Strong CP? No problem!

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Formerly CP3 origins

The aim:

Challenge the conventional view of the strong CP problem by showing that **path integral** computations with a careful **infinite 4d volume** limit as well as calculations in **canonical quantization** imply that **QCD does not violate CP regardless** of the value of the θ **angle**

The plan:

1. The strong CP problem in the UV and the IR
2. Constraints from chiral symmetries
3. Fermion correlators from cluster decomposition and the index theorem
4. The θ parameter in canonical quantization

1. The strong CP problem in the UV and the IR

Charge conjugation and parity

Charge conjugation (C):

Exchanges particles with antiparticles

$$A_\mu \to -A_\mu, \quad \psi \to -i(\overline{\psi}\gamma^0\gamma^2)^\top$$

Parity (P):

Reverses vector quantities (electric fields), exchanges fermion chirality

Does not affect axial quantities (spin)

$$A_0 \to A_0, \quad A_i \to -A_i \quad \psi \to \gamma^0\psi$$

CP violation from the neutron dipole moment

Neutron dipole moment: coupling between the neutron's spin and electric fields

Spin operator $S^i = \frac{i}{8}\epsilon^{ijk}[\gamma^j, \gamma^k]$

Electric field $E_i = F_{0i}$

$$\mathcal{L}_{\text{eff}} \supset \frac{i}{4} f(q^2)\bar{N}\gamma^\mu\gamma^\nu\gamma_5 F_{\mu\nu} N \propto \frac{i}{8}\bar{N}[\gamma^\mu, \gamma^\nu]\gamma_5 F_{\mu\nu} N \propto \bar{N}[\gamma^\mu, \gamma^\nu]\tilde{F}_{\mu\nu} N \supset \bar{N}(\vec{S}\cdot\vec{E})N$$

CP-odd!

Neutron dipole moment $d_n = f(0)$

CP violation from the neutron dipole moment

As the neutron is a boundstate of the strong interactions, $d_n \neq 0$ would imply that the strong force violates CP

Experiments have not detected $d_n \neq 0$

Experimental bound

$$|d_n| < 1.8 \times 10^{-26} e \cdot cm$$

What are the theoretical expectations for d_n?

The UV perspective: QCD angle

$$S_{\rm QCD} = \int \mathrm{d}^4x \left[-\frac{1}{4g^2} F^a_{\mu\nu} F^a_{\mu\nu} + \frac{g^2\theta}{64\pi^2} \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma} + \sum_{i=1}^{N_f} \overline{\psi}_i \left(i\gamma^\mu D_\mu - m_i e^{i\alpha_i\gamma_5} \right) \psi_i \right] .$$

~~CP~~ ~~CP~~

θ **-term** is a total derivative and thus corresponds to a **boundary term**

→ it can never contribute in perturbation theory:

- effects of θ are nonperturbative

- S is **CP-odd!** Expected to contribute to the neutron dipole moment

Nonperturbative ‘t Hooft vertices in QCD

[‘t Hooft] derived an **effective Lagrangian** accounting for nonperturbative interactions arising from nontrivial saddle points (**instantons**) in the Euclidean path integral

$$\mathcal{L}^{\rm QCD}_{\rm eff,'t\,Hooft} \sim e^{-i\theta} \prod_{i=1}^{N_f} \bar{\psi}_i P_R \psi_i + e^{i\theta} \prod_{i=1}^{N_f} \bar{\psi}_i P_L \psi_i$$

According to [‘t Hooft] : **phases misaligned with fermion masses: CP violation**

- To link θ to the neutron dipole moment, we must **match** with **low-energy theory** that **includes the neutron**

The IR perspective: Chiral Lagrangian

Goldstones from $U(3)_L \times U(3)_R \to U(3)_V$ $\quad U = \langle U \rangle e^{\mathrm{i}\frac{\Pi^a \sigma^a}{\sqrt{2} f_\pi}} \sim \bar{\psi} P_R \psi$

Neutron-proton doublet $N = \begin{pmatrix} p \\ n \end{pmatrix}$

Quark masses $M = \begin{pmatrix} m_u e^{i\alpha_u} & & \\ & m_d e^{i\alpha_d} & \\ & & m_s e^{i\alpha_s} \end{pmatrix}$

CP-odd phases

Lagrangian

$$\mathcal{L}_{\pi,\mathrm{p,n}} \supset \frac{1}{4} f_\pi^2 \mathrm{Tr} D_\mu U D^\mu U^\dagger + (a f_\pi^3 \mathrm{Tr} M U + |b| e^{-i\xi} f_\pi^4 \mathrm{det} U + \mathrm{h.c.})$$

$$+ i \bar{N} \not{D} N - (m_N \bar{N} \tilde{U} P_L N + ic\, \bar{N} \gamma^\mu \tilde{U}^\dagger D_\mu \tilde{U} P_L N + d\, \bar{N} \tilde{M}^\dagger P_L N + e\, \bar{N} \tilde{U} \tilde{M} \tilde{U} P_L N + \mathrm{h.c.})$$

($\tilde{U}$: projection into u,d flavours)

CP-odd terms in the neutron interactions

Writing

$$\langle U \rangle = \begin{pmatrix} e^{i\varphi_u} & & \\ & e^{i\varphi_d} & \\ & & e^{i\varphi_s} \end{pmatrix}$$

Minimizing $\mathcal{L}_{\rm pion}[U = \langle U \rangle]$ w.r.t. angles:

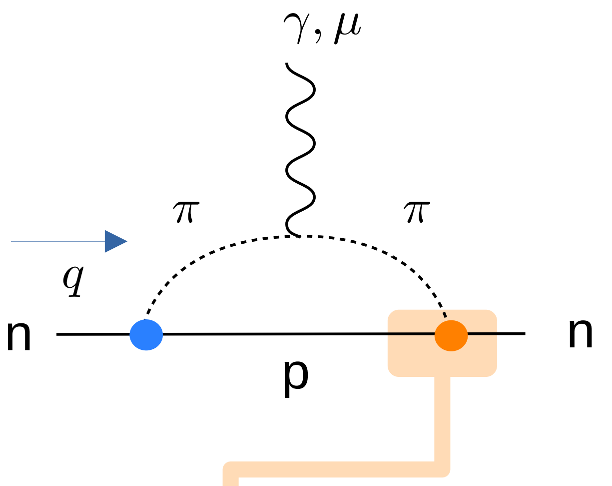
$$m_i(\varphi_i + \alpha_i) = \tilde{m}(m_u, m_d, m_s)(\xi + \alpha_u + \alpha_d + \alpha_s)$$

Substituting φ_i in $\mathcal{L}_{\rm neutron}$ and after appropriate field redefinition $N \to \mathcal{N}(N, U)$

$$\mathcal{L}_{\rm neutron} \supset -\frac{2c+1}{f_\pi}\partial_\mu \pi^a \bar{\mathcal{N}} T^a \gamma^\mu \gamma_5 \mathcal{N} + \frac{2(d+e)\tilde{m}}{f_\pi}(\xi + \alpha_u + \alpha_d + \alpha_s)\bar{\mathcal{N}}\, \pi^a T^a\, \mathcal{N}$$

CP-even **CP-odd**

Neutron dipole moment



$$\mathcal{L}_{\text{eff}} \supset (\xi + \alpha_u + \alpha_d + \alpha_s) f(q^2) \bar{N} (\vec{S} \cdot \vec{E}) N$$

- CP-odd term $\propto (\xi + \alpha_u + \alpha_d + \alpha_s)$ gives contribution to neutron dipole moment

Summary: d_n from chiral Lagrangian

ChPT result $d_n \propto (\xi + \alpha_u + \alpha_d + \alpha_s)$

Experimental bound $|d_n| < 1.8 \times 10^{-26} e \cdot cm$ [nEDM collaboration 2020]

What is the value of ξ in terms of fundamental parameters?

Central question of this talk

How to fix ξ in the low energy theory?

- Using **symmetry arguments** related to anomalous chiral symmetries

  $\xi = \theta$ thought to be the unique possibility **(→ CP violation)**

- **Matching correlators** with results from the fundamental **UV theory** (**QCD)**

  Only real computation that we know of is **`t Hooft**'s, using **dilute instanton gas** and yielding $\xi = \theta$ **(→ CP violation)**

Matching the UV and the IR a la ‘t Hooft

UV: ‘t Hooft vertices

$$\mathcal{L}^{\rm QCD}_{\rm eff,'t\,Hooft} \sim e^{-i\theta}\prod_{i=1}^{N_f}\bar{\psi}_i P_R \psi_i + e^{i\theta}\prod_{i=1}^{N_f}\bar{\psi}_i P_L \psi_i$$

IR: Chiral Lagrangian

$$\mathcal{L}_{\rm pion} \supset |b| e^{-i\xi} f_\pi^4 {\rm det} U + {\rm h.c.}, \qquad U \sim \bar{\psi} P_R \psi$$

Matching leads to

$$\xi = \theta$$

Neutron dipole moment:

$$|d_n| \propto (\xi + \alpha_u + \alpha_d + \alpha_s) = \theta + \sum_i \alpha_i \equiv \bar{\theta}$$

Experimental bounds:

$$\bar{\theta} < 10^{-10}$$

The strong CP problem:

Why does nature prefer $\bar{\theta} < 10^{-10}$ as opposed to $\mathcal{O}(1)$?

Why do we care?

Strong CP problem motivates searching for **new physics** that dynamically relaxes $\bar{\theta}$ to zero: e.g. **QCD axions**

Our work

- We have noted an ambiguity in the choices of ξ compatible with chiral symmetries

This talk

Using **path integral methods**:

- We have **recomputed Green's functions in the dilute instanton gas**, in Euclidean and Minkowski spacetime, and found $\xi = -\sum \alpha_i$ → **no CPV**

- We also have a UV **computation** of fermion correlators **which does not rely on instantons**, yielding the same conclusion

This talk

- Using **canonical quantization**, we have rederived how θ drops out of observables and P is conserved

This talk

Implications of our work

If we are right:

There would be **no strong CP problem**

QCD would directly **explain** the **lack of CP violation in the strong force**

There would be **no need for QCD axions**

2. Constraints from chiral symmetries

Spurious chiral symmetry

The QCD partition function changes under **chiral field redefinitions** due to **masses** and **anomaly**

$$\psi \to e^{i\beta\gamma_5}\psi$$
$$\bar\psi \to \bar\psi e^{i\beta\gamma_5}$$

$$Z(\theta, \alpha_j) \to Z(\theta - 2N_f\beta, \alpha_j + 2\beta)$$

fermion mass phases

Spurion symmetry: Z invariant under chiral transformations plus "spurion" transf:

$$\psi \to e^{i\beta\gamma_5}\psi$$
$$\bar\psi \to \bar\psi e^{i\beta\gamma_5}$$

$$\theta \to \theta + 2N_f\beta, \quad \mathfrak{m}_j = m_j e^{i\alpha_j} \to e^{-2i\beta}\mathfrak{m}_j$$

▶ **Effective Lagrangians** for QCD should **respect spurion symmetry**

Spurious symmetry in the chiral Lagrangian

$$\psi \to e^{i\beta\gamma_5}\psi \qquad \bar\psi \to \bar\psi e^{i\beta\gamma_5}$$

$$\Rightarrow \quad U \to e^{2i\beta} U \qquad N \to e^{i\beta\gamma_5} N \qquad \bar N \to \bar N e^{i\beta\gamma_5}$$

$$\mathfrak{m}_j = m_j e^{\mathrm{i}\alpha_j} \to e^{-2i\beta}\mathfrak{m}_j \quad \Rightarrow \quad M \to e^{-2i\beta} M$$

+2 -2 -2 +2 -2N_f +2N_f

$$\mathcal{L}_{\pi,\mathrm{p,n}} \supset \frac{1}{4} f_\pi^2 \mathrm{Tr} D_\mu U D^\mu U^\dagger + (a f_\pi^3 \mathrm{Tr} M U + |b| e^{-i\xi} f_\pi^4 \det U + \mathrm{h.c.})$$

∓1 ±1 -1 +2 -1 +1 -2 +2 -1 -1 +2 -1 -1+2-2+2 -1

$$+i\bar N \not{D} N - (m_N \bar N \tilde U P_L N + ic\,\bar N \gamma^\mu \tilde U^\dagger D_\mu \tilde U P_L N + d\,\bar N \tilde M^\dagger P_L N + e\,\bar N \tilde U \tilde M \tilde U P_L N + \mathrm{h.c.})$$

Spurious symmetry in the chiral Lagrangian

Spurious chiral symmetry requires

$$\xi \to \xi + 2N_f\beta$$

More than one possibility in terms of **fundamental parameters** θ, α_i !

- $\xi = \theta$ Usual option, **assumed** by [Baluni, Crewther et al] → **CP violation**

$$d_n \propto (\xi + \sum_i \alpha_i) = (\theta + \sum_i \alpha_i) \equiv \bar{\theta}$$

- $\xi = -\sum_i \alpha_i$ Alternative option → **CP conservation**

$$d_n \propto (\xi + \sum_i \alpha_i) = 0$$

What is the correct value of ξ?

3. Fermion correlators from cluster decomposition and the index theorem

- In order to resolve the ambiguity, we must **match effective** $\det U$ **term** in the chiral Lagrangian with **results for correlators in QCD, paying special attention to complex phases**

- We will derive an **effective Lagrangian** capturing this correlators and match to

$$\mathcal{L}_{\text{eff}}^{\text{QCD}} \sim e^{-\mathrm{i}\xi} \prod_{i=1}^{N_f} \bar{\psi}_i P_R \psi_i + e^{\mathrm{i}\xi} \prod_{i=1}^{N_f} \bar{\psi}_i P_L \psi_i \leftrightarrow \mathcal{L}_{ChPT} \propto e^{-\mathrm{i}\xi} \mathrm{det}U + e^{\mathrm{i}\xi} \mathrm{det}U^\dagger$$

Read ξ from phases in effective vertices derived from QCD

- Next we proceed to calculate the **phase of QCD correlators** starting from the **path integral** and using **clustering** and the **index theorem**.

Towards correlators: vacuum path integral

$$\int_{\phi_i,\phi_f,T} \left(\prod \mathcal{D}\phi\right) e^{\mathrm{i}S_T} = \langle\phi_f|e^{-\mathrm{i}HT}|\phi_i\rangle = \sum_n e^{-\mathrm{i}E_nT}\langle\phi_f|n\rangle\langle n|\phi_i\rangle$$

To get a **vacuum transition amplitude** we can take the **infinite *T* limit,**

$$Z = \lim_{T\to\infty e^{-i0+}} \int_T \left(\prod \mathcal{D}\phi\right) e^{\mathrm{i}S_T} \sim \lim_{T\to\infty e^{-i0+}} \langle 0|e^{-\mathrm{i}HT}|0\rangle$$

To recover the vacuum amplitude for **finite *T***, one would **needs to know the wave functional of the vacuum**

$$\langle 0|e^{-iHT}|0\rangle = \int [\mathcal{D}\phi_f]_{T/2}[\mathcal{D}\phi_i]_{-T/2}\langle 0|\phi_f\rangle\langle\phi_f|e^{-iHT}|\phi_i\rangle\langle\phi_i|0\rangle$$

$$= \int [\mathcal{D}\phi_f]_{T/2}[\mathcal{D}\phi_i]_{-T/2}\langle 0|\phi_f\rangle\langle\phi_i|0\rangle \int_{\phi_i,\phi_f,T} \left(\prod \mathcal{D}\phi\right) e^{\mathrm{i}S}$$

Wrap-up: the importance of boundary conditions

Infinite *T* method

$$Z = \lim_{T \to \infty e^{-i0_+}} \int_T \left(\prod \mathcal{D}\phi \right) e^{\mathrm{i}S_T} \sim \lim_{T \to \infty e^{-i0_+}} \langle 0 | e^{-\mathrm{i}HT} | 0 \rangle$$

Boundary conditions remain arbitrary!

Wave functional method

$$\langle 0 | e^{-iHT} | 0 \rangle = \int [\mathcal{D}\phi_f]_{T/2} [\mathcal{D}\phi_i]_{-T/2} \langle 0 | \phi_f \rangle \langle \phi_i | 0 \rangle \int_{\phi_i, \phi_f, T} \left(\prod \mathcal{D}\phi \right) e^{\mathrm{i}S}$$

Boundary conditions are fixed by wave functional, need additional reweighting

Wrap-up: the importance of boundary conditions

To ensure projection into vacuum, we first use the Euclidean path integral for infinite V *T*, without the need to enforce particular b.c.s

This is in contrast with **lattice simulations** at **finite volume** with **periodic b.c.s**

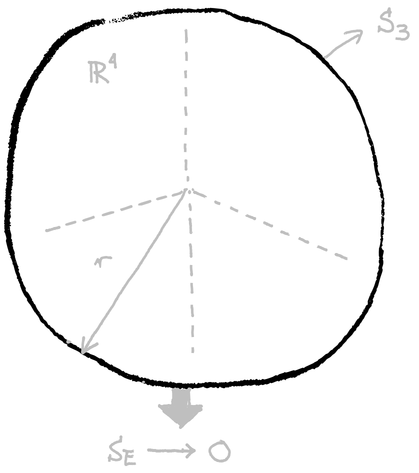
This requires to **subtract contamination** from **excited states**!

Later in the talk we will use **canonical quantization** to determine the θ dependence of the wave functional

Finite action constraints and topology

According to **Picard-Lefschetz theory,** Euclidean path integral can be formulated in terms of a sum of integrations over **steepest descent flows** that start from **finite action saddles** [Witten]

In **infinite spacetime, gauge fields at saddles** must be **pure gauge transf. at ∞**



$$A_m(r,\theta,\varphi,\xi) = \frac{i}{g} U_r(\theta,\varphi,\xi)\,\partial_m U_r^\dagger(\theta,\varphi,\xi), \quad U_r(\theta,\varphi,\xi) \in SU(3)$$

Finite action constraints and topology

This leads to maps S_3 → SU(3) that fall into equivalence or **homotopy classes**

"wrappings" of SU(3) over S_3 that cannot be connected by continuous deformations

The **steepest descent flows** are **continuous**

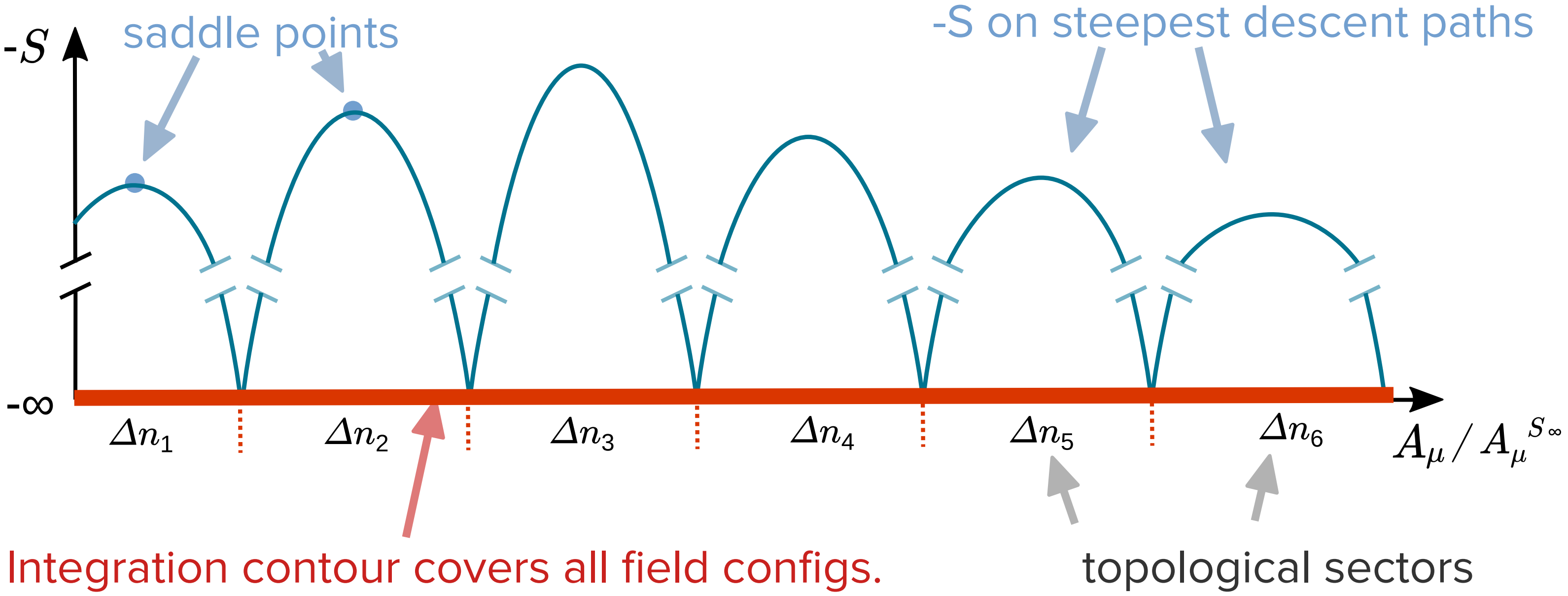
▶ the full flow from a saddle point falls into the same homotopy class

Homotopy classes characterized by **integer topological charge** Δn

▶ In an **infinite spacetime** $Z = \sum_{\Delta n} Z_{\Delta n}$

Path integral a la Picard-Lefschetz

$$Z = \sum_{\Delta n} Z_{\Delta n}$$

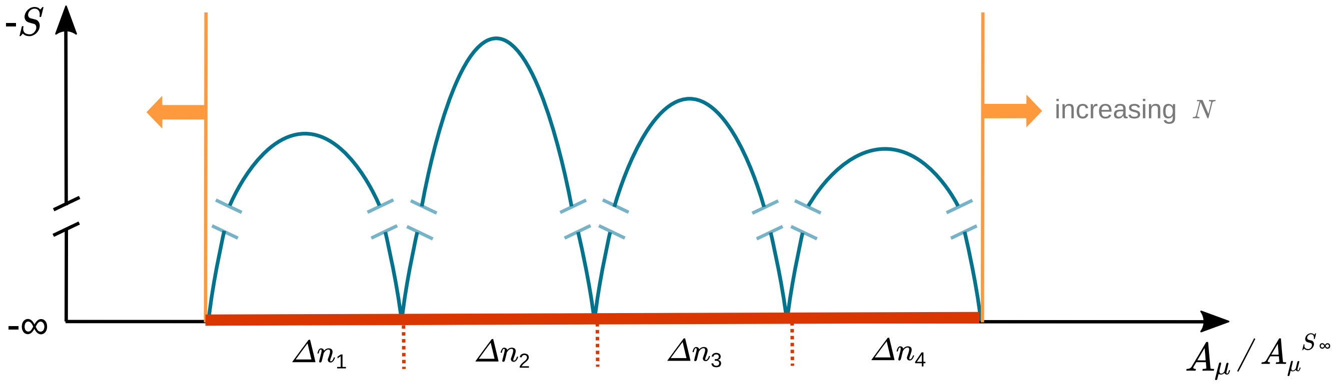


Ordering of limits

$$Z = \lim_{N\to\infty} \sum_{|\Delta n|<N} \lim_{\Omega\to\infty} Z_{\Delta n}(\Omega)$$

Need infinite spacetime volume to project into vacuum

Δn **required to be integer only in infinite volume → take infinite volume first**



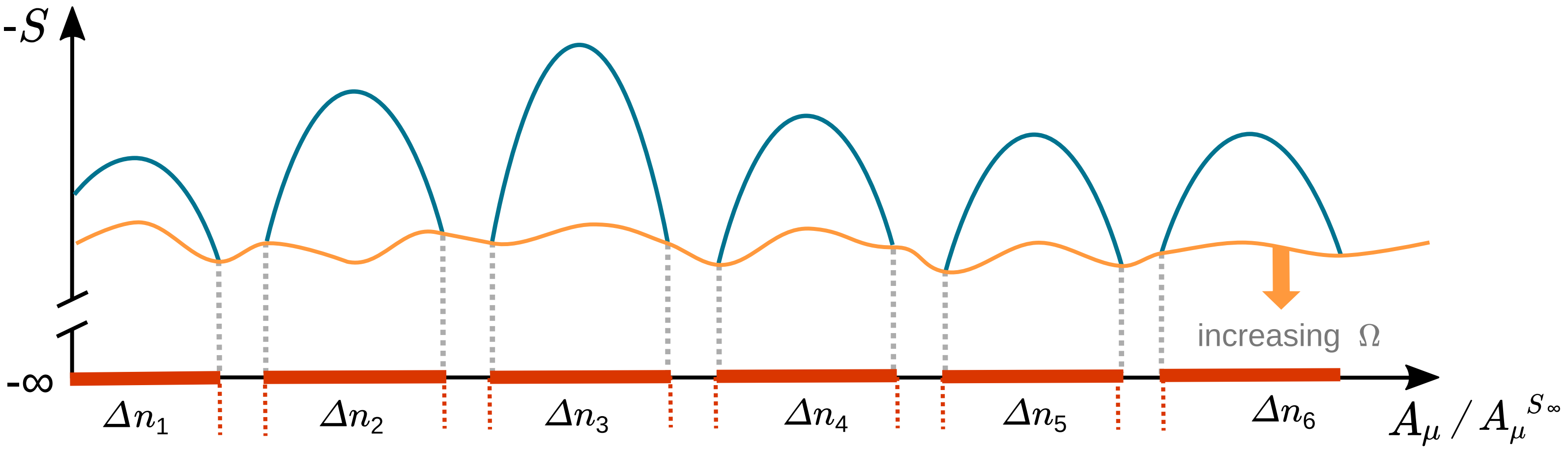
Integration contour remains continuous
Exponential suppression of large N contributions

Alternative ordering of limits

$$Z = \lim_{\Omega\to\infty} \lim_{N\to\infty} \sum_{|\Delta n|<N} Z_{\Delta n}(\Omega)$$

For finite spacetime volume, **topological charge is not necessarily quantized**

Insisting on integer charge means that one **misses configurations**



Integration contour not connected → does not capture full path integral

Topological charge and the index theorem

Atiyah-Singer's index theorem relates the **topological charge** to the **eigenspectrum** of the **Euclidean Dirac operator**

$$\not{D}\varphi^\lambda = \lambda\varphi^\lambda$$

Zero modes:

$$\not{D}\varphi^0 = 0$$

Index theorem

$$\Delta n = \#(\text{Right-handed zero modes of } \not{D}) - \#(\text{Left-handed zero modes of } \not{D})$$

$$\not{D}\psi_R = 0 \qquad \not{D}\psi_L = 0$$

The θ term and the topological charge

The θ **term** turns out to be **proportional to the topological charge**

$$-S_\theta^E = i\theta \int d^4x \frac{g^2}{64\pi^2} \epsilon_{mnrs} F^a_{mn} F^a_{rs} = i\theta \Delta n$$

- In an **infinite spacetime** $Z = \sum_{\Delta n} e^{i\Delta n \theta} \tilde{Z}_{\Delta n}$

- **Remember: Integer topological charge only enforced for infinite volume**

Strategy to compute correlators

The aim is to **constrain** the **functional dependence** of the partition functions $Z_{\Delta n}$ on $VT \equiv \Omega,\ \Delta n,\ \mathfrak{m}_j = m_j e^{i\alpha_j}$

Fermion masses can be understood as **sources** for the integrated fermion correlators [Leutweyler & Smilga]

$$\mathcal{L} \supset \sum_j \left(\bar{\psi}_j (\mathfrak{m}_j^* P_L + \mathfrak{m}_j P_R)\psi_j\right)$$

These correlators should be **sensitive to global CP-violating phases**

$$\frac{\partial}{\partial \mathfrak{m}_i} Z_{\Delta n} = -\int d^4x \, \langle \bar{\psi}_i P_R \psi_i \rangle_{\Delta n}, \qquad \frac{\partial}{\partial \mathfrak{m}_i^*} Z_{\Delta n} = -\int d^4x \, \langle \bar{\psi}_i P_L \psi_i \rangle_{\Delta n}.$$

Cluster decomposition

$$Z(\Omega) = \sum_{\Delta n=-\infty}^{\infty} \int_{\Delta n} \mathcal{D}\phi e^{-S_\Omega[\phi]+i\Delta n\theta} \equiv \sum_{\Delta n=-\infty}^{\infty} e^{i\Delta n\theta} \tilde{Z}_{\Delta n}(\Omega)$$

4D volume

Factorizing path integral a la [Weinberg]

$$\tilde{Z}_{\Delta n}(\Omega = \Omega_1 + \Omega_2) = \int_{\Delta n} \mathcal{D}\phi e^{-S_{\Omega_1+\Omega_2}[\phi]} = \sum_{\Delta n_1} \int_{\Delta n_1} \mathcal{D}\phi e^{-S_{\Omega_1}[\phi]} \int_{\Delta n - \Delta n_1} \mathcal{D}\phi e^{-S_{\Omega_2}[\phi]}$$

$$\tilde{Z}_{\Delta n}(\Omega = \Omega_1 + \Omega_2) = \sum_{\Delta n_1=-\infty}^{\infty} \tilde{Z}_{\Delta n_1}(\Omega_1) \tilde{Z}_{\Delta n - \Delta n_1}(\Omega_2)$$

Cluster decomposition

Want to **solve** this **infinite number of relations** by following these steps:

- **Factorize all complex phases** and obtain a set of relations for real functions
- Find a suitable **Ansatz**
- Assuming **analiticity** in Ω, we can construct the full function by computing all the derivatives at $\Omega = 0$.

Factorizing out complex phases

Fermionic path integration corresponds to $\det(-\not{D} - \mathfrak{m} P_R - \mathfrak{m}^* P_L)$

Zero modes of $\not{D}$ contribute phases to

$$\not{D}\varphi^0 = 0, \quad \varphi^0 = P_{R/L}\varphi^0 \Rightarrow (\not{D} + \mathfrak{m} P_R + \mathfrak{m}^* P_L)\varphi^0 = |\mathfrak{m}| e^{\pm i\alpha} P_{R/L}\varphi^0$$

Total phase of $\det(-\not{D} - \mathfrak{m} P_R - \mathfrak{m}^* P_L)$ follows from index theorem

$$\begin{aligned}
&\det(-\not{D} - \mathfrak{m} P_R - \mathfrak{m}^* P_L) \\
&= (-\mathrm{e}^{\mathrm{i}\alpha})^{\#(\text{r.h. zero modes}) - \#(\text{l.h. zero modes})} |\det(-\not{D} - \mathfrak{m} P_R - \mathfrak{m}^* P_L)| \\
&= (-e^{\mathrm{i}\alpha})^{\Delta n} |\det(-\not{D} - \mathfrak{m} P_R - \mathfrak{m}^* P_L)|
\end{aligned}$$

Factorizing out complex phases

Finally considering all fermion flavours, and defining $\bar{\alpha} \equiv \sum_i \alpha_i$

$$\tilde{Z}_{\Delta n}(\Omega) = e^{i\Delta n \bar{\alpha}} g_{\Delta n}(\Omega) \Rightarrow g_{\Delta n}(\Omega_1 + \Omega_2) = \sum_{\Delta n_1 = -\infty}^{\infty} g_{\Delta n_1}(\Omega_1) g_{\Delta n - \Delta n_1}(\Omega_2)$$

Real

Parity properties and the $\Omega_2 = 0$ limit of the previous equation motivate **Ansatz**

$$g_{\Delta n}(\Omega) = g_{|\Delta n|}(\Omega) = \Omega^{|\Delta n|} f_{|\Delta n|}(\Omega^2), \quad f_{|\Delta n|}(0) \neq 0, \quad f_0(0) = 1$$

A unique solution to the infinite tower of eqns

Taking derivatives of cluster relations and **proceeding recursively** leads to

$$\frac{d^m}{d\Omega^m} g_{\Delta n}(\Omega) = (f_1(0))^m \sum_{k=0}^{m} \binom{m}{k} g_{\Delta n - m + 2k}(\Omega).$$

Taylor expansion

$$g_{\Delta n}(\Omega) = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m}{d\Omega^m} g_{\Delta n}(0)\, \Omega^m = \sum_{k=0}^{\infty} \frac{1}{(|\Delta n| + 2k)!} (f_1(0))^{|\Delta n| + 2k} \binom{|\Delta n| + 2k}{k} \Omega^{|\Delta n| + 2k}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!(|\Delta n| + k)!} \left(\frac{2 f_1(0)\, \Omega}{2} \right)^{|\Delta n| + 2k} = I_{\Delta n}(2 f_1(0) \Omega).$$

Final result of partition function

There is a **unique solution** with a single **real parameter** $f_1(0) \equiv \beta$

$$f_{\Delta n}(\Omega) = I_{\Delta n}(2\beta\Omega)$$

$$Z_{\Delta n} = \mathrm{e}^{\mathrm{i}\Delta n(\theta+\bar{\alpha})} I_{\Delta n}(2\beta\Omega)$$

Matches results of [Leutweyler & Smilga] achieved in a different way!

Making **dependence on complex masses explicit**:

$$Z_{\Delta n}(\Omega) = e^{\mathrm{i}\Delta n(\theta - \mathrm{i}/2 \sum_j \log(\mathfrak{m}_j/\mathfrak{m}_j^*))} I_{\Delta n}(2\beta(\mathfrak{m}_k \mathfrak{m}_k^*)\,\Omega)$$

Mass dependence and correlators

Taking derivatives with respect to $\mathfrak{m}, \mathfrak{m}^*$ gives **averaged integrated correlators**

Spurion chiral charge -2

+2Nf -2N$_f$ -2

$$\frac{1}{VT}\int d^4x\, \langle \bar{\psi}_i P_L \psi_i \rangle_{\Delta n} = -\, e^{\mathrm{i}\Delta n(\theta+\bar{\alpha})} \left(\frac{\beta}{2\mathfrak{m}_i^*} \left(I_{\Delta n+1}(2\beta\Omega) - I_{\Delta n-1}(2\beta\Omega) \right) + \mathfrak{m}_i \left(I_{\Delta n+1}(2\beta\Omega) + I_{\Delta n-1}(2\beta\Omega) \right) \frac{\partial}{\partial(\mathfrak{m}_i \mathfrak{m}_i^*)} \beta(\mathfrak{m}_k \mathfrak{m}_k^*) \right)$$

-2

spurion chiral charges match!

Summing over topological sectors

$$\frac{1}{VT}\int d^4x\,\langle\bar\psi_i P_L\psi_i\rangle = \lim_{N\to\infty}\lim_{VT\to\infty}\frac{\sum_{\Delta n=-N}^{N}\frac{1}{VT}\int d^4x\,\langle\bar\psi_i P_L\psi_i\rangle_{\Delta n}}{\sum_{\Delta m=-N}^{N} Z_{\Delta m}} = -2\mathfrak{m}_i\,\partial_{\mathfrak{m}_i\mathfrak{m}_i^*}\beta(\mathfrak{m}_k\mathfrak{m}_k^*),$$

-2 -2

Topological classification only **enforced in infinite volume**, which **fixes ordering**

Result due to all **Bessel functions** having a **common asymptotic behaviour**

$$I_{\Delta n}(2\beta\Omega) = I_0(2\beta\Omega)(1+\mathcal{O}((\beta\Omega)^{-1}))$$

Phase of correlator fixed by masses → **CP conservation**

Summing over topological sectors

$$\frac{1}{VT}\int d^4x\,\langle\bar{\psi}_i P_L \psi_i\rangle = \lim_{N\to\infty}\lim_{VT\to\infty}\frac{\sum_{\Delta n=-N}^{N}\frac{1}{VT}\int d^4x\,\langle\bar{\psi}_i P_L\psi_i\rangle_{\Delta n}}{\sum\limits_{\Delta m=-N}^{N} Z_{\Delta m}} =$$

$$\lim_{N\to\infty}\lim_{VT\to\infty}\frac{1}{\sum_{|\Delta m|<N}\mathrm{e}^{\mathrm{i}\Delta m(\theta+\bar{\alpha})}I_{\Delta m}(2\beta\Omega)}\sum_{|\Delta n|<N}\left(-e^{\mathrm{i}\Delta n(\theta+\bar{\alpha})}\left(\frac{\beta}{2\mathfrak{m}_i^*}\left(I_{\Delta n+1}(2\beta\Omega)-I_{\Delta n-1}(2\beta\Omega)\right)\right.\right.$$

$$\left.\left.+\,\mathfrak{m}_i\left(I_{\Delta n+1}(2\beta\Omega)+I_{\Delta n-1}(2\beta\Omega)\right)\frac{\partial}{\partial(\mathfrak{m}_i\mathfrak{m}_i^*)}\beta(\mathfrak{m}_k\mathfrak{m}_k^*)\right)\right)$$

Summing over topological sectors

$$\frac{1}{VT}\int d^4x\,\langle\bar{\psi}_i P_L \psi_i\rangle = \lim_{N\to\infty}\lim_{VT\to\infty}\frac{\sum_{\Delta n=-N}^{N}\frac{1}{VT}\int d^4x\,\langle\bar{\psi}_i P_L\psi_i\rangle_{\Delta n}}{\sum\limits_{\Delta m=-N}^{N} Z_{\Delta m}} =$$

$$= \lim_{N\to\infty}\lim_{VT\to\infty}\frac{1}{\sum_{|\Delta m|<N} \mathrm{e}^{\mathrm{i}\Delta m(\theta+\bar{\alpha})} I_0(2\beta\Omega)}\sum_{|\Delta n|<N}\left(-e^{\mathrm{i}\Delta n(\theta+\bar{\alpha})}\left(\frac{\beta}{2\mathfrak{m}_i^*}(I_0(2\beta\Omega)-I_0(2\beta\Omega))\right.\right.$$

$$\left.\left. +\,\mathfrak{m}_i(I_0(2\beta\Omega)+I_0(2\beta\Omega))\frac{\partial}{\partial(\mathfrak{m}_i\mathfrak{m}_i^*)}\beta(\mathfrak{m}_k\mathfrak{m}_k^*)\right)\right)$$

$$I_{\Delta n}(2\beta\Omega) = I_0(2\beta\Omega)(1+\mathcal{O}((\beta\Omega)^{-1})$$

Summing over topological sectors

$$\frac{1}{VT}\int d^4x\,\langle\bar{\psi}_i P_L \psi_i\rangle = \lim_{N\to\infty}\lim_{VT\to\infty}\frac{\sum_{\Delta n=-N}^{N}\frac{1}{VT}\int d^4x\,\langle\bar{\psi}_i P_L\psi_i\rangle_{\Delta n}}{\sum\limits_{\Delta m=-N}^{N} Z_{\Delta m}} =$$

$$-\lim_{N\to\infty}\lim_{VT\to\infty}\frac{\sum_{|\Delta n|<N} e^{\mathrm{i}\Delta n(\theta+\bar{\alpha})}}{\sum_{|\Delta m|<N} \mathrm{e}^{\mathrm{i}\Delta m(\theta+\bar{\alpha})}}\,2\,\mathfrak{m}_i\,\frac{\partial}{\partial(\mathfrak{m}_i\mathfrak{m}_i^*)}\,\beta(\mathfrak{m}_k\mathfrak{m}_k^*)$$

Summing over topological sectors

$$\frac{1}{VT}\int d^4x\,\langle\bar{\psi}_i P_L \psi_i\rangle = -2\mathfrak{m}_i \frac{\partial}{\partial(\mathfrak{m}_i\mathfrak{m}_i^*)}\beta(\mathfrak{m}_k\mathfrak{m}_k^*)$$

Opposite order of limits:

$$\lim_{VT\to\infty}\lim_{N\to\infty}\frac{\sum_{\Delta n=-N}^{N}\frac{1}{VT}\int d^4x\,\langle\bar{\psi}_i P_L \psi_i\rangle_{\Delta n}}{\sum\limits_{\Delta m=-N}^{N} Z_{\Delta m}} \to \frac{2i\beta}{\mathfrak{m}^*}\sin(\theta+\bar{\alpha}) - 2\mathfrak{m}_i\cos(\theta+\bar{\alpha})\frac{\partial}{\partial(\mathfrak{m}_i\mathfrak{m}_i^*)}\beta(\mathfrak{m}_k\mathfrak{m}_k^*)$$

Phases not fixed by masses! → **CP violation**

General correlators

Taking **higher-order derivatives** w.r.t. $\mathfrak{m}_i, \mathfrak{m}_i^*$, yields general integrated correlators

-2Nf -2Nf

$$\langle \left(\prod_{j=1}^{N_f} \int d^4x_j (\bar\psi_j P_L \psi_j) \right) \rangle = e^{\mathrm{i} \sum_j \alpha_j} f(\mathfrak{m}_k^* \mathfrak{m}_k)$$

Reproduced by the following **effective interaction** (after factoring out ordinary props/)

-2Nf +2Nf

$$\mathcal{L}_{\rm eff} \supset e^{\mathrm{i} \sum_j \alpha_j} \Gamma \prod_j \bar\psi_j P_R \psi_j$$

To be **matched** to **chiral Lagrangian** with $U \sim \bar\psi P_R \psi$

$$\xi = -\sum_j \alpha_j$$

$$\mathcal{L}_{\rm pion} \supset |b| e^{-i\xi} f_\pi^4 {\rm det} U$$

Consequences for d_n and CP violation

$$d_n \propto \xi + \alpha_u + \alpha_d + \alpha_s = 0$$

- All phases of all fermion correlators are fixed by the α_i
- θ effects cancel
- All phases can be eliminated with chiral field redefinitions

No CP violation in fermion correlators

Where we did depart from the usual results?

- **Only in the ordering of limits!**

- **Opposite order of limits yields traditional picture of CP-violation**

3. The θ parameter in canonical quantization

Goal

$$\langle 0|e^{-iHT}|0\rangle = \int [\mathcal{D}A_f]_{T/2}[\mathcal{D}A_i]_{-T/2}\langle 0|A_f\rangle\langle A_i|0\rangle \int_{A_i,A_f,T} \left(\prod \mathcal{D}A\right) e^{\mathrm{i}S}$$

Understand the θ-**dependence of wave functionals** such that we can carry out **computations of partition function** and **expectation values** in the vacuum **without needing to take the infinite volume limit**

If the previous path integral calculations for $VT \to \infty$ are meaningful, we expect a **cancellation of the** θ**-dependence**

For simplicity we focus on the case of a **pure gauge theory**

Chern-Simons Number

In **pure gauge theory**

$$\langle 0|e^{-iHT}|0\rangle = \int [\mathcal{D}A_f]_{T/2}[\mathcal{D}A_i]_{-T/2}\langle 0|A_f\rangle\langle A_i|0\rangle \int_{A_i,A_f,T} \left(\prod \mathcal{D}A\right) e^{iS}$$

$$S = \int d^4x \left(-\frac{1}{2g^2}\mathrm{tr}F_{\mu\nu}F^{\mu\nu} + \frac{1}{16\pi^2}\theta\,\mathrm{tr}F_{\mu\nu}\widetilde{F}^{\mu\nu}\right) = S_0 + S_\theta.$$

With **appropriate gauge-fixing**

$$S_\theta = \theta(W[\mathbf{A}_f] - W[\mathbf{A}_i]) \qquad W[\mathbf{A}] = \frac{1}{4\pi^2}\varepsilon_{ijk}\int_S \mathrm{d}^3x\,\mathrm{tr}\left[\frac{1}{2}A_i\partial_j A_k - \frac{\mathrm{i}}{3}A_iA_jA_k\right]$$

Chern-Simons number (CSN)

$W[\mathbf{A}]$ changes by integer numbers under spatial gauge transf with $\mathrm{U}(\mathbf{x}) \underset{|\mathbf{x}|\to\infty}{\longrightarrow} 1$

We single out $U^{(1)}(\mathbf{x})$ with $W[\mathbf{A}_{U^{(1)}}] = W[\mathbf{A}] + 1$

Wave functionals in canonical quantization

$$\langle 0|e^{-iHT}|0\rangle = \int [\mathcal{D}A_f]_{T/2}[\mathcal{D}A_i]_{-T/2}\langle 0|A_f\rangle\langle A_i|0\rangle \int_{A_i,A_f,T} \left(\prod \mathcal{D}A\right) e^{\mathrm{i}S}$$

Wave functional $\Psi_0[A_i]$ satisfying Schrödinger equation

$$H\Psi_0[A] = E_0\Psi_0[A]$$

We quantize in the gauge $A_0 = 0$, which still allows gauge transformations $U(\mathbf{x})$

$$[A^{i,a}(\mathbf{x}), \Pi^{j,b}(\mathbf{x})] = \mathrm{i}\delta^{ij}\delta^{ab}\delta^3(\mathbf{x}-\mathbf{y}) \Rightarrow \mathbf{\Pi}^a = \frac{\delta}{\mathrm{i}\delta\mathbf{A}^a}.$$

$$\mathcal{H} = \frac{1}{2}\left[(\mathbf{E}^a)^2 + (\mathbf{B}^a)^2\right] = \frac{1}{2}\left[\left(g\frac{\delta}{\mathrm{i}\delta\mathbf{A}^a} - \frac{g^2}{8\pi^2}\theta\mathbf{B}^a\right)^2 + (\mathbf{B}^a)^2\right].$$

The hidden symmetry of the Hamiltonian

Under **parity**

$$\mathbf{A}(\mathbf{x}) \to \mathbf{A}^P(\mathbf{x}) = -\mathbf{A}(-\mathbf{x}) \qquad W[\mathbf{A}^P] \to -W[\mathbf{A}]$$

Within class of gauge transformations $U^{(1)}$, one can find $\tilde{U}^{(1)}$ such that

$$\widetilde{U}^{(1)}(\widetilde{U}^{(1)}\mathbf{A}^P)^P = \mathbf{A}$$

$$\text{CSN}: -W[\mathbf{A}] + 1$$

$$\text{CSN}: 1 - (-W[\mathbf{A}] + 1) = W[\mathbf{A}]$$

While we think of the θ term breaking parity, there is still a **discrete symmetry**!

$$\hat{P}\,\Psi[\mathbf{A}] = e^{i\theta(2W[\mathbf{A}]-1)}\Psi[\widetilde{U}^{(1)}\mathbf{A}^P] \qquad \hat{P}^2 = \mathbb{I} \qquad [H, \hat{P}] = 0$$

Hidden symmetry = parity + gauge transf.

Under a **change of basis**

$$\Psi[\mathbf{A}] = \mathrm{e}^{\mathrm{i}\theta W[\mathbf{A}]}\Psi'[\mathbf{A}]$$

The Hamiltonian and discrete symmetry become

$$\mathcal{H}' = \frac{1}{2}\left[\left(g\frac{\delta}{\mathrm{i}\delta\mathbf{A}^a}\right)^2 + (\mathbf{B}^a)^2\right] .$$

$$\hat{P}'\Psi'[\mathbf{A}] = \Psi'[\widetilde{U}^{(1)}\mathbf{A}^P] \Leftrightarrow \Psi'[\mathbf{A}^P] = (\hat{P}'\Psi')[(\widetilde{U}^{(1)P})^{-1}\mathbf{A}]$$

Ordinary parity matches symmetry *P'* up to a large gauge transformation

Physical states

Gauge transformations are a **redundancy**

Expect **physical states** to be **eigenstates of U** with **eigenvalues pure phases**

Can be shown using **separation of variables** in **Schrödinger equation**

$$\Psi'[\mathbf{A}_{U(\mathbf{x})}] = e^{\mathrm{i}\theta_{\Psi,U}}\Psi'[\mathbf{A}] \quad \Rightarrow \quad \begin{aligned}\hat{P}'\tilde{U}^{(1)}\Psi'[\mathbf{A}] &= e^{\mathrm{i}\left(\theta_{\Psi,U^{(1)}}+\theta_{\Psi,U^{(1)P}}\right)}\Psi'[\mathbf{A}^P],\\ \tilde{U}^{(1)}\hat{P}'\Psi'[\mathbf{A}] &= e^{2\mathrm{i}\theta_{\Psi,U^{(1)}}}\Psi'[\mathbf{A}^P]\end{aligned}$$

If some $\theta_{\Psi,\tilde{U}^{(1)}} \neq \theta_{\Psi,\tilde{U}^{(1)P}}$ $\quad [U, \hat{P}'] \neq 0$

Physical states are not eigenstates of $\hat{P}'$ and ordinary parity P

Parity is **not a good quantum number** and is violated

P violation in QCD is directly related to how **gauge transformations** act on **physical states** / how the physical Hilbert space is defined

Physical states in simple Lie Group

$$U\Psi'[\mathbf{A}] = \Psi'[\mathbf{A}_{U(\mathbf{x})}] = e^{i\theta_{\Psi,U}}\Psi'[\mathbf{A}]$$

The $\theta_{\Psi,U}$ are a **1D representation** of the gauge group

$$\theta_{\Phi,U_1U_2} = \theta_{\Phi,U_1} + \theta_{\Phi,U_2}, \quad \theta_{\Phi,U_1^{-1}} = -\theta_{\Phi,U_1}$$

In a **simple Lie Group**, any element can be written as

$$U = U_1U_2U_1^{-1}U_2^{-1}$$

$$\theta_{\Psi,U} = \theta_{\Psi,U_1} + \theta_{\Psi,U_2} - \theta_{\Psi,U_1} - \theta_{\Psi,U_2} = 0$$

In QCD, $\Psi'[\mathbf{A}]$ are invariant under gauge transformations

$$P' = \hat{P}', \quad [H', P'] = 0, \quad [P', U] = 0$$

Parity is a **good quantum number** and it is conserved!

Back to original basis and partition function

$$\Psi[\mathbf{A}] = \mathrm{e}^{\mathrm{i}\theta W[\mathbf{A}]}\Psi'[\mathbf{A}]$$ gauge invariant

$\Psi'[\mathbf{A}]$ are eigenstates of $\hat{P}'$ with $\hat{P}'^2 = \mathbb{I}$

$$\Psi[\mathbf{A}^P] = \pm \mathrm{e}^{-2\mathrm{i}\theta W[\mathbf{A}]}\Psi[\mathbf{A}]$$

Under parity transformations with CSN changing sign,

$$\mathrm{i}S \underset{P}{\longrightarrow} \mathrm{i}S - 2\mathrm{i}\theta(W[\mathbf{A}_f] - W[\mathbf{A}_i])$$

$+2\mathrm{i}\theta W[\mathbf{A}_f]$ $-2\mathrm{i}\theta W[\mathbf{A}_i]$ $-2\mathrm{i}\theta W[\mathbf{A}_f] + 2\mathrm{i}\theta W[\mathbf{A}_i]$

$$\langle 0|e^{-iHT}|0\rangle = \int [\mathcal{D}\phi_f]_{T/2}[\mathcal{D}\phi_i]_{-T/2}\Psi_0[\phi_f]^*\Psi_0[\phi_i]\int_{\phi_i,\phi_f,T}\left(\prod \mathcal{D}\phi\right) e^{\mathrm{i}S}$$

The **partition function** is **parity invariant!**

Cancellation of θ

θ **disappears** from the **partition function**

$$e^{-i\theta W[\mathbf{A}_f]}\Psi_0'[\mathbf{A}_f] \qquad e^{i\theta W[\mathbf{A}_i]}\Psi_0'[\mathbf{A}_i] \qquad e^{i\theta(W[\mathbf{A}_i]-W[\mathbf{A}_f])}$$

$$\langle 0|e^{-iHT}|0\rangle = \int [\mathcal{D}\mathbf{A}_f]_{T/2}[\mathcal{D}\mathbf{A}_i]_{-T/2}\Psi_0[\mathbf{A}_f]^*\Psi_0[\mathbf{A}_i]\int_{\mathbf{A}_i,\mathbf{A}_f,T}\left(\prod \mathcal{D}\phi\right)e^{iS+iS_\theta}$$

$$= \int [\mathcal{D}\mathbf{A}_f]_{T/2}[\mathcal{D}\mathbf{A}_i]_{-T/2}\Psi_0'[\mathbf{A}_f]^*\Psi_0'[\mathbf{A}_i]\int_{\mathbf{A}_i,\mathbf{A}_f,T}\left(\prod \mathcal{D}\phi\right)e^{iS}$$

This leads again to θ**-independent correlators**

Our limiting procedure for $T\to\infty$ in the first part of the talk can be interpreted as **selecting the correct vacuum state**, leading to a cancellation of all θ dependence

Where do we depart from the usual results?

In the allowed eigenvalues under gauge transformations!

In the literature, it is assumed that **large gauge transformations**

$$U^{(n)}(\mathbf{x}) \quad \text{with} \quad W[\mathbf{A}_{U^{(n)}}] = W[\mathbf{A}] + n$$

can have **eigenvalues** $e^{in\theta'} \neq 1$ leading to **CPV.** E.g. **theta vacua**

$$|0\rangle = \sum_{m=-\infty}^{\infty} e^{im\theta'} |m\rangle, \quad U^{(n)}|m\rangle = |m+n\rangle$$

This is **not consistent** with the **full group structure**, only possible when **artificially restricting** to $\mathrm{U}(\mathbf{x}) \underset{|\mathbf{x}|\to\infty}{\longrightarrow} 1$

The resulting states are not eigenstates of $\mathrm{U}(\mathbf{x}) \underset{|\mathbf{x}|\to\infty}{\not\longrightarrow} 1$ and so **unphysical**

Where do we depart from the usual results?

We can have normalizable physical states

Note that the usual **theta vacua** are **not normalizable,** which goes **against postulates of quantum mechanics**

$$\langle\theta|\theta'\rangle = \delta(\theta - \theta')$$

We can define normalizable states by **restricting inner products** to a **gauge-fixed field hypersurface**

Conclusions

In the path integral formalism in infinite volume (ensuring projection into vacuum):

QCD with an arbitrary θ **does not predict CP violation**, as long as the sum over topological sectors is performed at **infinite volume**

This **ordering of limits** is the correct one because the topological classification is only enforced for an infinite volume

In canonical quantization in **pure gauge** theory:

The **phase** of the **wave functionals** of stationary states is **correlated** with θ

This leads to **parity** being **conserved**, and leads to θ-**independent path integrals**

Thank you!

Additional material

Finite volumes from an infinite spacetime

- We aim to derive an **effective finite-volume description starting from an infinite-volume path integral** guaranteed to capture the vacuum state

- The finite volume description can can help make **contact with results from canonical quantization and with lattice computations**

Finite volumes in an infinite spacetime

Assume **local operator** $\mathcal{O}_1$ with **support** in finite spacetime volume Ω_1

$$\langle \mathcal{O}_1 \rangle_\Omega = \frac{\sum_{\Delta n=-\infty}^{\infty} e^{i\Delta n \theta} \int_{\Delta n} \mathcal{D}\phi\, \mathcal{O}_1\, \mathrm{e}^{-S_\Omega[\phi]}}{\sum_{\Delta n=-\infty}^{\infty} e^{i\Delta n \theta} \int_{\Delta n} \mathcal{D}\phi\, \mathrm{e}^{-S_\Omega[\phi]}}$$

$$= \frac{\sum_{\Delta n=-\infty}^{\infty} \sum_{\Delta n_1=-\infty}^{\infty} e^{i\Delta n \theta} \int_{\Delta n_1} \mathcal{D}\phi\, \mathcal{O}_1\, \mathrm{e}^{-S_{\Omega_1}[\phi]} \int_{\Delta n_2 = \Delta n - \Delta n_1} \mathcal{D}\phi\, \mathrm{e}^{-S_{\Omega_2}[\phi]}}{\sum_{\Delta n=-\infty}^{\infty} \sum_{\Delta n_1=-\infty}^{\infty} e^{i\Delta n \theta} \int_{\Delta n_1} \mathcal{D}\phi\, \mathrm{e}^{-S_{\Omega_1}[\phi]} \int_{\Delta n_2 = \Delta n - \Delta n_1} \mathcal{D}\phi\, \mathrm{e}^{-S_{\Omega_2}[\phi]}} .$$

[Note: Integer Δn_1 is only an approximation, carried out in a surface kept finite, with reduced impact in full path integral.]

Finite volumes in an infinite spacetime

Path integrations over Ω_2 give just the **partition functions** we calculated before

In the **infinite volume** limit the **Bessel functions tend to common value** and dependence on Δn factorizes out and cancels:

$$\langle \mathcal{O}_1 \rangle_\Omega = \frac{\sum_{\Delta n_1=-\infty}^{\infty} \int_{\Delta n_1} \mathcal{D}\phi \, (-1)^{-N_f \Delta n_1} \mathrm{e}^{-\mathrm{i}\,\alpha \Delta n_1} \mathcal{O}_1 \, \mathrm{e}^{-S_{\Omega_1}[\phi]}}{\sum_{\Delta n_1=-\infty}^{\infty} \int_{\Delta n_1} \mathcal{D}\phi \, (-1)^{-N_f \Delta n_1} \mathrm{e}^{-\mathrm{i}\,\alpha \Delta n_1} \mathrm{e}^{-S_{\Omega_1}[\phi]}} \, .$$

We recover a **path integration** over a **finite volume, without θ dependence as in canonical quantization: CP is conserved**

Extra phases precisely **cancel those from fermion determinants in Ω_1**

Baluni's CP-violating effective Lagrangian

Baluni's CP-violating Lagrangian (used by [Crewther et al]) is based on searching for field redefinitions that minimize the QCD mass term

$$\mathcal{L}_M(U_{R,L}) = \bar{\psi} U_R^\dagger M U_L \psi_L + \text{h.c.}, \quad U_{R,L} \in SU_{R,L}(3)$$

$$\langle 0|\delta\mathcal{L}|0\rangle = \min_{U_{R,L}} \langle 0|\mathcal{L}_M(U_{R,L})|0\rangle$$

However, there is an **extra assumption**: that the **phase of the fermion condensate is aligned with**

$$\langle \bar{\psi}_R \psi_L \rangle = \Delta e^{\mathrm{i}c\theta} \mathbb{I}$$

This assumption **does not hold** for the chiral Lagrangian with $\xi = -\alpha$, but is valid for $\xi = \theta$

Crewther et al's calculations

Using [Baluni]'s CP-violating Lagrangian and current algebra [Crewther et al] get

$$\langle 0|\delta\mathcal{L}|\eta'\pi^0\pi^0\rangle = \frac{m_u m_d}{(m_u+m_d)^2}\frac{m_\pi^2}{f_\pi}\bar{\theta}\,.$$

From our general Chiral Lagrangian we get

$$\mathcal{L}_{\rm M}^{\rm EFT} \supset \frac{B_0 \sin(\xi+\alpha_u+\alpha_d)}{f_\pi\sqrt{\frac{1}{m_u^2}+\frac{1}{m_d^2}+\frac{2\cos(\xi+\alpha_u+\alpha_d)}{m_u m_d}}}\left[(\pi^0)^2+2\pi^+\pi^-\right]\eta'$$

$$\langle 0|\delta\mathcal{L}|\eta'\pi^0\pi^0\rangle = \frac{m_u m_d}{(m_u+m_d)^2}\frac{m_\pi^2}{f_\pi}(\xi+\bar{\alpha}),\,.$$

Match for $\xi=\theta$

So once more, traditional results are built on **(hidden) assumption** $\xi=\theta$

The η' mass

Chiral Lagrangian with alignment in the phases of mass terms and anomalous terms still predicts a **nonzero value of the ' mass**

$$\mathcal{L} = f_\pi^2 \mathrm{Tr}\partial_\mu U \partial^\mu U^\dagger + a f_\pi^3 \mathrm{Tr} M U + |b| e^{i \arg \det M} f_\pi^4 \mathrm{det} U + \mathrm{h.c.}$$

$$m_{\eta'}^2 = 8|b| f_\pi^2$$

Can be seen to be **proportional** to the **topological susceptibility** over **finite volumes** of the **pure gauge theory**, in line with [Witten, Di Vecchia & Veneziano]

Classic arguments linking topological susceptibility to CP violation ([Shifman et al]) rely on analytic expansions in which **don't apply** with our limiting procedure

Z **from infinite-volume partition function becomes non-analytic in** θ.
This possibility has been mentioned by [Witten]

[Witten, Nucl. Phys. B 156 (1979)]

the physics is of order e^{-N}, contrary to the basic assumptions of this paper, or else the physics is non-analytic as a function of θ, In the latter case, which is quite plausible, the singularities would probably be at $\theta = \pm\pi$, as Coleman found for the massive Schwinger model [10]. It is also quite plausible that θ is not really an angular variable.)

To write a formal expression for $d^2E/d\theta^2$, let us think of the path integral formulation of the theory:

$$Z = \int dA_\mu \exp i \int \mathrm{Tr}\left[-\tfrac{1}{4}F_{\mu\nu} + \frac{g^2\theta}{16\pi^2 N} F_{\mu\nu}\tilde{F}_{\mu\nu}\right]. \quad (5)$$

Partition function and analiticity

Usual partition function is analytic in θ

$$Z_{\text{usual}} = \lim_{VT\to\infty} \lim_{\substack{N\to\infty \\ N\in N}} \sum_{\Delta n=-N}^{N} Z_{\Delta n} = \mathrm{e}^{2\mathrm{i}\kappa_{N_f} VT\cos(\bar{\alpha}+\theta+N_f\pi)}$$

θ-dependence of observables (giving CP violation) usually relies on expansion. e.g.

$$\frac{\langle \Delta n\rangle}{\Omega} = \mathrm{i}\,(\theta-\theta_0)\left.\frac{\langle \Delta n^2\rangle}{\Omega}\right|_{\theta_0} + \mathcal{O}(\theta-\theta_0)^2$$

topological susceptibility

[Shifman et al]

In our limiting procedure the former is not valid, as Z becomes nonanalytic in θ

$$Z = \lim_{\substack{N\to\infty \\ N\in N}} \lim_{VT\to\infty} \sum_{\Delta n=-N}^{N} Z_{\Delta n} = I_0(2\mathrm{i}\kappa_{N_f} VT) \lim_{\substack{N\to\infty \\ N\in N}} \sum_{|\Delta n|\leq N} \mathrm{e}^{\mathrm{i}\Delta n(\bar{\alpha}+\theta+N_f\pi)}$$

θ drops out from observables, there is no CP violation

Dvali's footnote

[2] The 3-form language of [14] clarifies the claim of [24] that by changing the order of limits in ordinary instanton calculation, one ends up with $\vartheta = 0$. In this approach one performs calculation in the finite volume and then takes it to infinity. In 3-form language the meaning of this is rather transparent. The finite volume is equivalent of introducing an infrared cutoff in form of a shift of the massless pole in (28) away from zero. This effectively gives a small mass to the 3-form. For any non-zero value of the cutoff, the unique vacuum is $E_0 = 0$ which is equivalent to $\vartheta = 0$. Other states $E \neq 0$ (corresponding to $\vartheta \neq 0$) have finite lifetimes which tend to infinity when cutoff is taken to zero. In this way the $\vartheta \neq 0$ vacua are of course present but one is constrained to $\vartheta = 0$ by the prescription of the calculation. Thus, changing the order of limits by no means eliminates the ϑ-vacua. As usual, when taking the limit properly, one must keep track of states that become stable in that limit. These are the states with $\vartheta \neq 0$ ($E \neq 0$), which become the valid vacua in the infinite volume limit. The effect is in certain sense equivalent to introducing an auxiliary axion and then decoupling it.

Dvali's 3-form formalism

[Dvali] has the following line of reasoning from which he concludes that QCD violates CP

Nonzero topological susceptibility at zero momentum / full volume → Massless pole in CS current-current correlators → EFT for massless 3-form with CP violating vacua

With **our ordering of limits**, we have that the **topological susceptibility** is:

zero at zero momentum/full volume

nonzero at finite volume/nonzero momentum (matching lattice)

▶ **Dvali's first premise is violated and his argument does not apply**

Dvali's criticism

[Dvali] argues that in a calculation at finite volume which is then sent to infinity, CP violation can't be captured because the infrared regulation gives a mass to the 3 form.

We make the following observations:

- [\`t Hooft]'s original calculations (at finite volume, taken to ∞ in the end) lead to CP violation for arbitrary θ, in conflict with Dvali's argument
- If finite volume is problematic, more reason to take the infinite volume limit as soon as possible, as we do, leading to no CP violation for arbitrary
- Dvali's formalism has no explicit/direct link to UV parameter
- Dvali's critique of finite volumes can be turned against his own construction, as it is based on assuming nonzero topological susceptibility, while the only nonperturbative evidence for it comes from lattice results at finite volume

Dvali's criticism

[Dvali]'s construction can be seen to imply boundary conditions that do not correspond to vanishing physical fields at the boundary, and so does not capture the standard partition functions

$$\tilde{F}F \propto \partial_\mu K^\mu$$

[Dvali] argues

$$\partial_\mu K^\mu = \sqrt{\chi}\,\theta_L, \quad \text{const.}$$

- This implies a single frozen topological sector as $\Delta n \propto \int d^4x\, \partial_\mu K^\mu = \text{const}$
- Constant, gauge-invariant $\partial_\mu K^\mu$ does not vanish at the boundary
- No reason for periodicity in θ_L so no clear relation to usual θ angle
- Does not correspond to QCD partition function