UNIVERSITÄT BERN

AEC
ALBERT EINSTEIN CENTER
FOR FUNDAMENTAL PHYSICS

Factorization & Glauber gluons

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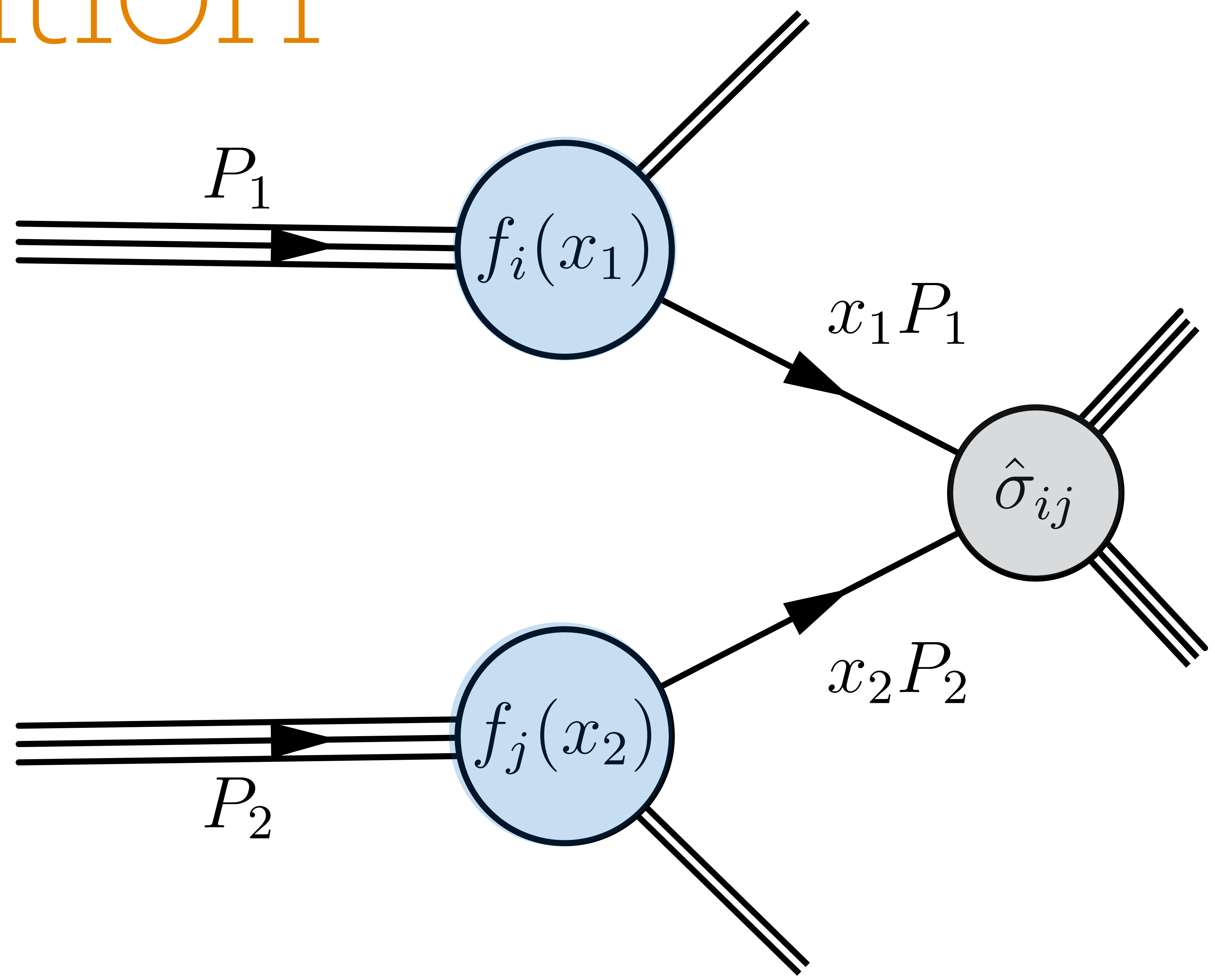
Based on 2408.10308, Phys.Rev.Lett. 134 (2025) 6, 061901

with Thomas Becher, Patrick Hager, Matthias Neubert & Sebastian Jaskiewicz

PDF factorization



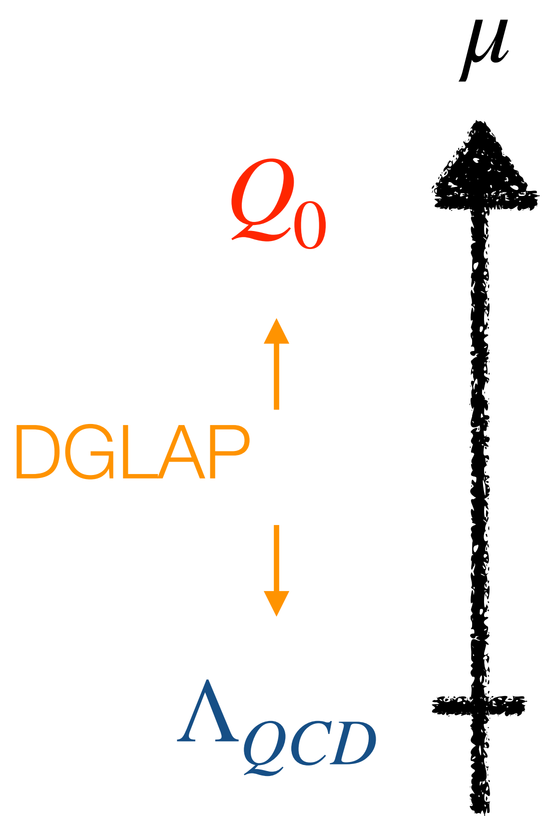
$$\sigma \sim f_{i,H}\, f_{j,H}\, \hat{\sigma}_{i,j}$$

- Factorization of long- and short–range physics
- Only proof for inclusive Drell-Yan

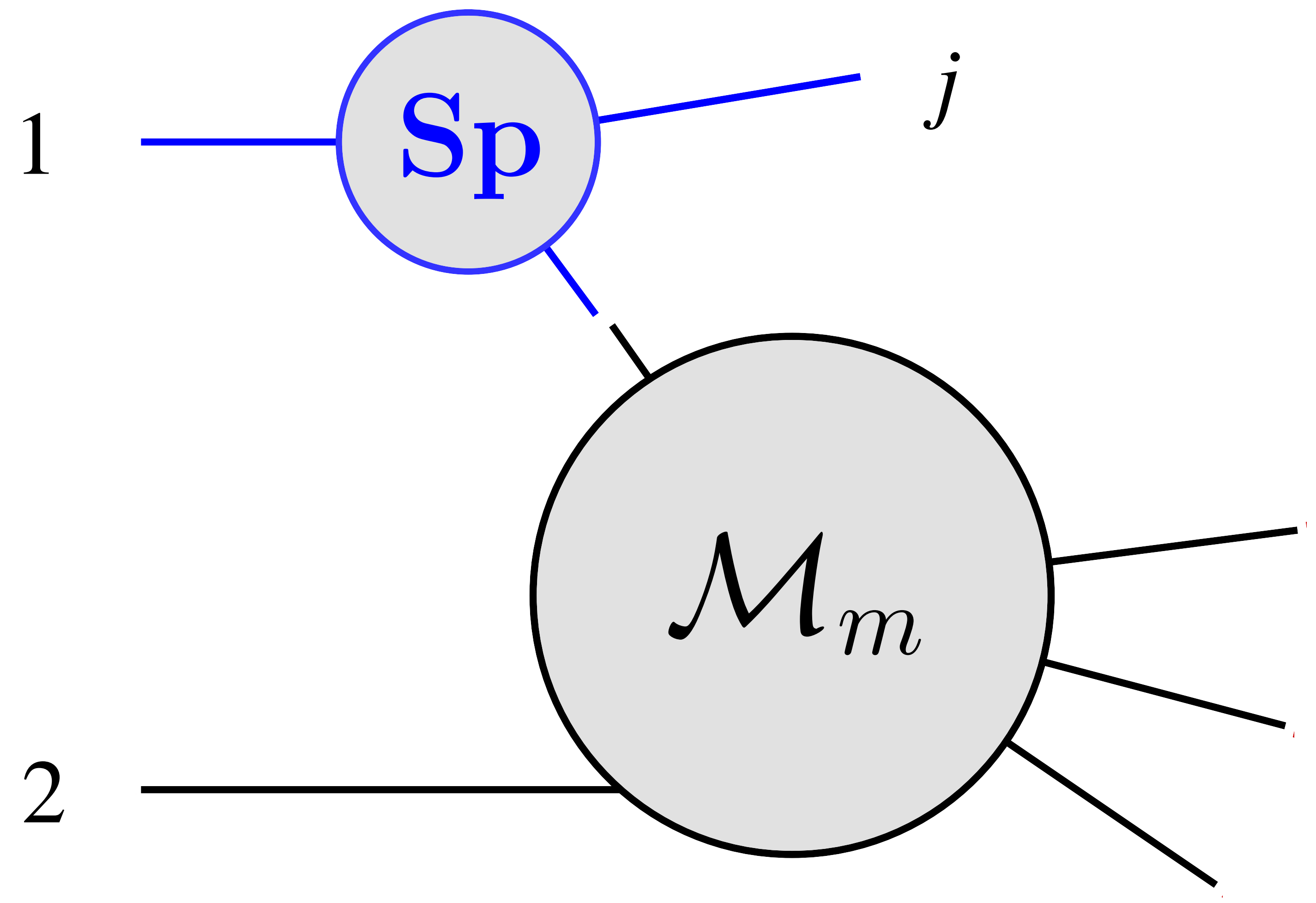
  CSS, '85/'88
- Crucially, Glauber phases cancel in this specific case

Splitting functions & DGLAP

- Take PDFs at scale Q_0 and evolve down to Λ_{QCD}
    - DGLAP evolution
- Use splitting functions P_{ij}

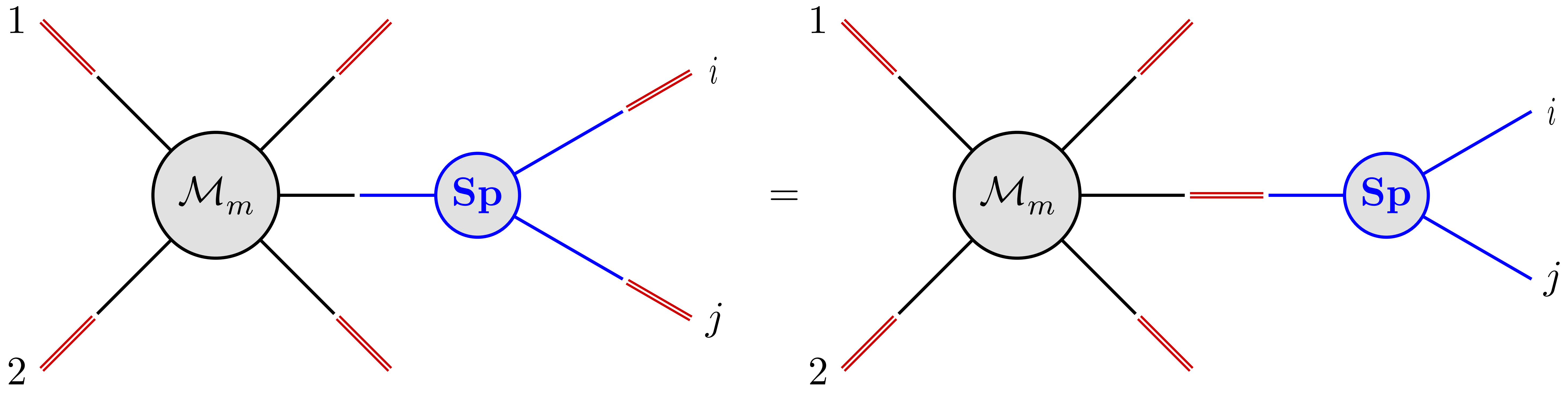


$$\mu^2 \frac{\partial}{\partial \mu^2} f_i(z,\mu) = \sum_{j=q,\bar{q},g} \int_z^1 \frac{dy}{y} P_{ij}\left(\frac{z}{y}\right) f_j(y,\mu)$$



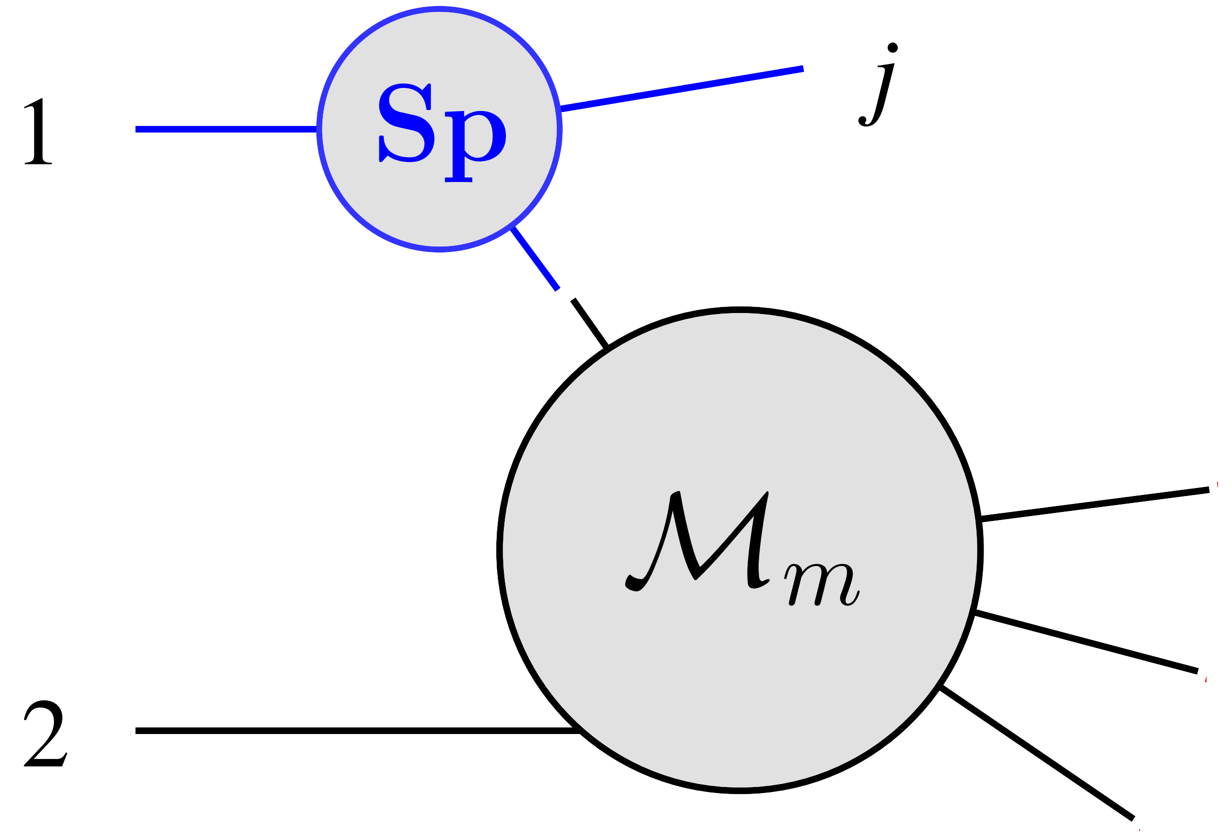
Collinear factorization

time-like splitting



factorization works !

space-like splitting



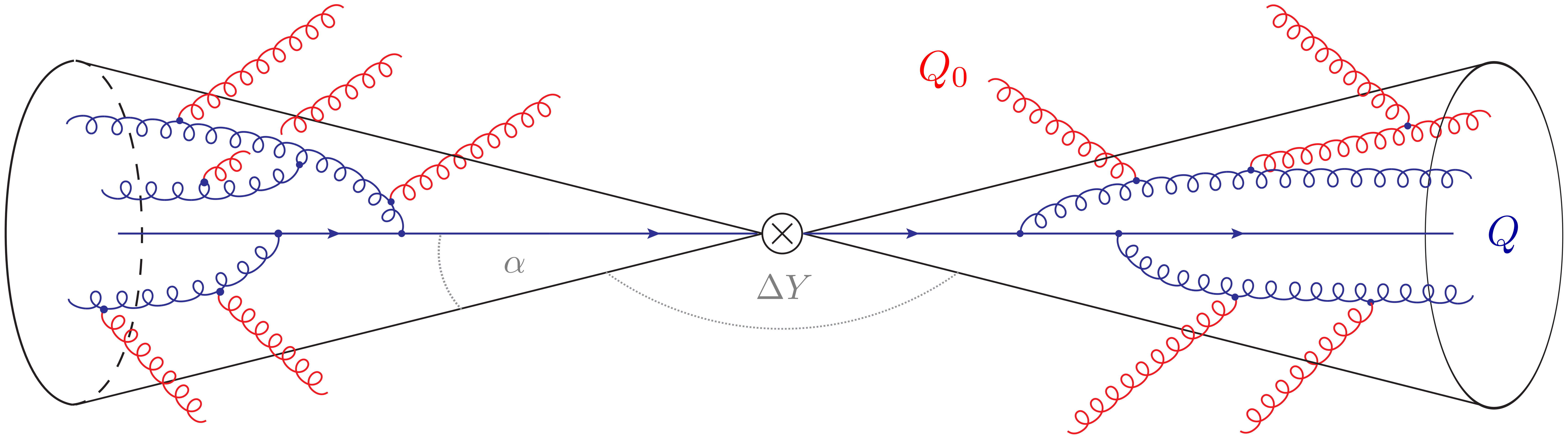
factorization violated!

Catani, de Florian, Rodrigo, '12

Observable

- Look at gap-in-between-jets cross section with veto scale Q_0
- Only soft radiation into the gap
- Jet scale Q and gap ΔY

Factorization sensitive observable!



Resummation

- For $Q_0 \ll Q$ in case of hadron colliders we expect

$$\sigma \sim \sigma_B + \alpha_s \ln((Q/Q_0)) + \alpha_s^2 \ln^2(Q/Q_0) + \ldots + \alpha_s^4 \ln^5(Q/Q_0) + \ldots$$

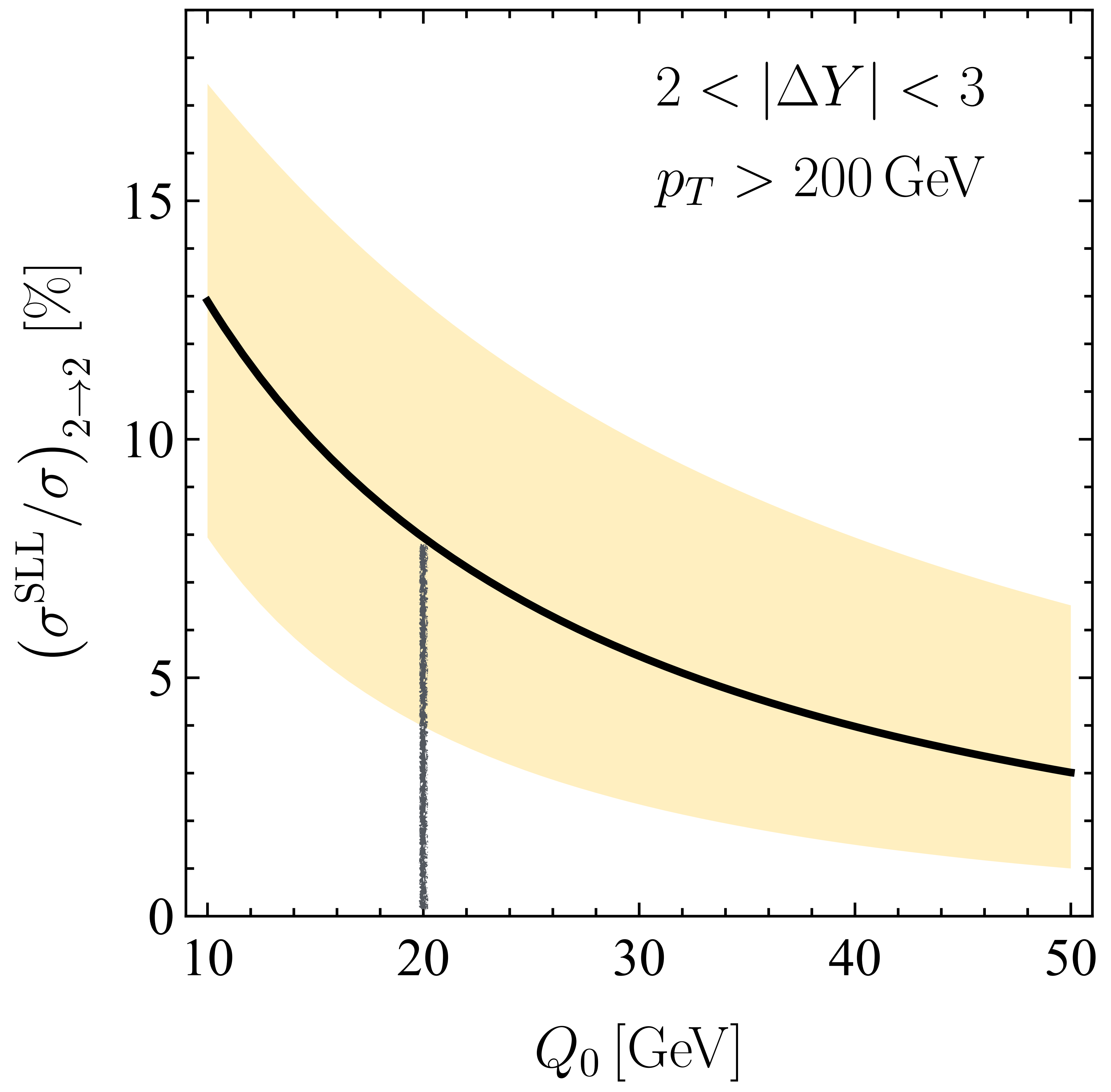
Forshaw, Kyrieleis, Seymour '06 '08

- Super-leading logarithms appear at hadron colliders due to Glauber phases
- Formally leading logarithmic effect but
- suppressed in color & loop order
- Size?

SLLs are directly connected to factorization violation!

Phenomenological relevance

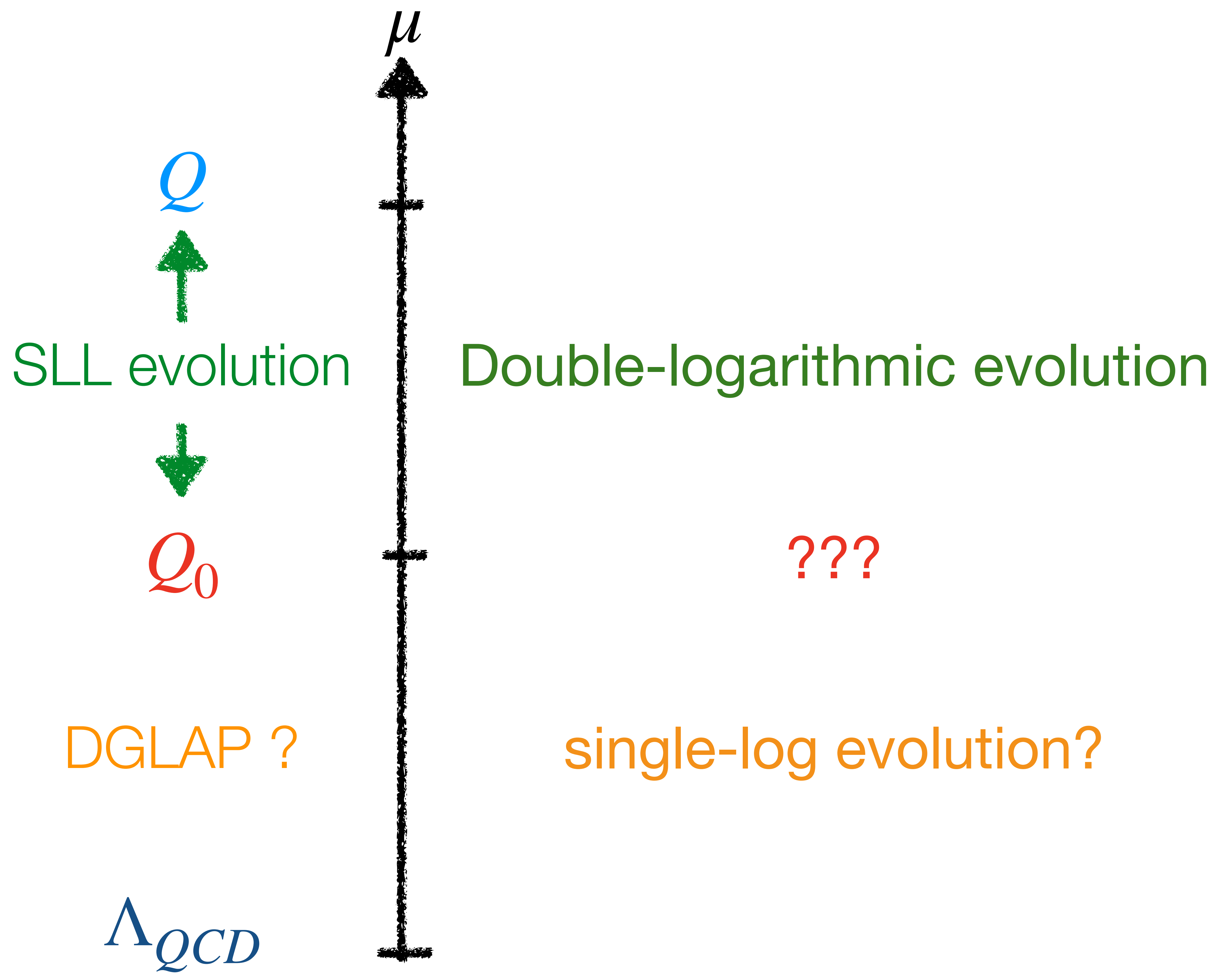
Becher, Hager, Martinelli, Neubert, DS, Stillger '25



process	$\sigma_{2\to2}$ [pb]	$\sigma^{\rm SLL}_{2\to2}$ [pb]	process	$\sigma_{2\to2}$ [pb]	$\sigma^{\rm SLL}_{2\to2}$ [pb]
$qq \to qq$	231.5	12.0	$q\bar{q} \to gg$	12.4	-0.9
$qq' \to qq'$	454.4	22.2	$qg \to qg$	4104.6	403.3
$q\bar{q} \to q\bar{q}$	142.0	7.4	$gg \to q\bar{q}$	57.5	-4.4
$q\bar{q}' \to q\bar{q}'$	372.9	18.0	$gg \to gg$	2281.1	150.6
$q\bar{q} \to q'\bar{q}'$	3.6	<0.1			
$\sum$	1204.4	59.6	$\sum$	6455.6	548.6
$\sum_{\text{all channels}}$		7660.0		608.2	

- >10% for small values of Q_0
- Biggest contribution through gluonic channels
- Full cross section for the first time

Evolution



Requires highly non-trivial interplay for consistency with DGLAP!

Upshot of the talk

Collinear factorization breaking at $\mu = Q$

x

soft-collinear factorization breaking by Glauber modes at $\mu = Q_0$

=

PDF factorization for $\mu < Q_0$

"factorization restoration"

Outline

- Explain method of regions using a basic example and SCET

  → Important for analysis later on

- Factorization theorem and appearance of SLL
- RG-consistency check for the low-energy matrix element
- Consistency with DGLAP evolution

Method of regions

Beneke,Smirnov '98, Smirnov '02

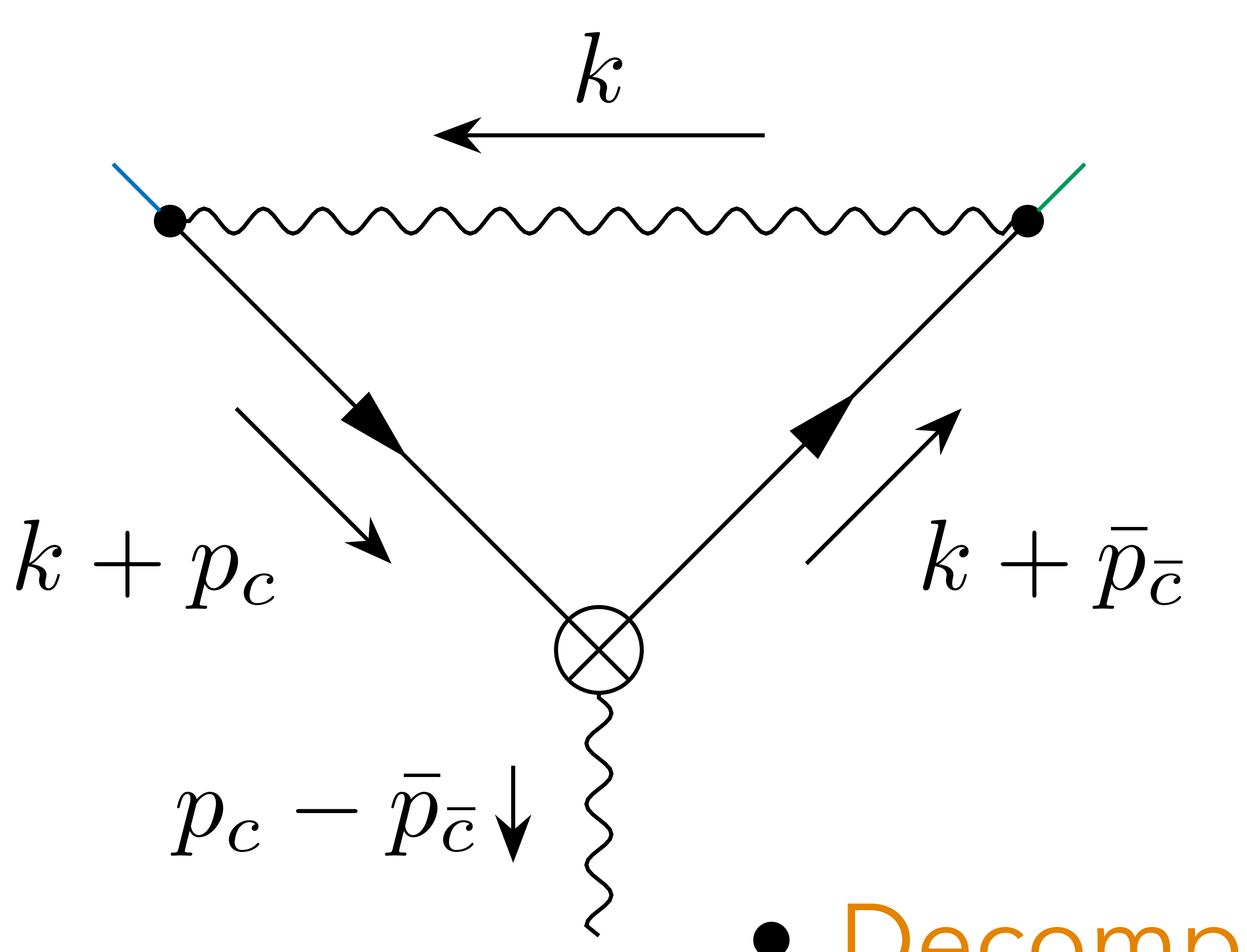
- Introduce light-cone vectors

$$n_\mu = (1,0,0,1) \quad \text{and} \quad \bar{n}_\mu = (1,0,0,-1) \qquad p^\mu = (n\cdot p)\frac{\bar{n}^\mu}{2} + (\bar{n}\cdot p)\frac{n^\mu}{2} + p_\perp^\mu \equiv p_+^\mu + p_-^\mu + p_\perp^\mu$$

- And expansion parameter λ such that

$$\bar{p}^\mu_{\bar{c}} = \underbrace{\bar{p}^\mu_{\bar{c}+}}_{\lambda^2} + \underbrace{\bar{p}^\mu_{\bar{c}-}}_{1} + \underbrace{\bar{p}^\mu_{\bar{c}\perp}}_{\lambda} \qquad p^\mu_c = \underbrace{p^\mu_{c+}}_{\lambda^2} + \underbrace{p^\mu_{c-}}_{1} + \underbrace{p^\mu_{c\perp}}_{\lambda}$$

collinear anti-collinear



$$I = i\pi^{-d/2}\mu^{4-d}\int d^dk \frac{1}{\left(k^2+i0\right)\left[(k+\bar{p}_{\bar{c}})^2+i0\right]\left[(k+p_c)^2+i0\right]}$$

- Decompose loop momentum

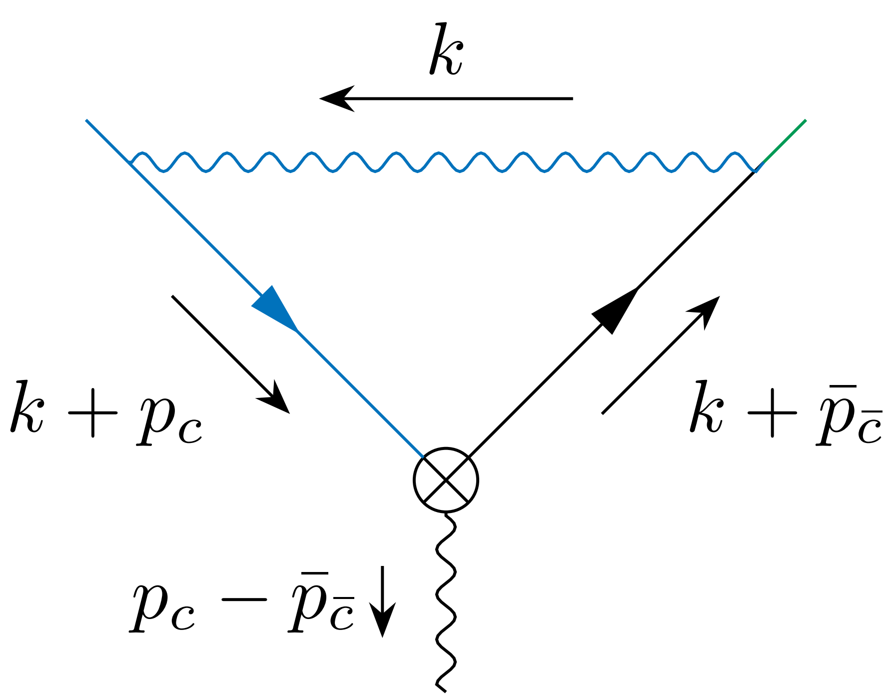
$$k^\mu = \underbrace{k_+^\mu}_{?} + \underbrace{k_-^\mu}_{?} + \underbrace{k_\perp^\mu}_{?}$$

Method of regions

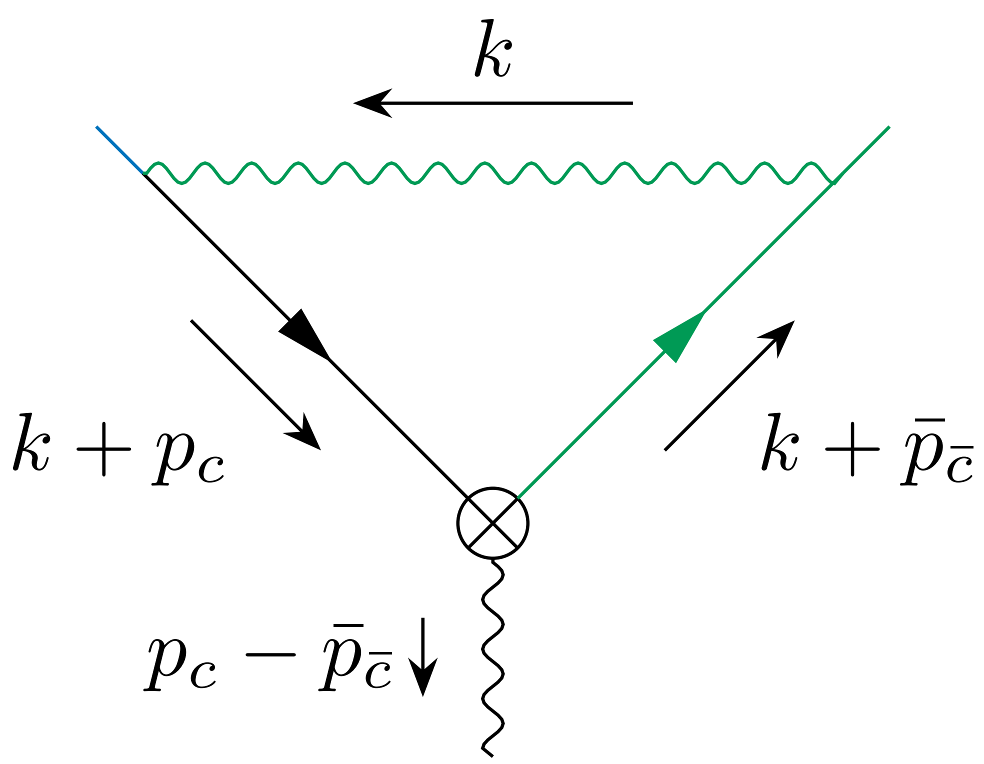
- Expand loop momentum k in different regions

$(+, -, \perp)$

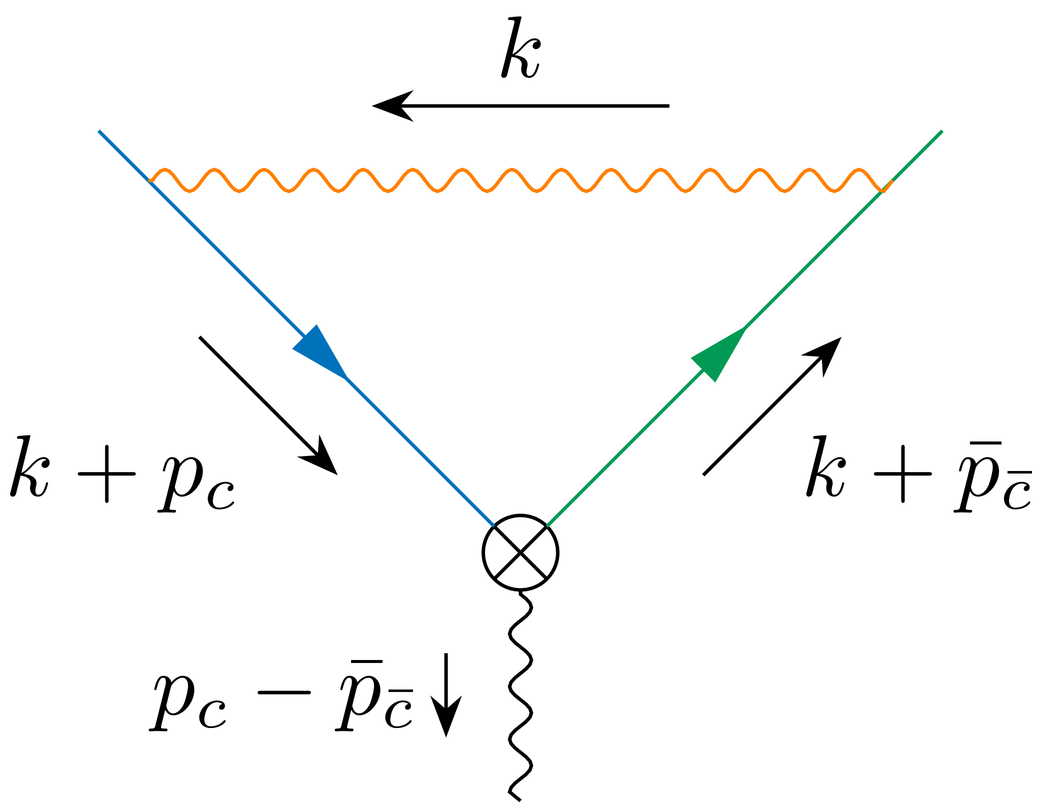
$k \sim$ collinear



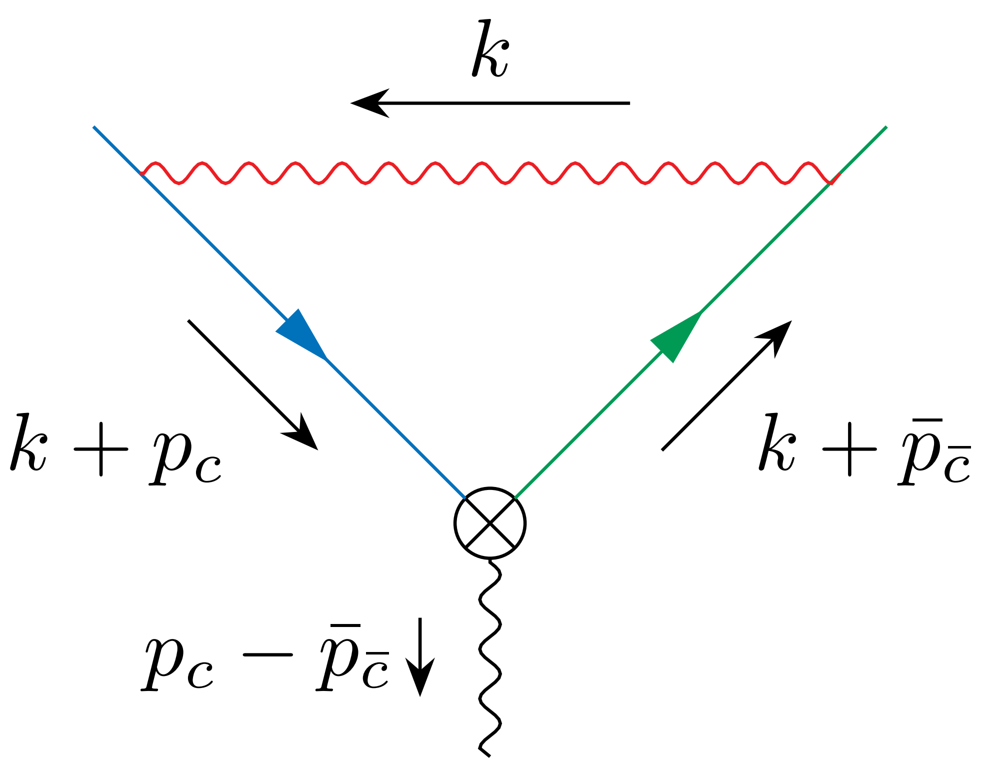
$k \sim$ anti-collinear



$k \sim$ soft $(\lambda^2, \lambda^2, \lambda^2)$



$k \sim$ glauber $(\lambda^2, \lambda^2, \lambda)$



vanishes!

- In collinear region this becomes e.g.

$$I_c = i\pi^{-d/2}\mu^{4-d}\int d^dk \frac{1}{\left(k^2 + i0\right)} \frac{1}{\left(2k_- \cdot \bar{p}_{\bar{c}+} + i0\right)} \frac{1}{\left[(k + p_c)^2 + i0\right]}$$

- After evaluating all regions

$$I = I_h + I_c + I_s + I_{\bar{c}}$$

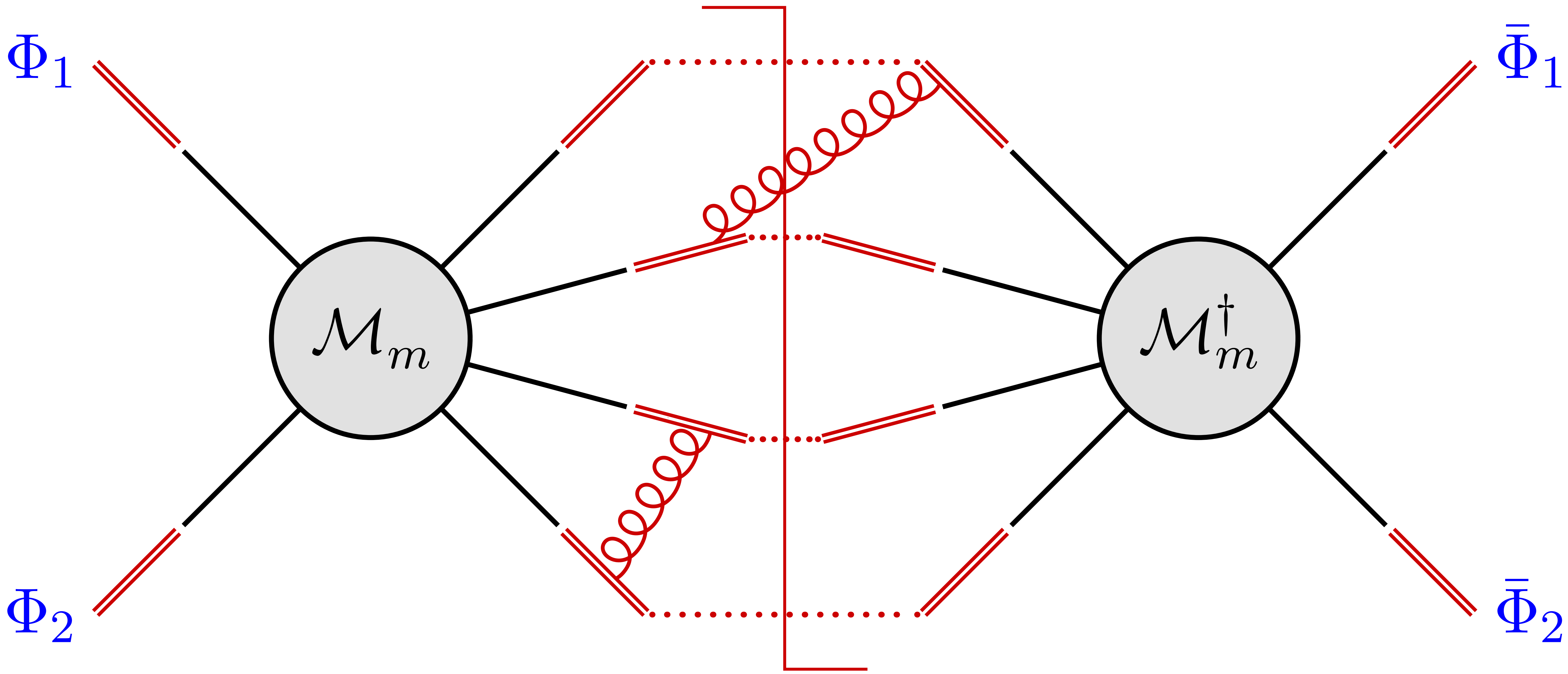
SCET

Soft collinear effective theory

Bauer, Pirjol, Stewart '02,+Fleming, Rothstein '02, Beneke,Chapovsky,Diehl,Feldmann'02

- Based on the method of regions
  - Integrate out „hard" region via Wilson coefficient
  - Left is collinear and soft physics
- Split fields according to MoR
  - $\mathscr{A} \to \mathscr{A}_c + \mathscr{A}_s$ and $\psi \to \psi_c + \psi_s$
  - Each collinear sector has its own gauge symmetry, one single soft background
    - Especially useful for factorization proofs & resummation

Resummation



- For $Q_0 \ll Q$ we can derive

$$\sigma_{2\to M}(Q_0) = \int dx_1 \int dx_2 \sum_{m=2+M}^{\infty} \langle \mathcal{H}_m(\{\underline{n}\}, s, x_1, x_2, \mu) \otimes \mathcal{W}_m(\{\underline{n}\}, Q_0, x_1, x_2, \mu) \rangle$$

Becher, Neubert, Shao, '21+Stillger'23

- Factorization between hard and soft-collinear physics

RG- evolution

- Renormalized hard functions fulfill RG equation

$$\frac{d}{d\ln\mu}\mathcal{H}_m = -\sum_{l=m_0}^{m} \mathcal{H}_l \, \boldsymbol{\Gamma}^H_{lm}$$

matrix in multiplicity and color space

- One-loop hard anomalous dimension:

$$\boldsymbol{\Gamma}^H = \gamma_{cusp}(\alpha_s)\left(\boldsymbol{\Gamma}^c \ln\frac{\mu^2}{Q^2} + \boldsymbol{V}^G\right) + \frac{\alpha_s}{4\pi}\overline{\boldsymbol{\Gamma}} + \boldsymbol{\Gamma}^C$$

cusp-piece soft+collinear

generates SLLs

$\propto i\pi$ Glauber

purely soft

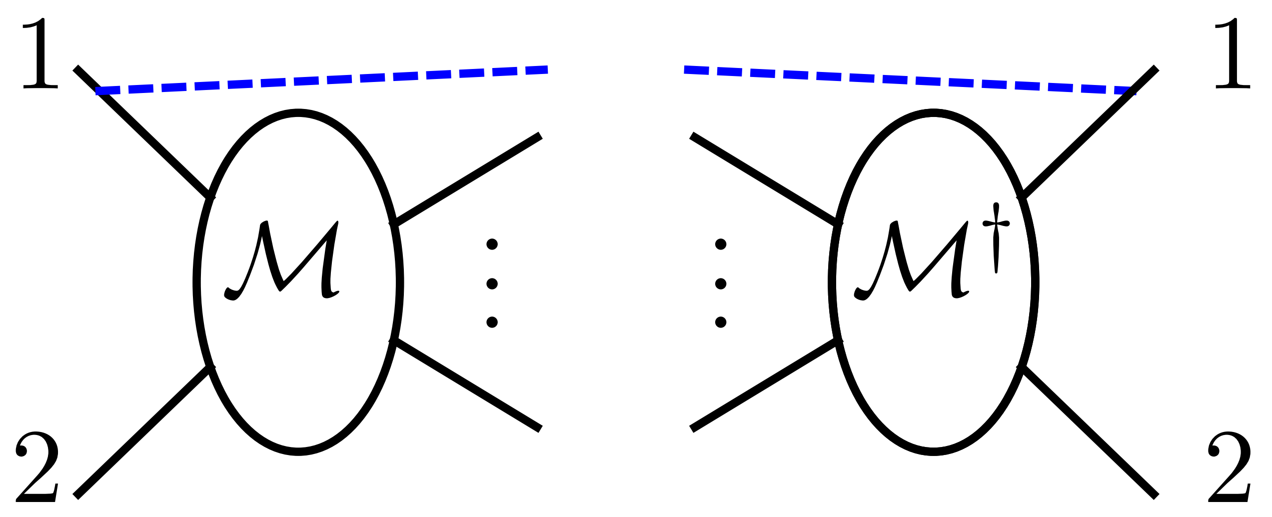
generates NGLs

purely collinear

generates DGLAP

Cusp terms

$$\mathcal{H}_m \, \boldsymbol{R}_1^c \; = $$



SLLs are directly connected to factorization violation!

$$\boldsymbol{R}_i^c = -4\boldsymbol{T}_{i,L} \circ \boldsymbol{T}_{i,R} \, \delta(n_{m+1} - n_i)$$

$$\boldsymbol{V}_i^c = 4C_i \, \mathbf{1}$$

space-like splitting from before

time-like splitting from before

- These are only present for initial states $i = 1,2$, for final states they cancel
- They are multiplied by $\ln \frac{\mu^2}{Q^2}$ and give rise to double-logarithmic running!

SLLs from RG evolution

- Evolve hard function from $\mu_h \sim Q$ to $\mu_s \sim Q_0$

$$\frac{d}{d\ln\mu}\mathscr{H}_m = -\sum_{l=m_0}^{m}\mathscr{H}_l\,\boldsymbol{\Gamma}^H_{lm}$$

$$\sigma(Q,Q_0) = \sum_{m,l=m_0}^{\infty}\int d\xi_1 d\xi_2\,\langle \mathscr{H}_m(Q,\mu_h)\,U_{ml}(\mu_h,\mu_s)\otimes \mathscr{W}_l(Q_0,\mu_s)\rangle$$

$$U(\mu_h,\mu_s) = \boldsymbol{P}\,exp\left[\int_{\mu_s}^{\mu_h}\frac{d\mu}{\mu}\boldsymbol{\Gamma}^H\right]$$

$$= \boldsymbol{1} + \int_{\mu_s}^{\mu_h}\frac{d\mu_1}{\mu_1}\boldsymbol{\Gamma}^H + \int_{\mu_s}^{\mu_h}\frac{d\mu_1}{\mu_1}\int_{\mu_1}^{\mu_h}\frac{d\mu_2}{\mu_2}\boldsymbol{\Gamma}^H(\mu_1)\,\boldsymbol{\Gamma}^H(\mu_2)$$

„tower" of anomalous dimensions

$U(\mu_h,\mu_s)$ achieves resummation of logarithms

SLLs

- For finite N_c Glauber phases spoil collinear cancellations
  - Appearance of super-leading logarithms
  - Only very few structures possible

$$[\boldsymbol{\Gamma}^c, \boldsymbol{V}^G] \neq 0$$

$$[\boldsymbol{\Gamma}^c, \overline{\boldsymbol{\Gamma}}] = 0$$

$$\langle \mathscr{H}\, \boldsymbol{V}^G \rangle = \langle \mathscr{H}\, \boldsymbol{\Gamma}^c \rangle = 0$$

Glauber phase $\sim i\pi$

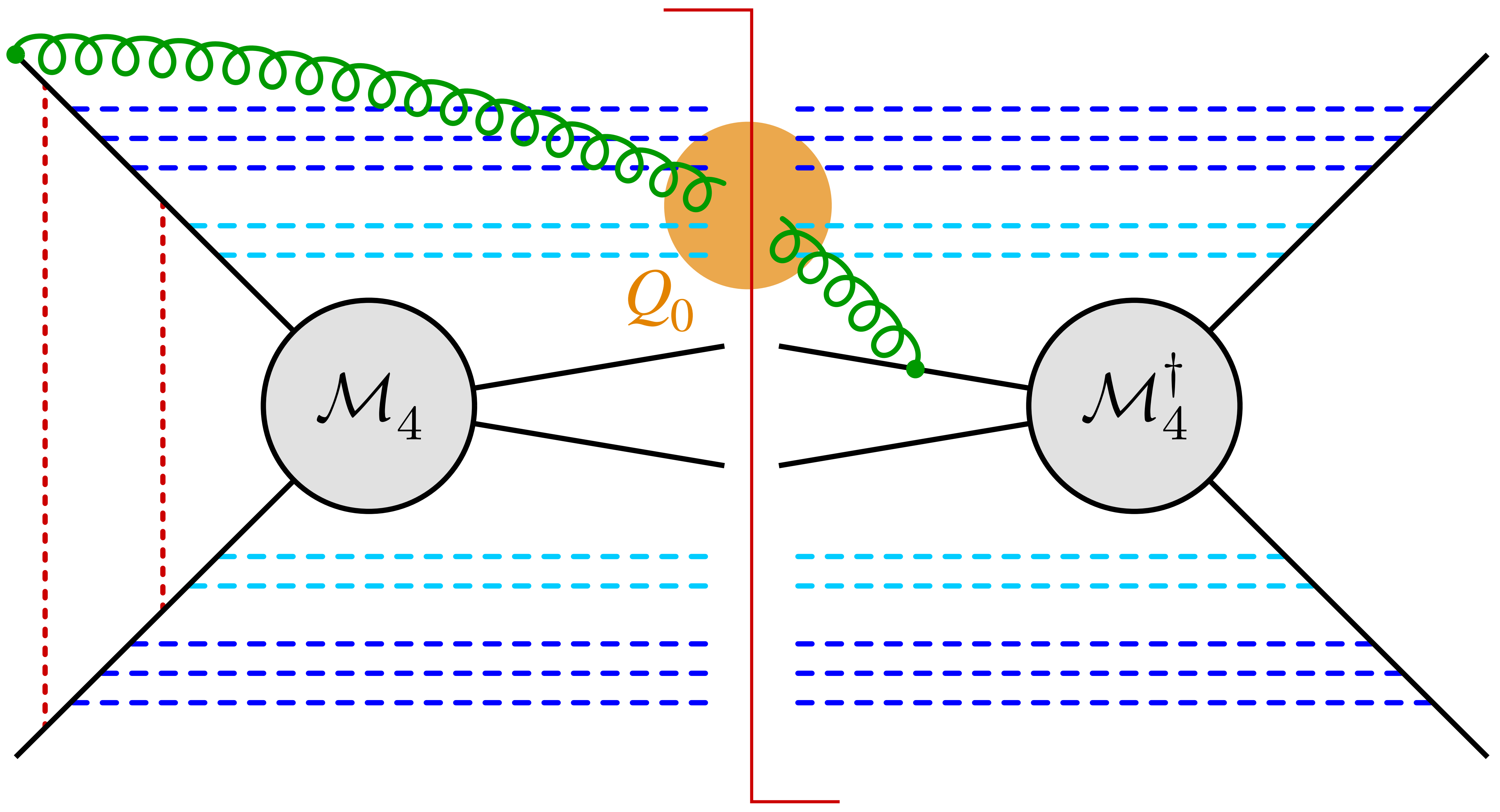
$$\sigma_{SLL} \sim \langle \mathscr{H} \, (\boldsymbol{\Gamma}^c)^{n-r} \, \boldsymbol{V}^G \, (\boldsymbol{\Gamma}^c)^r \, \boldsymbol{V}^G \, \overline{\boldsymbol{\Gamma}} \rangle$$

n insertions of cusp term

single emission into the gap

SLLs

$$\sigma_{SLL} \sim \langle \mathscr{H} \, (\boldsymbol{\Gamma}^c)^{n-r} \, \boldsymbol{V}^G \, (\boldsymbol{\Gamma}^c)^r \, \boldsymbol{V}^G \, \overline{\boldsymbol{\Gamma}} \rangle$$



Outline

- Explain method of regions using a basic example and SCET ✓

  Important for analysis later on (soon)

- Factorization theorem and appearance of SLL ✓

- RG-consistency check for the low-energy matrix element

  - Know that $\langle \mathscr{H}_m(\{\underline{n}\}, s, x_1, x_2, \mu) \otimes \mathscr{W}_m(\{\underline{n}\}, Q_0, x_1, x_2, \mu) \rangle$ must be finite!

    - Use $\langle \mathscr{H}_m^{\text{bare}} \mathscr{W}_m^{\text{bare}} \rangle = \langle \underbrace{(H_m^{\text{bare}} \mathbf{Z}^{-1})}_{\text{finite}} \underbrace{(\mathbf{Z} \mathscr{W}_m^{\text{bare}})}_{\text{finite}} \rangle$

RG-consistency

- Renormalization factor Z given by

derivative w.r.t. μ

$$\mathbf{Z} = \mathbf{1} + \frac{\alpha_s}{4\pi}\left(-\frac{\boldsymbol{\Gamma}_0'}{4\varepsilon^2} - \frac{\boldsymbol{\Gamma}_0}{2\varepsilon}\right) + \left(\frac{\alpha_s}{4\pi}\right)^2\left(\frac{\boldsymbol{\Gamma}_0'\boldsymbol{\Gamma}_0}{32\varepsilon^3} + \frac{\boldsymbol{\Gamma}_0^2}{8\varepsilon^2} + \dots\right) + \left(\frac{\alpha_s}{4\pi}\right)^3\left(-\frac{\boldsymbol{\Gamma}_0'\boldsymbol{\Gamma}_0^2}{288\varepsilon^4} - \frac{\boldsymbol{\Gamma}_0^3}{48\varepsilon^3} + \dots\right) + \mathcal{O}(\alpha_s^4)$$

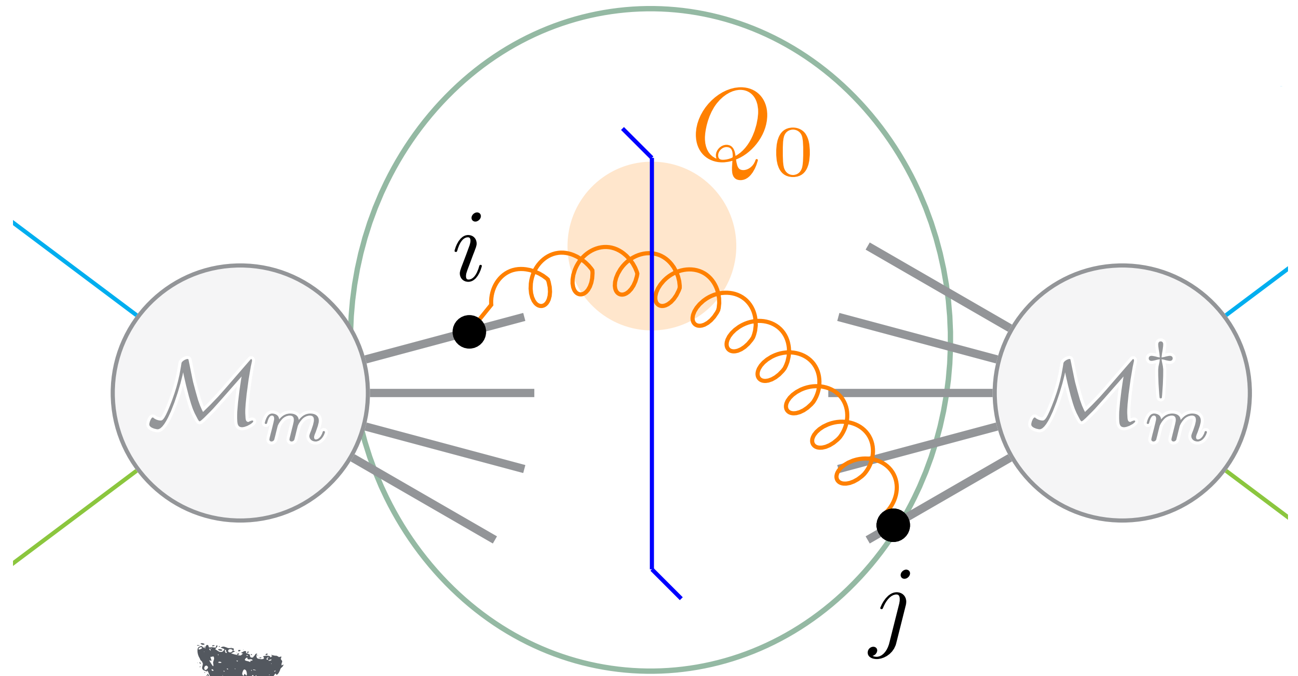
Becher, Neubert'09

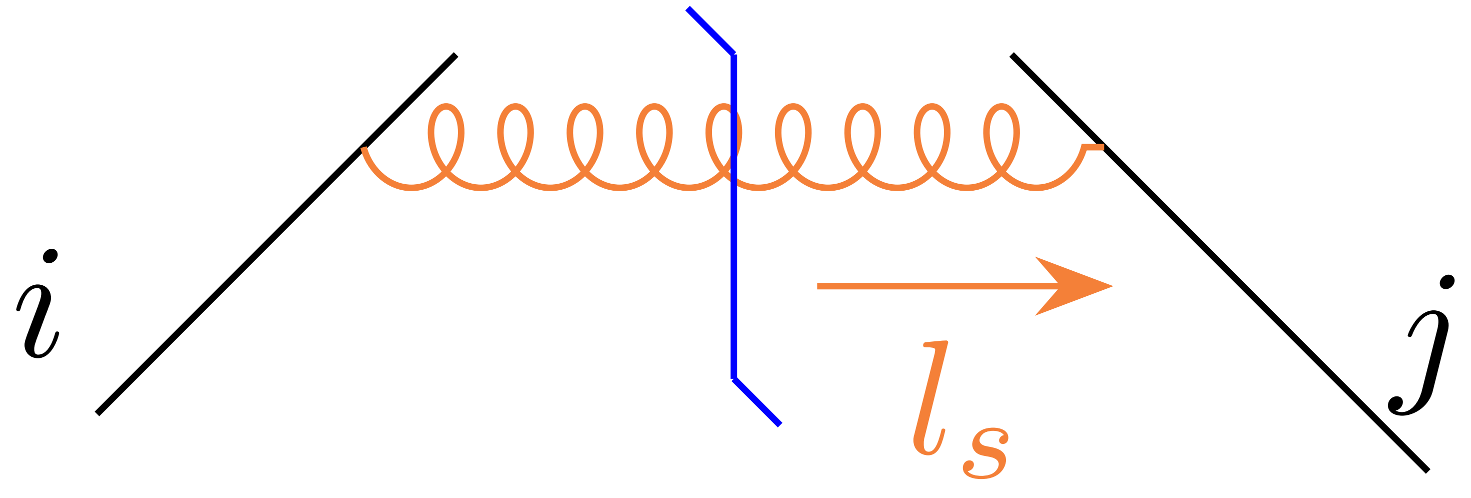
- Soft-collinear matrix element has to be rendered finite $\mathscr{W}_m(\mu) = Z \cdot \mathscr{W}_m^{\text{bare}}$ such that

$$\mathscr{W}_m^{\text{bare}} = \mathbf{1} + \frac{\alpha_s}{4\pi}\frac{\overline{\boldsymbol{\Gamma}}}{2\varepsilon} + \left(\frac{\alpha_s}{4\pi}\right)^2\left(\frac{\boldsymbol{V}^G\overline{\boldsymbol{\Gamma}}}{2\varepsilon^2} + \dots\right) + \left(\frac{\alpha_s}{4\pi}\right)^3\left[\frac{\boldsymbol{\Gamma}^c\boldsymbol{V}^G\overline{\boldsymbol{\Gamma}}}{3\varepsilon^3}\left(\frac{11}{6\varepsilon} + \ln\frac{\mu_s^2}{Q^2} + \frac{9}{2}\ln\frac{\mu_s^2}{Q_0^2}\right) + \frac{\boldsymbol{V}^G\boldsymbol{V}^G\overline{\boldsymbol{\Gamma}}}{3\varepsilon^3} + \dots\right] + \mathcal{O}(\alpha_s^4)$$

RG-consistency

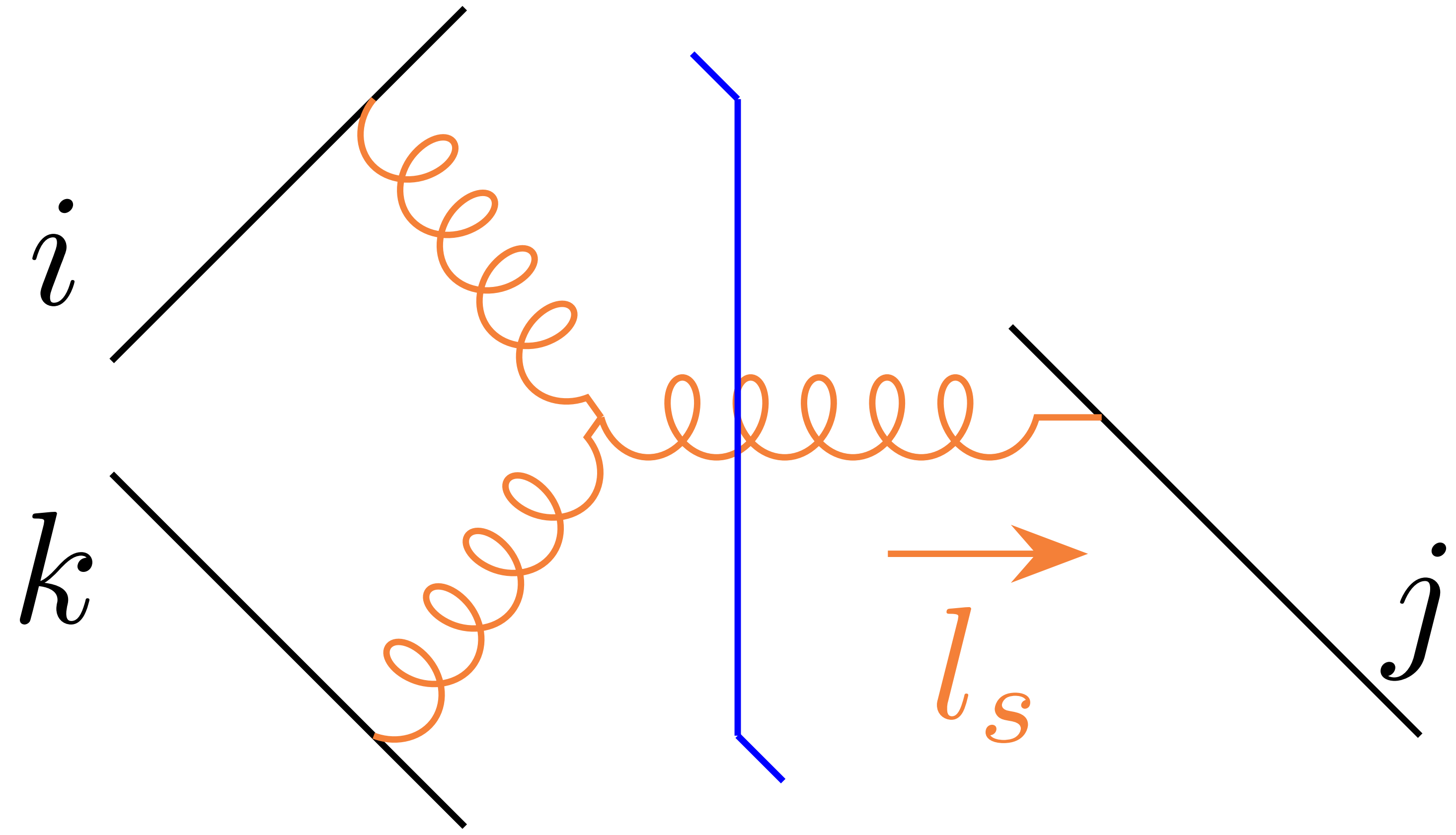
- Look at tree-level and one-loop diagrams first





eikonal factor

$$\sim \boldsymbol{J}^{\mu,a(0)} \boldsymbol{J}_\mu^{a(0)\dagger} \quad \text{with} \quad \boldsymbol{J}^{\mu,a(0)} = \sum_{i=1}^{m} \boldsymbol{T}_{iL}^a \frac{n_i^\mu}{n_i \cdot l_s}$$



one-loop soft current

Catani, Grazzini '00

- Matches structure

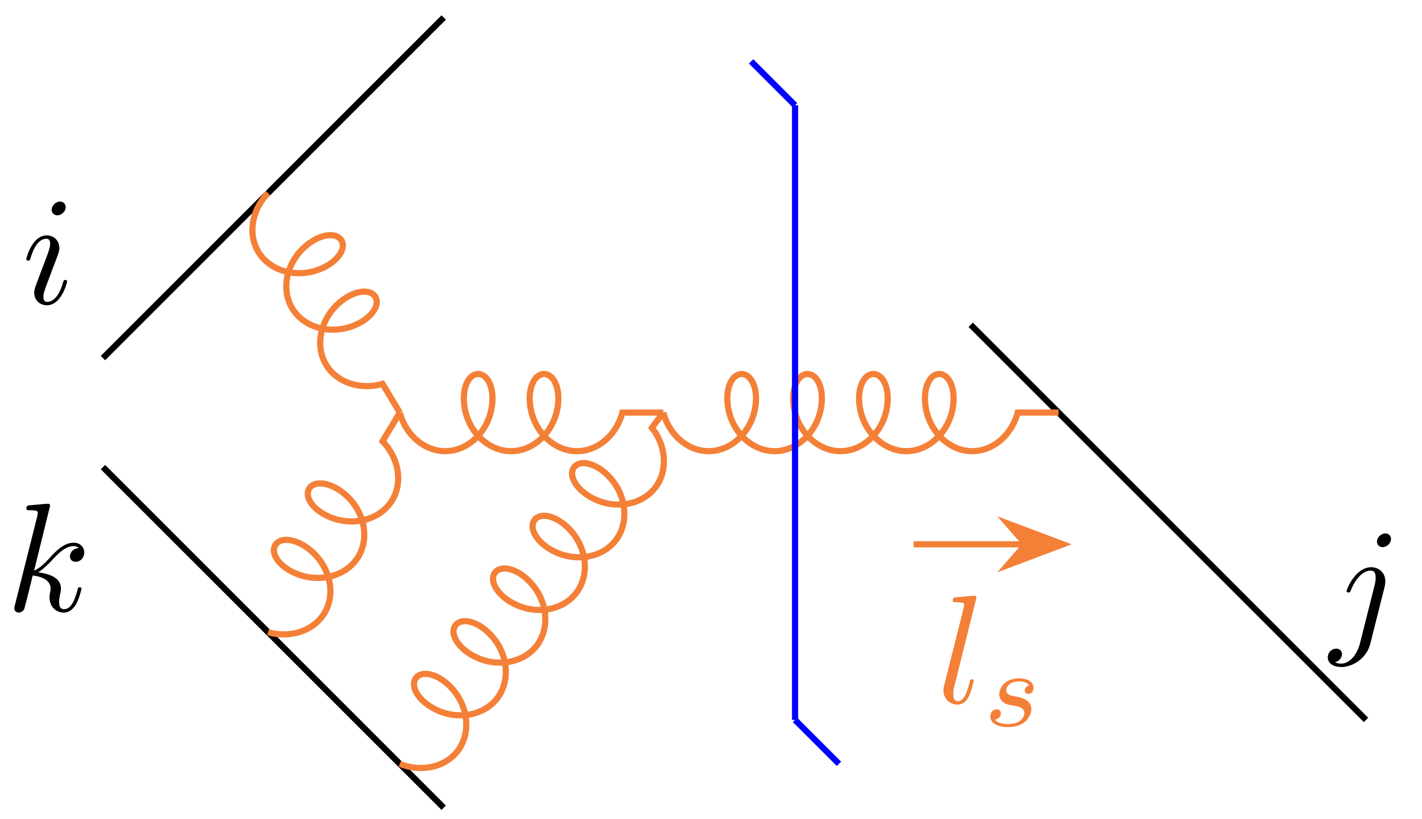
$$\mathscr{W}_m^{\text{bare}} = \mathbf{1} + \frac{\alpha_s}{4\pi} \frac{\overline{\boldsymbol{\Gamma}}}{2\varepsilon} + \left(\frac{\alpha_s}{4\pi}\right)^2 \left(\frac{\boldsymbol{V}^G \overline{\boldsymbol{\Gamma}}}{2\varepsilon^2} + \ldots\right) + \left(\frac{\alpha_s}{4\pi}\right)^3 \left[\frac{\boldsymbol{\Gamma}^c \boldsymbol{V}^G \overline{\boldsymbol{\Gamma}}}{3\varepsilon^3}\left(\frac{11}{6\varepsilon} + \ln\frac{\mu_s^2}{Q^2} + \frac{9}{2}\ln\frac{\mu_s^2}{Q_0^2}\right) + \frac{\boldsymbol{V}^G \boldsymbol{V}^G \overline{\boldsymbol{\Gamma}}}{3\varepsilon^3} + \ldots\right] + \mathcal{O}(\alpha_s^4)$$

✔ ✔

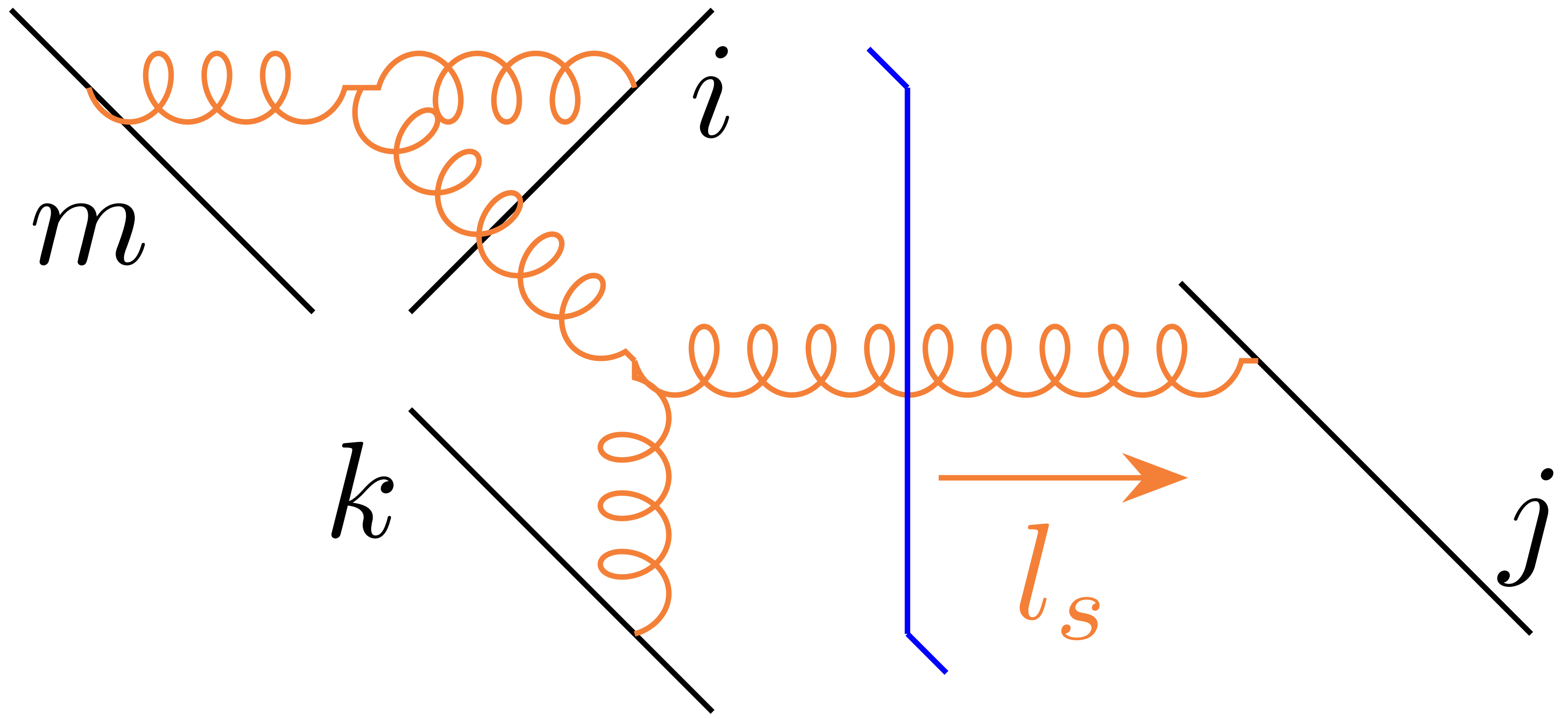
RG-consistency

- Now, go one loop further

Duhr, Gehrmann '13 / Dixon, Herrmann, Yan, Zhu '20



dipole terms



tripole terms

- Does not match all terms

Large logarithm

$$\mathscr{W}_m^{\text{bare}} = \mathbf{1} + \frac{\alpha_s}{4\pi}\frac{\overline{\boldsymbol{\Gamma}}}{2\varepsilon} + \left(\frac{\alpha_s}{4\pi}\right)^2\left(\frac{\boldsymbol{V}^G\overline{\boldsymbol{\Gamma}}}{2\varepsilon^2} + \dots\right) + \left(\frac{\alpha_s}{4\pi}\right)^3\left[\frac{\boldsymbol{\Gamma}^c\boldsymbol{V}^G\overline{\boldsymbol{\Gamma}}}{3\varepsilon^3}\left(\frac{11}{6\varepsilon} + \ln\frac{\mu_s^2}{Q^2} + \frac{9}{2}\ln\frac{\mu_s^2}{Q_0^2}\right) + \frac{\boldsymbol{V}^G\boldsymbol{V}^G\overline{\boldsymbol{\Gamma}}}{3\varepsilon^3} + \dots\right] + \mathcal{O}(\alpha_s^4)$$

✓ ✓ ✗ ✓

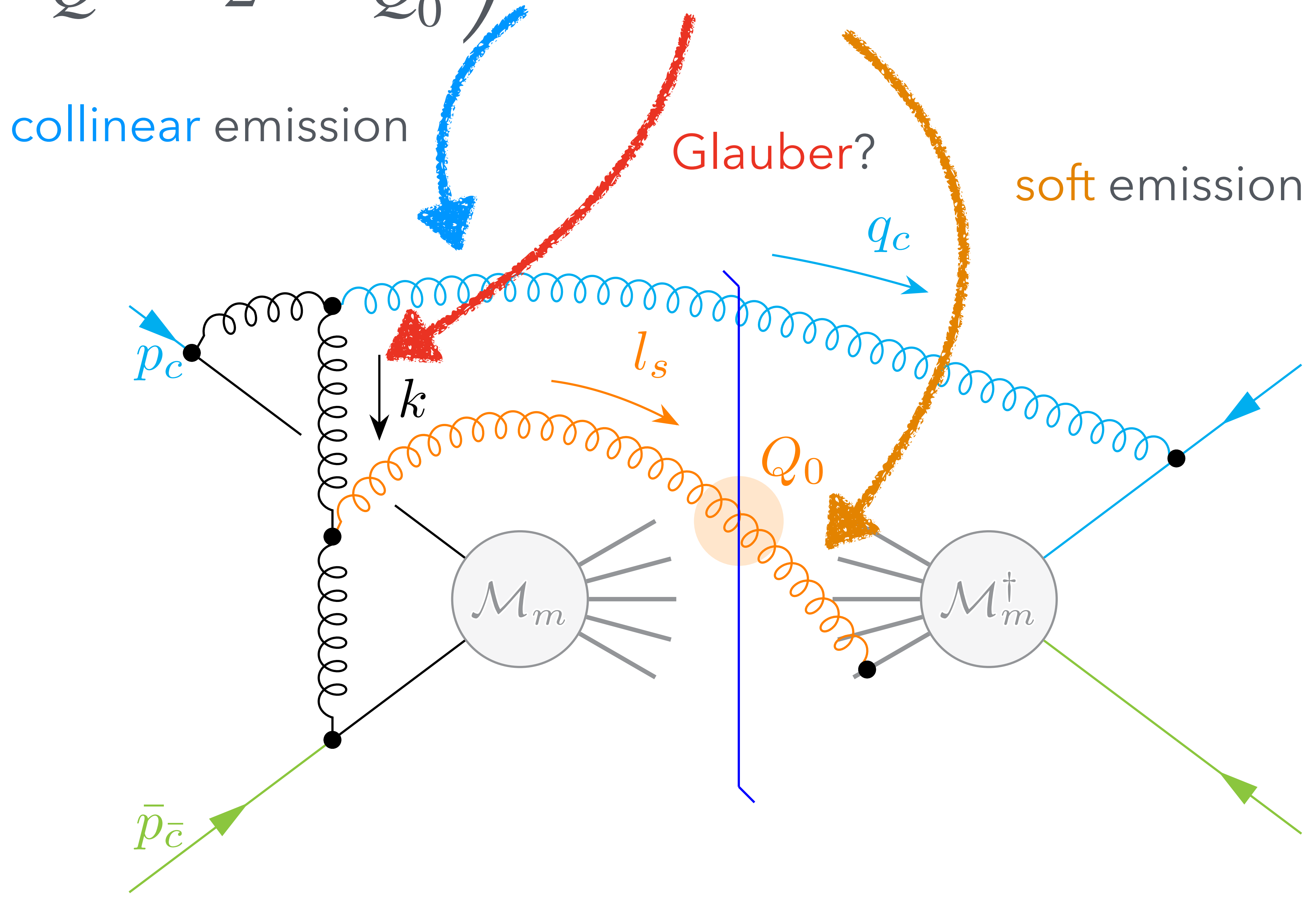
Three options to get $\ln Q$

1. Perturbative on-shell modes with virtuality below Q_0
   - e.g. Ultra-soft modes
2. A collinear anomaly inducing rapidity logarithms
   - In our case, the collinear alone is scaleless ⟶ Glauber is needed
3. Non-Perturbative low-energy interactions among incoming hadrons
   - Complete breaking of PDF factorization! Non perturbative two-nucleon matrix elements

MoR analysis

- Look at terms we do not match

$$\mathscr{W}_m^{\text{bare}} \ni \left(\frac{\alpha_s}{4\pi}\right)^3 \frac{1}{3\varepsilon^3}\left(\frac{11}{6\varepsilon} + \ln\frac{\mu_s^2}{Q^2} + \frac{9}{2}\ln\frac{\mu_s^2}{Q_0^2}\right) \boldsymbol{\Gamma}^c \, \boldsymbol{V}^G \, \bar{\boldsymbol{\Gamma}}$$

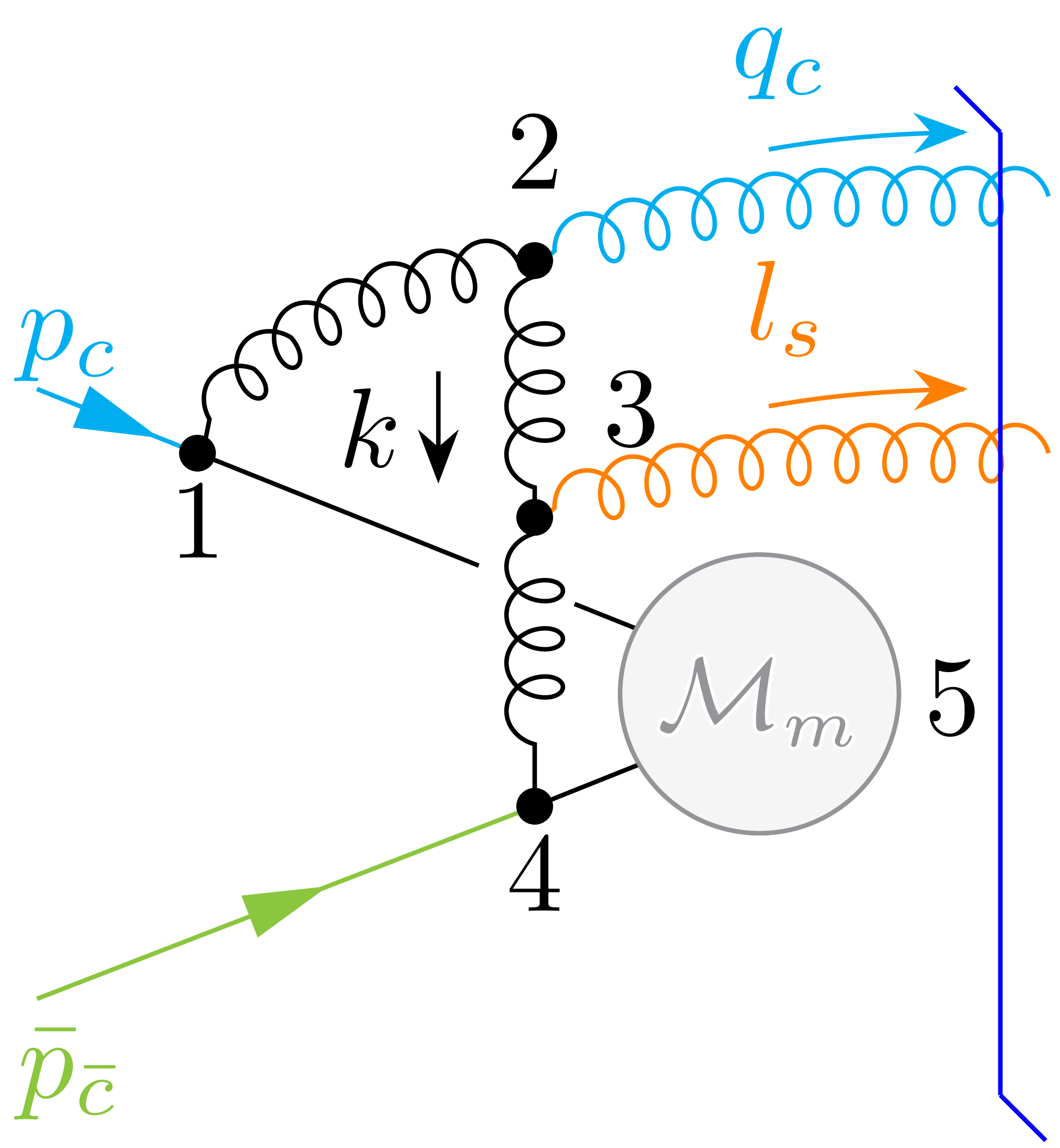


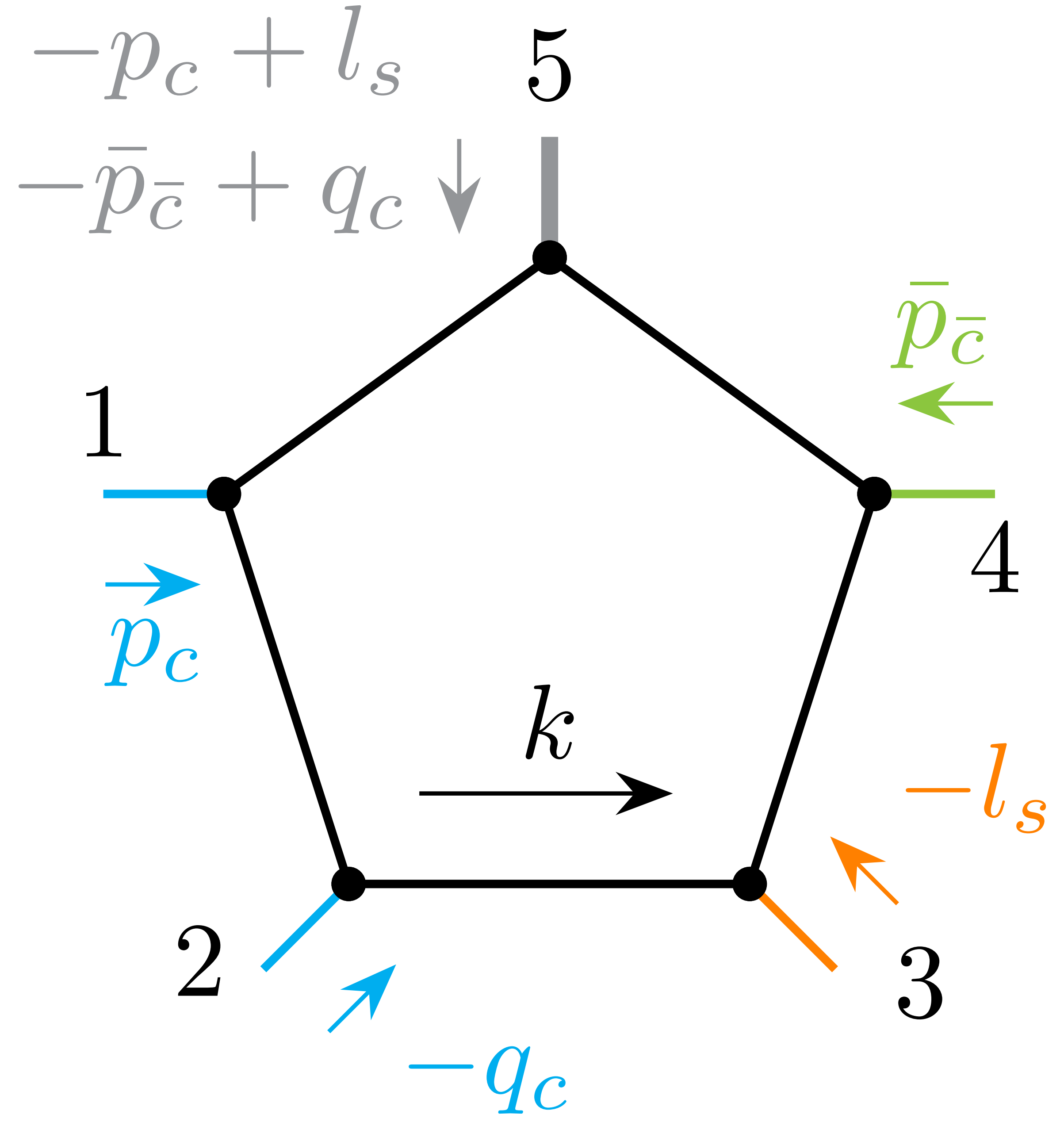
MoR analysis

- Make certain we are not missing any regions
  - Map to well-known scalar pentagon result

Bern,Dixon,Kosower '94

    - Expand pentagon result in λ
    - Ascertain that full result is recovered using all possible regions





Euclidean region

- Introduce kinematics

$$s_{12} = -p_c^- q_c^+\,, \quad s_{23} = q_c^- l_s^+\,, \quad s_{45} = -(p_c^- - q_c^-) l_s^+\,,$$
$$s_{34} = -\bar{p}_{\bar{c}}^+ l_s^-\,, \quad s_{51} = -q_c^- \bar{p}_{\bar{c}}^+\,, \quad p_5^2 = (p_c^- - q_c^-)\bar{p}_{\bar{c}}^+$$

- In Euclidean region $s_{ij} = (p_i + p_j)^2 < 0, p_5^2 < 0$ only soft-collinear region with $k \sim (\lambda^2, \lambda, \lambda^{3/2})$
  - Also found by Asy2.1 & pySecDec
  - Compatible with option 1)
  - But decouples completely after q_c integration

Physical region

- For physical region extra terms due to cancellation

$$\underbrace{s_{45}\, s_{51}}_{\lambda} - \underbrace{p_5^2 s_{23}}_{\lambda} = \underbrace{p_c^- \bar{p}_{\bar{c}}^+ \left(q_{cT} + l_{sT}\right)^2}_{\lambda^2} > 0$$

- Terms (proportional to a prefactor) arise

$$P = \underbrace{\frac{s_{45}\, s_{51}}{s_{45}\, s_{51} - p_5^2 s_{23}}}_{\lambda^{-1}} \left[1 - e^{i\pi\varepsilon\,\Theta}\left(1 + \underbrace{\frac{p_5^2\, s_{23} - s_{45}\, s_{51}}{s_{45}\, s_{51}}}_{\lambda}\right)^{-\varepsilon}\right]$$

$$P \sim \begin{cases} 1 & \text{for } \Theta = 0 \\ \lambda^{-1} & \text{for } \Theta \neq 0 \end{cases}$$

Power enhancement in physical region, due to complex phase!

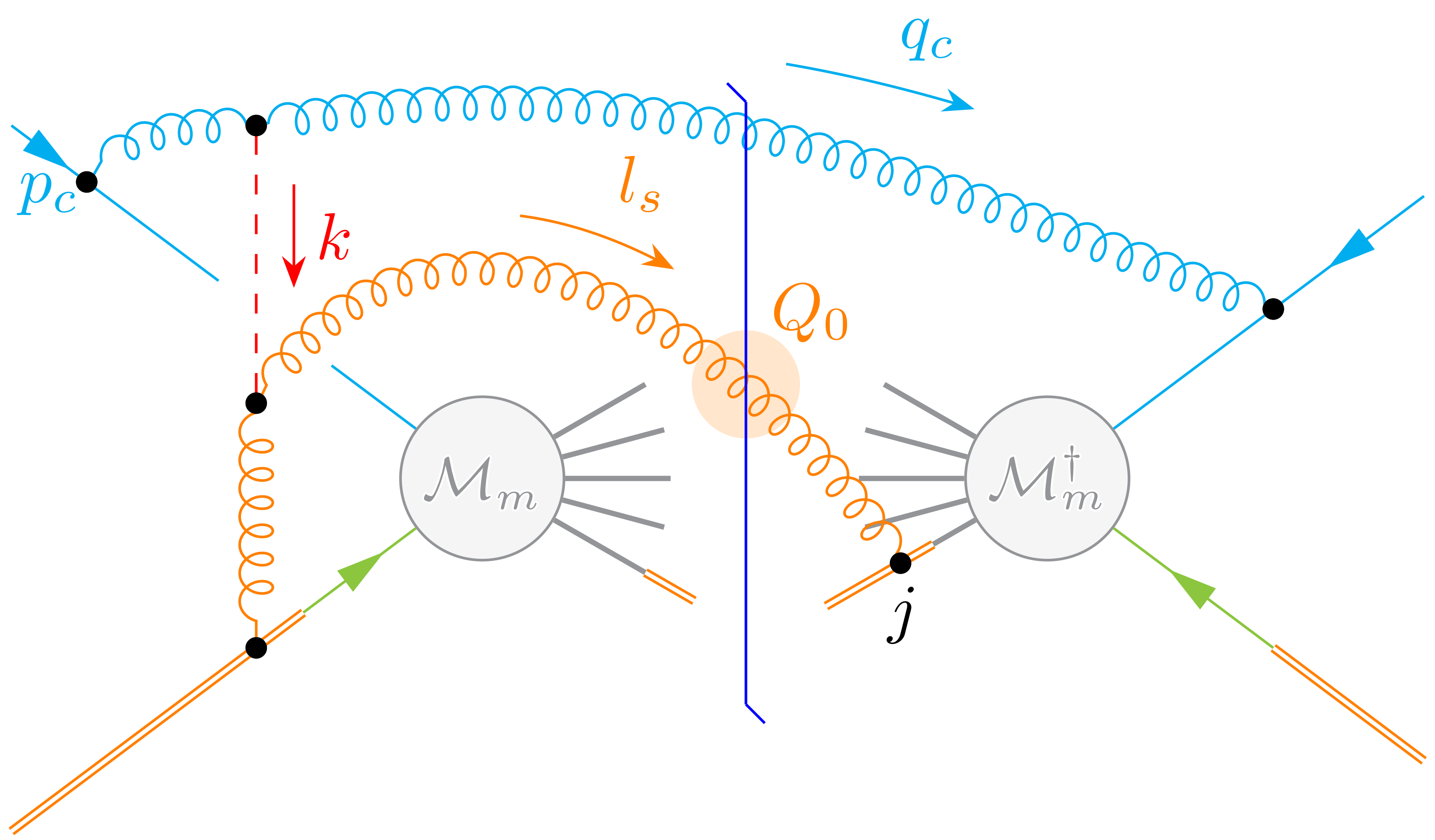
Glauber contribution

- Leads to a „hidden" region with $k \sim (\lambda^2, \lambda, \lambda)$
  - Off-shell potential gluon
  - Couples soft and collinear sectors → collinear factorization breaking
- Perform k_+ and k_- integral via residues
  - Well-defined without additional regulators

$$I^{\mathrm{g}} = i(4\pi)^{2-\varepsilon} \int \frac{d^d k}{(2\pi)^d} \frac{1}{-k_T^2} \frac{1}{k^+ q_c^- - k_T^2 - 2k_T \cdot q_{cT}} \times \frac{1}{\left[-k^+(p_c^- - q_c^-) - q_c^+ p_c^- - k_T^2 - 2k_T \cdot q_{cT}\right]} \times \frac{1}{\bar{p}_{\bar{c}}^+ (k^- - l_s^-)} \frac{1}{-l_s^+ k^- - k_T^2 + 2k_T \cdot l_{sT}}$$

← Euclidean of-shell triangle in $d - 2\varepsilon$

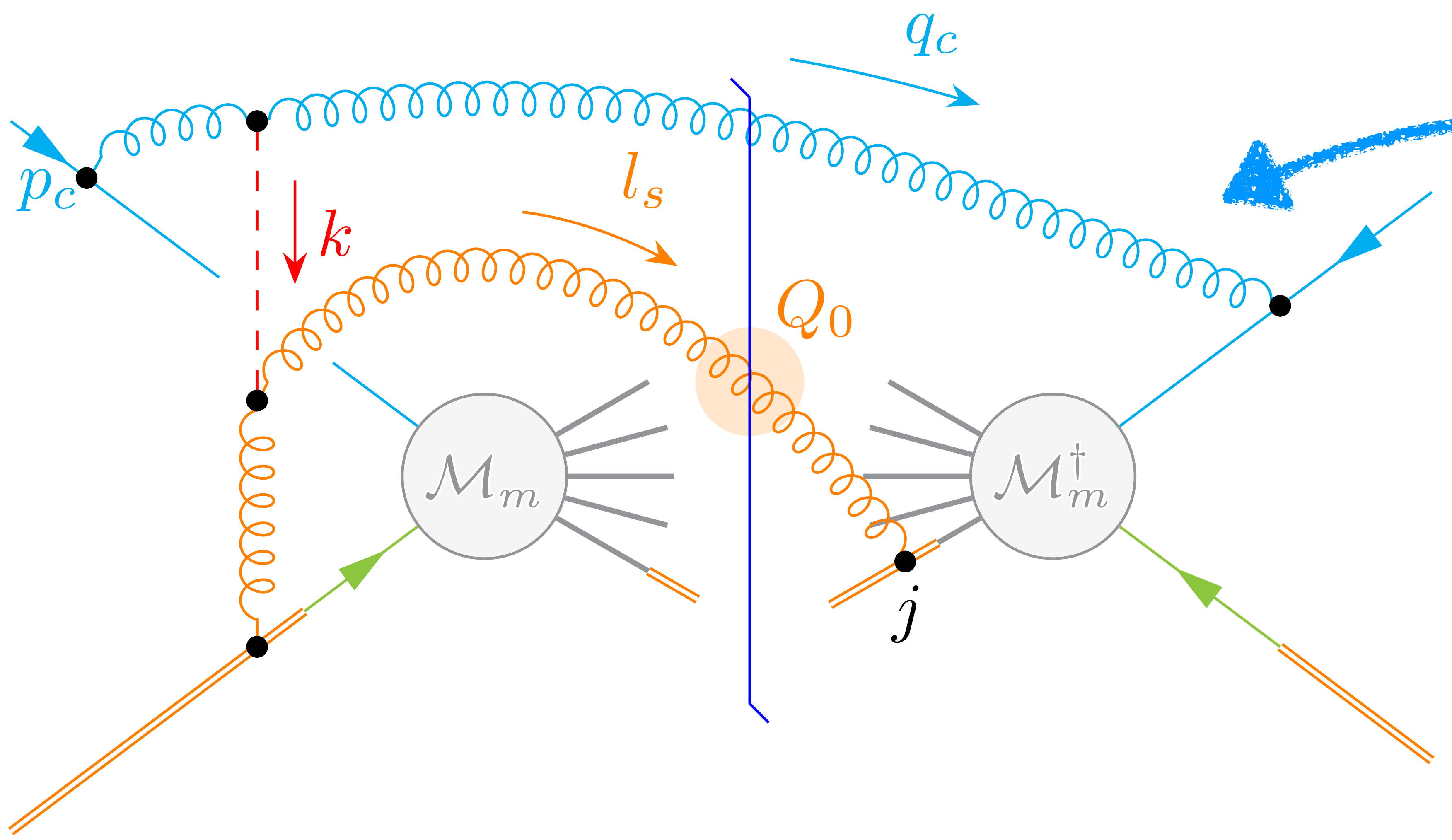
Effective Lipatov vertex



- Soft-collinear region vanishes!
- Only contribution due to Glauber exchange
    - Either use expanded QCD or effective Lipatov vertex using Glauber SCET

Rothstein, Stewart '16

Calculate the diagram & matching



q_c integral not well-defined in dim. reg. alone

Introduce rapidity regulator $\left(\frac{\nu}{q_{c-}}\right)^{\alpha}$

$$\mathscr{W}_m^{\text{bare}} \ni \frac{i\alpha_s^3}{12\pi^2\varepsilon^3}\left(\frac{\mu_s^2}{Q_0^2}\right)^{3\varepsilon} f^{abc} f^{ade} \sum_{j>2} J_j \left[\boldsymbol{T}_{2L}^d \boldsymbol{T}_{2R}^e \boldsymbol{T}_{1L}^b \boldsymbol{T}_{jR}^c \left(\frac{1}{\alpha} + \ln\frac{\nu \bar{p}_{\bar{c}}^+}{Q_0^2} - \frac{11}{6\varepsilon} + \ldots\right) + \boldsymbol{T}_{1L}^d \boldsymbol{T}_{1R}^e \boldsymbol{T}_{2L}^b \boldsymbol{T}_{jR}^c \left(-\frac{1}{\alpha} - \ln\frac{\nu}{p_c^-} + \ldots\right)\right] - (L \leftrightarrow R)$$

- α-poles cancel in between collinear and anti-collinear sector

$$\langle \mathscr{H}_m(\{\underline{n}\}, s, x_1, x_2, \mu) \otimes \mathscr{W}_m(\{\underline{n}\}, Q_0, x_1, x_2, \mu) \rangle$$

Under color trace we arrive at

$$\mathscr{W}_m^{\text{bare}} \ni \frac{iN_c \alpha_s^3}{12\pi^2 \varepsilon^3} \, if^{abc} \sum_{j>2} J_j \, \boldsymbol{T}_1^a \boldsymbol{T}_2^b \boldsymbol{T}_j^c \left(\ln \frac{Q_0^2}{Q^2} + \frac{11}{6\varepsilon} + \frac{11}{2} \ln \frac{\mu_s^2}{Q_0^2} + \ldots \right)$$

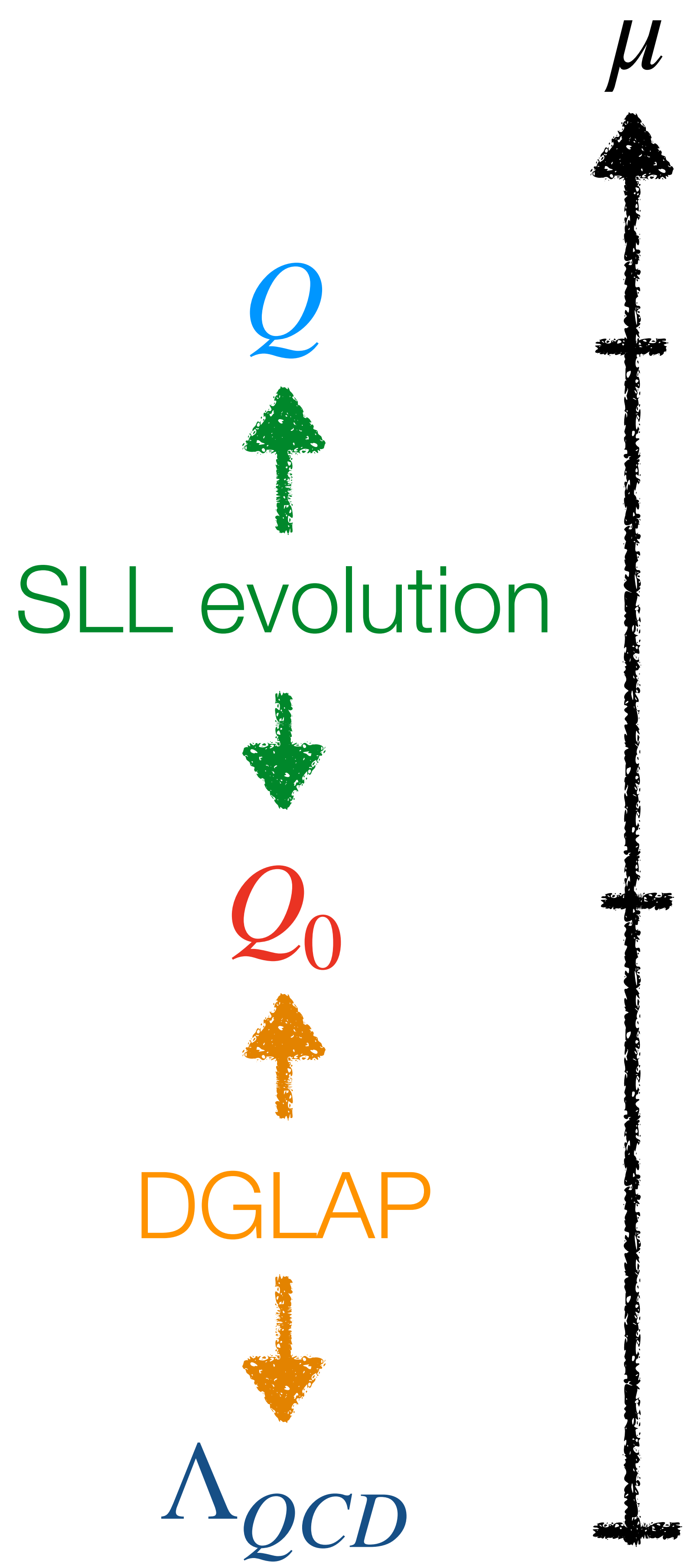
$$= \left(\frac{\alpha_s}{4\pi} \right)^3 \frac{1}{3\varepsilon^3} \left(\frac{11}{6\varepsilon} + \ln \frac{\mu_s^2}{Q^2} + \frac{9}{2} \ln \frac{\mu_s^2}{Q_0^2} \right) \boldsymbol{\Gamma}^c \; \boldsymbol{V}^G \; \overline{\boldsymbol{\Gamma}}$$

Perturbative Glauber contribution yields

- Correct $\ln Q$ term
- Correct $\frac{1}{\varepsilon^4}$ pole

$\Big\}$ $\mathscr{W}_m^{\text{bare}}$ consistent with both SLL *and* DGLAP evolution!

Conclusion



phase factors soft+collinear contributions
double-log evolution

soft-collinear interaction Glauber contribution

“factorization restoration”

single-log evolution

Outlook

- Showed consistency of PDF factorization at least up to 3 loops
  - All elements of factorization breaking are present but cancel in exactly the right way
- Show consistency of non log-enhanced terms with DGLAP as well

  → In the meantime also checked this!
- Look at higher loops e.g. $\mathscr{W}_m^{(4)}$?
- All-order structure of Glauber terms
  - Proof of factorization?

Backup

SLLs

$$C_{rn} = -\,256\pi^2 \left(4N_c\right)^{n-r} \left[\sum_{j=3}^{M+2} J_j \sum_{i=1}^{4} c_i^{(r)} \langle \mathscr{H}_{2\to M}\, O_i^{(j)} \rangle - J_{12} \sum_{i=1}^{6} d_i^{(r)} \langle \mathscr{H}_{2\to M}\, S_i \rangle \right] ,$$

- Able to perform whole computation analytically
  - Color structure closes under repeated operation
  - S_i and O_i have to be computed for each channel separately

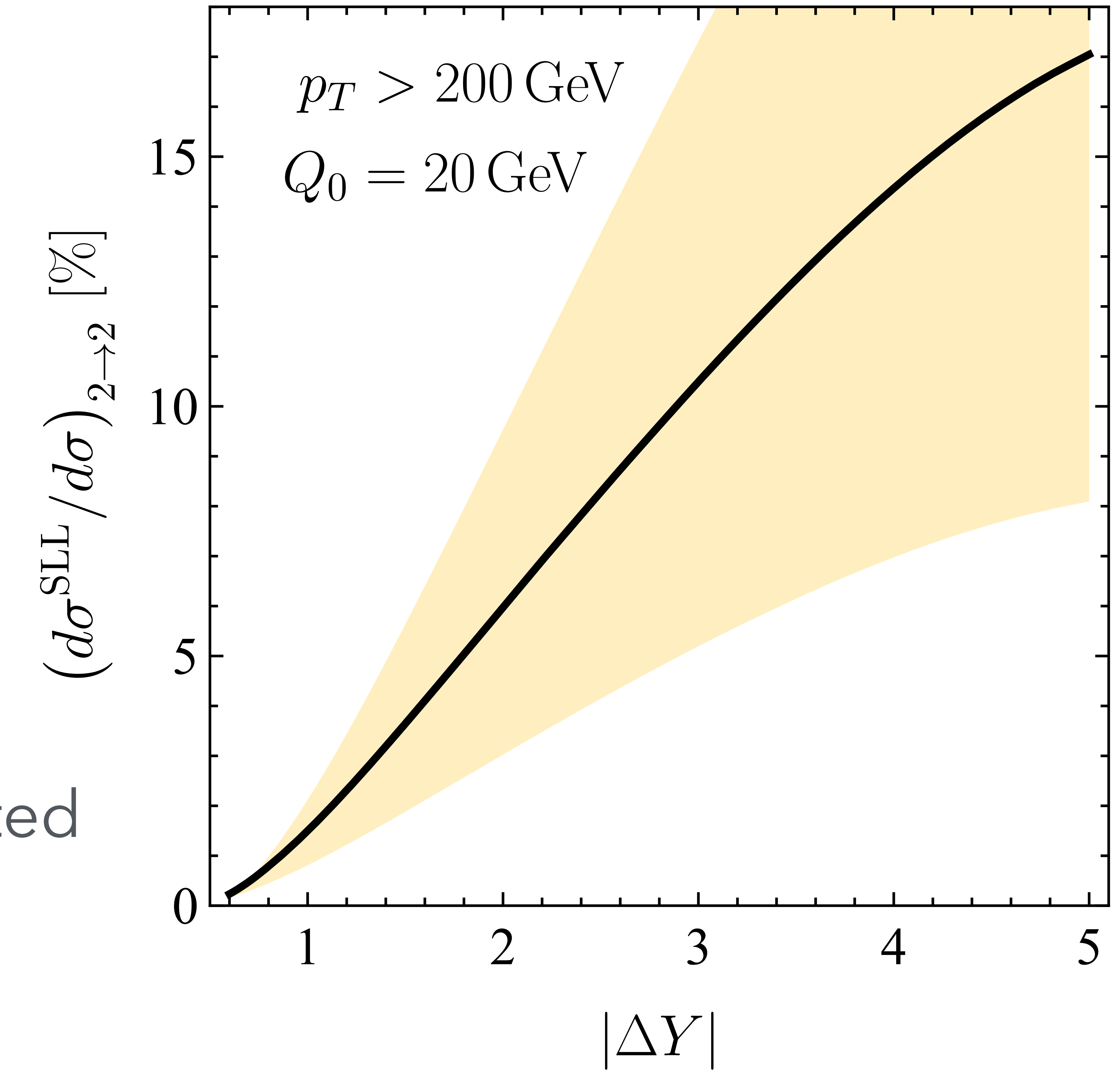
$$S_1 = C_F \begin{pmatrix} -2N_c^2 & N_c - \frac{N_c^3}{4} \\ N_c - \frac{N_c^3}{4} & -\frac{1}{4}\left(N_c^2+2\right) \end{pmatrix}$$

$$\sum_{j=3}^{4} O_1^{(j)} J_j = \frac{C_F N_c J_{43}}{2} \begin{pmatrix} -2N_c & 1 \\ 1 & C_F \end{pmatrix}$$

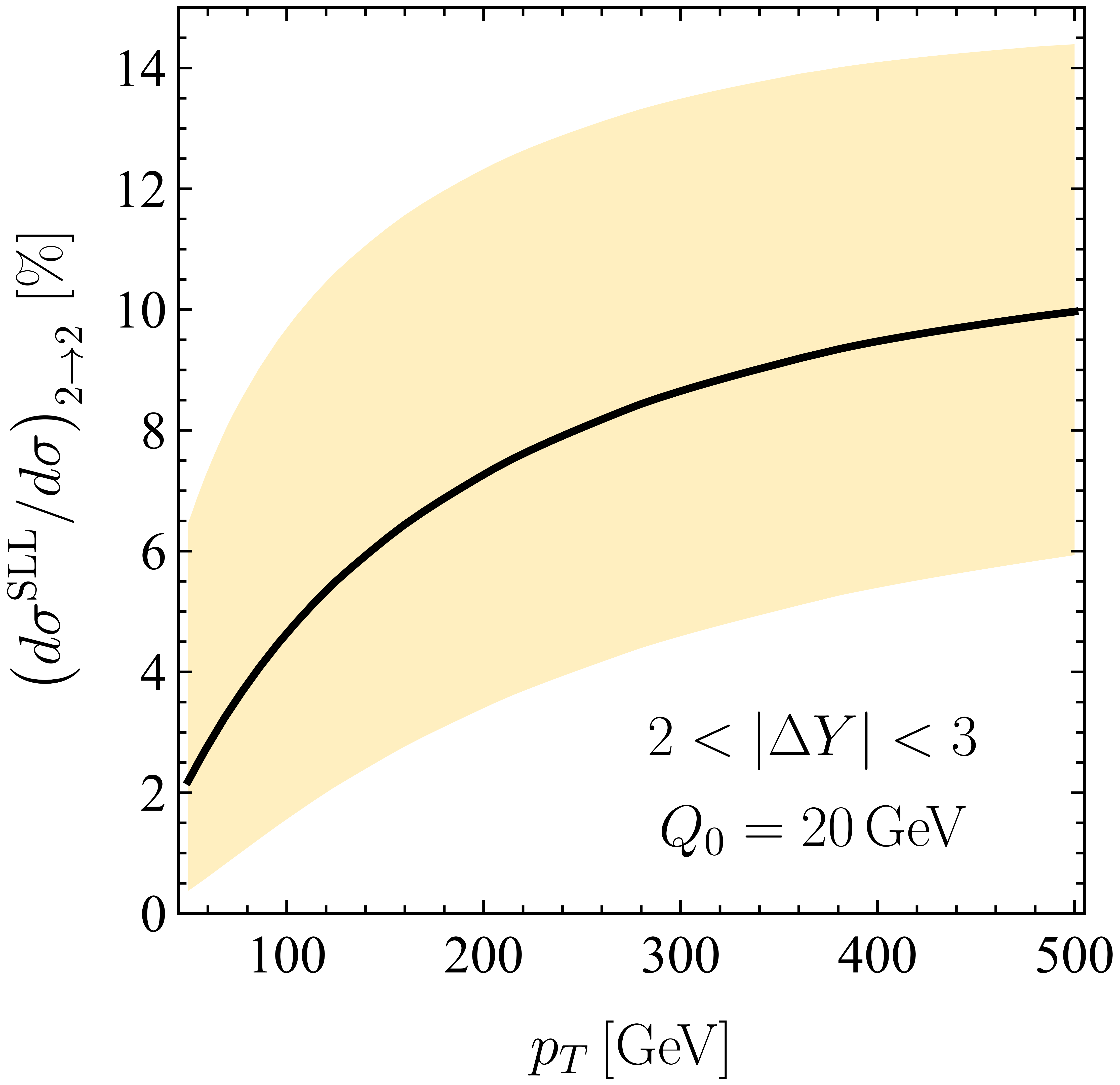
Numerical results

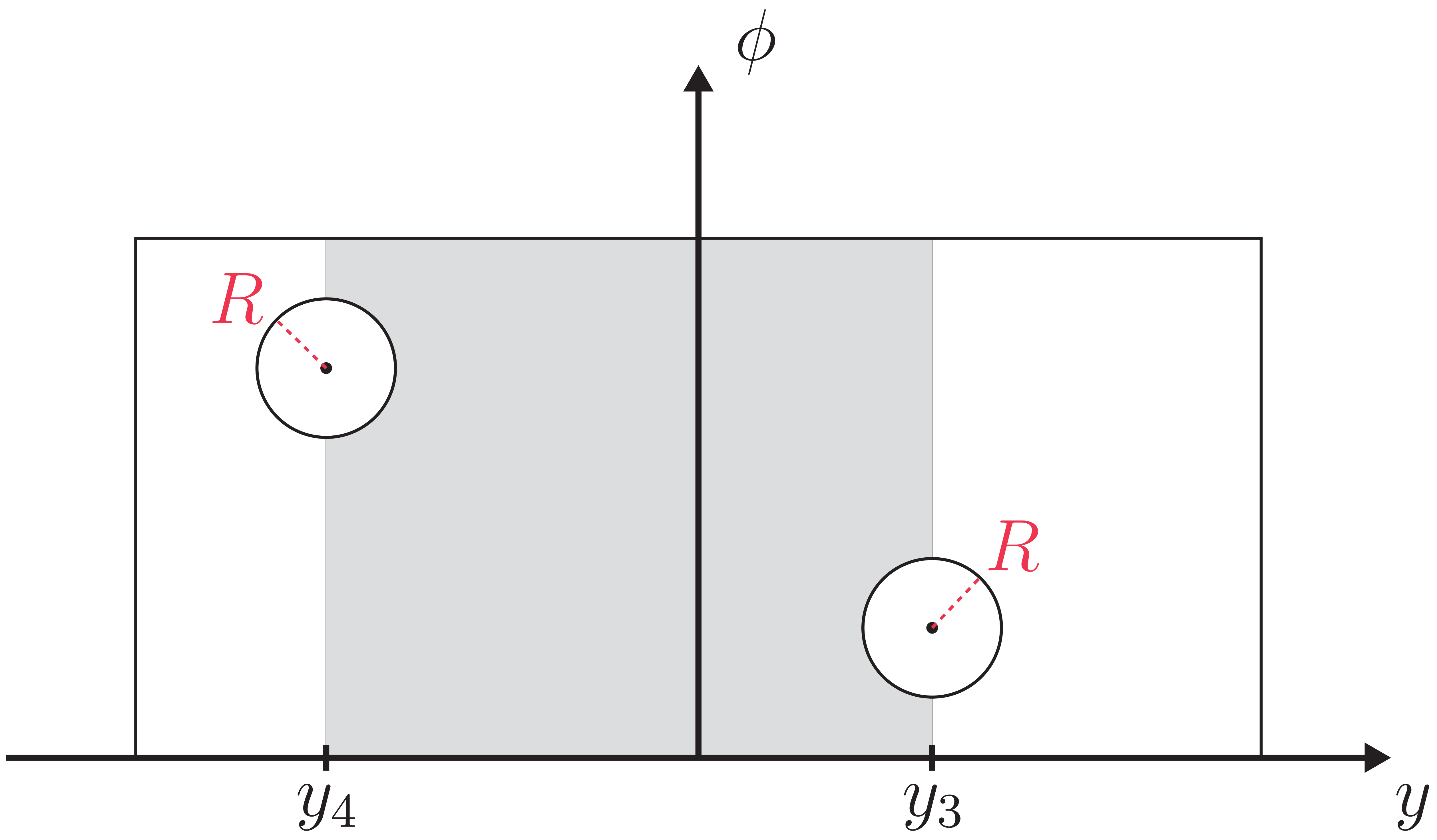
- Large contributions
  - Should be accounted for in precision physics

- Not yet complete
  - Large scale uncertainty
    - Only leading logarithmic effect
    - Purely collinear emissions only approximated



Numerical results





$$J_4 = \Delta Y - \operatorname{sgn}(\Delta Y)\frac{R}{\pi}\int_0^1 dx \left[\ln \frac{\cosh\left(R\sqrt{1-x^2}\right) - \cos Rx}{1-\cos Rx} + \ln \frac{\cosh \Delta Y + \cos Rx}{\cosh\left(|\Delta Y| - R\sqrt{1-x^2}\right) + \cos Rx} \right]$$

$$= \Delta Y - \frac{R^2}{4}\tanh\frac{\Delta Y}{2} - \operatorname{sgn}(\Delta Y)\left[\frac{2R}{\pi} + \frac{R^3}{6\pi}\left(\tanh^2\frac{\Delta Y}{2} - \frac{2}{3}\right) + \mathcal{O}(R^5)\right]$$

One loop soft current

$$\boldsymbol{J}^{\mu,a(1)} = -\frac{1}{(4\pi)^2}\frac{\Gamma^3(1-\varepsilon)\,\Gamma^2(\varepsilon)}{\Gamma(1-2\varepsilon)} \times i f^{abc} \sum_{i\neq j} \boldsymbol{T}^b_{iL}\boldsymbol{T}^c_{jL}\left(\frac{n_i^\mu}{n_i\cdot l_s} - \frac{n_j^\mu}{n_j\cdot l_s}\right)\left[\frac{2\pi\ n_i\cdot n_j\ e^{-i\lambda_{ij}\pi}}{n_i\cdot l_s\, n_j\cdot l_s\, e^{-i\lambda_{il}\pi}\ e^{-i\lambda_{jl}\pi}}\right]^\varepsilon$$

Glauber region in parameter space

Can perform region analysis in Schwinger or Lee-Pomeransky parameter space (like **Asy** and **PySecDec**)

$$(x_1, x_2, x_3, x_4, x_5) \sim (\lambda^{-2}, \lambda^{-2}, \lambda^{-2}, \lambda^{-1}, \lambda^{-2})$$

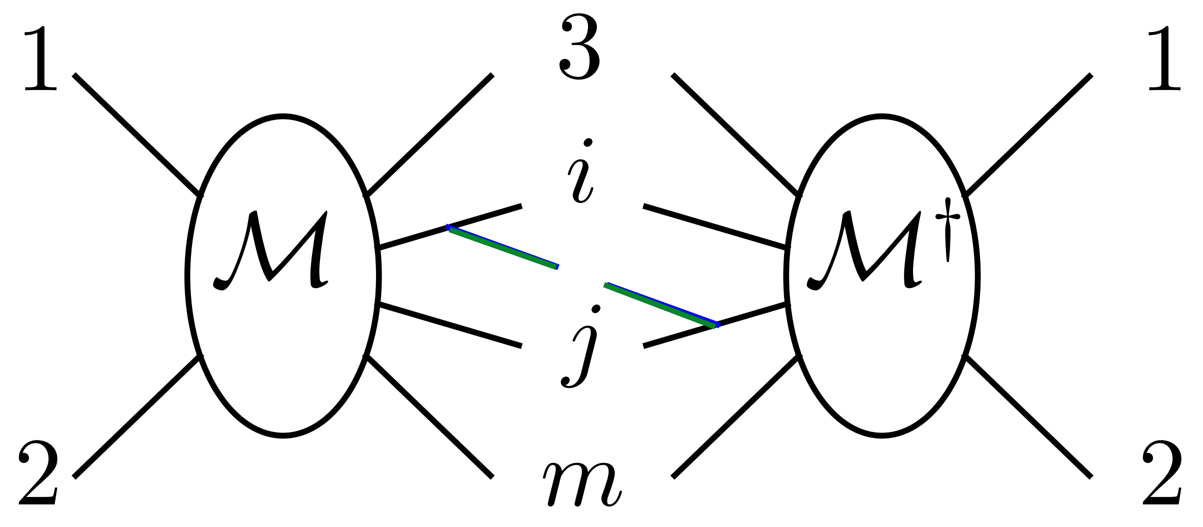
$$\begin{aligned}\mathcal{F} = &- \underbrace{x_1 x_3\, s_{23}}_{\lambda^{-3}} - \underbrace{x_1 x_4\, s_{51}}_{\lambda^{-3}} - \underbrace{x_3 x_5\, s_{45}}_{\lambda^{-3}} \\ &- \underbrace{x_4 x_5\, m^2}_{\lambda^{-3}} - \underbrace{x_2 x_4\, s_{34}}_{\lambda^{-2}} - \underbrace{x_2 x_5\, s_{12}}_{\lambda^{-2}}\end{aligned}$$

The Glauber region corresponds to a pinch due to cancellations in the $\mathcal{F}$ polynomial

$$\mathcal{F} = \underbrace{\left(-q_c^-\, x_1 + (p_c^- - q_c^-)\, x_5\right)}_{\lambda^{-2}} \underbrace{\left(l_s^+\, x_3 - \bar{p}_{\bar{c}}^+\, x_4\right)}_{\lambda^{-1}}$$

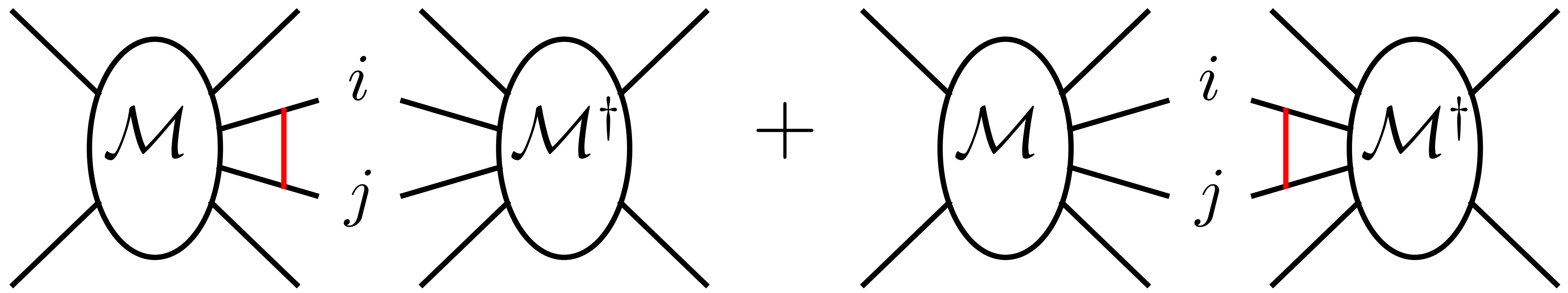
Purely soft contributions

$$\boldsymbol{\mathcal{H}}_m \overline{\boldsymbol{R}}_m = \sum_{(ij)}$$



$$\overline{\boldsymbol{R}}_m = -4 \sum_{(ij)} \boldsymbol{T}_{i,L} \circ \boldsymbol{T}_{j,R} \, \overline{W}_{ij}^{m+1} \, \Theta_{\text{hard}}(n_{m+1})$$

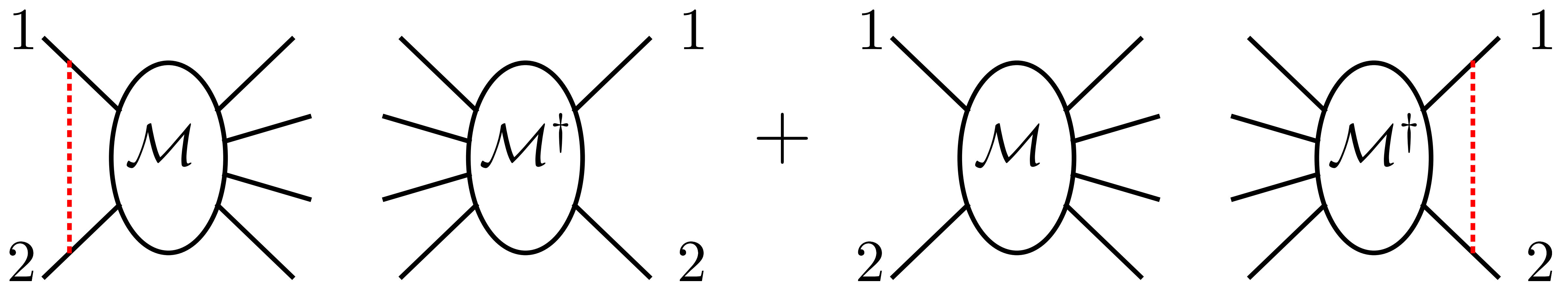
$$\boldsymbol{\mathcal{H}}_m \overline{\boldsymbol{V}}_m = \sum_{(ij)}$$



$$\overline{\boldsymbol{V}}_m = 2 \sum_{(ij)} \left(\boldsymbol{T}_{i,L} \cdot \boldsymbol{T}_{j,L} + \boldsymbol{T}_{i,R} \cdot \boldsymbol{T}_{j,R} \right) \int \frac{d\Omega(n_k)}{4\pi} \, \overline{W}_{ij}^k$$

$$\overline{W}_{ij}^q = W_{ij}^q - \frac{1}{n_i \cdot n_q} \delta(n_i - n_q) - \frac{1}{n_j \cdot n_q} \delta(n_j - n_q)$$

Glauber contributions

$\mathcal{H}_m \, \boldsymbol{V}^G =$ 

$$\sim \sum_{(ij)} \left(\boldsymbol{T}_{i,L} \cdot \boldsymbol{T}_{j,L} - \boldsymbol{T}_{i,R} \cdot \boldsymbol{T}_{j,R}\right) \Pi_{ij} = 4\left(\boldsymbol{T}_{1,L} \cdot \boldsymbol{T}_{2,L} - \boldsymbol{T}_{1,R} \cdot \boldsymbol{T}_{2,R}\right)$$

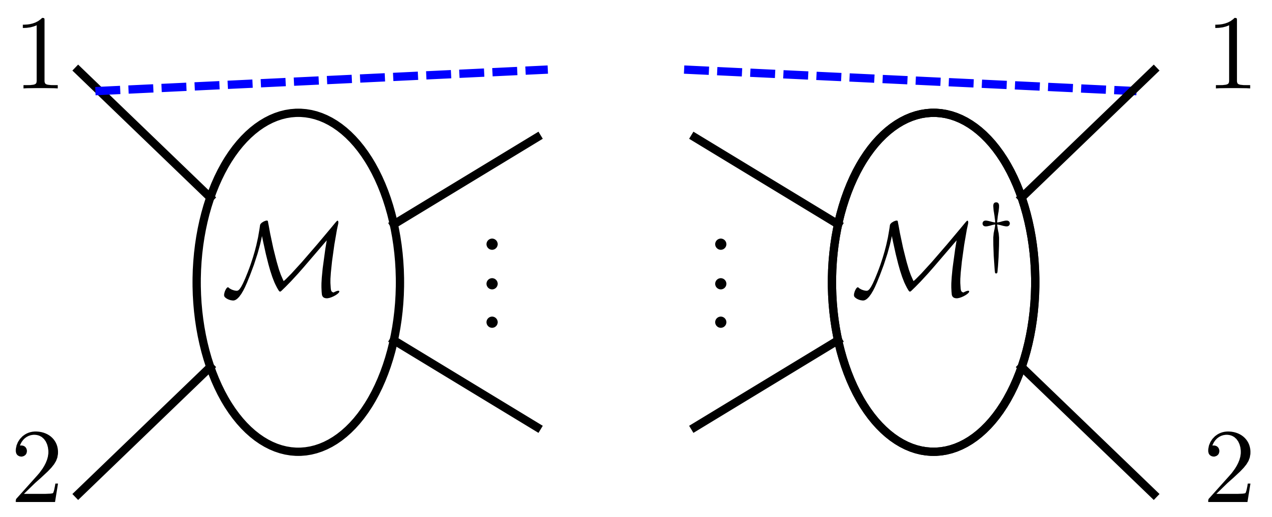
$$\Pi_{ij} = \begin{cases} 1 \text{ if both in- or outgoing} \\ 0 \text{ else} \end{cases}$$

use $\sum_i \boldsymbol{T}_i = 0$

$$\boldsymbol{V}^G = -8i\pi\left(\boldsymbol{T}_{1,L} \cdot \boldsymbol{T}_{2,L} - \boldsymbol{T}_{1,R} \cdot \boldsymbol{T}_{2,R}\right)$$

Cusp terms

$$\mathcal{H}_m \, \boldsymbol{R}_1^c \; = $$



SLLs are directly connected to factorization violation!

$$\boldsymbol{R}_i^c = -4\boldsymbol{T}_{i,L} \circ \boldsymbol{T}_{i,R}\,\delta(n_{m+1} - n_i)$$

$$\boldsymbol{V}_i^c = 4C_i\,\mathbf{1}$$

space-like splitting from before

time-like splitting from before

- These are only present for initial states $i = 1{,}2$, for final states they cancel

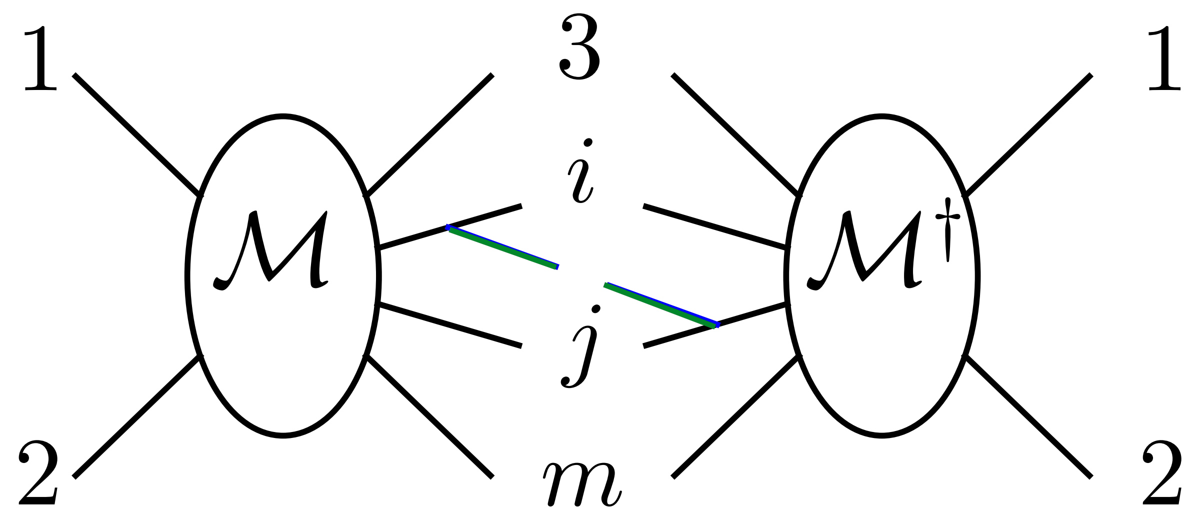
- They are multiplied by $\ln\frac{\mu^2}{Q^2}$ and give rise to double-logarithmic running!

$$U(\mu_h, \mu_s) = \boldsymbol{P}\,exp\left[\int_{\mu_s}^{\mu_h} \frac{d\mu}{\mu}\boldsymbol{\Gamma}^H\right]$$

$$= \mathbf{1} + \int_{\mu_s}^{\mu_h} \frac{d\mu_1}{\mu_1}\boldsymbol{\Gamma}^H + \int_{\mu_s}^{\mu_h} \frac{d\mu_1}{\mu_1}\int_{\mu_1}^{\mu_h} \frac{d\mu_2}{\mu_2}\boldsymbol{\Gamma}^H(\mu_1)\,\boldsymbol{\Gamma}^H(\mu_2)$$

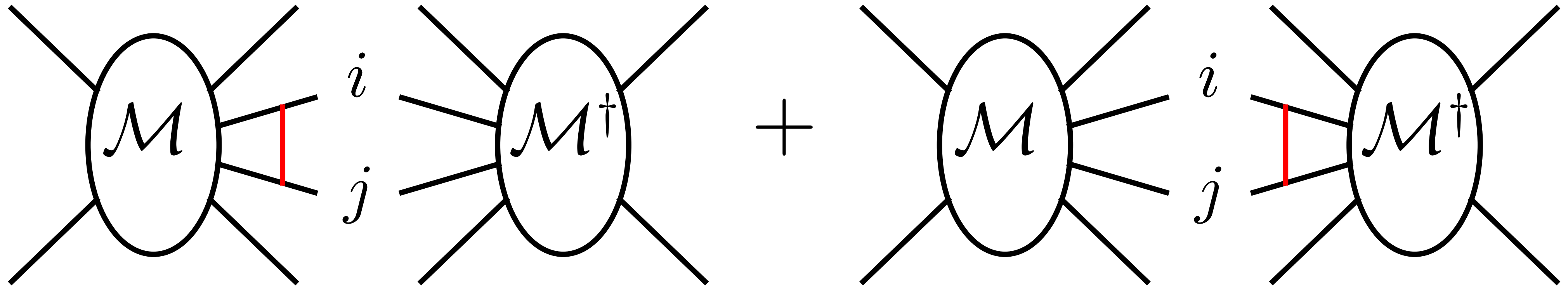
Purely soft contributions

$$\boldsymbol{\mathcal{H}}_m \overline{\boldsymbol{R}}_m = \sum_{(ij)}$$



$$\overline{\boldsymbol{R}}_m = -4 \sum_{(ij)} \boldsymbol{T}_{i,L} \circ \boldsymbol{T}_{j,R}\, \overline{W}_{ij}^{m+1}\, \Theta_{\text{hard}}(n_{m+1})$$

$$\boldsymbol{\mathcal{H}}_m \overline{\boldsymbol{V}}_m = \sum_{(ij)}$$



$$\overline{\boldsymbol{V}}_m = 2 \sum_{(ij)} \left(\boldsymbol{T}_{i,L} \cdot \boldsymbol{T}_{j,L} + \boldsymbol{T}_{i,R} \cdot \boldsymbol{T}_{j,R}\right) \int \frac{d\Omega(n_k)}{4\pi}\, \overline{W}_{ij}^{k}$$

$$\overline{W}_{ij}^{q} = W_{ij}^{q} - \frac{1}{n_i \cdot n_q}\delta(n_i - n_q) - \frac{1}{n_j \cdot n_q}\delta(n_j - n_q)$$