Field theory of the Fermi function

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BNL theory seminar
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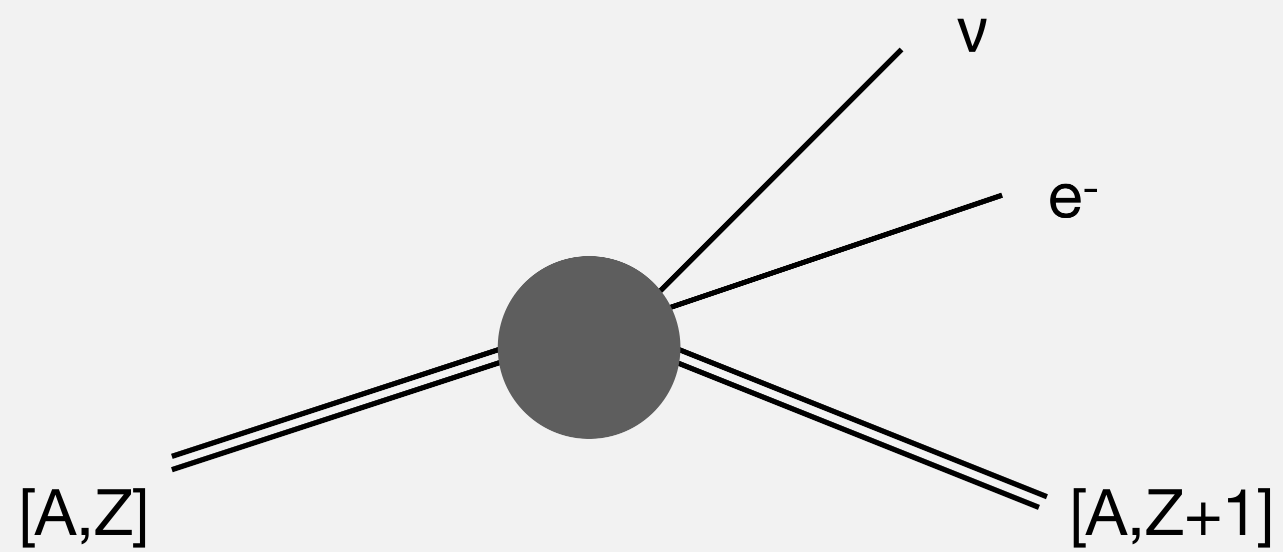
based on
- 2309.07343 with R. Plestid
- 2309.15929 with R. Plestid
- 2402.13307 with K. Borah and R. Plestid
- work in progress with R. Plestid and P. Vander Griend

Outline

- Definition of classical Fermi function
- Effective theory and factorization
- Renormalization and anomalous dimension
- Neutron beta decay and large pi resummation
- Summary

Fermi function

The classical Fermi function describes enhanced (huge!) QED corrections for electron/positron emission from large-Z nucleus



$$\sigma \approx F(Z, E)\,\sigma_0$$

$$F(Z,E) = \frac{2(1+\eta)}{[\Gamma(2\eta+1)]^2}\,|\Gamma(\eta+i\xi)|^2\,e^{\pi\xi}(2pr)^{2(\eta-1)}$$

$$\eta = \sqrt{1-(Z\alpha)^2} \qquad \xi = \frac{Z\alpha}{\beta}$$

Fermi function

In NR limit (not applicable to neutron and nuclear beta decay) the Fermi function reduces to the Sommerfeld factor

$$\xi = \frac{Z\alpha}{\beta} \qquad \eta = \sqrt{1 - \beta^2\xi^2} \approx 1$$

$$F(Z, E) \to \frac{2\pi\xi}{1 - \exp(-2\pi\xi)} = 1 + \pi\xi + \frac{\pi^2}{3}\xi^2 + \dots$$

Fermi function

Several questions naturally arise:

- what is the quantity r appearing in F(Z,E)? Approximately the nuclear radius, but how to go beyond this qualitative model? (answer: $r^{-1}e^{\gamma_E} = \mu_{\overline{\rm MS}}$)

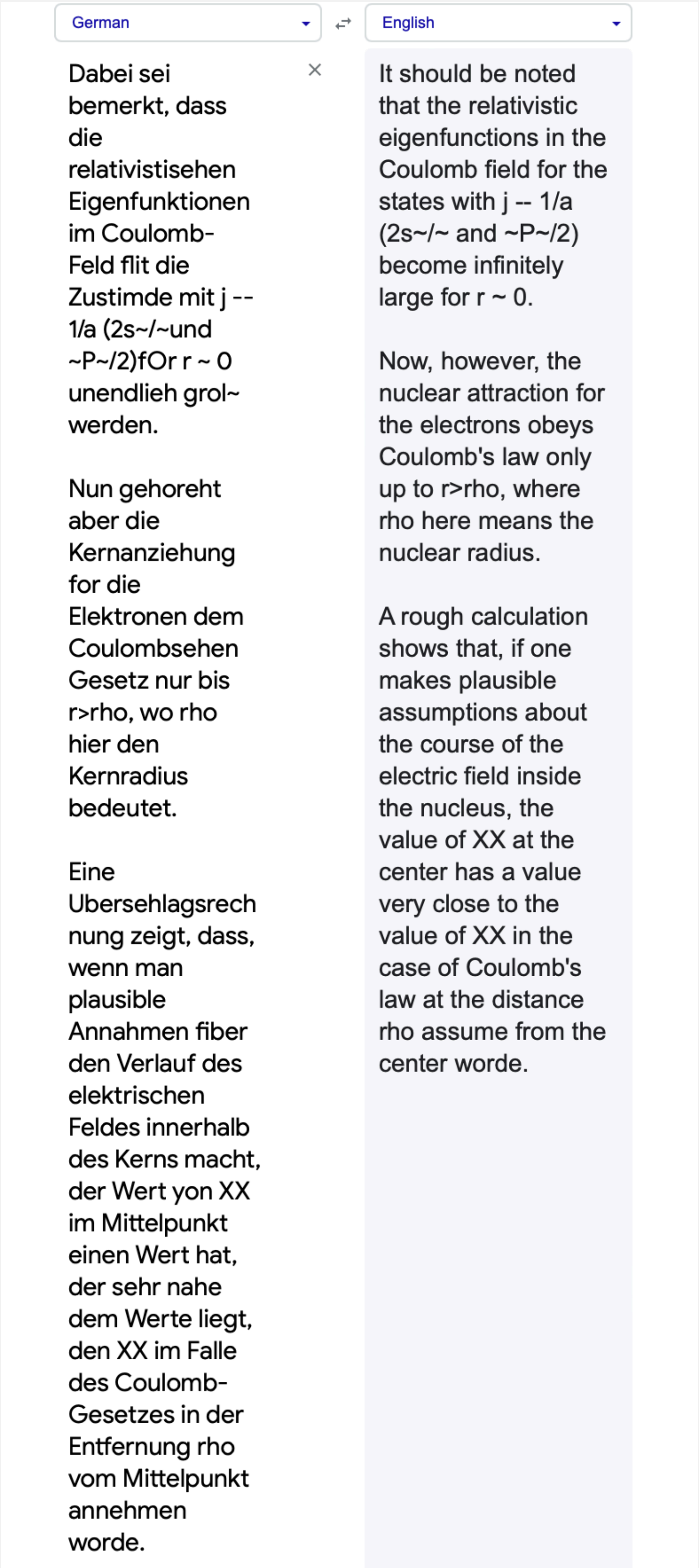
- how to combine with phenomenologically important subleading corrections? (answer: factorization, EFT, symmetry relating different powers of Z at the same power of α)

- what is the “Fermi function” for neutron beta decay, for which neither Z nor β^{-1} is large? (answer: RG analysis to resum $\left(i\pi \log \frac{-\vec{p}^2 - i0}{\vec{p}^2} \right)^n = \pi^{2n}$)

Fermi function

- what is the quantity r appearing in F(Z,E)?

$$F(Z,E) = \frac{2(1+\eta)}{[\Gamma(2\eta+1)]^2}\,|\,\Gamma(\eta+i\xi)\,|^2\, e^{\pi\xi}(2pr)^{2(\eta-1)}$$



E. Fermi, An attempt of a theory of beta radiation. 1. Z.Phys. 88 (1934)

“a rough estimate shows that … “

⇒need a systematic understanding

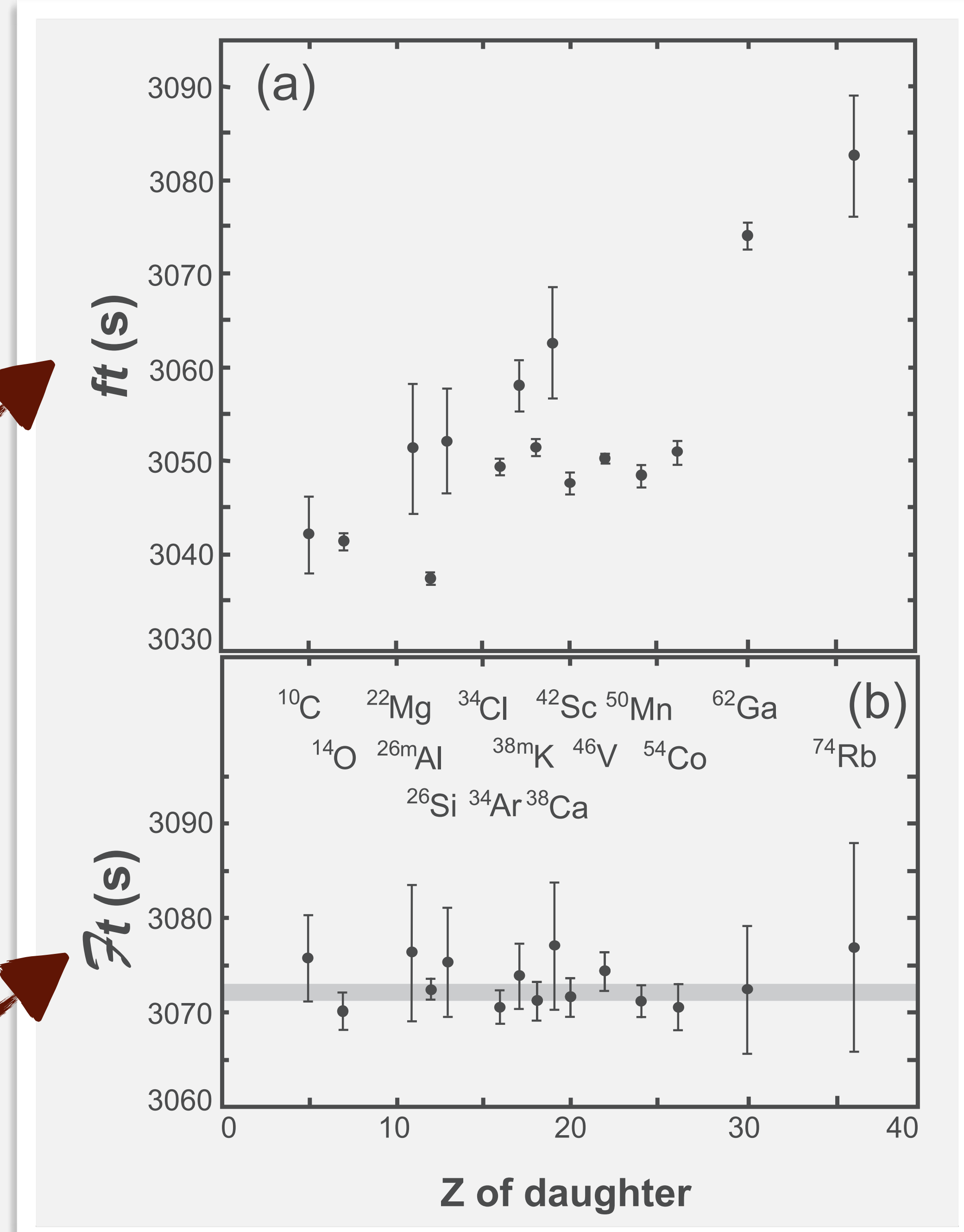
Fermi function

- Subleading corrections are critical at the sub-per mille experimental precision. How are these incorporated?

$$\mathscr{F}t = \frac{\text{const}\,.}{G_F^2 |V_{ud}|^2}$$

including just Fermi function

including other Z-dependent radiative corrections

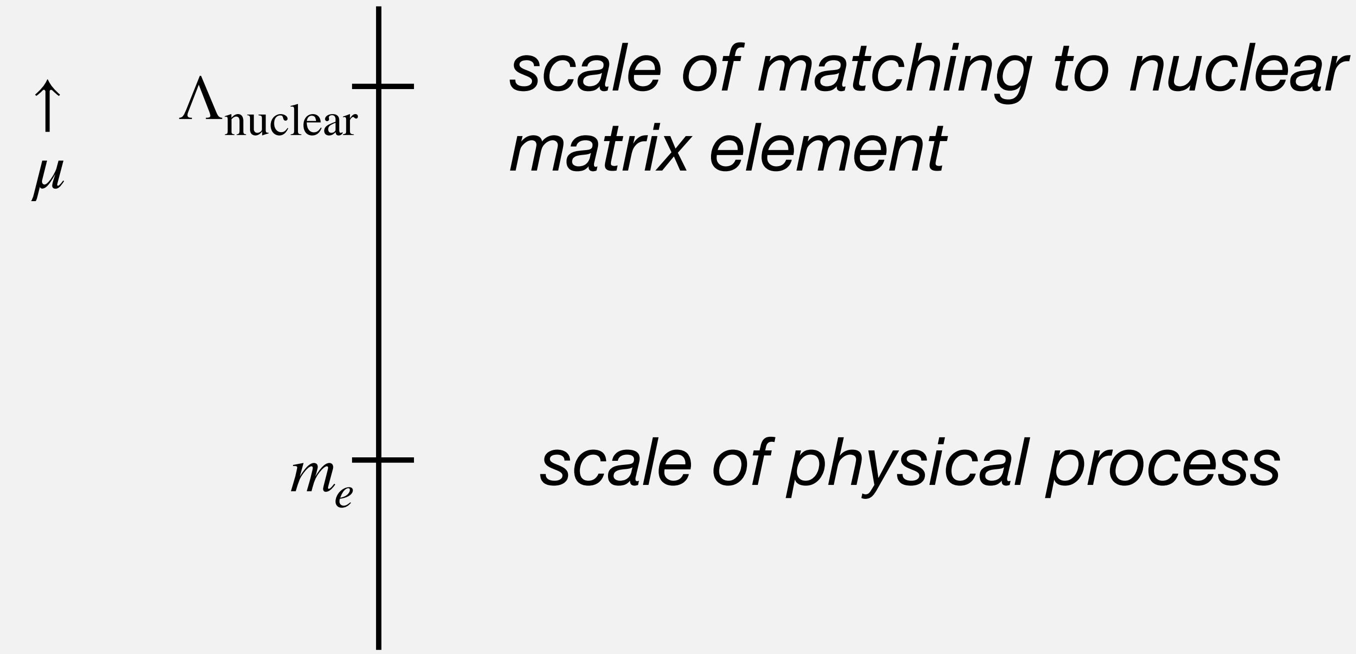


Hardy and Towner summary PRC (2020)

and Z-independent radiative corrections
Czarnecki, Marciano, Sirlin 1907.06737, …

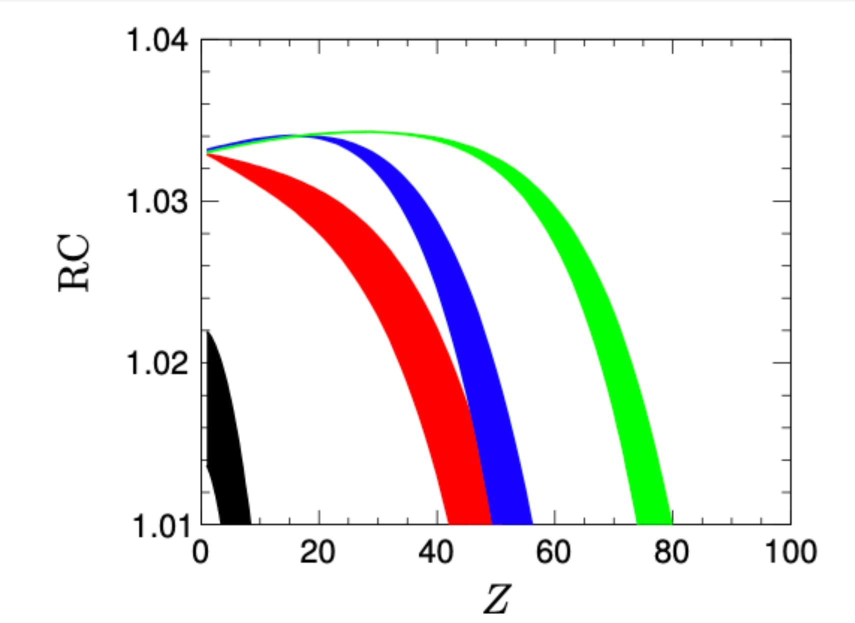
Fermi function

- Subleading corrections are critical at the sub-per mille experimental precision. How are these incorporated?



Systematically perform renormalization analysis from scale of matching ($\mu = \Lambda_{\text{nuclear}}$) to scale of process ($\mu \sim p \sim m_e$)

Fermi function



convenient counting for nuclei determining V_{ud}:

$$Z^2 \sim L^2 \equiv \log^2(\Lambda/m) \sim \alpha^{-1}$$

- ➡ $\alpha L, Z^2\alpha^2 L \sim \alpha^{\frac{1}{2}}$
- ➡ $Z\alpha^2 L \sim \alpha$
- ➡ $\alpha^2 L, Z^2\alpha^3 L, Z^4\alpha^4 L \sim \alpha^{\frac{3}{2}}$

⇒ need three and four loops for per mille precision

Factorization

When matrix elements are computed for the beta decay process, large perturbative coefficients appear

$$\alpha^{-1} \sim \log^2 \frac{\Lambda_{\text{nuc}}}{m_e} \sim Z^2 \sim 100$$

For example, super allowed nuclear beta decay provides most precise determination of V_{ud}

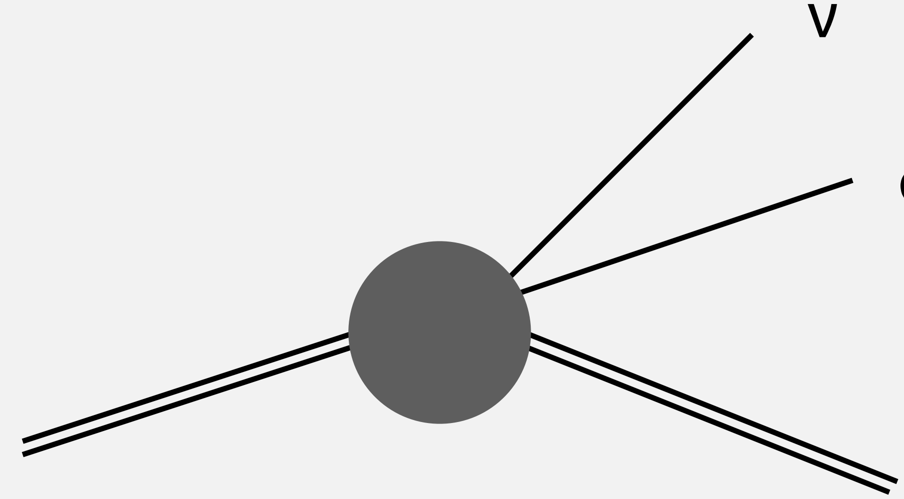
$$\delta |V_{ud}| \sim 3 \times 10^{-4}$$

Require high-order perturbative corrections, e.g.

$$Z^2 \alpha^3 \log \frac{\Lambda_{\text{nuc.}}}{m_e} \sim 10^{2-6+1} = 10^{-3}$$

Factorization

Map the problem to effective field theory



$$\mathscr{L}_{\rm eff} = -\,\mathscr{C}\,(\phi_v^{[A,Z+1]})^* \phi_v^{[A,Z]}\,\bar{e} v_\mu \gamma^\mu (1-\gamma_5)\nu_e + \mathrm{H.c.}$$

$v^\mu = (1,0,0,0)$ is four-velocity of the heavy nucleus, $\mathscr{C} \sim G_F$

Factorization

Resum large logarithms by renormalization group within a sequence of effective field theories

$$\mathscr{M} = \mathscr{M}_S(\lambda_{\mathrm{IR}})\mathscr{M}_H(m,p)\mathscr{M}_{\mathrm{UV}}(\Lambda_{\mathrm{UV}})$$

matching coefficient onto heavy-heavy-light EFT

matching coefficient onto heavy-heavy-heavy EFT

soft matrix element in heavy-heavy-heavy EFT

Factorization

Consider the factorization at leading power in Z

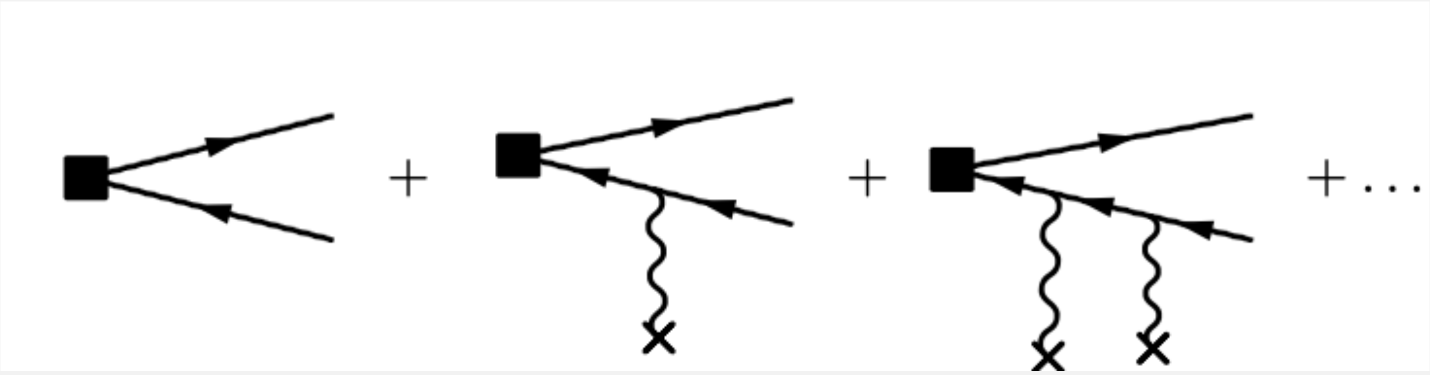
Start with the Schrodinger Coulomb problem (i.e., NR limit)

$$\mathcal{M} = \sum_{n=0}^{\infty} \mathcal{M}^{(n)} = \sum_{n=0}^{\infty} (2mZe^2)^n \int \frac{d^D L_1}{(2\pi)^D} \int \frac{d^D L_2}{(2\pi)^D} \cdots \int \frac{d^D L_n}{(2\pi)^D}$$

$$\frac{1}{\vec{L}_1^2 + \lambda^2} \frac{1}{(\vec{L}_1 - \vec{p})^2 - \vec{p}^2 - i0} \frac{1}{(\vec{L}_1 - \vec{L}_2)^2 + \lambda^2} \frac{1}{(\vec{L}_2 - \vec{p})^2 - \vec{p}^2 - i0}$$

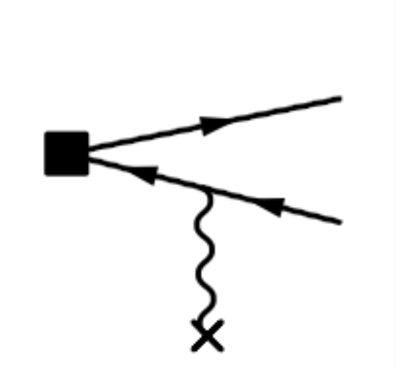
$$\cdots \frac{1}{(\vec{L}_{n-1} - \vec{L}_n)^2 + \lambda^2} \frac{1}{(\vec{L}_n - \vec{p})^2 - \vec{p}^2 - i0}$$

$D = 3 - 2\epsilon$



Factorization

At one loop:



$$\mathcal{M}^{(1)} = 2mZe^2 \int \frac{d^D L_1}{(2\pi)^D} \frac{1}{\vec{L}^2 + \lambda^2} \frac{1}{(\vec{L} - \vec{p})^2 - \vec{p}^2 - i0} \rightarrow \frac{im}{p} \frac{Ze^2}{4\pi} \left(\log \frac{2p}{\lambda} - \frac{i\pi}{2} \right)$$

Decompose into momentum regions

$$\mathcal{M}_S^{(1)} = \int \frac{d^d L_1}{(2\pi)^d} \frac{Ze^2}{L^2 + \lambda^2} \frac{2m}{-2\vec{p} \cdot \vec{L} - i0} = \frac{iZ\alpha}{\beta} \left(\frac{\lambda^2}{\mu^2} \right)^{-\epsilon} \frac{1}{2\epsilon}$$

$$\mathcal{M}_H^{(1)} = \int \frac{d^d L}{(2\pi)^d} \frac{Ze^2}{L^2} \frac{2m}{(\vec{L} - \vec{p})^2 - p^2 \cdot \vec{L} - i0} = \frac{iZ\alpha}{\beta} \left(\frac{-4p^2}{\mu^2 - i0} \right)^{-\epsilon} \frac{-1}{2\epsilon}$$

Readily see that factorization holds through one-loop order

$$\mathcal{M} = \mathcal{M}_S \mathcal{M}_H = 1 + \mathcal{M}_S^{(1)} + \mathcal{M}_H^{(1)} + \dots$$

Factorization

Remember an integration identity

$$\int d^n x \, \frac{\partial}{\partial x^i} f(x_1, x_2, \ldots, x_n) = 0$$

Apply this to Feynman diagram integrals,

$$J(a_1,a_2,a_3,a_4,a_5) = \int \frac{d^dK}{(2\pi)^d} \int \frac{d^dL}{(2\pi)^d} \frac{1}{[\mathbf{K}^2]^{a_1}} \frac{1}{[(\mathbf{p}-\mathbf{K})^2-\mathbf{p}^2]^{a_2}} \frac{1}{[\mathbf{L}^2]^{a_3}} \frac{1}{[(\mathbf{p}-\mathbf{L})^2-\mathbf{p}^2]^{a_4}} \frac{1}{[(\mathbf{L}-\mathbf{K})^2]^{a_5}}$$

Insert: $\frac{\partial}{\partial K^i} K^i \qquad \frac{\partial}{\partial K^i} L^i$

$$0 = d - a_1 - a_2 - 2a_5 - a_1 \mathbf{1}^+(\mathbf{5}^- - \mathbf{3}^-) - a_2 \mathbf{2}^+(\mathbf{5}^- - \mathbf{4}^-)$$

E.g., apply this to J(0,1,1,1,1):

$$J(0,1,1,1,1) = \frac{1}{d-3}\left[J(0,2,1,1,0) - J(0,2,1,0,1)\right]$$

simpler integrals

Factorization

We can continue to higher order,

$$\mathcal{M}_H^{(2)} = \int \frac{d^d L_1}{(2\pi)^d} \int \frac{d^d L_2}{(2\pi)^d} \frac{Ze^2}{L_1^2} \frac{2m}{(\vec{L}_1 - \vec{p})^2 - p^2 - i0} \frac{Ze^2}{(\vec{L}_1 - \vec{L}_2)^2} \frac{2m}{(\vec{L}_2 - \vec{p})^2 - p^2 - i0}$$

$$= (2mZe^2)^2 J(0,1,1,1,1)$$

$$= \left[\frac{iZ\alpha}{\beta}(-4p^2/\mu^2 - i0)^{-\epsilon}\right]^2 \left[\frac{1}{8\epsilon^2} + \frac{\pi^2}{12}\right]$$

$$\mathcal{M}_H^{(3)} = \left[\frac{iZ\alpha}{\beta}(-4p^2/\mu^2 - i0)^{-\epsilon}\right]^3 \left[\frac{-1}{48\epsilon^3} - \frac{\pi^2}{24\epsilon} - \frac{13\zeta(3)}{6}\right]$$

- Even with the tricks of dimensional regularization, these integrals become increasing difficult at high loop order: 4 loops, 5, loops, etc.
- How about 48 loops?

Factorization

Return to the position-space picture

$$H = \frac{p^2}{2m} - \frac{Z\alpha}{r} e^{-\lambda r}$$

Solve for wavefunction

$$\mathcal{M} = [\psi^{(-)}(0)]^* = \Gamma\left(1 - \frac{iZ\alpha}{\beta}\right) \exp\left[\frac{Z\alpha}{\beta}\left(\frac{\pi}{2} + i\log\frac{2p}{\lambda} - i\gamma_{\mathrm{E}}\right)\right] + \mathcal{O}\left(\frac{\lambda}{p}\right)$$

Since we know the soft function to all orders (exponentiation), we also know the hard function to all orders:

$$\mathcal{M}_H(\mu) = \frac{\mathcal{M}}{\mathcal{M}_S(\mu)} = \Gamma\left(1 - \frac{iZ\alpha}{\beta}\right) \exp\left[\frac{Z\alpha}{\beta}\left(\frac{\pi}{2} + i\log\frac{2p}{\mu} - i\gamma_{\mathrm{E}}\right)\right]$$

$$= 1 + \frac{Z\alpha}{\beta}\left(\frac{\pi}{2} + i\log\frac{2p}{\mu}\right) + \left(\frac{Z\alpha}{\beta}\right)^2\left(\frac{\pi^2}{24} + \frac{i\pi}{2}\log\frac{2p}{\mu} - \frac{1}{2}\log^2\frac{2p}{\mu}\right)$$

(check: matches with order by order results)

Explicit, all-orders factorization.

Factorization

Wavefunction computation

Recall the Lippmann-Schwinger equation and Born series,

$$\psi_{\vec{p}}^{(\pm)}(\vec{x}) = \langle \vec{x} | \left(1 + \frac{1}{E - \hat{H}_0 \pm i0} \hat{V} + \frac{1}{E - \hat{H}_0 \pm i0} \hat{V} \frac{1}{E - \hat{H}_0 \pm i0} \hat{V} + \ldots \right) | \vec{p} \rangle$$

Coulomb with massive photon,

$$V(\vec{x}) = (-Ze^2) \frac{\exp(-\lambda |\vec{x}|)}{4\pi |\vec{x}|} \qquad \tilde{V}(\vec{L}) = \frac{-Ze^2}{\vec{L}^2 + \lambda^2}$$

$$\psi_{\vec{p}}^{(\pm)}(\vec{x}) = e^{i\vec{p}\cdot\vec{x}} \left[1 + \int \frac{d^3L}{(2\pi)^3} e^{i\vec{L}\cdot\vec{x}} \frac{-2m}{2\vec{p}\cdot\vec{L} + \vec{L}^2 \mp i0} \tilde{V}(\vec{L}) \right.$$

$$\left. + \int \frac{d^3L_1}{(2\pi)^3} \frac{d^3L_2}{(2\pi)^3} e^{i\vec{L}_2\cdot\vec{x}} \frac{-2m}{2\vec{p}\cdot\vec{L}_2 + \vec{L}_2^2 \mp i0} \tilde{V}(\vec{L}_2 - \vec{L}_1) \frac{-2m}{2\vec{p}\cdot\vec{L}_1 + \vec{L}_1^2 \mp i0} \tilde{V}(\vec{L}_1) + \ldots \right]$$

We require $\psi_{\vec{p}}^{(-)}(\vec{0})$

Factorization

Let us solve the Schrodinger equation,

$$\left[-\frac{1}{2m}\nabla^2 - \frac{Z\alpha}{r}e^{-\lambda r}\right]\psi(\vec{x}) = \frac{\vec{p}^2}{2m}\psi(\vec{x})$$

Write

$$\psi_{\vec{p}}(\vec{x};\lambda) = e^{i\vec{p}\cdot\vec{x}}F_{\vec{p}}(\vec{x},\lambda)$$

Then

$$\left[-\frac{1}{2}\nabla^2 - i\vec{p}\cdot\nabla - \frac{mZ\alpha}{r}e^{-\lambda r}\right]F(\vec{x}) = 0$$

For $r \ll \lambda^{-1}$

$$\left[-\frac{1}{2}\frac{\nabla^2}{p^2} - i\frac{\hat{\vec{p}}\cdot\vec{\nabla}}{p} - \frac{\xi}{pr}\right]F_< = 0 \qquad F_<^{(+)}(\vec{x}) = N(p,\lambda)\,{}_1F_1(\,i\xi,1,ip(r-z)\,)$$

For $r \gg p^{-1}$

$$\left[-i\frac{\hat{\vec{p}}\cdot\vec{\nabla}}{\lambda} - \frac{\xi}{\lambda r}e^{-\lambda r}\right]F_> = 0 \qquad F_>^{(+)}(\vec{x}) = \exp\left[i\xi\int_{-\infty}^{z} dz'\,,\frac{e^{-\lambda\sqrt{z'^2+r^2-z^2}}}{\sqrt{z'^2+r^2-z^2}}\right]$$

Factorization

In the overlap region, $p^{-1} \ll r \ll \lambda^{-1}$, both solutions apply,

$$F_{<}^{(+)} \to N(p,\lambda)\frac{1}{\Gamma(1-i\xi)}\exp\left\{-\frac{\pi\xi}{2} - i\xi\log[p(r-z)]\right\},$$

$$F_{>}^{(+)} \to \exp\left\{i\xi\left[-\log\frac{\lambda(r-z)}{2} - \gamma_{\rm E}\right]\right\}$$

Enforcing equality determines $N(p,\lambda)$, and then

$$\psi_{\vec{p}}^{(+)}(\vec{x}=0) = N(p,\lambda) = \Gamma(1-i\xi)\exp\left\{\frac{\pi}{2}\xi + i\xi\left[\log\frac{2p}{\lambda} - \gamma_{\rm E}\right]\right\}$$

Recall that this all-orders amplitude, combined with the all-orders soft function, determines the all-orders hard function

Factorization

Extend the factorization formalism to field theory: full QED with relativistic and quantum corrections for the electron

$$\mathcal{M}^{(1)} = 2EZe^2 \int \frac{d^d L}{(2\pi)^d} \frac{1}{\mathbf{L}^2 + \lambda^2} \frac{1}{(\mathbf{L}-\mathbf{p})^2 - \mathbf{p}^2 - i0} \left[1 - \frac{1}{2E}\gamma^0 \not{L}\right]$$
$$= \frac{iZ\alpha}{\beta} \left[\left(\log\frac{2p}{\lambda} - \frac{i\pi}{2}\right) + \left(\frac{m\gamma^0}{E} - 1\right)\left(-\frac{1}{2}\right)\right].$$

at one loop: two relevant momentum regions, $L \sim p$ (hard) and $L \sim \lambda$ (soft)

at two loops: contributions from $L \gg p$

General factorization formula:

$$\mathcal{M} = \mathcal{M}_S \mathcal{M}_H \mathcal{M}_{\rm UV}$$

Factorization

Using asymptotic wavefunction methods, can extract hard function to all orders:

$$\mathscr{M}_H(\mu_S, \mu_H) = \mathscr{M}_S^{-1}(\mu_S)\, \mathscr{M}\, \mathscr{M}_{\mathrm{UV}}^{-1}(\mu_H)$$

$$\mathscr{M}_H = \exp\left[i\xi \log \frac{2pe^{-\gamma_E}}{\mu_S} - i(\eta - 1)\frac{\pi}{2} \right] \frac{2\Gamma(\eta - i\xi)}{\Gamma(2\eta + 1)} \sqrt{\frac{\eta - i\xi}{1 - i\xi\frac{m}{E}}} \sqrt{\frac{E + \eta m}{E + m}} \sqrt{\frac{2\eta}{1 + \eta}} \times$$
$$\times \left(\frac{2pe^{-\gamma_E}}{\mu_H} \right)^{\eta - 1} \left[\frac{1 + \gamma^0}{2} + \frac{E + m}{E + \eta m} \left(1 - i\xi \frac{m}{E} \right) \frac{1 - \gamma^0}{2} \right]$$

$$\xi = \frac{Z\alpha}{\beta}$$

$$\eta = \sqrt{1 - (Z\alpha)^2}$$

⇒ explicit, all-orders factorization (!)

$$\mathscr{M} = \mathscr{M}_S(\lambda/\mu)\mathscr{M}_H(p/\mu)\mathscr{M}_{\mathrm{UV}}(\Lambda/\mu)$$

Factorization

With the explicit factorization formula,

- what is F(Z,E) as a field theory object? *answer: leading-in-Z hard function*

$$\left\langle |\mathscr{M}_H|^2 \right\rangle = F(Z,E)\Big|_{r_H} \times \frac{4\eta}{(1+\eta)^2}$$

- what is the quantity r appearing in F(Z,E)? Approximately the nuclear radius, but how to beyond this qualitative model? *answer: renormalization scale*

$$r^{-1} e^{\gamma_E} = \mu_{\overline{\rm MS}}$$

$$\mathscr{M}_H = \exp\left[i\xi \log \frac{2pe^{-\gamma_E}}{\mu_S} - i(\eta-1)\frac{\pi}{2} \right] \frac{2\Gamma(\eta - i\xi)}{\Gamma(2\eta+1)} \sqrt{\frac{\eta - i\xi}{1 - i\xi\frac{m}{E}}} \sqrt{\frac{E+\eta m}{E+m}} \sqrt{\frac{2\eta}{1+\eta}} \times$$
$$\times \left(\frac{2pe^{-\gamma_E}}{\mu_H} \right)^{\eta-1} \left[\frac{1+\gamma^0}{2} + \frac{E+m}{E+\eta m}\left(1 - i\xi\frac{m}{E}\right)\frac{1-\gamma^0}{2} \right]$$

Renormalization

When matrix elements are computed for the beta decay process, large perturbative coefficients appear

$$\alpha^{-1} \sim \log^2 \frac{\Lambda_{\rm nuc}}{m_e} \sim Z^2 \sim 100$$

Account for log enhancements by RG evolution

$$\frac{d \log C}{d \log \mu} \sim \begin{array}{c} \alpha (Z+1) \\ +\alpha^2 \left(Z^2 + Z + 1\right) \\ +\alpha^3 \left(Z^3 + Z^2 + Z + 1\right) \\ +\alpha^4 \left(Z^4 + Z^3 + Z^2 + Z + 1\right) \end{array}$$

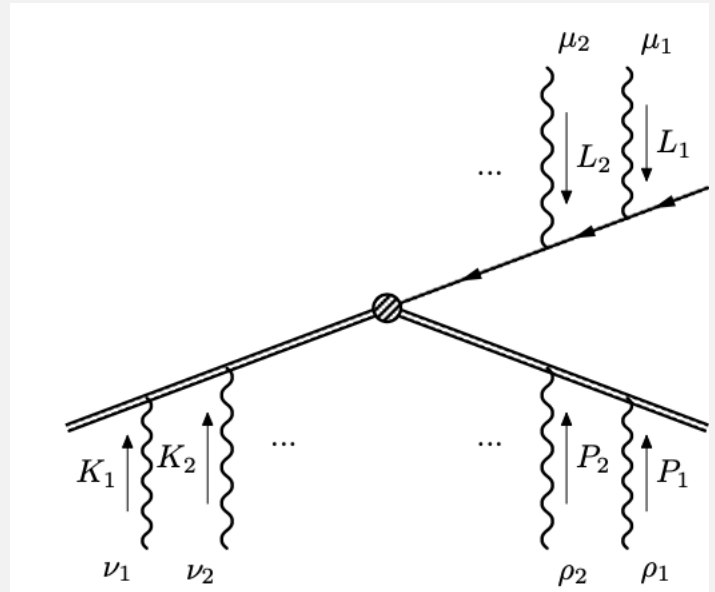
Leading Z given by Dirac equation/Fermi function analysis

Z=0 limit given by heavy-light current operator

Subleading Z determined by leading Z (new!) using heavy particle symmetry

Renormalization

Symmetry argument



$$= \left[\cdots \frac{(-eQ_e)(\not{L}_1 + \not{L}_2)\gamma^{\mu_2}}{(L_1 + L_2)^2 + i0} \frac{(-eQ_e)\not{L}_1\gamma^{\mu_1}}{L_1^2 + i0} \right]$$

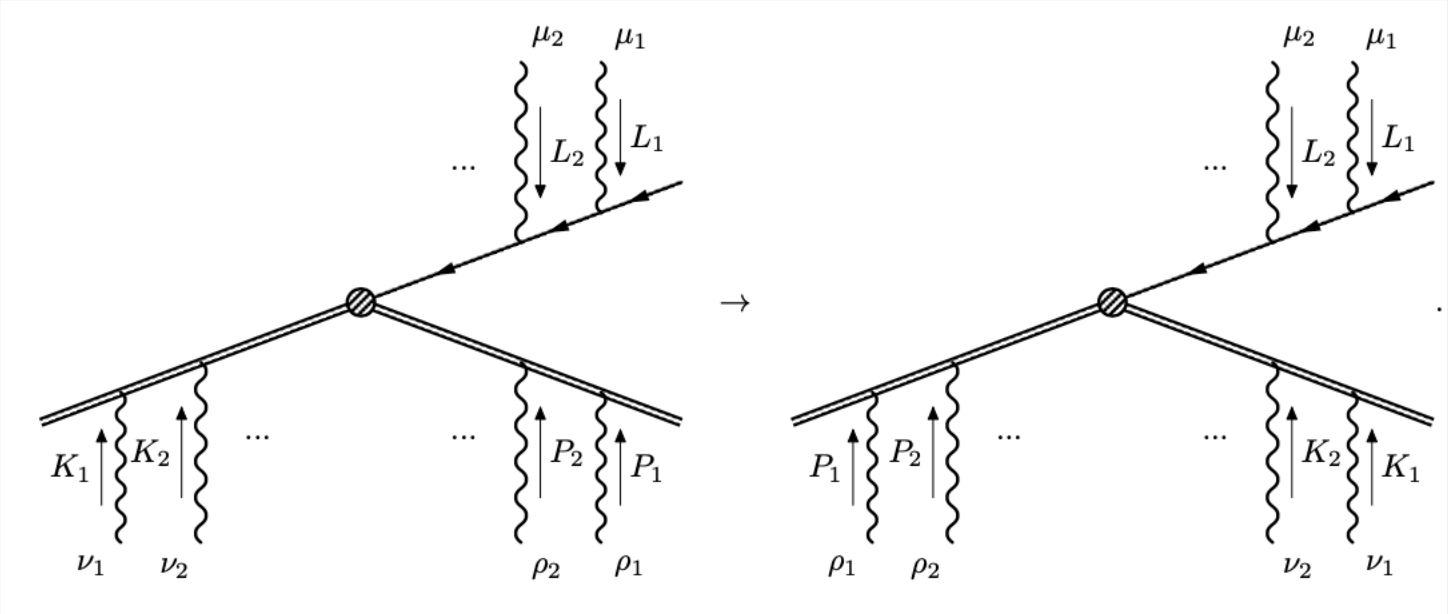
$$\left[\cdots \frac{-eQ_A v^{\nu_2}}{v \cdot (K_1 + K_2) + i0} \frac{-eQ_A v^{\nu_1}}{v \cdot K_1 + i0} \right]$$

$$\left[\cdots \frac{-eQ_B v^{\rho_1}}{v \cdot (-P_1) + i0} \frac{-eQ_B v^{\rho_2}}{v \cdot (-P_1 - P_2) + i0} \right]$$

sum over diagrams is invariant under $Q_A \leftrightarrow Q_B$, $Q_e \to -Q_e$

Renormalization

See that e.g.



and in the sum over diagrams at a fixed order in α, the amplitude is invariant

Can see that the anomalous dimension must be built from

$$Q_e^2\,, \quad Q_A Q_B\,, \quad Q_e(Q_A - Q_B)$$

In particular, with $Q_e = -1\,, \quad Q_A = Z+1\,, \quad Q_B = Z$

the anomalous dimension at n loop order is a linear combination of

$$Z^i(Z+1)^i\,, \quad 2i \leq n$$

Renormalization

$$\frac{d\log C}{d\log\mu} \sim \begin{array}{l} \alpha\,(Z+1) \\ +\alpha^2\left(Z^2+Z+1\right) \\ +\alpha^3\left(Z^3+Z^2+Z+1\right) \\ +\alpha^4\left(Z^4+Z^3+Z^2+Z+1\right) \end{array}$$

$$\to \begin{array}{l} \alpha\,(1) \\ +\alpha^2\left(Z(Z+1)+1\right) \\ +\alpha^3\left(\#Z(Z+1)+1\right) \\ +\alpha^4\left(Z^2(Z+1)^2+\#Z(Z+1)+1\right) \end{array}$$

we calculate this number

remaining undetermined coefficient at 4 loops

$$\gamma = \frac{d \log \mathscr{C}}{d \log \mu} = \sum_{n=0}^{\infty} \sum_{i=0}^{n+1} \left(\frac{\alpha}{4\pi} \right)^{n+1} \gamma_n^{(i)} Z^{n+1-i} \equiv \gamma^{(0)}(Z\alpha) + \alpha\gamma^{(1)}(Z\alpha) + \ldots$$

Z^n \ Loops	1-loop	2-loop	3-loop	4-loop
Z^0				
Z^1				
Z^2				
Z^3				
Z^4				

$$\gamma = \frac{d\log \mathscr{C}}{d\log \mu} = \sum_{n=0}^{\infty}\sum_{i=0}^{n+1}\left(\frac{\alpha}{4\pi}\right)^{n+1}\gamma_n^{(i)} Z^{n+1-i} \equiv \gamma^{(0)}(Z\alpha) + \alpha\gamma^{(1)}(Z\alpha) + \ldots$$

Loops / Z^n	1-loop	2-loop	3-loop	4-loop
Z^0				
Z^1				
Z^2				
Z^3				
Z^4				

- no contributions $Z^m\alpha^n$ with $m > n$

$$\gamma = \frac{d \log \mathscr{C}}{d \log \mu} = \sum_{n=0}^{\infty} \sum_{i=0}^{n+1} \left(\frac{\alpha}{4\pi}\right)^{n+1} \gamma_n^{(i)} Z^{n+1-i} \equiv \gamma^{(0)}(Z\alpha) + \alpha\gamma^{(1)}(Z\alpha) + \ldots$$

Z^n \ Loops	1-loop	2-loop	3-loop	4-loop
Z^0				
Z^1				
Z^2	–			
Z^3	–	–		
Z^4	–	–	–	

- no contributions $Z^m \alpha^n$ with $m > n$

$$\gamma = \frac{d \log \mathscr{C}}{d \log \mu} = \sum_{n=0}^{\infty} \sum_{i=0}^{n+1} \left(\frac{\alpha}{4\pi} \right)^{n+1} \gamma_n^{(i)} Z^{n+1-i} \equiv \gamma^{(0)}(Z\alpha) + \alpha\gamma^{(1)}(Z\alpha) + \ldots$$

Loops / Z^n	1-loop	2-loop	3-loop	4-loop
Z^0				
Z^1				
Z^2	–			
Z^3	–	–		
Z^4	–	–	–	

- no contributions $Z^m \alpha^n$ with $m > n$
- leading Dirac solution

$$\gamma = \frac{d\log\mathscr{C}}{d\log\mu} = \sum_{n=0}^{\infty}\sum_{i=0}^{n+1}\left(\frac{\alpha}{4\pi}\right)^{n+1}\gamma_n^{(i)}Z^{n+1-i} \equiv \gamma^{(0)}(Z\alpha) + \alpha\gamma^{(1)}(Z\alpha) + \ldots$$

Loops / Z^n	1-loop	2-loop	3-loop	4-loop
Z^0				
Z^1	$\gamma_0^{(0)} = 0$			
Z^2	–	$\gamma_1^{(0)} = -8\pi^2$		
Z^3	–	–	$\gamma_2^{(0)} = 0$	
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- leading Dirac solution

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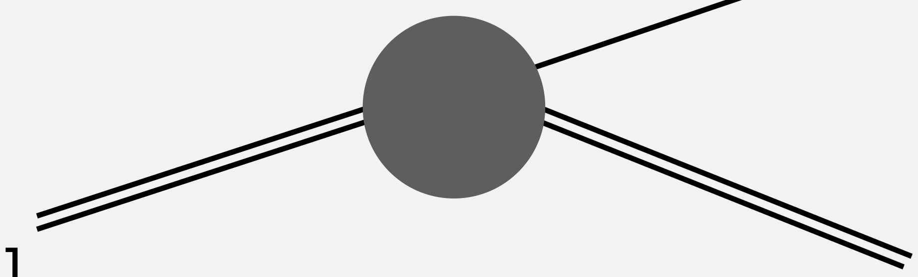
$\Rightarrow$ need $\gamma_2^{(1)}$ as missing ingredient for permille level analysis of beta decay

Renormalization

To isolated powers of Z, it is convenient to rearrange the perturbation series



$$\mathscr{L} = \bar{h}_v^{(A)}(iv \cdot \partial + e(Z+1)v \cdot A)h_v^{(A)} + \bar{h}_v^{(B)}(iv \cdot \partial + eZv \cdot A)h_v^{(B)}$$

$$+G_F \; h_v^{(B)}\Gamma h_v^{(A)} \; \bar{\nu}\Gamma' e$$

Introduce Wilson line field redefinition to (almost) decouple photons from heavy particles

$$h_v^{(A)} = S_v h_v^{(A0)} \qquad h_v^{(B)} = S_v h_v^{(B0)}$$

$$\mathscr{L} = \bar{h}_v^{(A0)}(iv \cdot \partial + ev \cdot A)h_v^{(A0)} + \bar{h}_v^{(B)} iv \cdot \partial h_v^{(B)}$$

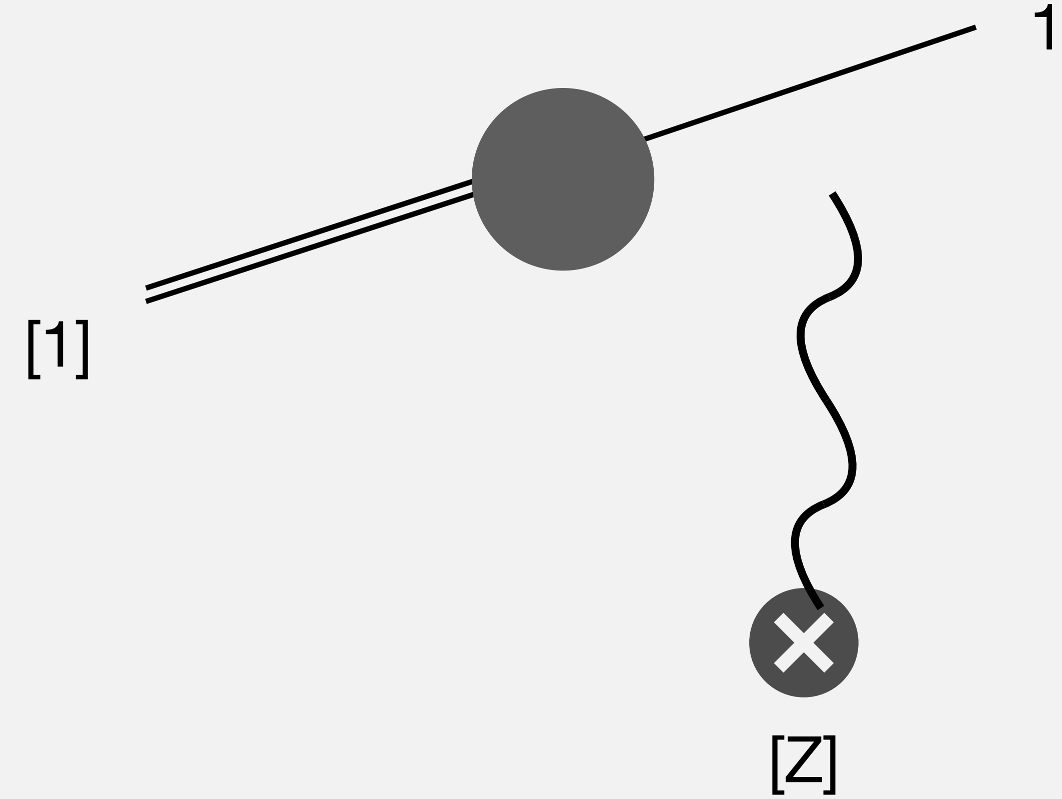
$$+G_F \; S_v^\dagger S_v \; h_v^{(B0)}\Gamma h_v^{(A0)} \; \bar{\nu}\Gamma' e$$

Renormalization

Equivalent picture in terms of charge 1 heavy particle in background charge Z field

$$S_v(x) = \exp\left[iZe \int_{-\infty}^{0} ds\, v \cdot A(x+sv) \right] = \exp\left[iZe \left(\frac{i}{iv\cdot\partial + i0} v \cdot A \right) \right]$$

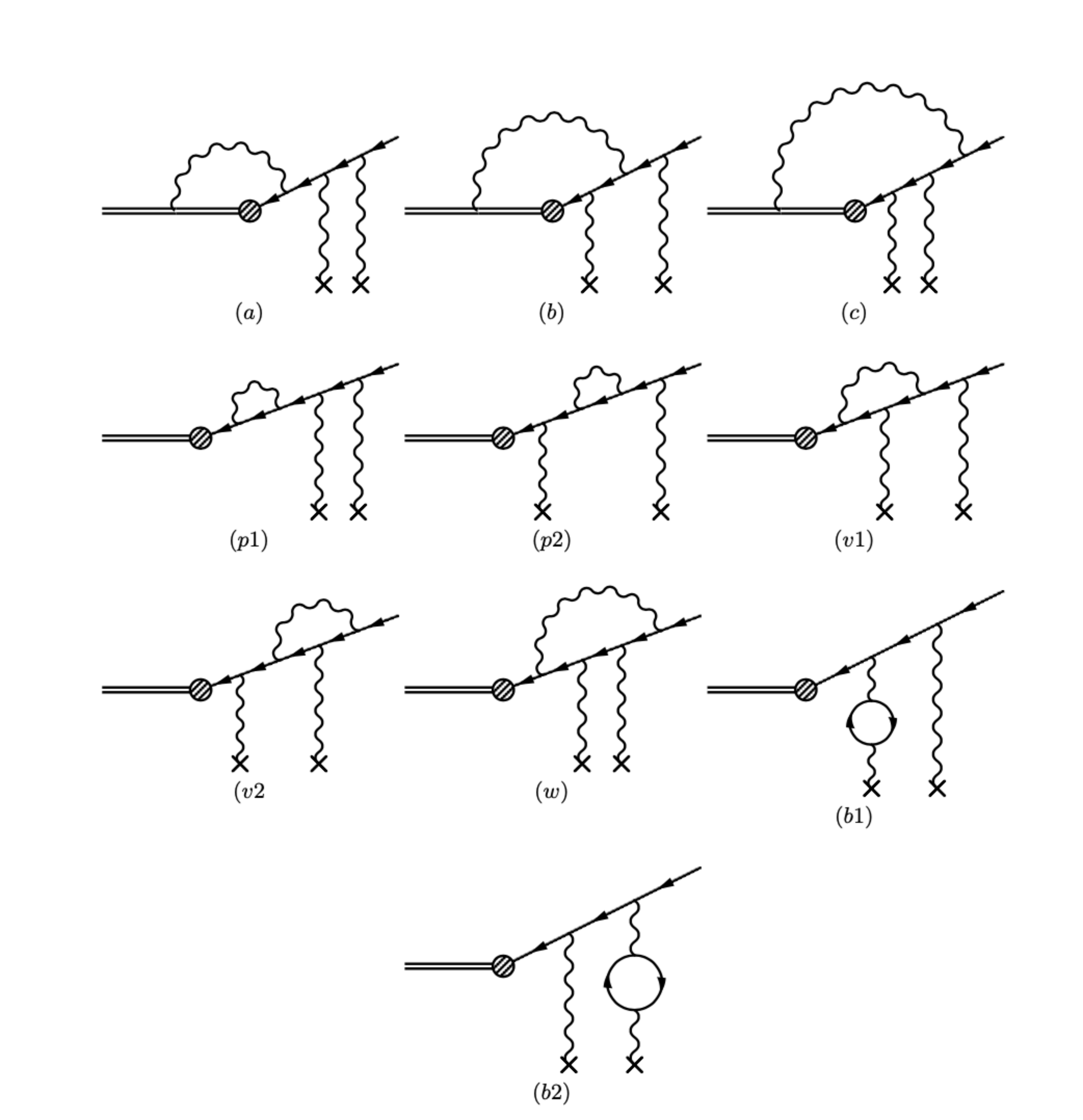
$$S_v^\dagger S_v = \exp\left[iZe \left(2\pi\delta(iv\cdot\partial) v \cdot A \right) \right]$$



Simplified calculation at fixed Z, and useful/interesting relations between different powers of Z obtained by equating the two pictures

Renormalization

With this rearrangement, 3-loop computation reduced to 10 diagrams



basic idea:

- reduce to basis integrals

$$I^{(b)}(a_1, a_2, a_3, a_4, a_5, a_6)$$
$$= \int (\mathrm{d}\omega)(\mathrm{d}k)(\mathrm{d}q)(\mathrm{d}p)\, \omega^b \frac{1}{[\omega^2 + (\boldsymbol{k}+\boldsymbol{p})^2]^{a_1}} \frac{1}{(\mathbf{p}^2)^{a_2}} \frac{1}{[(\boldsymbol{p}-\boldsymbol{q})^2]^{a_3}} \frac{1}{[\omega^2 + (\boldsymbol{k}+\boldsymbol{q})^2]^{a_4}} \frac{1}{(\mathbf{q}^2)^{a_5}} \frac{1}{\mathbf{q}^2+\lambda^2} \times$$
$$\times \frac{1}{(\omega^2 + \boldsymbol{k}^2)^{a_6}} \frac{1}{\omega^2 + \boldsymbol{k}^2 + \lambda^2},$$

- subset of integrals involving two momentum differences, evaluated by

1) use that λ (IR regulator) is the only scale

$$I \sim \lambda^{-6\epsilon} \quad \Longrightarrow \quad I = \frac{-1}{6\epsilon} \lambda \frac{d}{d\lambda} I$$

2) isolate and evaluate sub divergences

3) evaluate remaining finite coefficient of $1/\epsilon$ at $\epsilon \to 0$

- remaining integrals reduced to previous step using integration by parts identities

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Z^2	–	$\gamma_1^{(0)} = -8\pi^2$	$\gamma_2^{(1)} = 16\pi^2\left(6 - \frac{\pi^2}{3}\right)$	$\gamma_3^{(2)}$
Z^3	–	–	$\gamma_2^{(0)} = 0$	$\gamma_3^{(1)} = 2\gamma_3^{(0)}$
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- we disagree with an old result in the literature [Jaus and Rasche, PRD 35, 3420 (1987)]. Only diagrams (a), (b), (c) considered. Unregulated subdivergences.

Implications

Numerically important modifications to nuclear beta decay rates

transition	$(\Delta a) \times Z^2 \alpha^3 \log(\Lambda/m)$
$^{10}\mathrm{C} \to {}^{10}\mathrm{B}$	-0.6×10^{-4}
$^{14}\mathrm{O} \to {}^{14}\mathrm{N}$	-1.1×10^{-4}
$^{26m}\mathrm{Al} \to {}^{26}\mathrm{Mg}$	-3.2×10^{-4}
$^{46}\mathrm{V} \to {}^{46}\mathrm{Ti}$	-10.5×10^{-4}
$^{54}\mathrm{Co} \to {}^{54}\mathrm{Fe}$	-14.6×10^{-4}

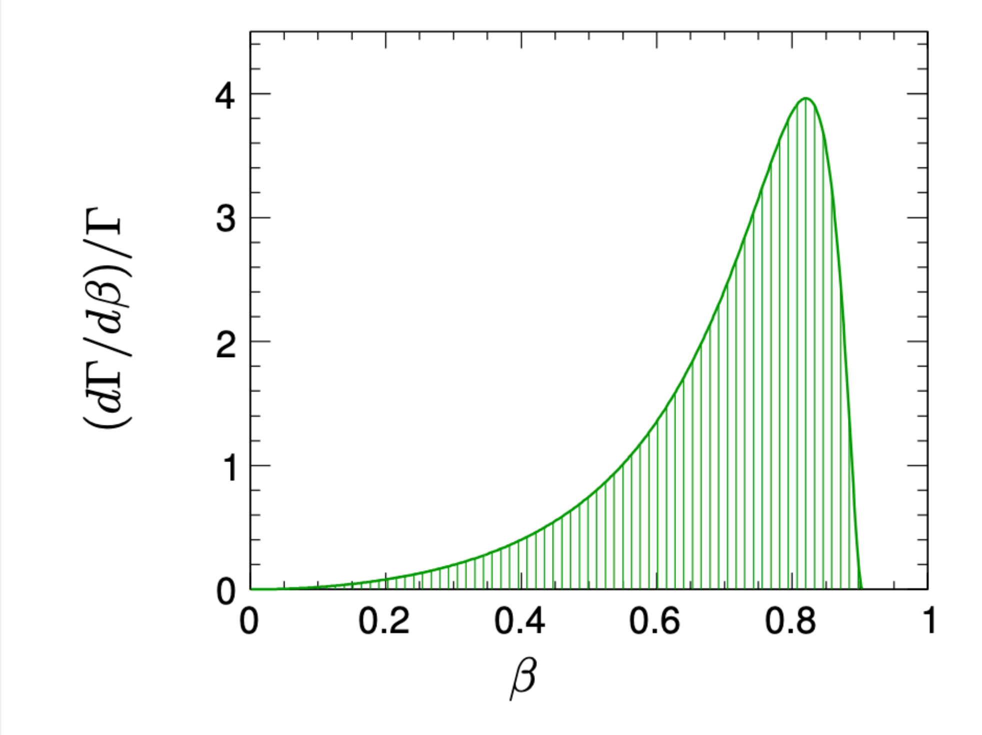
Current implementations of three-loop corrections based on “heuristic ansatz” of Sirlin and Zucchini, which incorporated (incorrect) log-enhanced term of Jaus and Rasche

$$RC = (1 + \delta'_R)(1 + \Delta_R^V)(1 + \delta_{\mathrm{NS}} - \delta_C)$$

table gives shift in “ δ'_R ”

The case Z=0

The Fermi function can be motivated by large Z or small velocity. Neutron beta decay has neither.

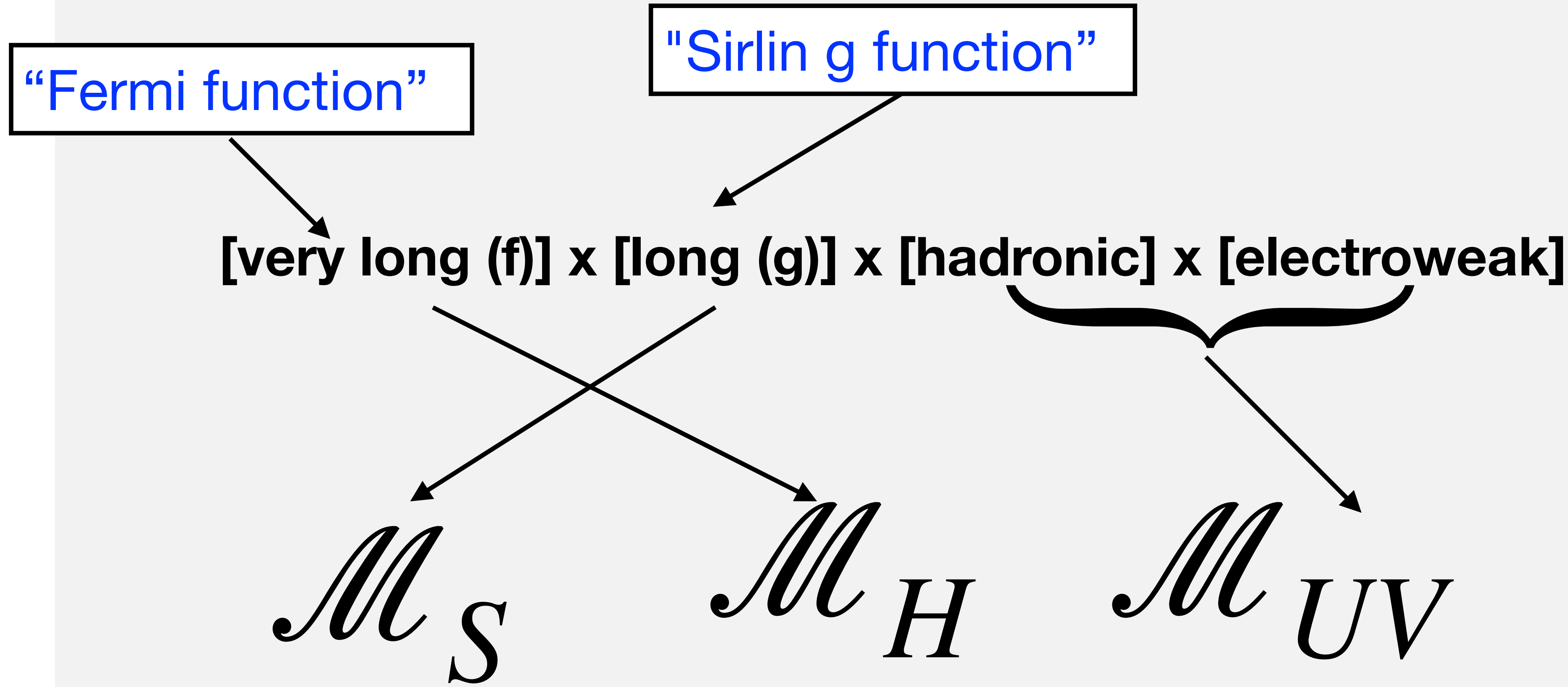


The one-loop correction, and a Fermi function ansatz for higher-order corrections, exhibit large perturbative corrections. Where would these come from?

$$1 + 4.6\,\alpha + 16\,\alpha^2 + 34\,\alpha^3 + \dots .$$

The case Z=0

Previous treatments have not separated scales



The usual Fermi function does not apply to neutron beta decay (Z=0). Differences starting at two loop order

The case Z=0

Recall the hard function for the Schrodinger-Coulomb problem (similar for Dirac-Coulomb),

$$\mathcal{M}_H^{(1)} = \left[\frac{iZ\overline{\alpha}}{\beta}(-4p^2/\mu^2 - i0)^{-\epsilon}\right]\left[\frac{-1}{2\epsilon}\right]$$

$$\mathcal{M}_H^{(2)} = \left[\frac{iZ\overline{\alpha}}{\beta}(-4p^2/\mu^2 - i0)^{-\epsilon}\right]^2\left[\frac{1}{8\epsilon^2} + \frac{\pi^2}{12} + 5\zeta(3)\epsilon + \mathcal{O}(\epsilon^2)\right]$$

$$\mathcal{M}_H^{(3)} = \left[\frac{iZ\overline{\alpha}}{\beta}(-4p^2/\mu^2 - i0)^{-\epsilon}\right]^3\left[\frac{-1}{48\epsilon^3} - \frac{\pi^2}{24\epsilon} - \frac{13\zeta(3)}{6} + \mathcal{O}(\epsilon)\right]$$

Note the presence of a new "scale", associated with the time-like process of electron+proton production

Idea: resum large logarithms associated with the ratio of scales:

$$i\pi \log\frac{-\vec{p}^2 - i0}{\vec{p}^2} = \pi^2 \approx 10$$

The case Z=0

These logarithms are associated with the dependence on soft factorization scale μ_S ,

$$\mathscr{M} = \mathscr{M}_S(\lambda/\mu_S)\mathscr{M}_H(p/\mu_S, p/\mu_H)\mathscr{M}_{\mathrm{UV}}(\Lambda/\mu_H)$$

Scale dependence determined by soft anomalous dimension, known to all orders (heavy-heavy cusp anomalous dimension for electron-proton system)

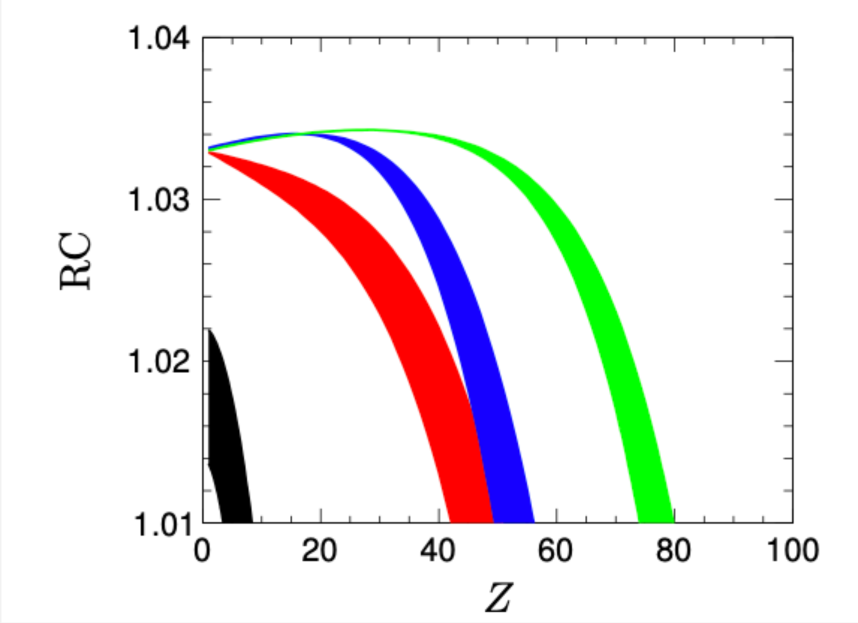
$$\mathscr{M}_H(\mu_{S+}) = e^{i\alpha\phi_\beta}\exp\left[\frac{\pi\alpha}{2\beta}\right]\mathscr{M}_H(\mu^2_{S-}) \qquad \mu_{S\pm} = \pm 4\vec{p}^2 - i0$$

$e^{i\alpha\phi_\beta}$: irrelevant phase

$\exp\left[\frac{\pi\alpha}{2\beta}\right]$: enhancement factor

$\mathscr{M}_H(\mu^2_{S-})$: “normal” perturbative series, without large logs

The case Z large



A different counting is needed when Z becomes large (e.g. Pb or U)

$$Z \sim L^2 \equiv \log^2(\Lambda/m) \sim \alpha^{-1}$$

$$\log\left(\frac{C(\mu_L)}{C(\mu_H)}\right) = \left[-\gamma^{(0)}(Z\alpha_L)L\right] + \left[b_0\alpha_L L^2 \frac{(Z\alpha_L)^2}{2\sqrt{1-(Z\alpha_L)^2}}\right] + \left[b_0^2\alpha_L^2 L^3 \frac{(Z\alpha_L)^2(3-2(Z\alpha_L)^2)}{6(1-(Z\alpha_L)^2)^{\frac{3}{2}}} - \alpha_L L \gamma^{(1)}(Z\alpha_L)\right]$$

$\alpha^{-\frac{1}{2}}$ α^0 $\alpha^{\frac{1}{2}}$

$$\gamma = \sum_{n=0}^{\infty}\sum_{i=0}^{n+1}\left(\frac{\alpha}{4\pi}\right)^{n+1}\gamma_n^{(i)}Z^{n+1-i} \equiv \gamma^{(0)}(Z\alpha) + \alpha\gamma^{(1)}(Z\alpha) + \dots$$

⇒ need all (or at least high) orders for $\gamma^{(0)}$ and $\gamma^{(1)}$ to go beyond O(1)

The case Z large

$\gamma^{(0)}$ is known to all orders (Dirac limit):

$$\gamma^{(0)} = \sqrt{1-(Z\alpha)^2} - 1$$

A simple “exploratory spirit” ansatz for $\gamma^{(1)}$ was previously used [Wilkinson 1997]*

$$\gamma^{(1)} = -\frac{1}{2}\left[(Z\alpha) + 0.57(Z\alpha)^2 + 0.50\frac{(Z\alpha)^3}{1-(Z\alpha)}\right]$$
$$= -\frac{1}{2}(Z\alpha) - 0.28(Z\alpha)^2 - 0.25\left[(Z\alpha)^3 + (Z\alpha)^5 + (Z\alpha)^7 + \ldots\right]$$
$$-0.25\left[(Z\alpha)^4 + (Z\alpha)^6 + (Z\alpha)^8 + \ldots\right]$$

We now know

$$\gamma^{(1)}_{\text{odd}} = \frac{1}{2}\frac{\partial}{\partial(Z\alpha)}\gamma^{(0)} = \frac{-Z\alpha}{2\sqrt{1-(Z\alpha)^2}}$$
$$= -\frac{1}{2}(Z\alpha) - \frac{1}{4}(Z\alpha)^3 - \frac{3}{16}(Z\alpha)^5 - \frac{5}{32}(Z\alpha)^7 + \ldots$$

$$\gamma^{(1)}_{\text{even}} = \frac{(Z\alpha)^2}{(4\pi)^3}\gamma^{(1)}_2 + \ldots = 0.216\,(Z\alpha)^2 + \ldots$$

* $0.50 \approx (0.48 + 1 + 0.57 + 0.16)/4$

Summary

- An all-orders explicit demonstration of factorization (implies answer for certain arbitrary loop order Feynman diagrams)
- Systematic high order evaluation of radiative corrections for neutron and nuclear beta decay for V_{ud}
- Potential applications for other beta decay observables, reactor neutrino cross sections and flux; muon conversion; …

Thank you