Random Coupling Models
&

Field Theory for Ocean Waves

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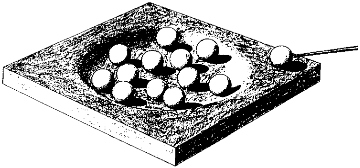
Random Matrix Theory

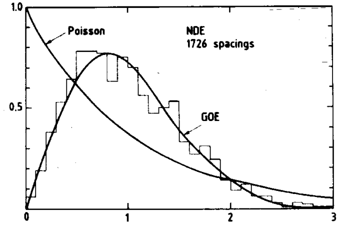
Problem: How to understand the energy levels of heavy nuclei. (Solving the Schrödinger equation is too difficult.)

Wigner (1951): replace the Hamiltonian by a totally random matrix,

$$P(H) \approx e^{-N\,\mathrm{tr} H^2}$$



Bohr's wooden toy model for compound nucleus scattering



Histogram of nearest neighbor level spacing (when all levels are arranged monotonically) for the "Nuclear data ensemble" representing 1726 spacings

$$p(s) = \frac{\pi s}{2} e^{-\pi s^2/4}$$

where s is the spacing between levels divided by the mean spacing.

This is in good agreement with random matrix theory.

Lead to "RMT" quantum chaos community.

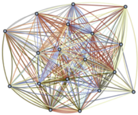
Random Couplings

Of course, the actual Hamiltonian is not a random matrix. How to improve?

Two-body random Hamiltonian (French & Wong '70, Bohigas & Flores, '71)

$$H = \sum_{i,j,k,l=1}^{N} J_{ijkl} c_i^\dagger c_j^\dagger c_k c_l$$

where $J_{ij;kl}$ is drawn from a Gaussian distribution.



Can consider q-body interaction. Sum over all q gives random matrix theory.

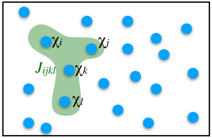
Sachdev-Ye-Kitaev Model

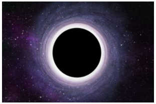
$$H = \sum_{i,j,k,l=1}^{N} J_{ijkl}\chi_i\chi_j\chi_k\chi_l$$

Large N solution: replica-trick, from studies of spin glasses,

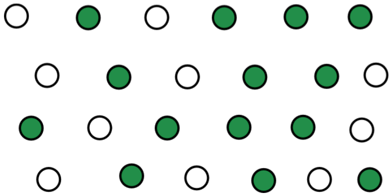
Sachdev-Ye ('93): $H = \sum_{ij} J_{ij} S_i \cdot S_j$ (spins in $SU(M)$, large M).

Kitaev ('15): in search of a model of many-body quantum chaos, in order to describe a black hole.

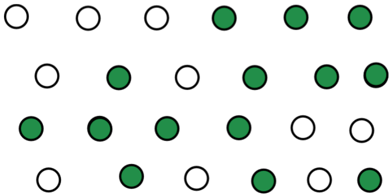




There are $N \gg 1$ sites, and many fermions. Each site may or may not be occupied



Under time evolution, any two occupied sites may become unoccupied, while two unoccupied sites become occupied



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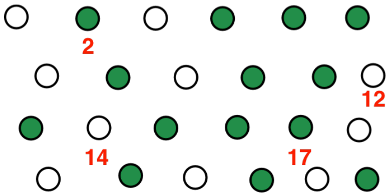

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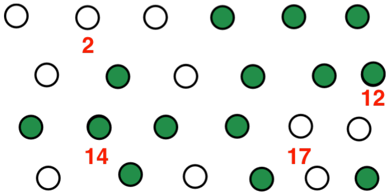


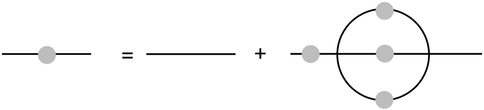
Can integrate out the coupling (Gaussian integrals). Left with a bilocal in time effective action,

$$\frac{I_{eff}}{N} = -\frac{1}{2}\log\det\left(\partial_\tau - \Sigma\right) + \frac{1}{2}\int d\tau_1 d\tau_2 \left(\Sigma(\tau_1,\tau_2)G(\tau_1,\tau_2) - \frac{J^2}{4}G(\tau_1,\tau_2)^4\right) .$$

At large N dominated by saddle,

$$\dot{G}(\tau) = \delta(\tau) + J^2 \int d\tau_1 \, G(\tau_1) G(\tau-\tau_1)^3 .$$

Equivalently, the dominant Feynman diagrams at large N are melons,

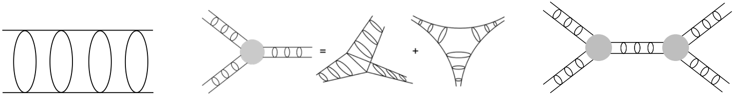


This is a context in which Dynamical Mean Field theory is exact: N interacting particles are replaced by one particle with a bilocal in time action.

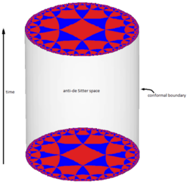
For large J (infrared) there is emergent conformal invariance, CFT$_1$, $G(\tau) \sim \frac{1}{|\tau|^{1/2}}$.

Dimension of χ is 0 in the UV, $1/4$ in the IR.

Large N correlation functions be computed (Kitaev '15; Polchinksi & V.R, '16; Maldacena & Stanford, '16; Gross & V.R., '17)



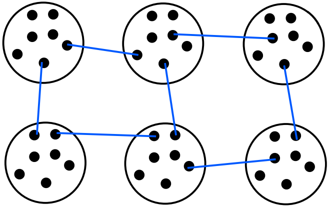
Belief: should be an AdS dual



	Matrix models	SYK	Vector models
Large N Feynman diagrams	planar	melons	bubbles
Difficulty	hard	easy/medium	easy
A CFT example	$\mathcal{N}$=4 Yang-Mills (4 dimensions)	SYK, at strong coupling (1 dimension)	free/critical O(N) vector model (3 dimensions)
AdS dual theory	string theory	?	Vasiliev higher spin
Gravitiational sector	stress-tensor $\longleftrightarrow$ Einsetin gravity	h=2 mode (Schwarzian) $\longleftrightarrow$ Jackiw-Teitelboim gravity	inapplicable
Gauge invariant operators	$\mathrm{tr}(X\partial^k XYY\cdots)$	$\chi_i\partial_\tau^{1+2n}\chi_i$	$\phi^a\partial_{(\mu_1}\cdots\partial_{\mu_s)}\phi^a$
Anomalous dimensions	large (at large 't Hooft coupling)	order 1	order 1/N
Dual of these operators	stringy modes	tower of scalars	tower of massless higher spin fields

"Harmonic Oscillator" of strongly coupled theories

Use a lattice of SYK "quantum dots" as a model of a strongly correlated metal (Gu, Xi, Stanford, '16; Song, Jian, Balents, '17)



Quartic all-to-all interactions within a dot, and quadratic all-to-all hopping terms between dots.

Open questions

- When does large N naturally occur?
- What is classical analogue?
- How to improve on q body random coupling to get a more realistic model? Have an effective action (of one fermion) that is bilocal in time. Want to do perturbation theory about maximally chaotic theory.

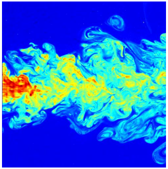
Random coupling model of turbulence

A random coupling model was known (and solved!) much earlier, in the turbulence literature (Kraichnan '59, Betchov '67, Hansen & Nicholson '81)

$$\dot{\phi}_i = \sum_{j,k,l=1}^{N} J_{ijkl}\phi_j\phi_k\phi_l \ .$$

where the sum of J_{ijkl} under cylic permutation vanishes. The index i is a Fourier mode.

Can solve like SYK, via path integral (Hu, V.R., '23).



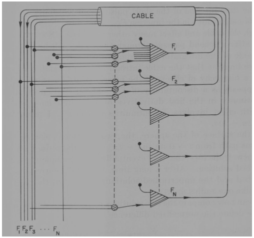
Direct interaction approximation

Assume $\langle\phi_i(t')\phi_j(t)\phi_k(t)\phi_l(t)\rangle$ vanishes if J_{ijkl} set to zero.
Compute $\Delta\phi_i$ when turn on J_{ijkl} using linear response theory,

$$\Delta\phi_i(t) = J_{ijkl} \int dt'\, R(t,t')\phi_j(t')\phi_k(t')\phi_l(t')$$

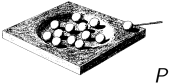
which gets inserted into,

$$\dot{G}(t_1,t_2) = \frac{1}{3!}\sum_{j,k,l} J_{ijkl}\langle\phi_j(t_1)\phi_k(t_1)\phi_l(t_1)\phi_i(t_2)\rangle \ ,$$



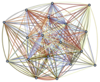
Imaginary analog computer that solves the equations of motion (Betchov '67)

Summary

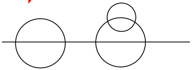


$$P(H) \approx e^{-N\,\mathrm{tr} H^2}$$

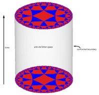
$$H = \sum_{i,j,k,l=1}^{N} J_{ijkl} c_i^\dagger c_j^\dagger c_k c_l$$

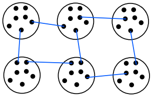


large N



$$G(\tau) \sim \frac{1}{|\tau|^{1/2}}$$





Tensor Models

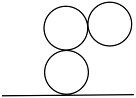
Closely related to SYK is the tensor model (Gurau '09, Witten '16, Klebanov & Tarnopolsky '16),

$$S = \int d\tau \left(\frac{1}{2} \psi_{abc} \partial_\tau \psi_{abc} + \frac{g}{4} \psi_{abc} \psi_{ade} \psi_{fbe} \psi_{fdc} \right)$$

At leading order in 1/N, it has the same Feynman diagrams as SYK, and so the same physical properties.

Large N Models

- Vector model: ϕ_a $\qquad (\vec{\phi}\cdot\vec{\phi})^2 \equiv \phi_a\phi_a\phi_b\phi_b$



Easy

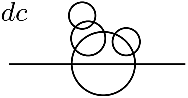
bubbles

- Matrix model: ϕ_{ab} $\qquad \mathrm{Tr}(\phi^4) \equiv \phi_{ab}\phi_{bc}\phi_{cd}\phi_{da}$

Hard

all planar

- Tensor model: ϕ_{abc} $\qquad \phi_{abc}\phi_{ade}\phi_{fbe}\phi_{fdc}$

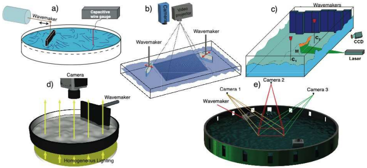


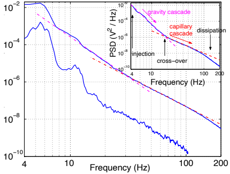
Medium

melons

Ocean Waves

- For small amplitude waves, within some range of momenta, it is both observed and theoretically predicted that $n_k \sim k^{-4}$.
- This is known as weak wave turbulence (unrelated to hydrodynamics turbulence).
- Similar effects occur in many other nonlinear systems, including reheating in the early universe and heavy ion collisions (prethermalization) (Micha & Tkachev, 04; Berges, Heller, Mazeliauskas, Venugopalan, '21)





(Falcon & Mordant '21)

Can describe by (Hasselmann '62, Zakharov '65),

$$H = \sum_p \omega_p a_p^* a_p + \sum_{p_1, p_2, p_3, p_4} \lambda_{p_1 p_2 p_3 p_4} a_{p_1}^* a_{p_2}^* a_{p_3} a_{p_4} + \ldots \ .$$

Assume weak nonlinearity to derive a wave kinetic equation (analog of Boltzmann equation). Find there is a far-from-equilibrium stationary solution $n_k \sim k^{-\gamma}$.

These are classical systems with many degrees of freedom, that are chaotic, and have a statistical description.

Claim: QFT is the correct framework to understand these systems. Gives novel and challenging QFTs, in a state that is stationary (constant energy flux), but is neither the vacuum nor thermal.

V.R., M. Smolkin, 2203.08168; V.R., M. Smolkin, 2212.02555; V.R, D. Schubring, Md. S. J. Shuvo, M. Smolkin, 2308.00740; V.R, G. Falkovich, 2308.00033

Classical stochastic = Quantum

$$\langle \mathcal{O}(a) \rangle = \int \mathcal{D}f \, P\left[f\right] \mathcal{O}(a) \ ,$$

where $a_k(t)$ satisfies the equations of motion for the particular f_k picked from $P\left[f\right]$.

Insert a delta function, to enforce the equations of motion,

$$\langle \mathcal{O}(a) \rangle = \int \mathcal{D}a\mathcal{D}f \, P\left[f\right] \mathcal{O}(a) \, \delta(\mathrm{eom}_f)$$

where $\mathrm{eom}_f = \dot{a}_k + i \frac{\delta H}{\delta a_k^*} - f_k(t) + \gamma_k a_k$. Write the delta functionals as an integral,

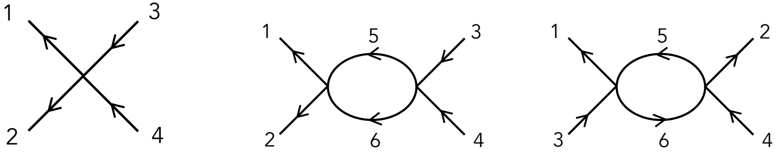
$$\delta(\mathrm{eom}_f) = \int D\eta e^{i \int dt \sum_k \eta_k(t) \mathrm{eom}_f^*}$$

then do the integral over forcing, then integrate out the Lagrange multiplier (η),

$$\langle \mathcal{O}(a) \rangle = \int \mathcal{D}a \, \mathcal{O}(a) \, e^{-\int dt \, L} \ , \qquad L = \sum_k \frac{\left|\mathrm{eom}_{f=0}\right|^2}{F_k} \ .$$

Wave kinetics, to one-loop

In standard framework, no effective tools for going to higher order. But it is necessary, since nonlinearity gets strong either in the UV or the IR.



$$\frac{dn_1}{dt} = 16\pi \sum_{2,\dots,4} \delta(\omega_{p_1 p_2; p_3 p_4}) \lambda_{1234}^2 \prod_{i=1}^{4} n_i \left(\frac{1}{n_1} + \frac{1}{n_2} - \frac{1}{n_3} - \frac{1}{n_4}\right)$$

$$\left[1 + 4 \sum_{5,6} \frac{\lambda_{1256}\lambda_{5634}}{\lambda_{1234}} \frac{n_5 + n_6}{\omega_{p_1 p_2; p_5 p_6}} + 16 \sum_{5,6} \frac{\lambda_{1635}\lambda_{2546}}{\lambda_{1234}} \frac{n_5 - n_6}{\omega_{p_4 p_6; p_2 p_5}}\right]$$

Interpretation

- Take a tank of water. No waves. Create and scatter two waves, which turn into two other waves. Interaction strength λ_{1234}.
- Now take a tank with a turbulent state. Repeat – create and scatter two waves. But now they are interacting with background of waves in turbulent state. Changes the effective interaction strength.

Summary

1. We can compute multimode correlation functions and the kinetic equation systematically, perturbatively in the nonlinearity
2. The corrections to the standard (leading order) kinetic equation can be much larger than naively expected (i.e. weak turbulence can break down earlier than expected)
3. The behavior (divergences) of corrections depend on the asymptotics of the interaction λ_{1234}
4. The appropriate renormalization framework for this setup – a far-from-equilibrium constant flux state, with forcing in the IR and dissipation in the UV (i.e. energy sinks in the UV and IR) – is an open problem