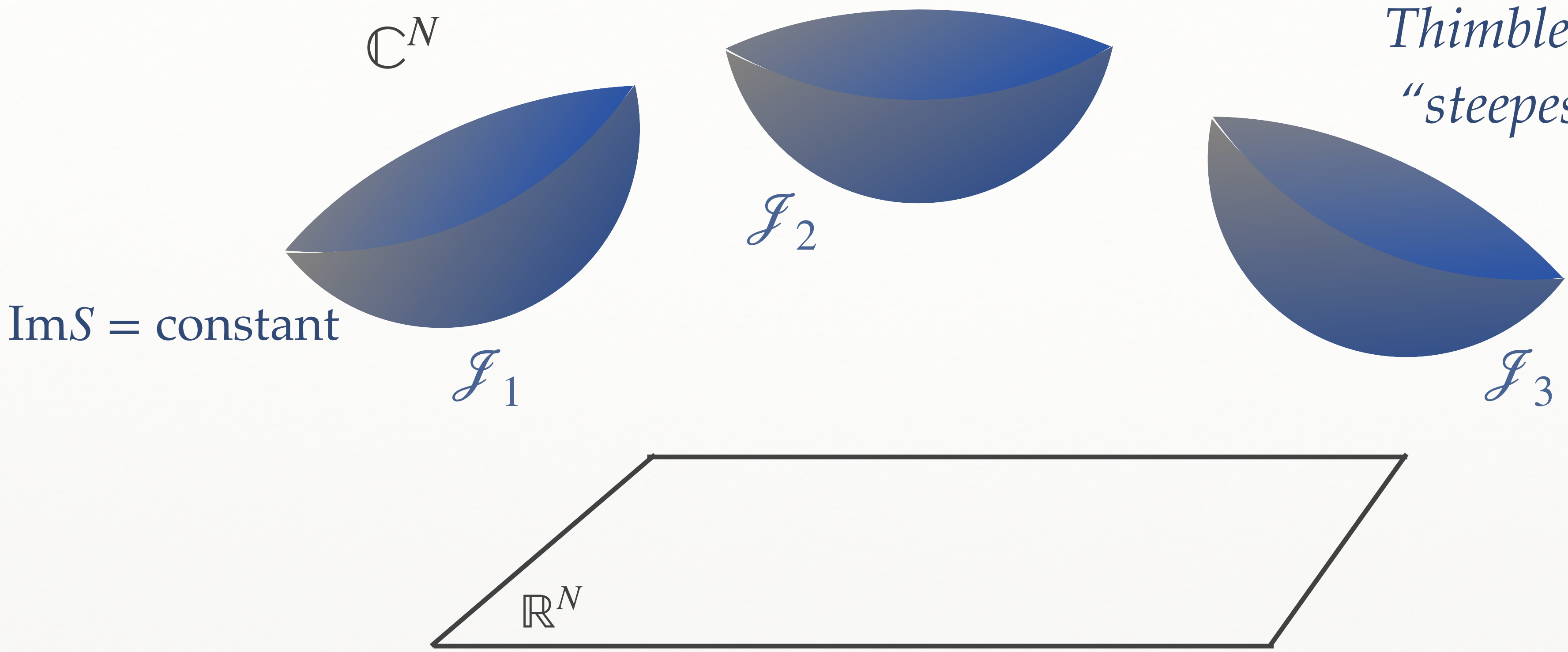
Im S= (piecewise) constant!

Thimbles can be used to mitigate the phase oscillations that arise at finite density, out-of-equilibrium (real time), nonzero theta angle, etc…

[Di Renzo et al '12; Fujii et al '13, GB et al '15]

Review article : "Complex paths around the sign problem"
[Alexandru, GB, Bedaque, Warrington, Rev.Mod.Phys. 94 (2022)]

Lefschetz thimbles and the sign problem



Thimbles: multi dimensional "steepest descent contours"

n_i: intersection # between steepest *ascent* surface and $\mathbb{R}^N$

Instead of $\mathbb{R}^N$ Sample the fields on $\sum_i n_i \mathcal{J}_i$ where oscillations are milder

Finding the relevant saddles and intersection numbers are challenging

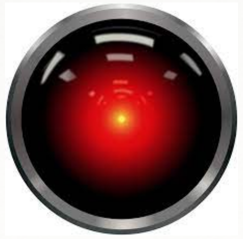
Different values of parameters can lead to different thimble decompositions (Stokes)

Lefschetz thimbles and the sign problem

Find complex path integration domains (not necessarily thimbles) where the phase oscillations are milder

[Alexandru, GB, Bedaque et al '15, …]

Many different ways find such domains : sign optimization, machine learning…



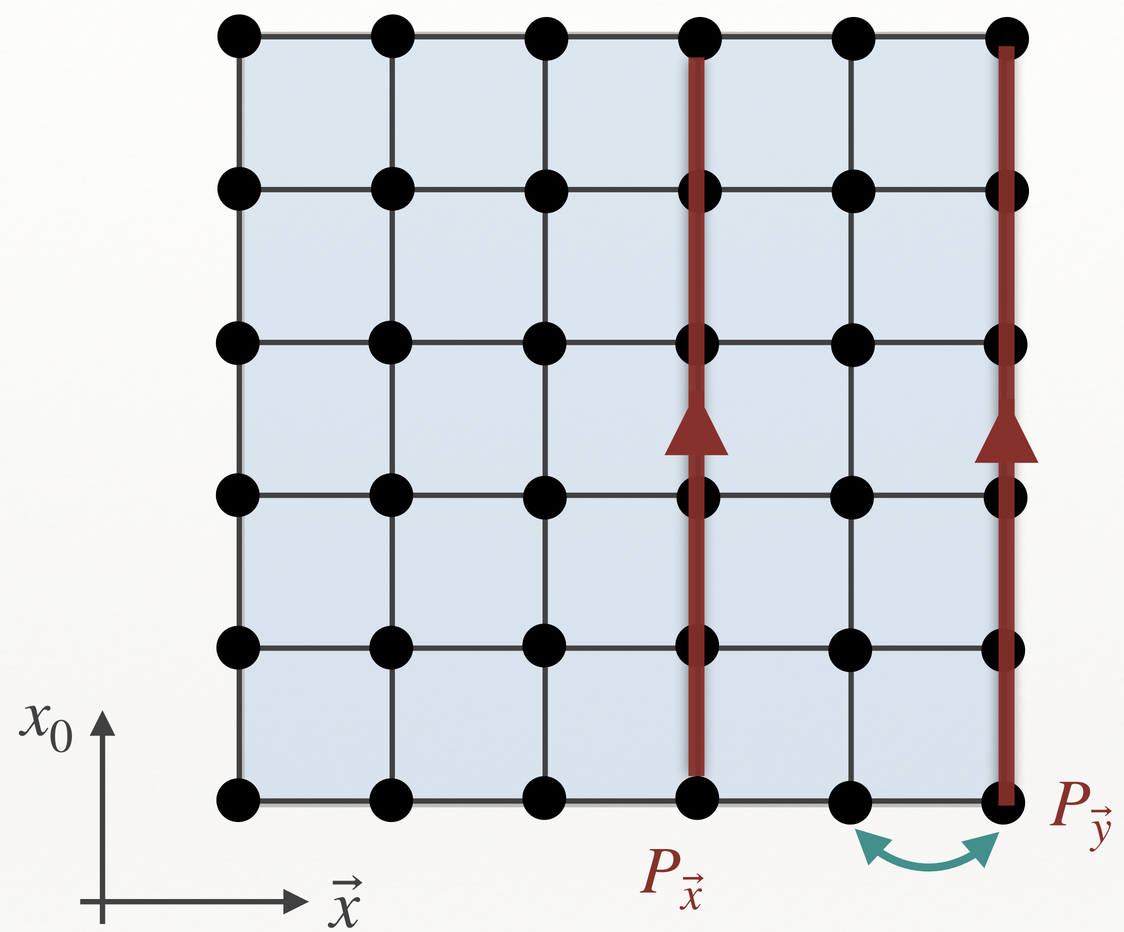
Review article : "Complex paths around the sign problem"
[Alexandru, GB, Bedaque, Warrington, Rev.Mod.Phys. 94 (2022)]

Example: Heavy-dense limit of QCD

QCD with heavy quarks at high density

3d effective theory of Polyakov loops

[Fromm, Langelage, Lottini, Philipsen, '11]



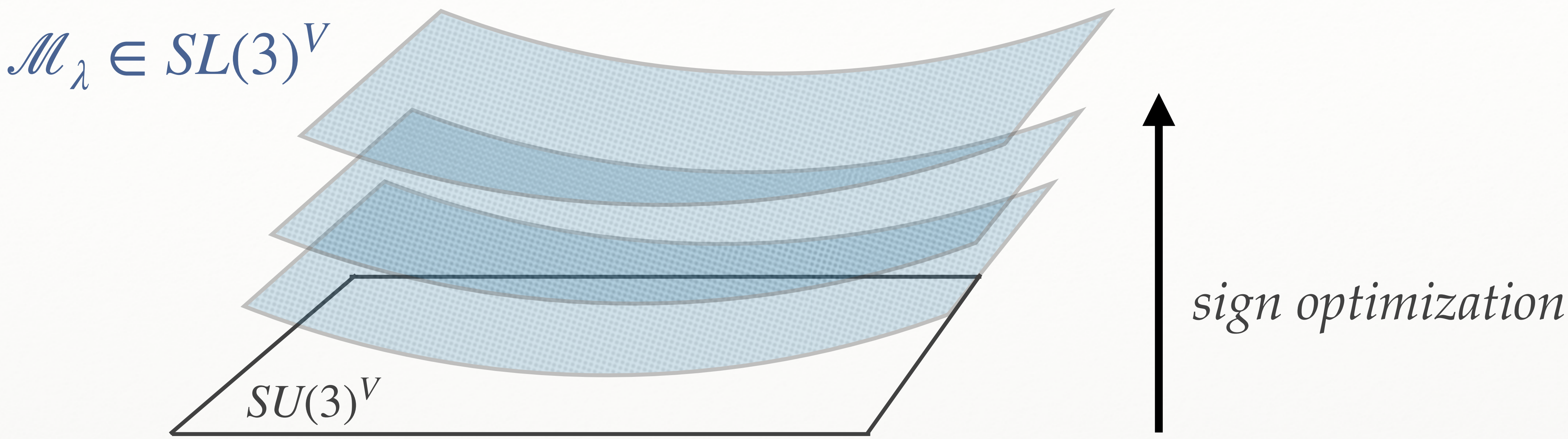
Inherits the sign problem from QCD

Idea: find a complex domain with milder phase oscillations via optimization

[GB, Marincel, 2310.xxxx]
[also Di Renzo et al via thimbles]

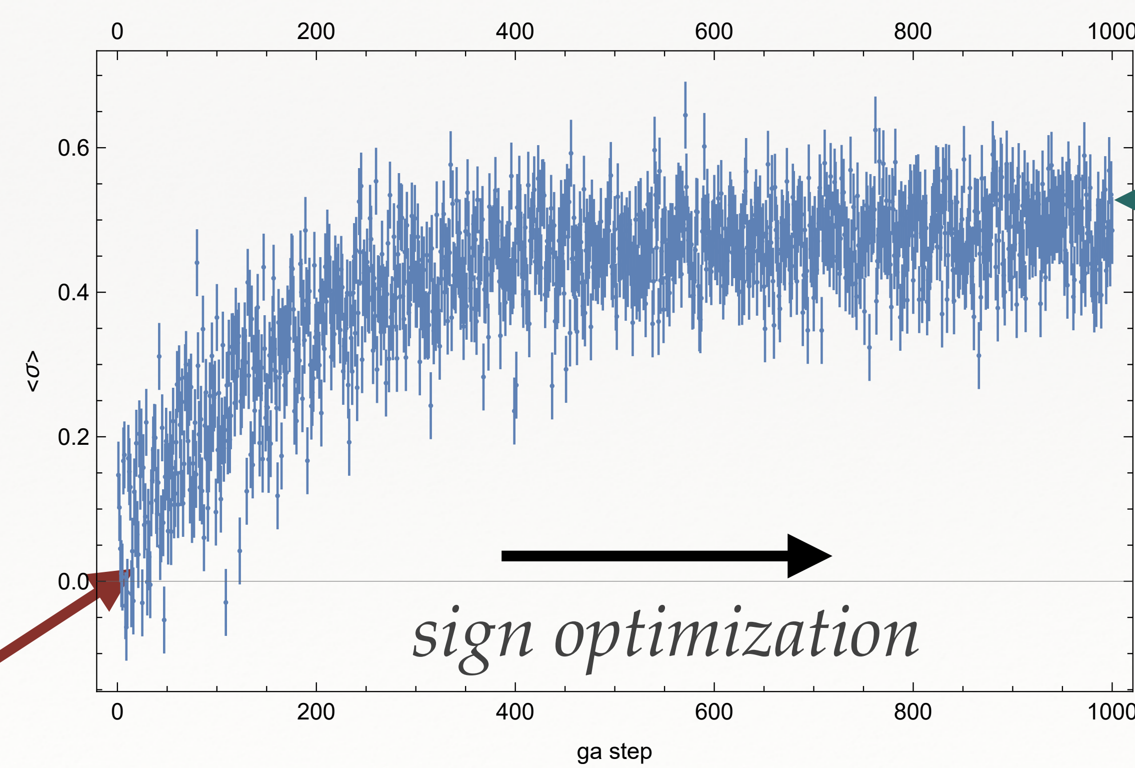
[Mori et al, Alexandru et al, Bursa et al., Kashiwa et al. Detmold et al. '20, ….]

Heavy-dense QCD



A measure for phase oscillations:

$$\langle \sigma \rangle = \frac{\int [dP] e^{-S[P]}}{\int [dP] e^{-\mathrm{Re} S[P]}}$$

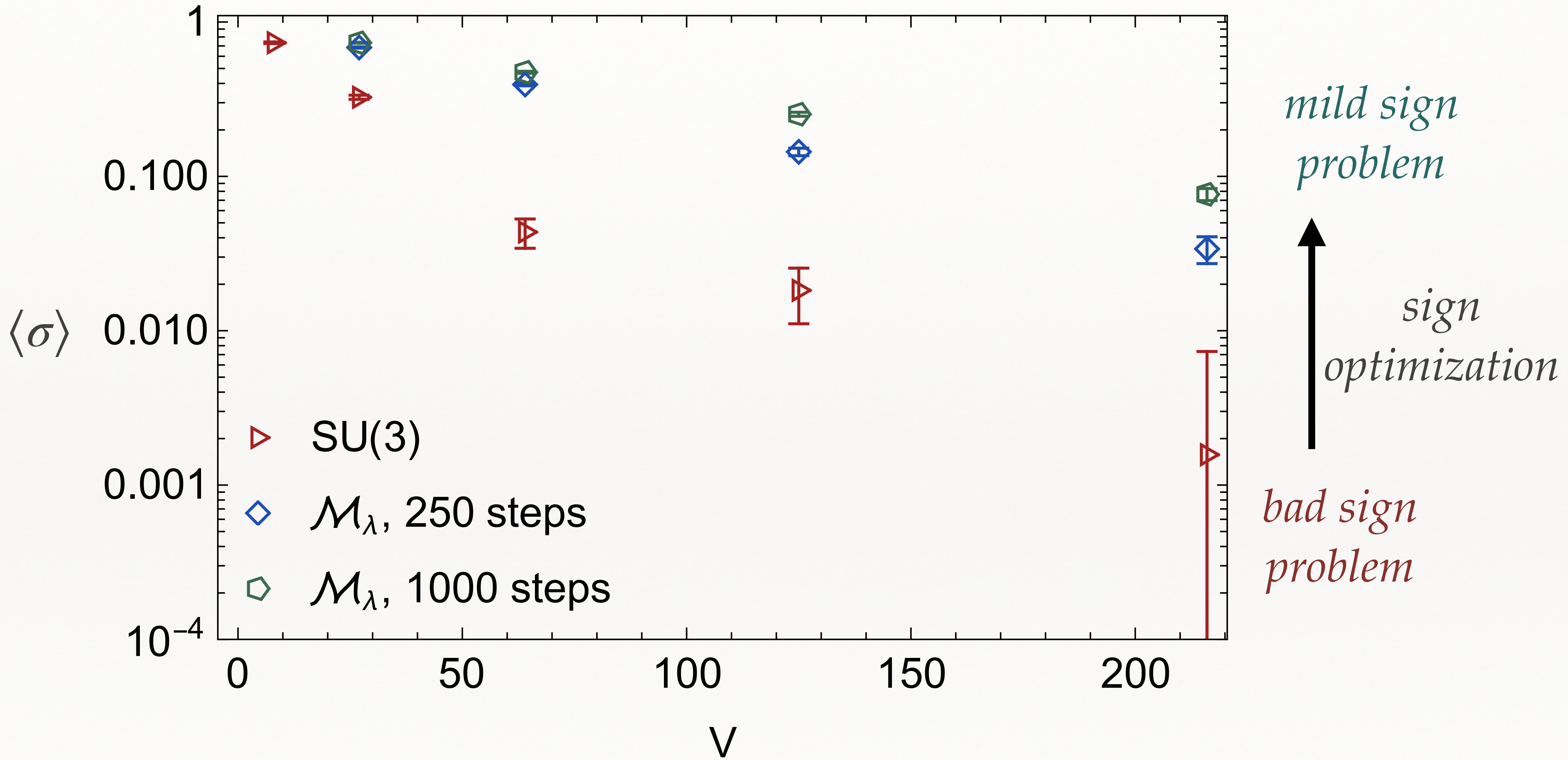


mild sign problem

bad sign problem

[GB, Marincel, 2310.xxxx]

Heavy-dense QCD



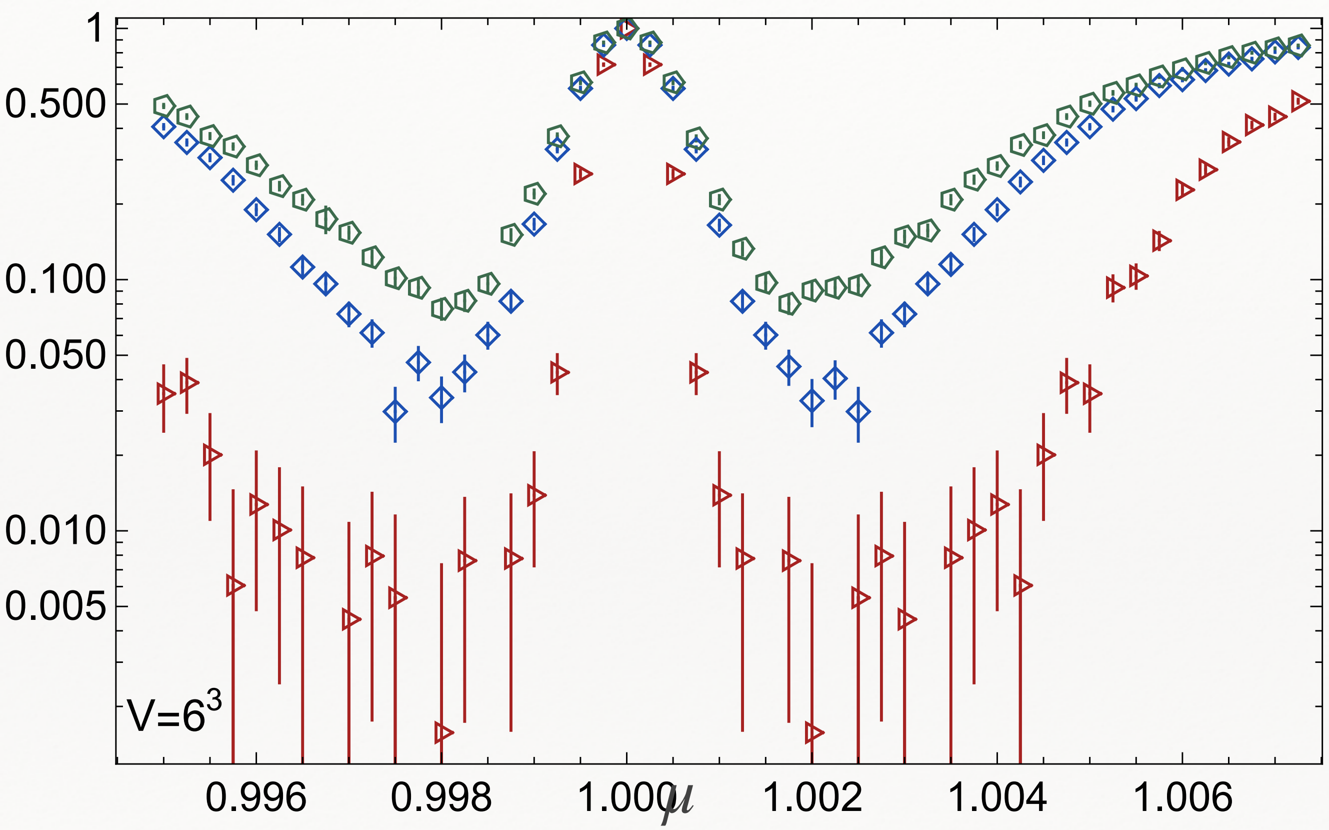
[GB, Marincel, 2310.xxxx]

Heavy-dense QCD

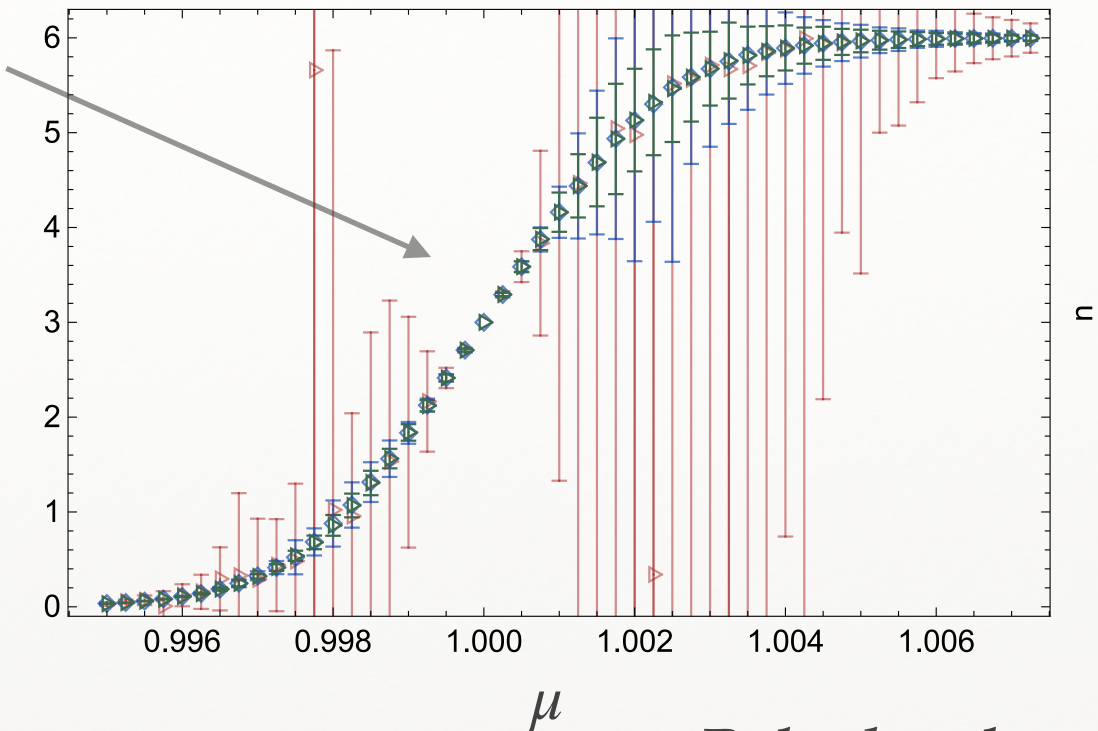
"nuclear saturation"

"Silver Blaze" [Cohen]

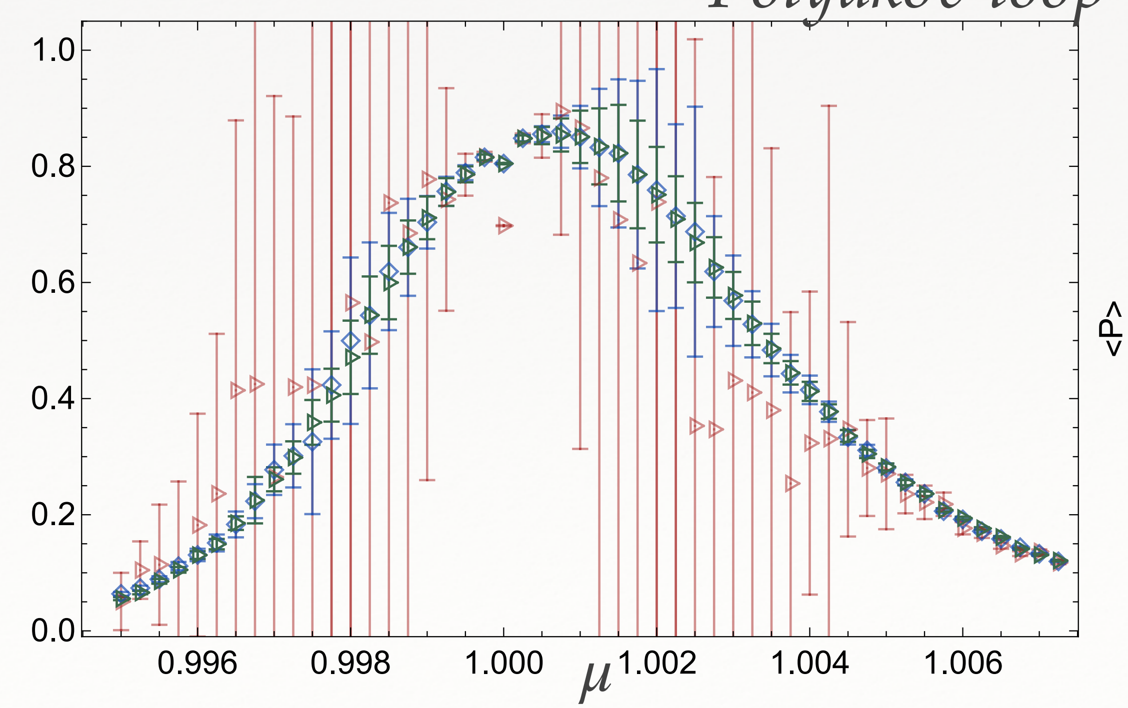
sign problem



equation of state n(μ)

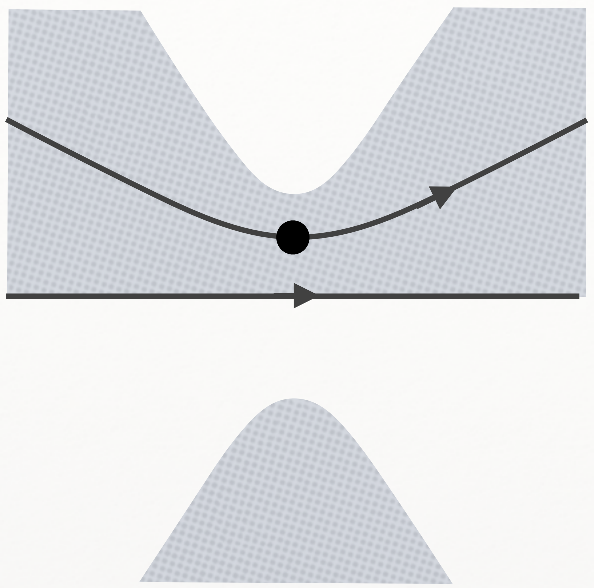


Polyakov loop



red: SU(3)
blue: optimization, 500 steps
green: optimization, 1000 steps

Real time path integrals



$$\int_{\mathbb{R}} e^{i\left(\frac{t^3}{3}+xt\right)} = \int_{\mathscr{C}} e^{i\left(\frac{t^3}{3}+xt\right)}$$

Pure phase
Minkowski path integral!

Im S= (piecewise) constant!

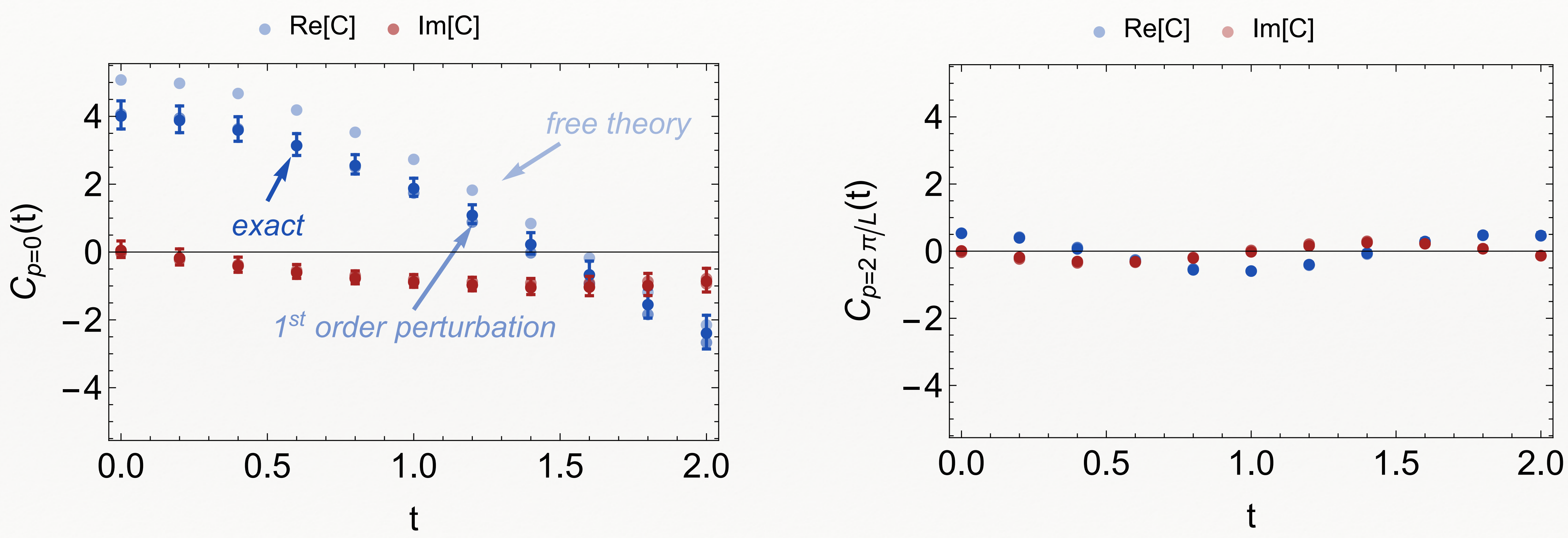
Can we simulate real-time path integrals via (generalized) thimbles?

[related ideas (not lattice)
Pham ‘83, Witten ‘10s]

Real time path integrals

interacting Bose gas: $\mathscr{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$

weak coupling $\boldsymbol{\lambda}$=0.1



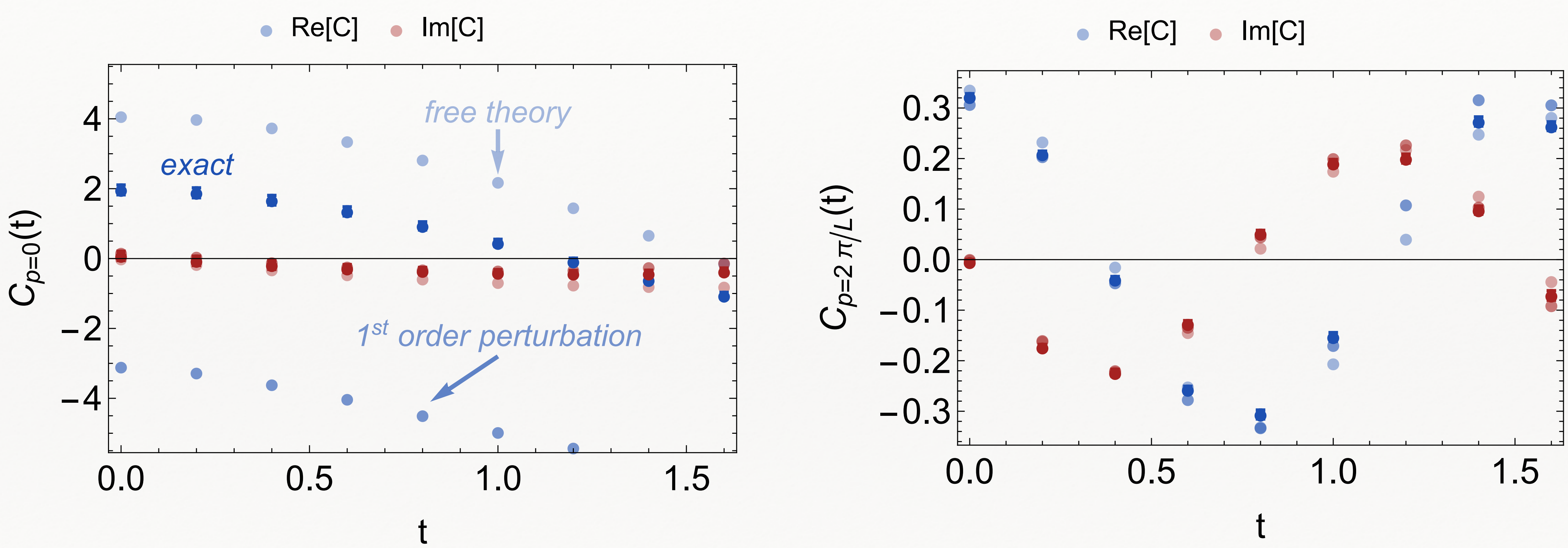
$$C_p(t) = \langle \phi(t, p)\phi(0,p)\rangle_\beta$$

[Alexandru, GB, Bedaque, Ridgway, Vartak, Warrington, PRL 117081602, PRD 95 114501]

Real time path integrals

interacting Bose gas: $\mathscr{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$

strong coupling $\boldsymbol{\lambda}$=1



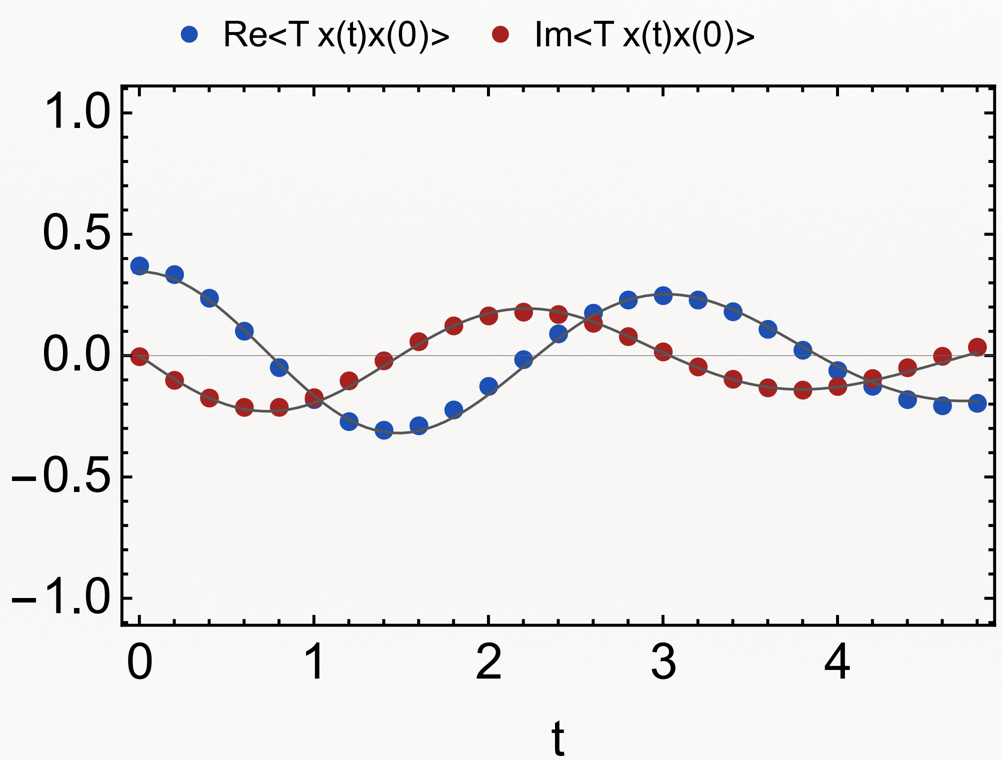
$$C_p(t) = \langle \phi(t,p)\phi(0,p) \rangle_\beta$$

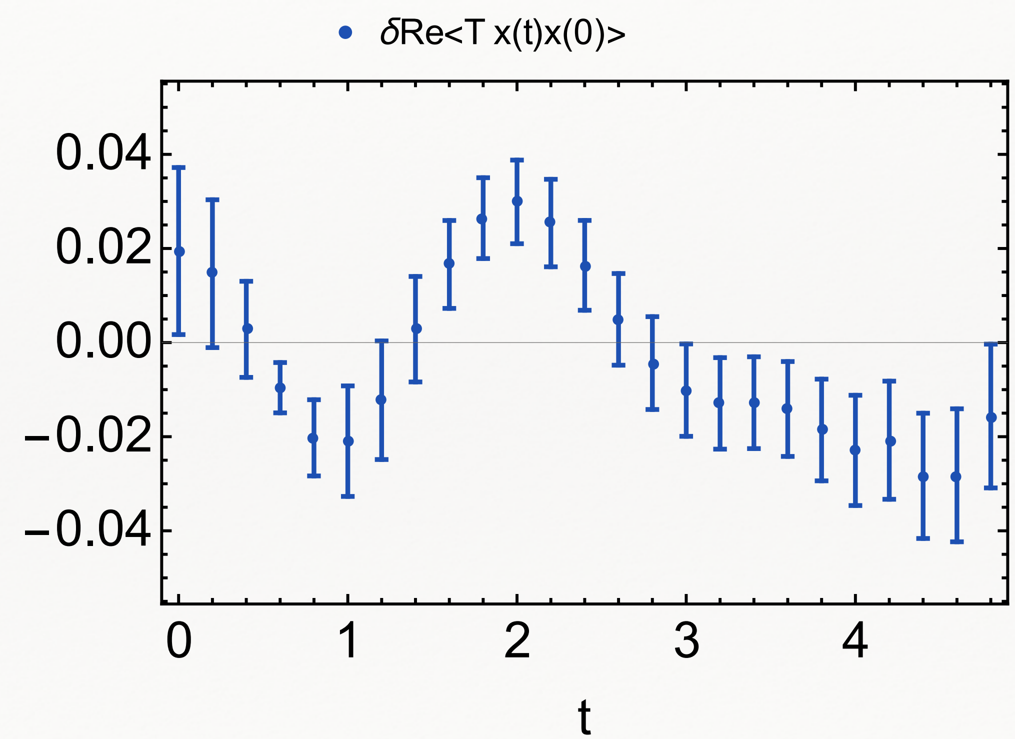
[Alexandru, GB, Bedaque, Ridgway, Vartak, Warrington, PRL 117081602, PRD 95 114501]

Real time path integrals -Hybrid Monte Carlo

Case Study : 0+1 d anharmonic oscillator $\mathscr{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$

$$N_t = 24, \quad N_\beta = 4, \quad \lambda = 24$$





in progress

Renormalons

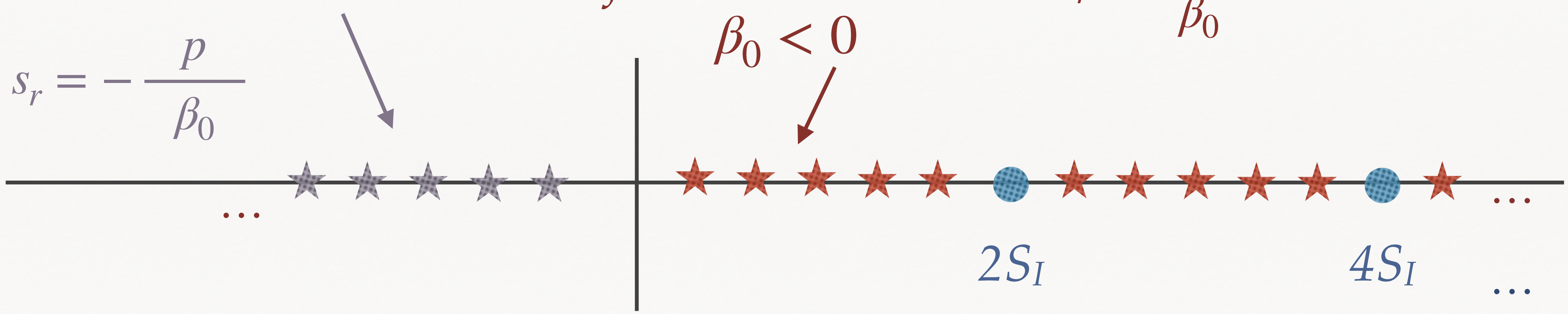
In QFT, perturbation theory has another source of divergence

[Parisi, 't Hooft, … late '70s]

$$g^n(\mu)\int_0^\mu dk\,k^{p-1}\left(\beta_0\log(k/\mu)\right)^n \propto g^n(\mu)\frac{n!}{(-p/\beta_0)^n}, \quad p = 2,4,\ldots$$

"ultraviolet renormalons" $\beta_0 > 0$ $\quad s_r = -\dfrac{p}{\beta_0}$

"infrared renormalons" $\beta_0 < 0$ $\quad s_r = \dfrac{p}{\beta_0}$



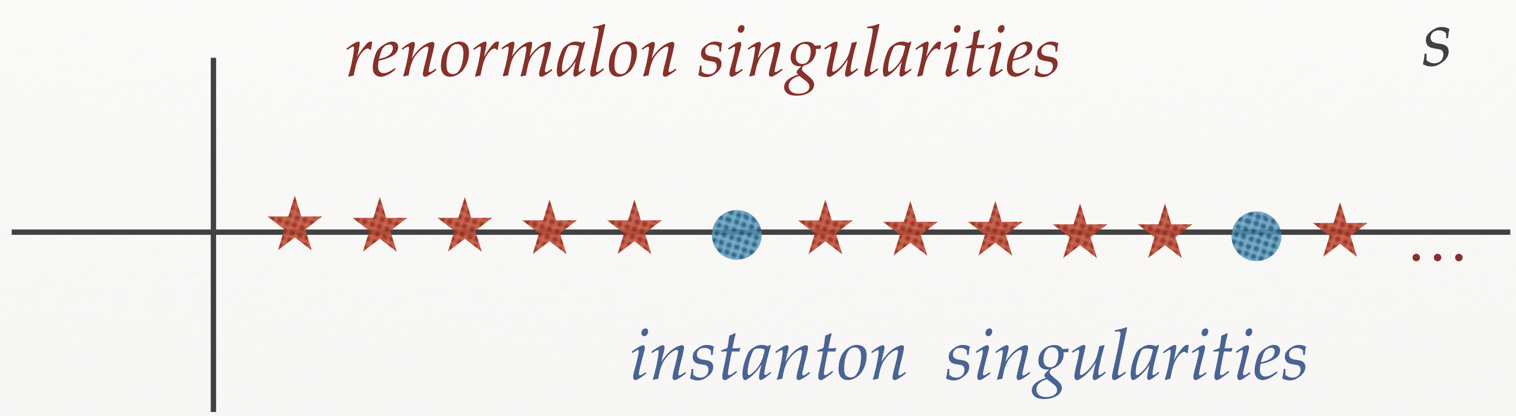
instanton singularities

QCD: $\beta_0 = \dfrac{1}{8\pi^2}\left(-\dfrac{11}{3}N_c + \dfrac{2}{3}N_f\right)$ $\qquad S_I = 8\pi^2$

Renormalons

For asymptotically free theories IR renormalons constitutes a puzzle

Semi-classical configurations that cancel the ambiguity??
not known in QCD



or gluon- channels. It is likely that these singularities are re-
lated to the quark confinement mechanism.

['t Hooft, "Can we make sense out of QCD?", '77]

Renormalons: recent developments

2d and 4d theories on $\mathbb{R} \times S^1$, $\mathbb{R}^3 \times S^1$ (*CP(N)*, principal chiral model, $\mathrm{QCD_{adj}}$,…)

twisted boundary conditions on S^1 *(preserve mixed 't Hooft anomaly if exists)*

small S^1: weakly coupled, but still confining semi-classically ``*adiabatic continuity*"

[Dunne, Ünsal, Cherman, Dorigoni, Argyres, Mismui, Sakai ,Tanizaki,….]

fractional instanton-like objects associated with confinement

e.g $S = \frac{2}{N_c} S_I \quad (\mathbb{R}^3 \times S^1)$ vs. $s_{renormalon} = \frac{12}{11 N_c} S_I \quad (\mathbb{R}^4)$

$\phi^4_{\bar{M}S}$ large order growth is dominated by instantons, not renormalons

[Dunne, Meying, '23]

Some open questions:

How does adiabatic continuity work in Borel plane?

New results on 2d theories via integrability, interpretation not obvious [Marino' 22]

Overview

Going back to the work of Stokes, making sense out of asymptotic series played a crucial role in many areas in physics and mathematics.

Resurgence: exact ``semi-classical" decomposition of the original function in terms of the basic elements $g^n, e^{-1/g}, \log g$ [Ecalle, 80s]

Some earlier parallel developments

Quantum mechanics Delabaere, Dillinger, Pham, Voros, Bogomolnyi, Zinn-Justin, Kawai, Takei,….
``Hyperasymptotics" 70s-90s Dingle, Berry, Howls

More recently

Strings, integrable models, Chern Simons ('07 - …) Aniceto, Marino, Schiappa, Weiss, Vonk, Gukov,…
QFT, QCD in semi-classical domain ('10 - …) Dunne, Unsal, Argyres, GB, Cherman, Dorigoni, …
Path integral, Lefschetz thimbles ('10 - ..) Witten, Kontsevich, …
Beyond semi-classics, Lefschetz thimbles, sign problem ('12 - onwards) Di Renzo et al., Alexandru, GB, Bedaque, Warrington, …

⋮

Still ongoing program, many open problems waiting to be tackled….

Other stuff…

Euler-Heisenberg

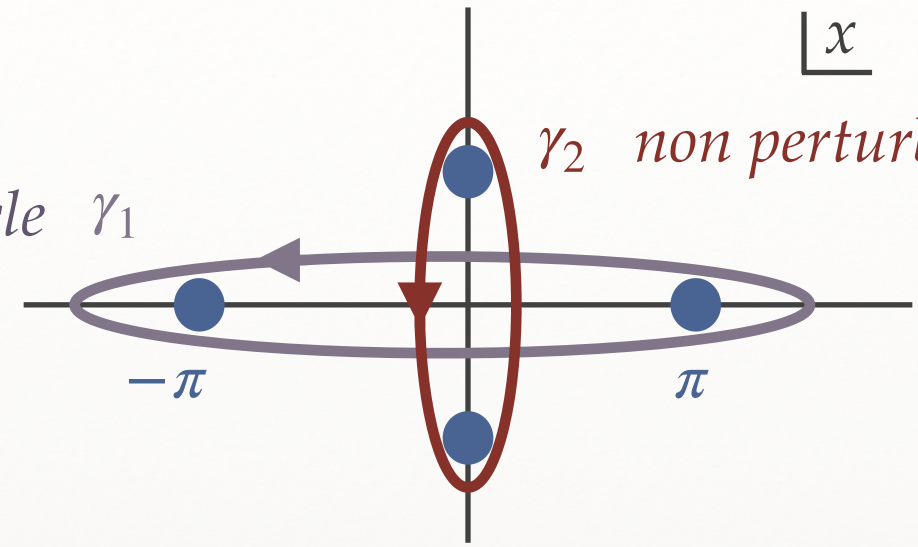
``worldline representation'': Borel-Laplace integral

$$\mathfrak{L} = \frac{1}{2}(\mathfrak{E}^2 - \mathfrak{B}^2) + 4\pi^2 m c^2 \left(\frac{m c}{h}\right)^3 \int_0^\infty e^{-\eta} \frac{d\eta}{\eta^3} \left\{ -a\eta \operatorname{ctg} a\eta \cdot b\eta \operatorname{Ctg} b\eta + 1 + \frac{\eta^2}{3}(b^2 - a^2) \right\}$$

$$= \frac{1}{2}(\mathfrak{E}^2 - \mathfrak{B}^2) + 4\pi^2 m c^2 \left(\frac{m c}{h}\right)^3 \int_0^\infty e^{-\eta} \frac{d\eta}{\eta^3} \left\{ -i a b \eta^2 \frac{\operatorname{Cos}(b + i a)\eta + \operatorname{Cos}(b - i a)\eta}{\operatorname{Cos}(b + i a)\eta - \operatorname{Cos}(b - i a)\eta} + 1 + \frac{\eta^2}{3}(b^2 - a^2) \right\}. \quad (45)$$

$$a^2 - b^2 = \vec{E}^2 - \vec{B}^2, \quad ab = \vec{E}\cdot\vec{B}$$

Complex instantons Mathieu



To leading order

$$E \approx \frac{\hbar^2 N^2}{2}$$

$$\mathrm{Im} S_{\gamma_2}(E) = \sqrt{2}\pi(1-E)\,{}_2F_1\left(\frac{1}{2},\frac{1}{2},1;1-E\right) \approx -2\sqrt{2E}\left(\log(16E)-2\right)$$

Gap width: $$\Delta E_N \approx \frac{1}{\pi}\frac{\partial E}{\partial N} e^{-\frac{1}{2\hbar}\mathrm{Im}\oint_{\gamma_2} P(x;\hbar)dx} \approx \frac{N\hbar^2}{\pi}\left(\frac{e}{2N^2\hbar^2}\right)^N$$

[GB, Dunne, '15]

Complex instantons in QFT

Vacuum pair production with monochromatic electric field

$$E(t) = \mathscr{E}\cos(\Omega t)$$

Mathieu problem with $\hbar \Leftrightarrow \frac{\omega^2}{\mathscr{E}}$ *~frequency* $N \Leftrightarrow \frac{m_e}{\Omega}$ *~number of photons*

$$\hbar N \leftrightarrow \frac{m\Omega}{\mathscr{E}} := \gamma$$ *~"Keldysh adiabaticity parameter"*

Pair production rate:

$$e^{-\frac{m^2\pi}{\mathscr{E}} f\left(\frac{m\Omega}{\mathscr{E}}\right)} \sim \begin{cases} e^{-\frac{m^2\pi}{\mathscr{E}}}, & \gamma \ll 1 \\ \left(\frac{\mathscr{E}}{4m\Omega}\right)^{\frac{4m}{\Omega}}, & \gamma \gg 1 \end{cases}$$

For $\gamma \ll 1$:

- Static limit
- Schwinger pair production
- Tunnelling from Dirac sea
- ~band width

For $\gamma \gg 1$:

- Multi-photon limit
- Brézin-Itzykson
- Tunnelling from Dirac sea
- ~gap width

[GB, Dunne, '15]

P=NP and number theory

Simple P=NP relation ⇔ Ramanujan's elliptic functions in alternative bases

[GB, Dunne, Ünsal]

Mathieu

Example 1

$$\exp\left(-\pi\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-x)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;x)}\right)=\frac{x}{16}\left(1+\frac{1}{2}x+\frac{21}{64}x^2+\cdots\right).$$

Triple well

Example 2

$$\exp\left(-\frac{2\pi}{\sqrt{3}}\frac{{}_2F_1(\frac{1}{3},\frac{2}{3};1;1-x)}{{}_2F_1(\frac{1}{3},\frac{2}{3};1;x)}\right)=\frac{x}{27}\left(1+\frac{5}{9}x+\cdots\right).$$

$x \leftrightarrow E$

Double well

Example 3

$$\exp\left(-\sqrt{2}\pi\frac{{}_2F_1(\frac{1}{4},\frac{3}{4};1;1-x)}{{}_2F_1(\frac{1}{4},\frac{3}{4};1;x)}\right)=\frac{x}{64}\left(1+\frac{5}{8}x+\cdots\right).$$

Cubic

Example 4

$$\exp\left(-2\pi\frac{{}_2F_1(\frac{1}{6},\frac{5}{6};1;1-x)}{{}_2F_1(\frac{1}{6},\frac{5}{6};1;x)}\right)=\frac{x}{432}\left(1+\frac{13}{18}x+\cdots\right).$$

[Berndt, Ramanujan's Notebooks Vol. II]

P=NP and number theory

Simple P=NP relation ⇔ Ramanujan's elliptic functions in alternative bases

[GB, Dunne, Ünsal]

Mathieu

Example 1

$$\exp\left(-\pi\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-x)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;x)}\right)=\frac{x}{16}\left(1+\frac{1}{2}x+\frac{21}{64}x^2+\cdots\right).$$

Triple well

Example 2

$$\exp\left(-\frac{2\pi}{\sqrt{3}}\frac{{}_2F_1(\frac{1}{3},\frac{2}{3};1;1-x)}{{}_2F_1(\frac{1}{3},\frac{2}{3};1;x)}\right)=\frac{x}{27}\left(1+\frac{5}{9}x+\cdots\right).$$

$x \leftrightarrow E$

Double well

Example 3

$$\exp\left(-\sqrt{2}\pi\frac{{}_2F_1(\frac{1}{4},\frac{3}{4};1;1-x)}{{}_2F_1(\frac{1}{4},\frac{3}{4};1;x)}\right)=\frac{x}{64}\left(1+\frac{5}{8}x+\cdots\right).$$

Cubic

Example 4

$$\exp\left(-2\pi\frac{{}_2F_1(\frac{1}{6},\frac{5}{6};1;1-x)}{{}_2F_1(\frac{1}{6},\frac{5}{6};1;x)}\right)=\frac{x}{432}\left(1+\frac{13}{18}x+\cdots\right).$$

[Berndt, Ramanujan's Notebooks Vol. II]

We do not know Ramanujan's intention in giving Examples 1–4.

P=NP and number theory

Simpl… P-NP relation ⇔ Ramanujan's elliptic functions in alternative bases

$$-\frac{\hbar^2}{2m}\psi'' + V(x)\psi = E\psi$$

[GB, Dunne, Ünsal]

Mathieu

$$\exp\left(-\pi \frac{{}_2F_1(\ldots;1;1-x)}{{}_2F_1(\ldots;1;x)}\right)$$

Triple well

Example 2

$$\exp\left(-\frac{2\pi}{\sqrt{3}}\frac{{}_2F_1(\frac{1}{3},\frac{2}{3};1;1\ldots}{{}_2F_1(\frac{1}{3},\frac{2}{3};1;\ldots}\right.$$

$x \leftrightarrow E$

Double well

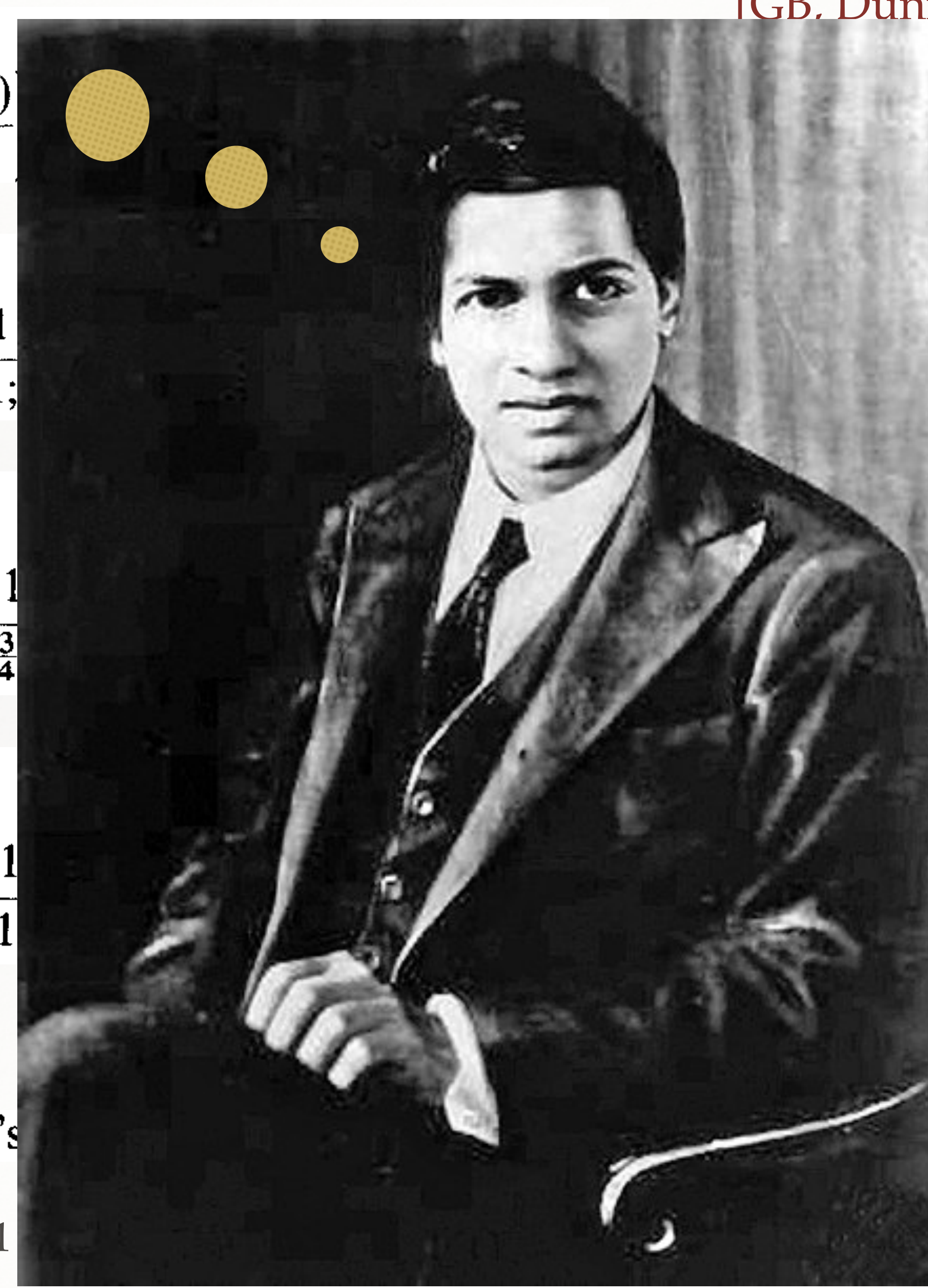
Example 3

$$\exp\left(-\sqrt{2}\pi\frac{{}_2F_1(\frac{1}{4},\frac{3}{4};1\ldots}{{}_2F_1(\frac{1}{4},\frac{3}{4}\ldots}\right.$$

Cubic

Example 4

$$\exp\left(-2\pi\frac{{}_2F_1(\frac{1}{6},\frac{5}{6};1;1\ldots}{{}_2F_1(\frac{1}{6},\frac{5}{6};1\ldots}\right.$$



…ks Vol. II]

We do not know Ramanujan's…