Precise computation of phase transition parameters

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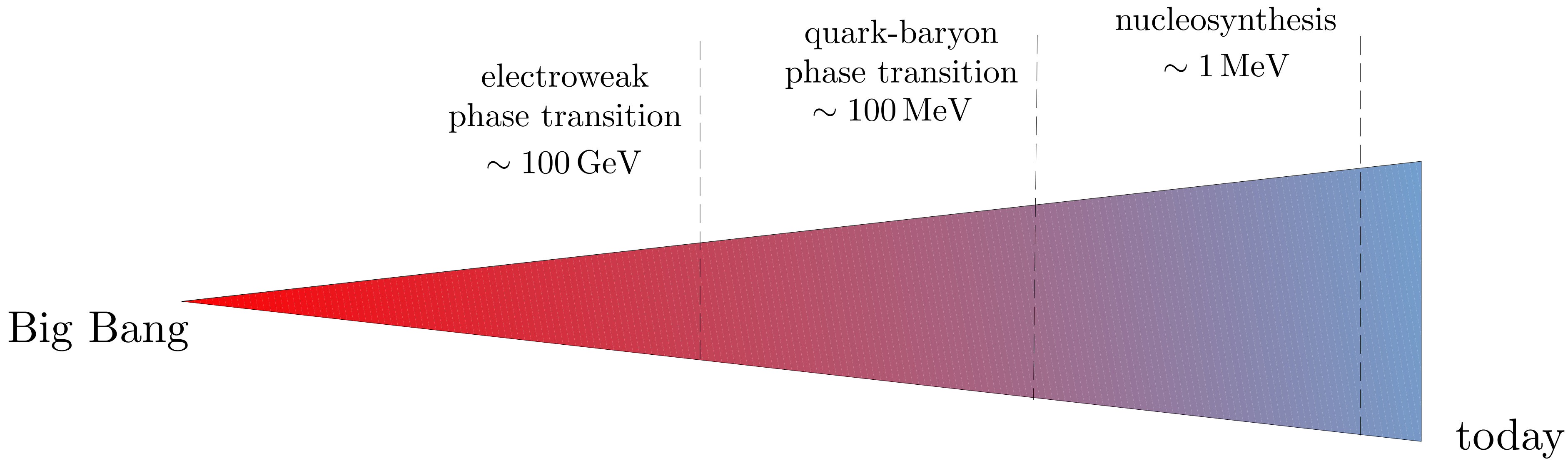
based on **MC**, JC. Criado, L. Gil, J. López-Miras; *2406.02667*

BNL HET Seminar; January 9, 2025


electroweak
phase transition
$\sim 100\,\mathrm{GeV}$
quark-baryon
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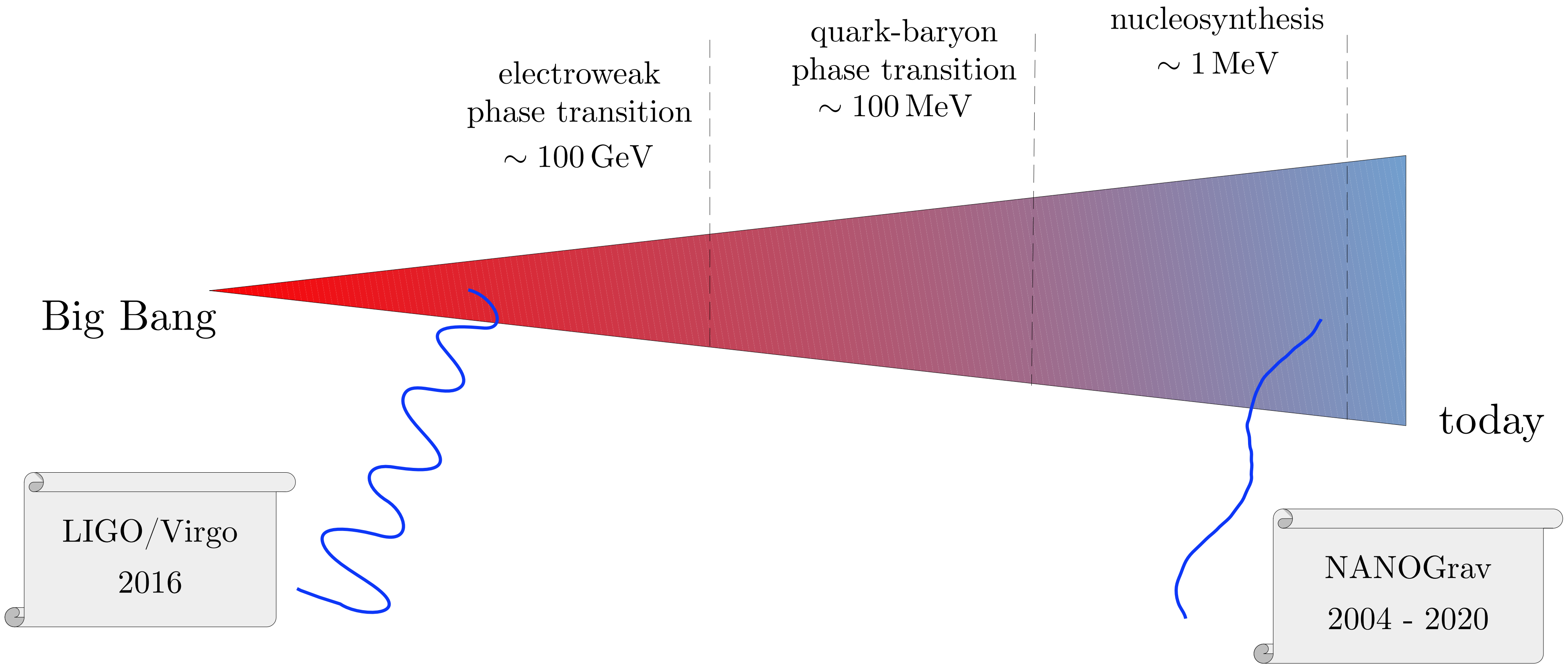
$\sim 100\,\mathrm{MeV}$
nucleosynthesis
$\sim 1\,\mathrm{MeV}$
Big Bang
today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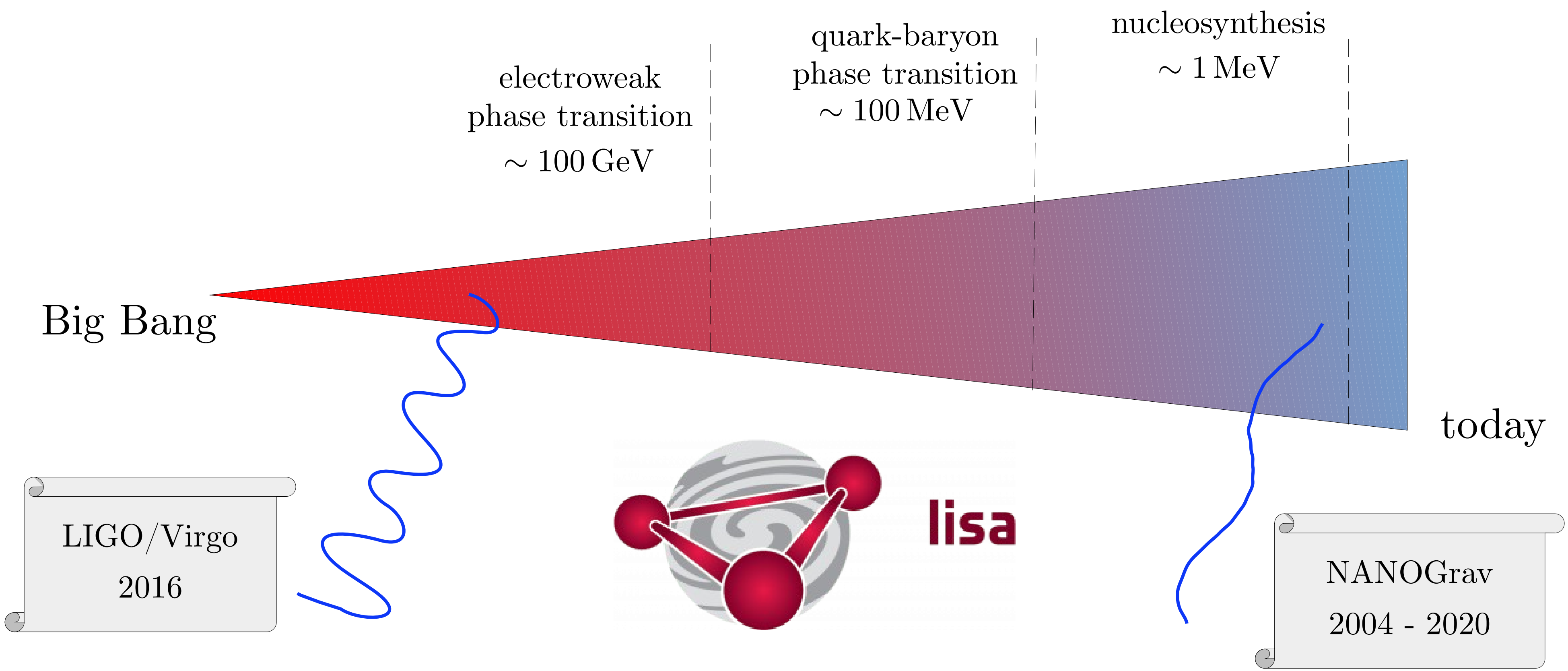
LIGO/Virgo
2016
NANOGrav
2004 - 2020

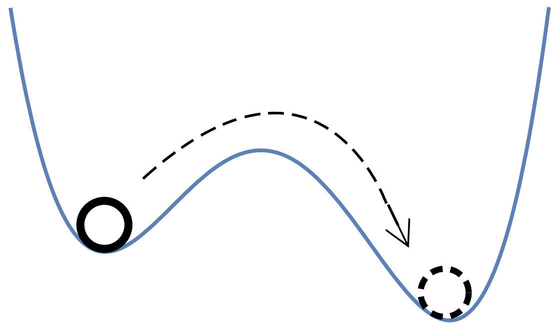


The next revolution is around the corner: LISA 2035

Compute reliably and accurately the spectrum of gravitational waves from phase transitions

Precise description of quantum field theory at finite temperature

Gravitational waves from phase transitions



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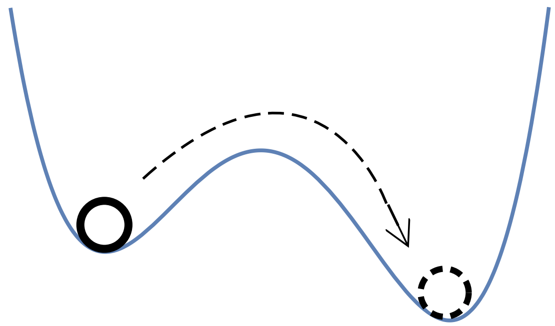

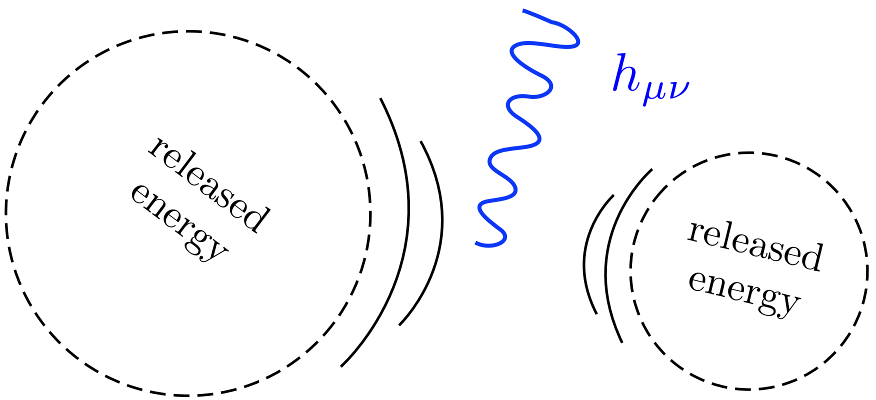

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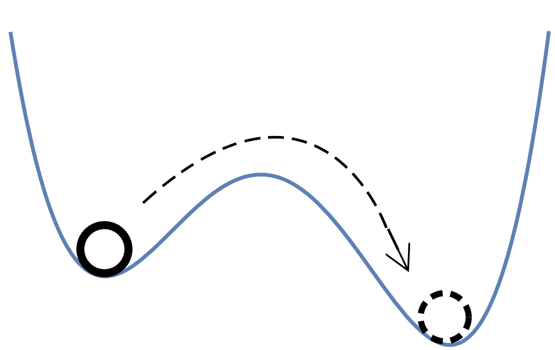
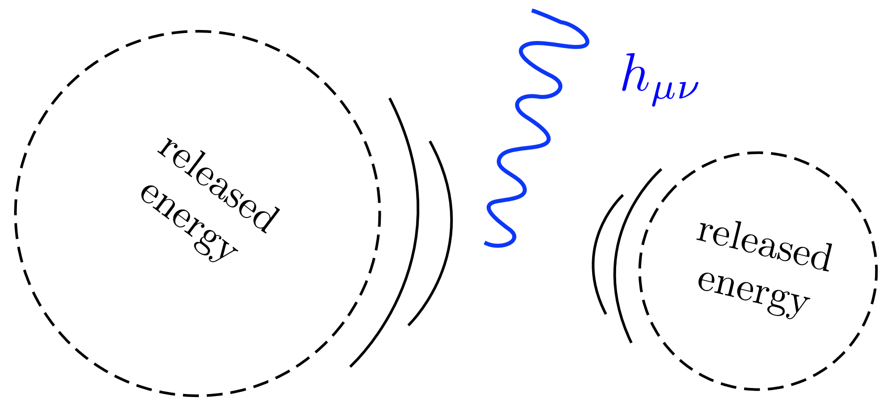


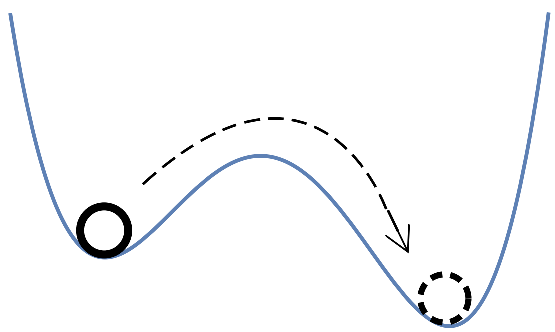


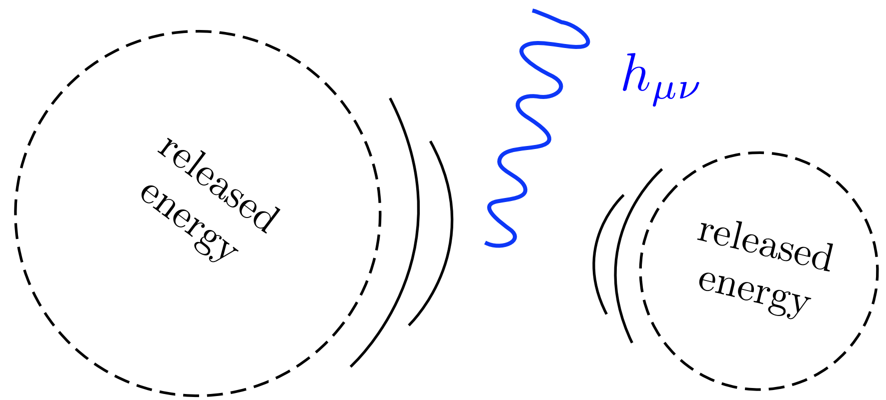
$S[m, T]$ → observables

field-theory techniques

- released energy
- duration of PT
- turbulence in plasma
- bubble velocity
- ...

Gravitational waves from phase transitions





$S[m, T]$ → observables

field-theory techniques

$m < T$ ↓

$$S_{\text{eff}} = S_{\text{eff}}^{(0)}[m] + \mathcal{O}\left(\frac{m}{T}\right)$$

released energy

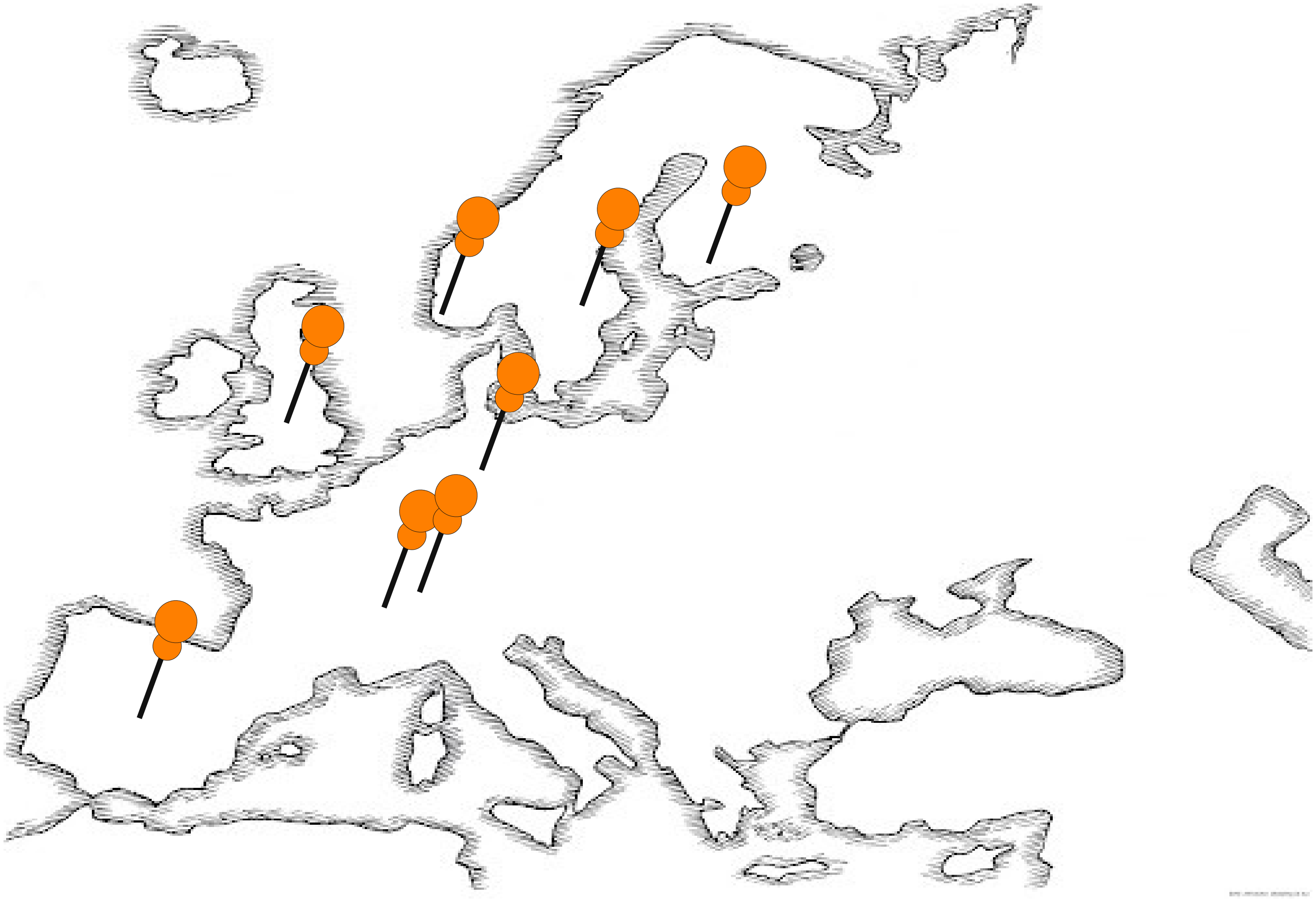
duration of PT

turbulence in plasma

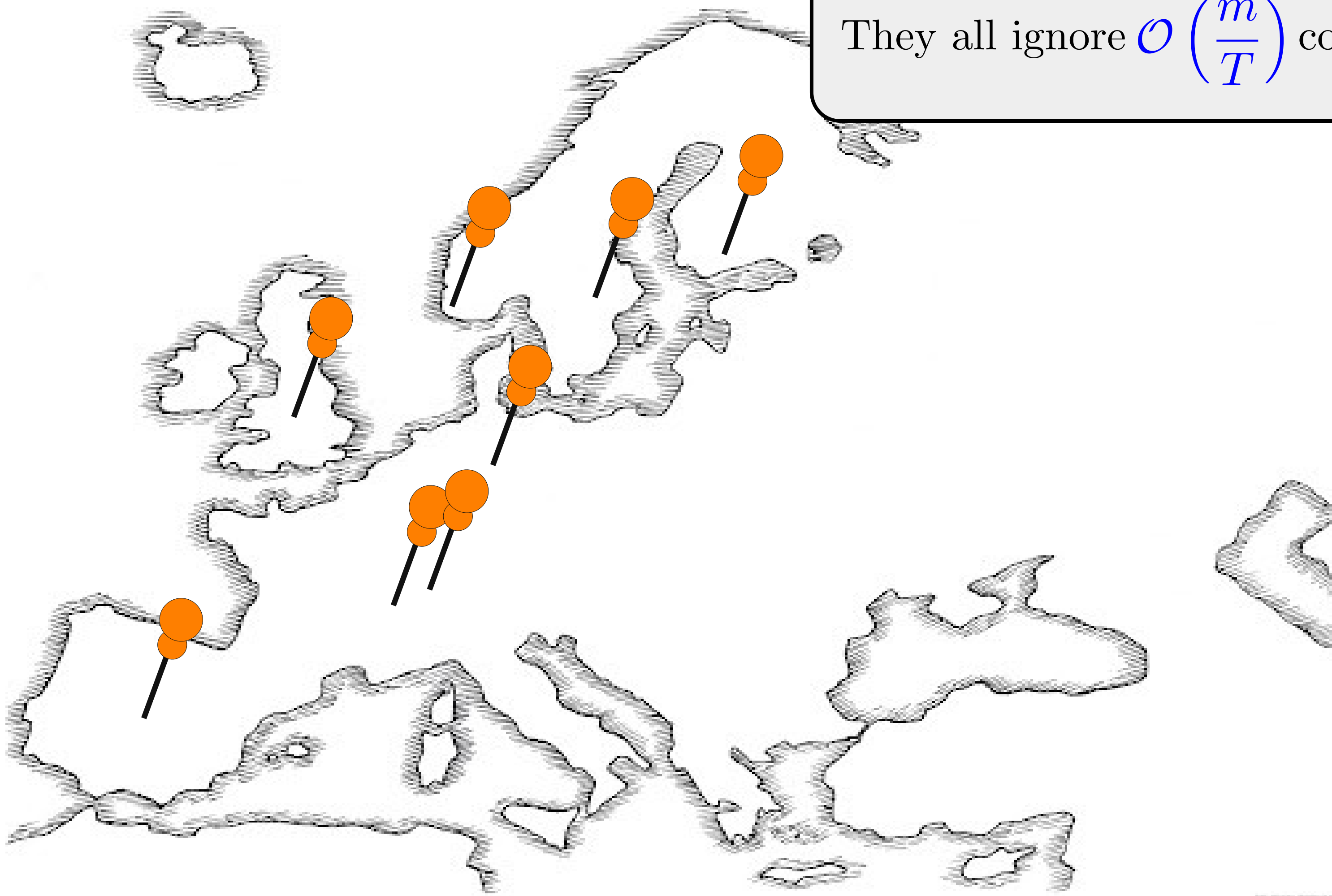
bubble velocity

...

A (lame) effort worldwide



A (lame) effort worldwide



They all ignore $\mathcal{O}\left(\frac{m}{T}\right)$ corrections!

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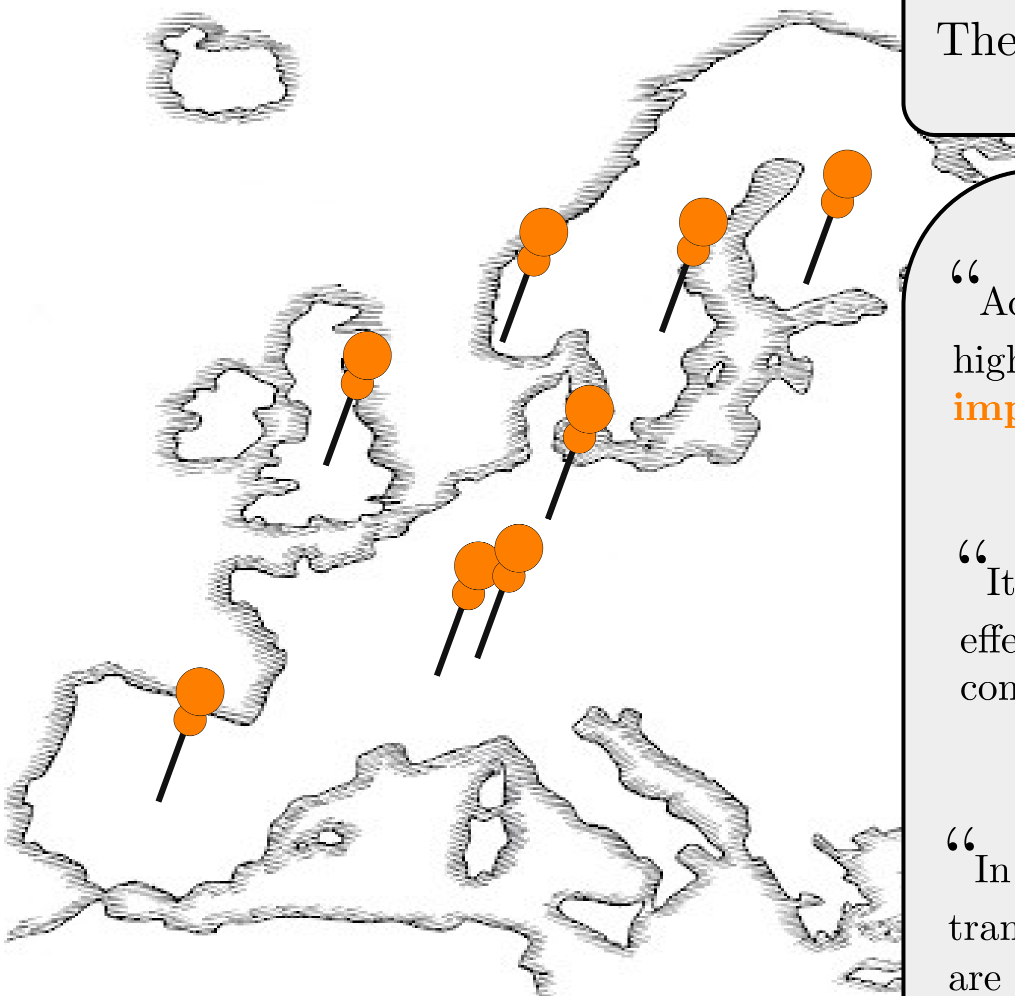


They all ignore $\mathcal{O}\left(\frac{m}{T}\right)$ corrections!

“Accurately determining [the effects of higher-dimensional operators] remains **important**

Ekstedt et al '24

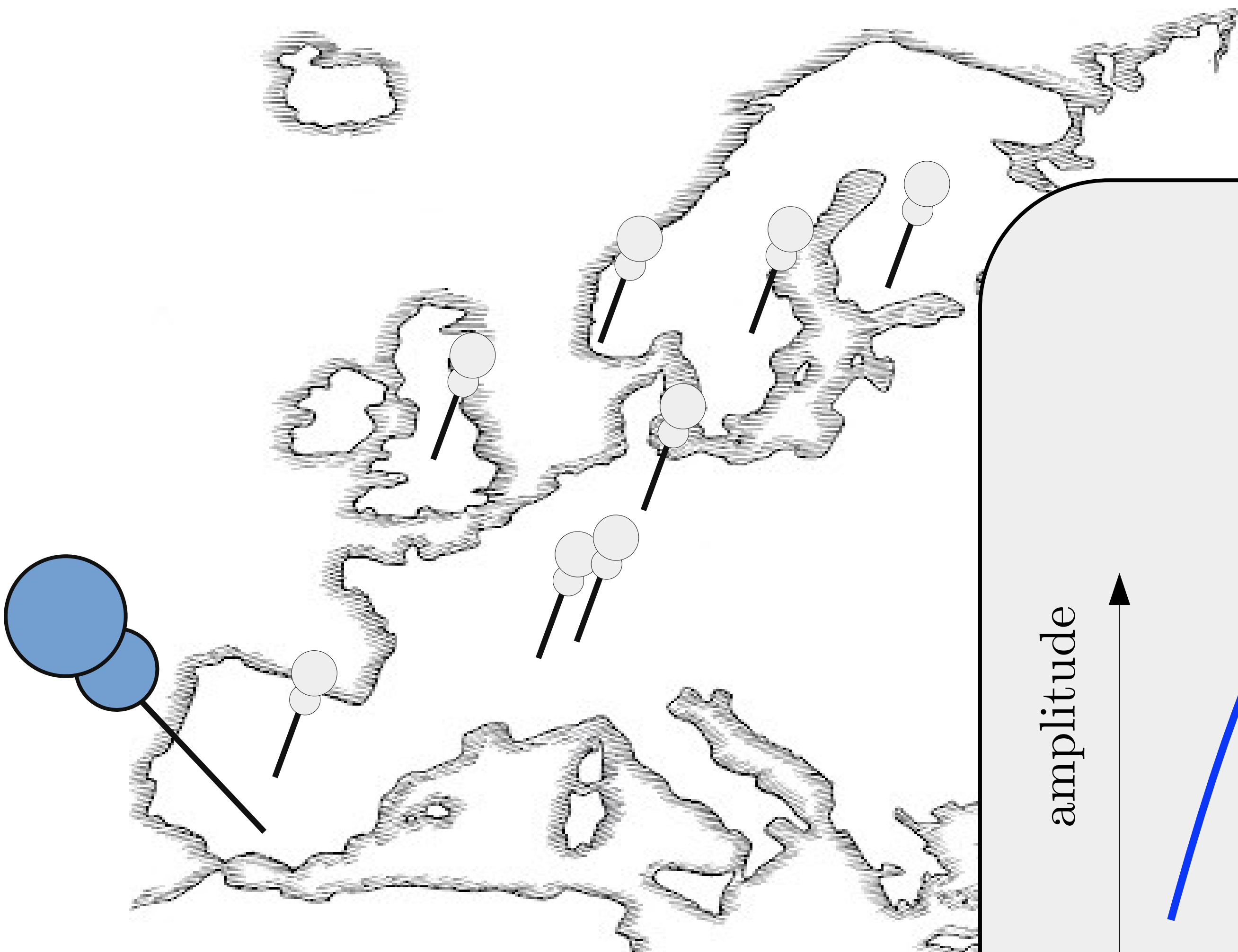
“It is rather **difficult** to estimate the effect of higher-dimensional operators comprehensively”

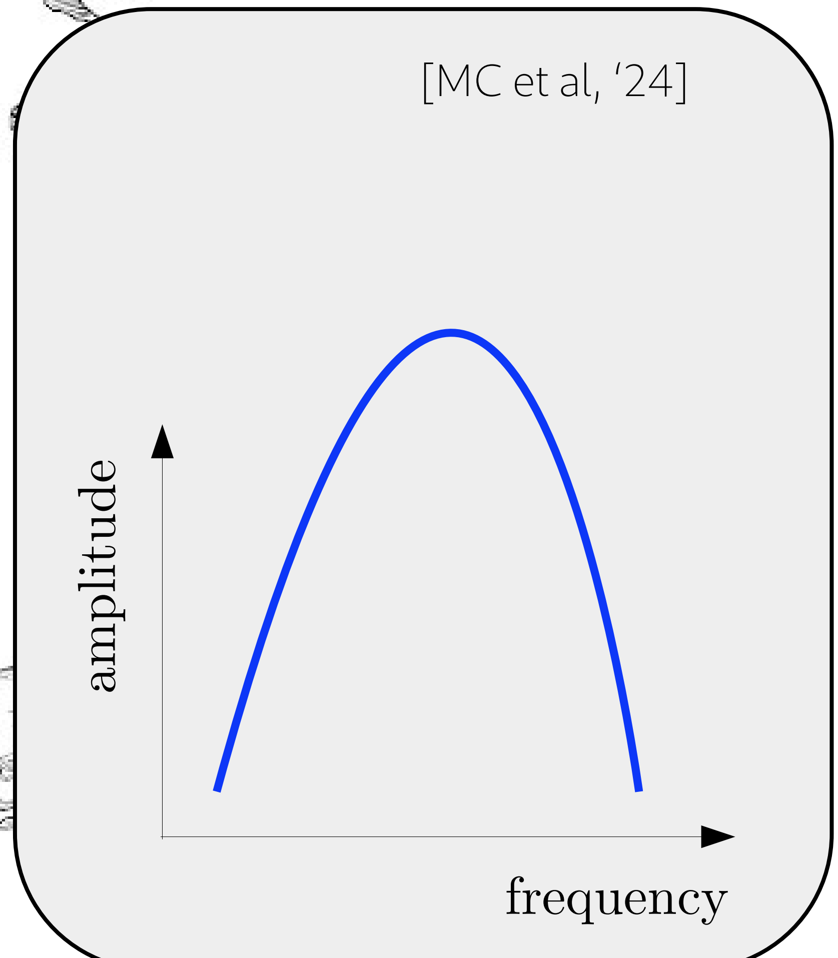
Kajantie et al '95

“In order to probe the [very strong transitions], **alternative approaches** are needed

Niemi et al '24

A (lame) effort worldwide





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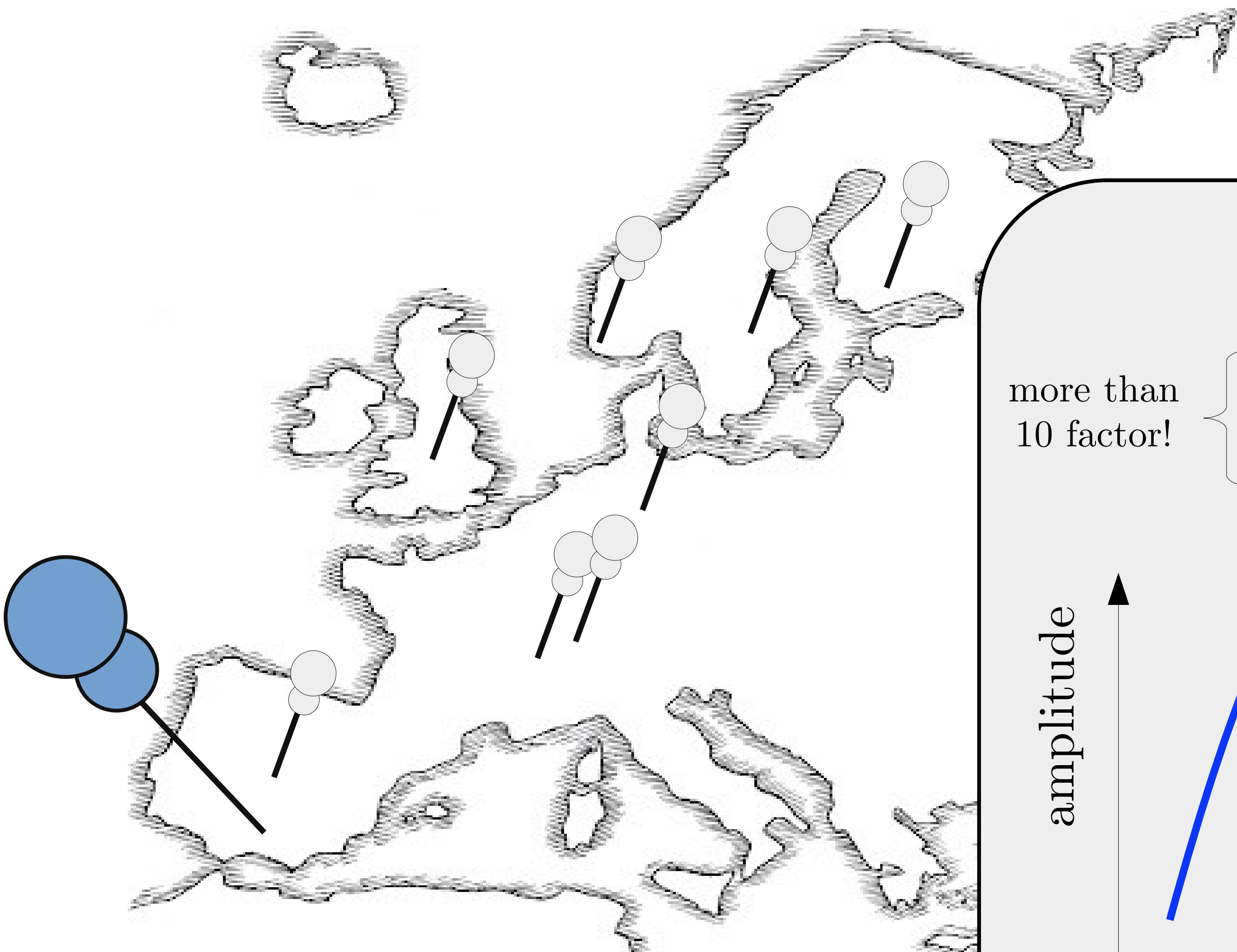
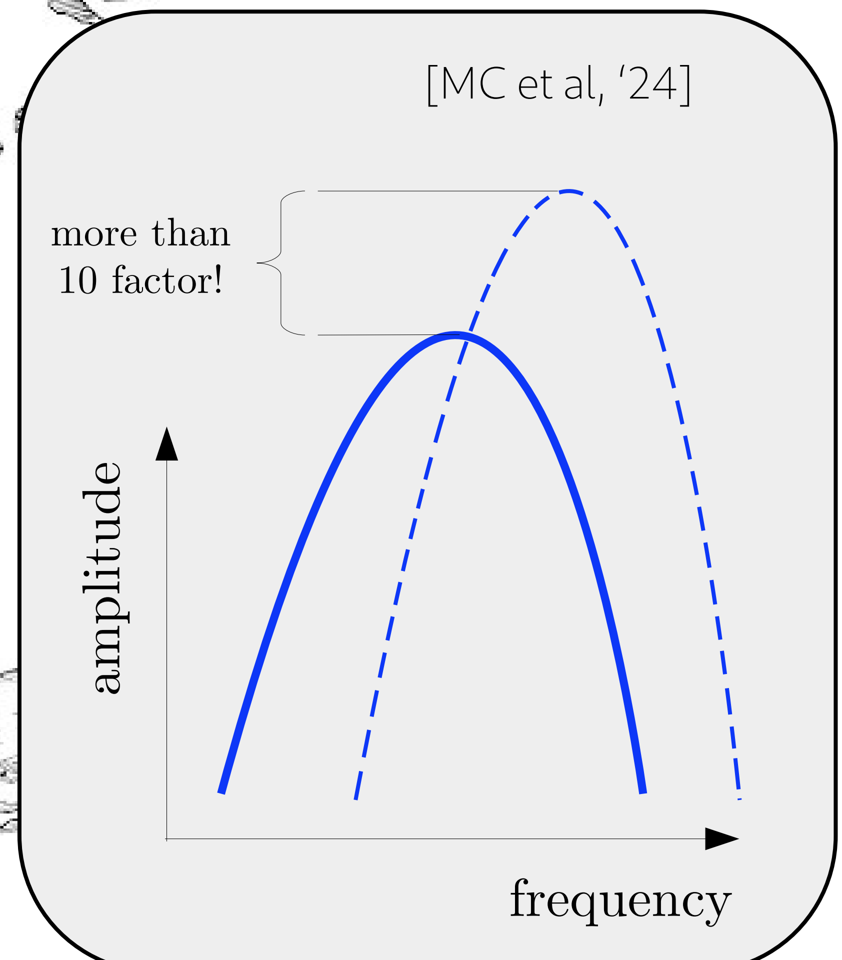




Short summary of field theory at finite temperature

$$\mathcal{Z} = \mathrm{Tr}(e^{-\beta H}) = \int \mathcal{D}\varphi \langle \varphi | e^{-\beta H} | \varphi \rangle$$

(For equilibrium physics) equivalent to a regular Euclidean field theory with periodic time

This gives rise to so-called Matsubara modes (equivalent to Kaluza-Klein excitations in extra-dimensions), which screen the masses and couplings of zero modes

This implies the presence of **large logarithms** in finite-temperature calculations:

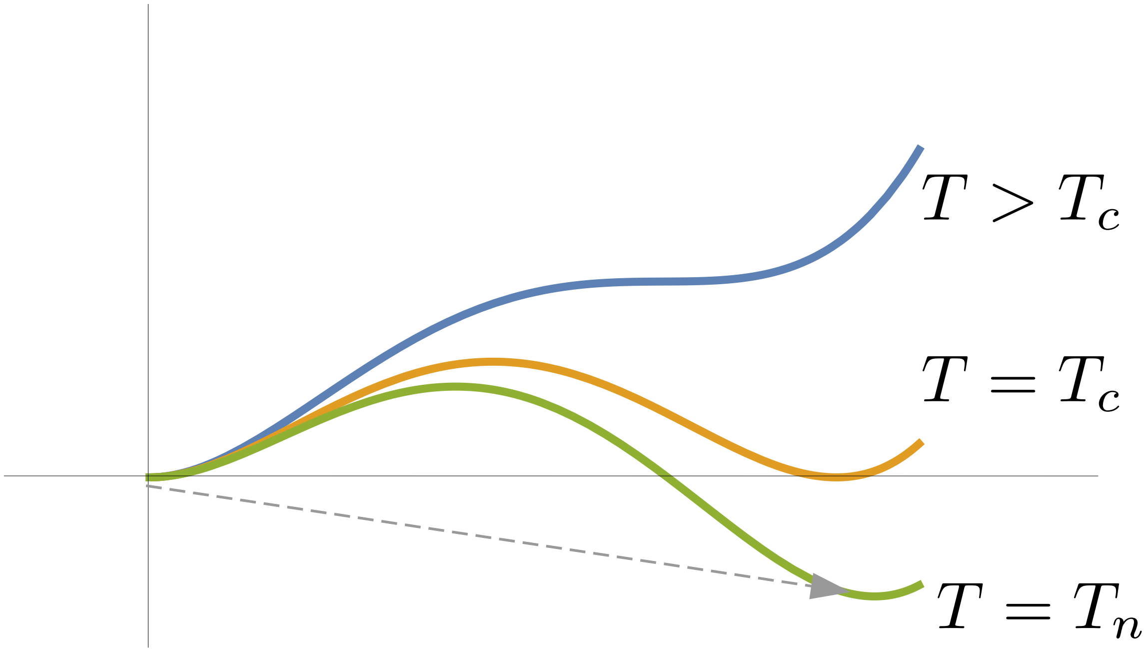
$$\log \frac{T}{m} \to \infty \quad \text{for} \quad m \to 0$$

Phase transitions

A scalar field gets an effective temperature-dependent mass in its interaction with a thermal bath

$$V(\varphi) = \frac{1}{2}(m^2 + g^2T^2)\varphi^2 + \kappa\varphi^3 + \lambda\varphi^4$$

Finite-temperature **loops compete with tree** level terms



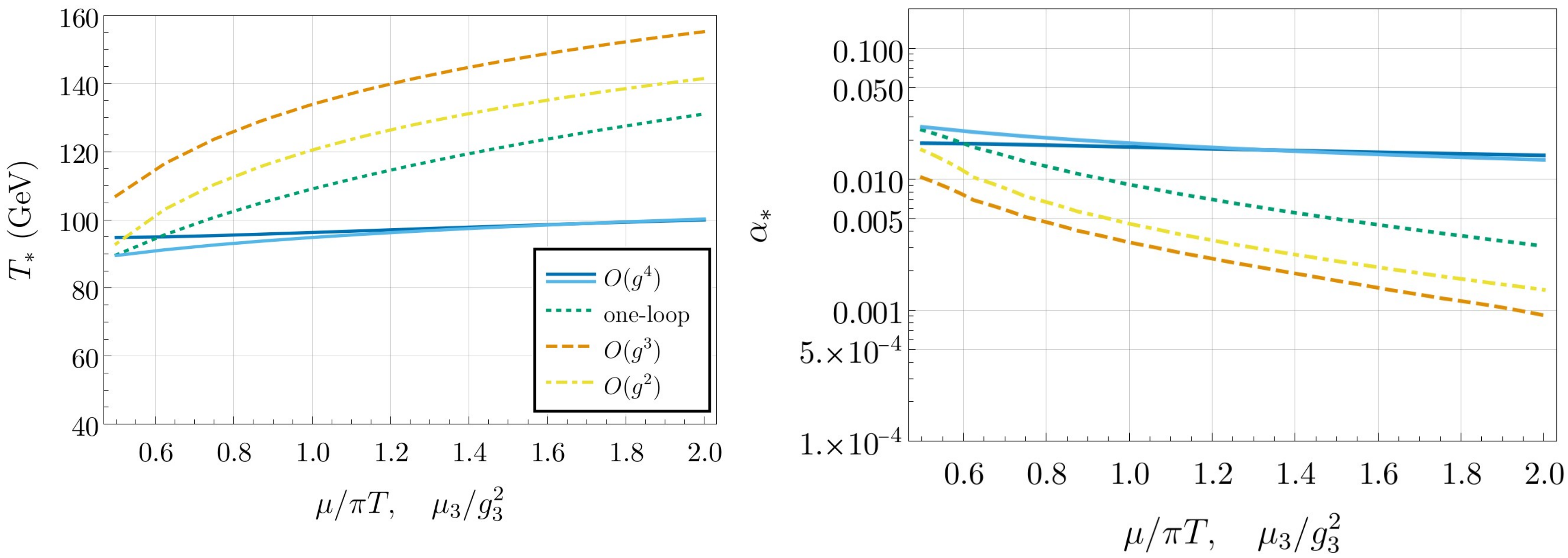
$$\mathcal{P} \sim e^{-\frac{S_3}{T}}$$

$$\alpha \sim \Delta V$$

$$\frac{\beta}{H} \sim \frac{dS}{dT}$$

Phase transitions

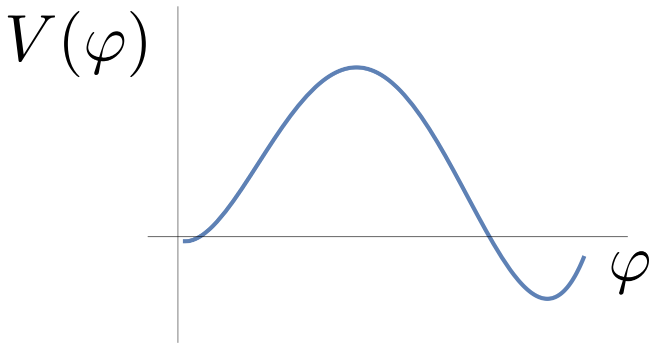
The perturbation theory at finite temperature is not organised in loops; **large renormalisation scale dependence** otherwise

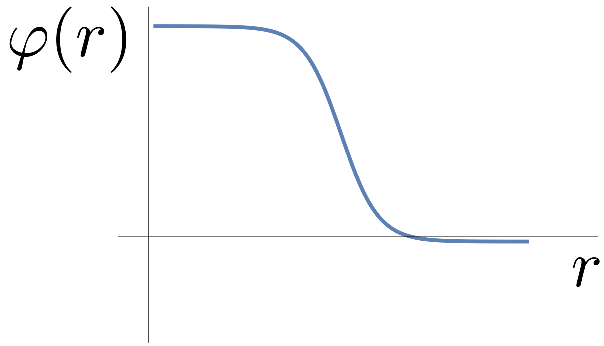


[Gould and Tenkanen '21]

Effective action on the bounce

The effective action must be evaluated at the inhomogeneous bounce solution [Coleman '77]





$$\dot{\varphi}(0) = 0$$

$$\lim_{r \to \infty} \varphi(r) = \varphi_F$$

This jeopardizes the computation of the effective action as an **expansion in derivatives** of the field [Langer '74]

Other issues: double counting, gauge dependence, ... [Langer '74, Strumia and Tetradis '98, Croon et al '21, ...]

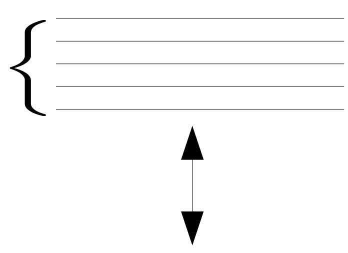
Dimensional-reduction solution

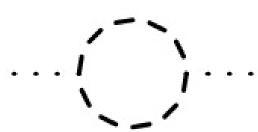
Build a 3D EFT integrating out all Matsubara modes. Capture the thermal effects in a local action, no derivative problems

Systematic resummation, renormalisation-scale dependence under control

Large logarithms avoided at the matching scale; summed within the EFT using renormalization group.

$M \sim \pi T$





$$= \frac{(-3!\kappa)^2}{2} \sum\!\!\!\!\!\!\!\int' \frac{1}{Q^2+m^2} \frac{1}{(Q+P)^2+m^2}$$

Properties of the 3D EFT

It involves only bosons (no zero fermionic Matsubara modes)

"Renormalisable" couplings are dimensionful

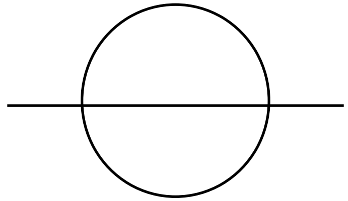
$$V(\varphi) = \frac{1}{2} m^2 \varphi^2 + \kappa \varphi^3 + \lambda \varphi^4$$

$$[m] = 1 \quad [\kappa] = 3/2 \quad [\lambda] = 1$$

It is renormalised first at two loops

$$\text{div}_{\text{1-loop}} \sim \int \frac{d^3 k}{k^3}$$

In the absence of operators of dimension larger than 3, only the mass (and tadpoles) gets renormalised



State-of-the-art of dimensional reduction

Automation of matching for "renormalizable" terms in arbitrary theories completed [Ekstedt, Schicho, Tenkanen '22]

Two-loop sum-integrals solved recently [Davydychev, Navarrete and Schroder '23]

Application to the study of phase transitions (including crossover of the SM): [Kajantie et al '95, Andersen '96, Niemi et al '05, D'Onofrio et al '16, Brauner et al '17, Croon et al '21, Gould '21, Hirvonen '22, ...]

Effective potential computed at N^3LO in the 3D EFT [Ekstedt, Schicho, Tenkanen '24]. Remarkable agreement with lattice results

Further progress in dimensional reduction

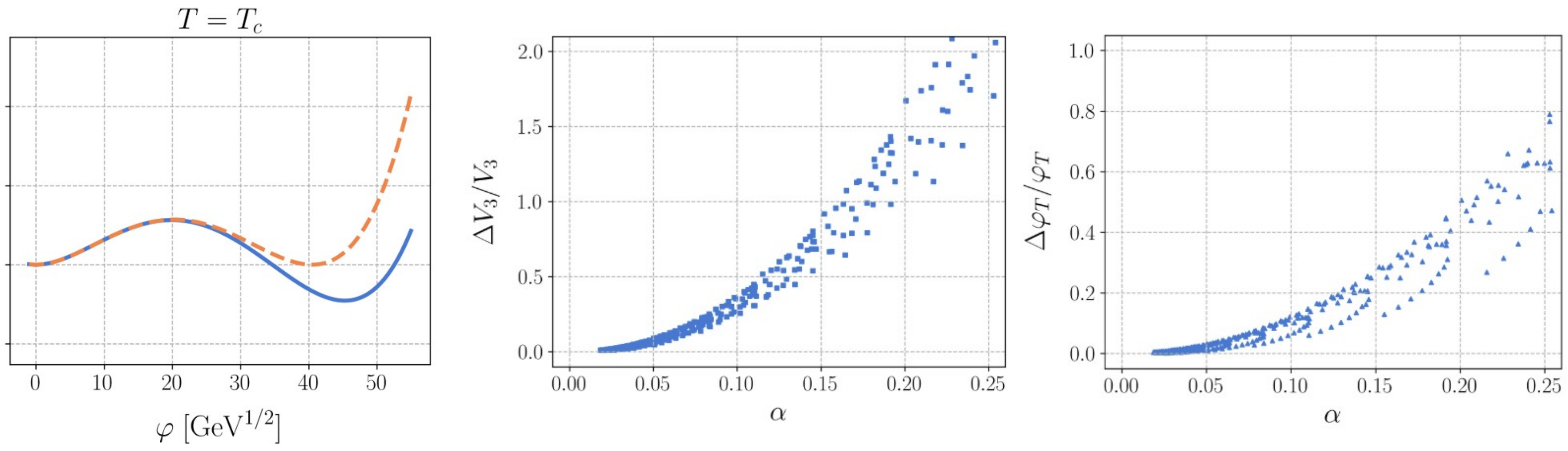
Strong phase transitions produce gravitational waves. Observable in the strong regime, $v/T \gtrsim 1$. Unavoidable effective operators not negligible! [**MC**, Criado, Gil and Miras; '24]

Further progress in dimensional reduction

Strong phase transitions produce gravitational waves. Observable in the strong regime, $v/T \gtrsim 1$. Unavoidable effective operators not negligible! [**MC**, Criado, Gil and Miras; '24]

Take, for example,

$$\mathscr{L}_3 = \frac{1}{2}K_3(\partial\varphi)^2 + \frac{1}{2}m_3^2\varphi^2 + \kappa_3\varphi^3 + \lambda_3\varphi^4 \; + \alpha_{61}\varphi^6$$

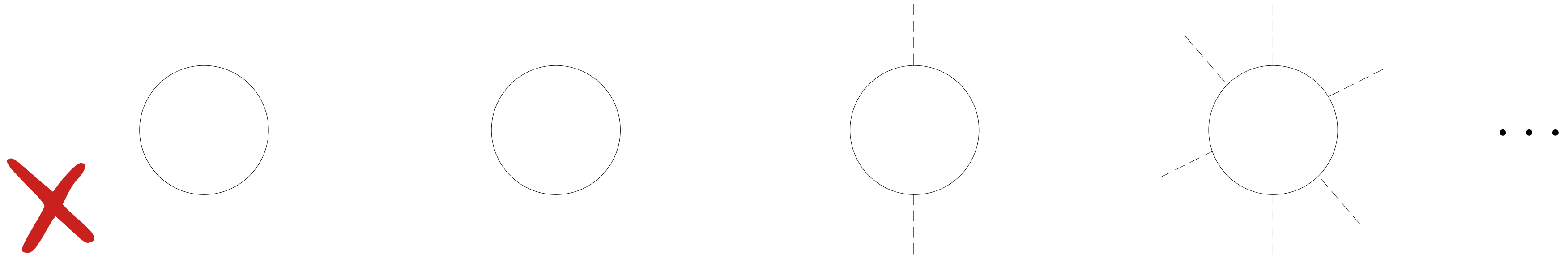


Further progress in dimensional reduction

Our goal: understanding the impact of unavoidable higher-dimensional operators on the estimation of phase-transition (and gravitational wave) parameters

Strategy: Consider a simple model, perform the matching to higher-order, compute the parameters, compare with results without higher-order terms

$$L = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2 - \kappa\varphi^3 - \lambda\varphi^4 + i\bar{\psi}D\psi + g\varphi\bar{\psi}\psi$$



Matching

We assume the standard power counting: $\partial^2 \sim p^2 \sim \mathcal{O}(g^2T^2)$

We compute off-shell (1PI) Green's functions in the full theory to order $\mathcal{O}(g^8)$ (so at most six external momenta)

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$$= \frac{(-3!\kappa)^2}{2} \sum\!\!\!\!\!\!\!\int{}' \frac{1}{Q^2 + m^2} \frac{1}{(Q+P)^2 + m^2} \simeq 18\kappa^2 \sum\!\!\!\!\!\!\!\int{}' \frac{1}{Q^2 (Q+P)^2}$$

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$$\simeq 18\kappa^2 \sum\!\!\!\!\!\!\!\int_Q' \frac{1}{Q^4}\left[1 - \frac{P^2}{Q^2} + \frac{P^4}{Q^4} + \frac{4\,(P\cdot Q)^2}{Q^4} - \frac{P^6}{Q^6} - \frac{12P^2\,(P\cdot Q)^2}{Q^6} \right.$$

$$\left. + \frac{24P^4\,(P\cdot Q)^2}{Q^8} + \frac{16\,(P\cdot Q)^4}{Q^8} - \frac{80P^2\,(P\cdot Q)^4}{Q^{10}} + \frac{64\,(P\cdot Q)^6}{Q^{12}}\right]$$

The effective field theory

The problem of building an off-shell basis of the effective field theory of a set of fields to a specified dimension is solved [Criado '19, Fonseca '19]

In our case, it reads:

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}(\partial_i\phi)^2 + \frac{1}{2}m_3^2\phi^2 + \kappa_3\phi^3 + \lambda_3\phi^4 \\ &+ \alpha_{61}\phi^6 + \beta_{61}\partial^2\phi\partial^2\phi + \beta_{62}\phi^3\partial^2\phi \\ &+ \alpha_{81}\phi^8 + \alpha_{82}\phi^2\partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi + \beta_{81}\phi\partial^6\phi + \beta_{82}\phi^3\partial^4\phi + \beta_{83}\phi^2\partial^2\phi\partial^2\phi + \beta_{84}\phi^5\partial^2\phi \\ &+ \cdots\end{aligned}$$

All the beta operators, however, can be removed upon field redefinitions

The effective field theory

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In our case, it reads:

$$K_3 = 1 + \frac{g^2}{12\pi^2}\,, \qquad m_3^2 = m^2 + \frac{g^2 T^2}{6}\,, \qquad \kappa_3 = \kappa\sqrt{T}\,, \qquad \lambda_3 = \lambda T\,;$$

$$\alpha_{61} = -\frac{7\zeta(3) g^6}{192\pi^4}\,, \qquad \beta_{61} = -\frac{7\zeta(3) g^2}{384\pi^4 T^2}\,, \qquad \beta_{62} = \frac{35\zeta(3) g^4}{576\pi^4 T}\,;$$

$$\alpha_{81} = \frac{31\zeta(5) g^8}{2048\pi^6 T}\,, \qquad \alpha_{82} = -\frac{31\zeta(5) g^4}{10240\pi^6 T^3}\,, \qquad \beta_{81} = -\frac{31\zeta(5) g^2}{10240\pi^6 T^4}\,,$$

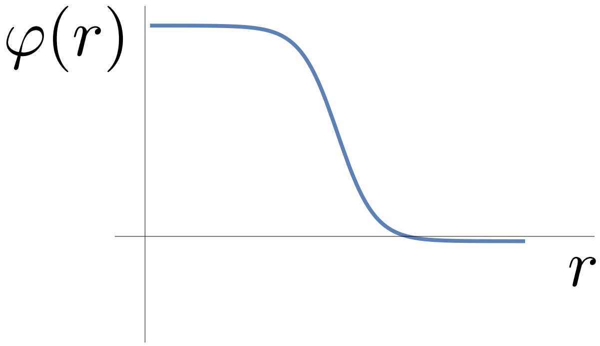
$$\beta_{82} = \frac{217\zeta(5) g^4}{20480\pi^6 T^3}\,, \qquad \beta_{83} = \frac{279\zeta(5) g^4}{20480\pi^6 T^3}\,, \qquad \beta_{84} = -\frac{217\zeta(5) g^6}{5120\pi^6 T^2}\,.$$

All the beta operators, however, can be removed upon field redefinitions

How to compute the bounce?

Well known methods/codes for computing the bounce solution in the presence of a "standard" kinetic term

$$S_3 = \int dr\, r^2 [\frac{1}{2} m^2 \varphi^2 + \frac{1}{2}(\partial\varphi)^2 + V(\varphi)]$$



$$\ddot{\varphi} + \frac{2}{r}\varphi = V'(\varphi)$$
$$\dot{\varphi}(0) = 0\,, \quad \lim_{r\to\infty} \varphi(r) = 0$$

Neither the bounce nor the effective action are physical; only the value of S_3 at extrema is (naive computations unphysical!)

Perturbative bounce solution

We have a perturbative expansion, so let's use it consistently

$$\varphi_c = \varphi_c^{(0)} + \epsilon\varphi_c^{(1)} + \epsilon^2\varphi_c^{(2)} + \cdots\,, \quad S_3 = S_3^{(0)} + \epsilon S_3^{(1)} + \epsilon^2 S_3^{(2)} + \cdots$$

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$$S_3[\varphi_c] = S_3^{(0)}[\varphi_c^{(0)}] + \epsilon S_3^{(1)}[\varphi_c^{0}]$$

$$+\epsilon \int \varphi_1 \frac{\delta S^{(0)}}{\delta\varphi}\bigg|_{\varphi_c^{(0)}}$$

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$$+\epsilon \int \varphi_1 \frac{\delta S^{(0)}}{\delta\varphi}\bigg|_{\varphi_c^{(0)}} \qquad +\epsilon^2 \int \varphi_2 \frac{\delta S^{(0)}}{\delta\varphi}\bigg|_{\varphi_c^{(0)}}$$

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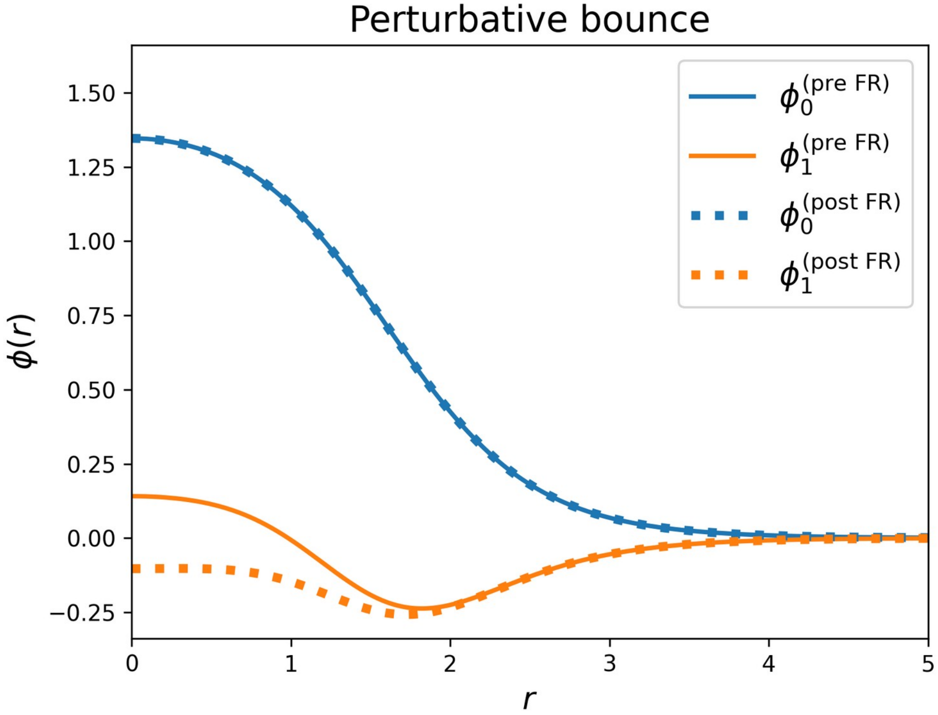
Then require that $\displaystyle\frac{\delta}{\delta\varphi}S_3\bigg|_{\varphi_c} = 0$

$$\frac{\delta}{\delta\varphi}S_3^{(0)}\bigg|_{\varphi_c^{(0)}} = 0 \qquad \ddot{\varphi}_c^{(1)} + \frac{2}{r}\dot{\varphi}_c^{(1)} - V_3^{(0)\prime\prime}(\varphi_c^{(0)})\;\varphi_c^{(1)} - \frac{1}{4\pi r^2}\frac{\delta S_3^{(1)}}{\delta\varphi}\bigg|_{\varphi_c^{(0)}} = 0$$

Perturbative bounce solution

$S_3[\varphi_c]$ computed this way is physical (i.e. invariant under field redefinitions); physical observables independent of how matching is performed



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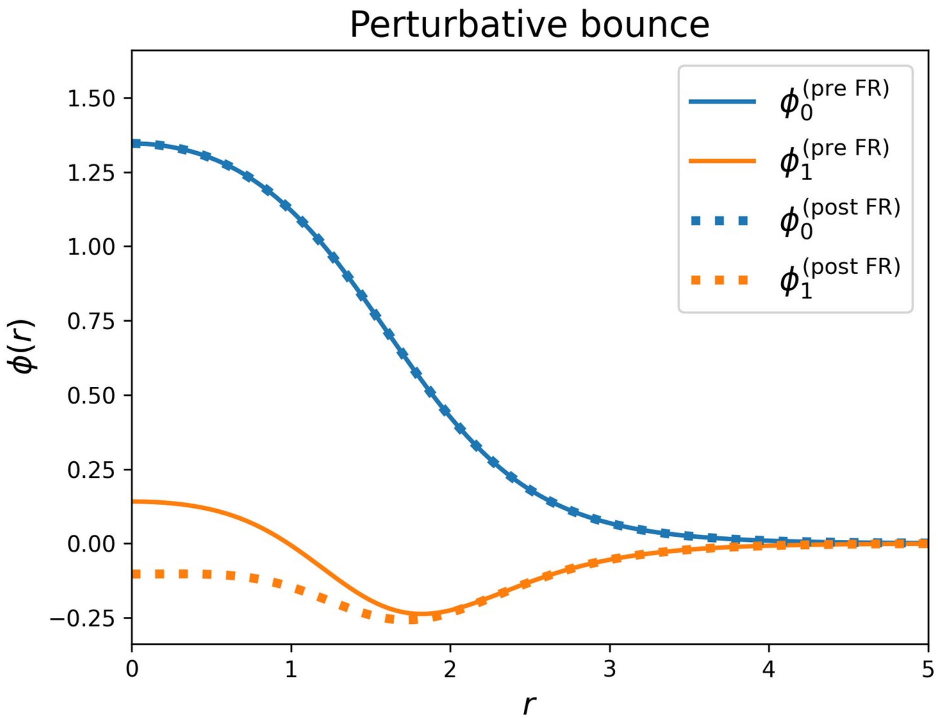
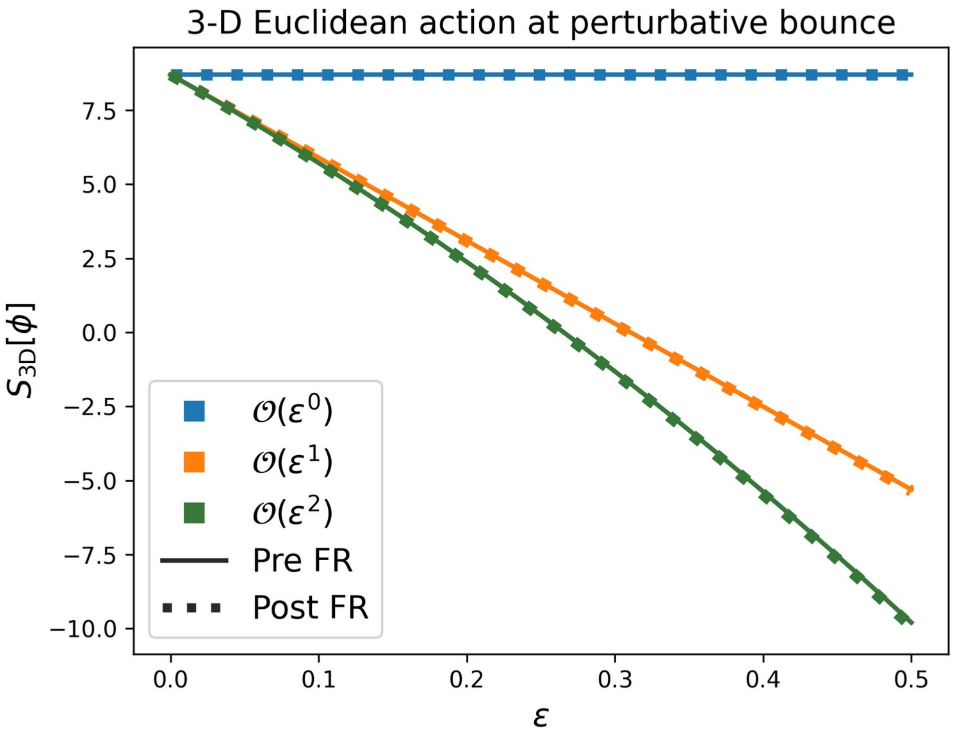
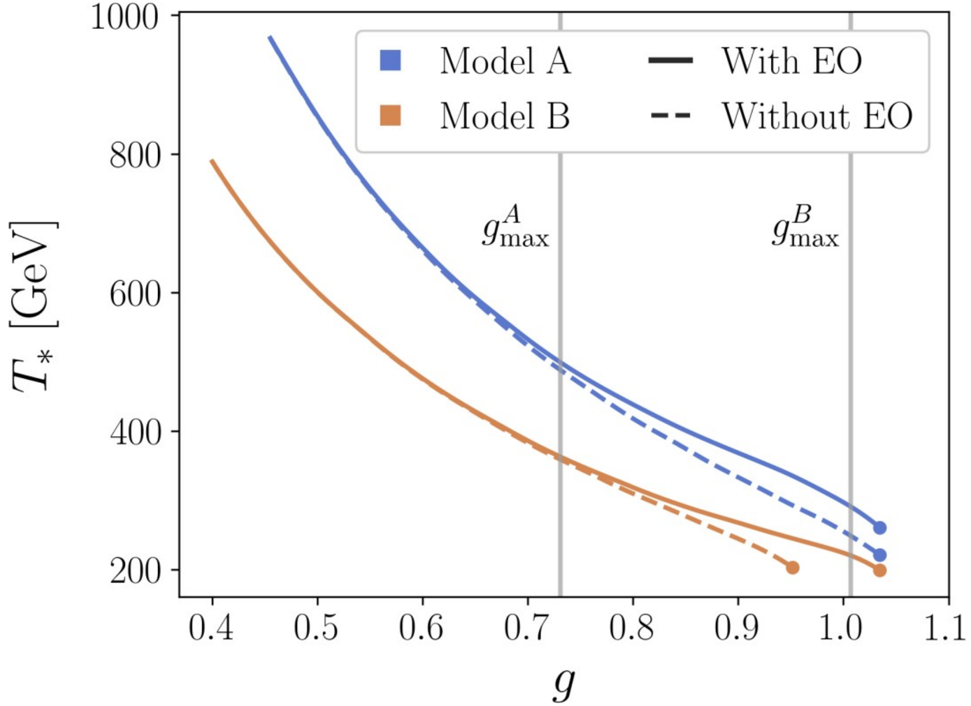




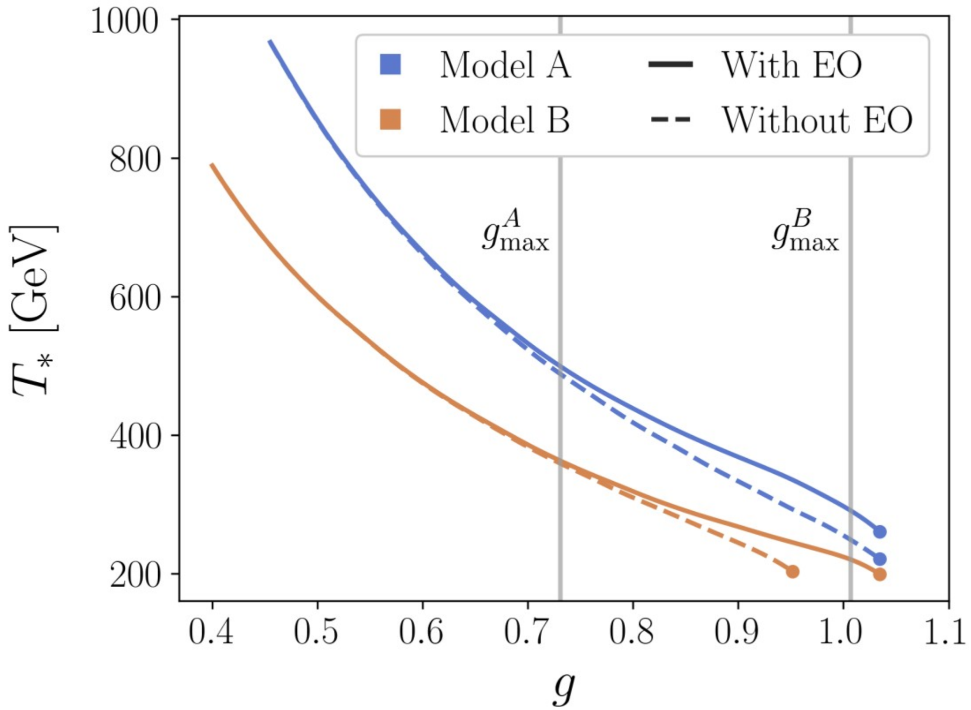
Phase transition parameters

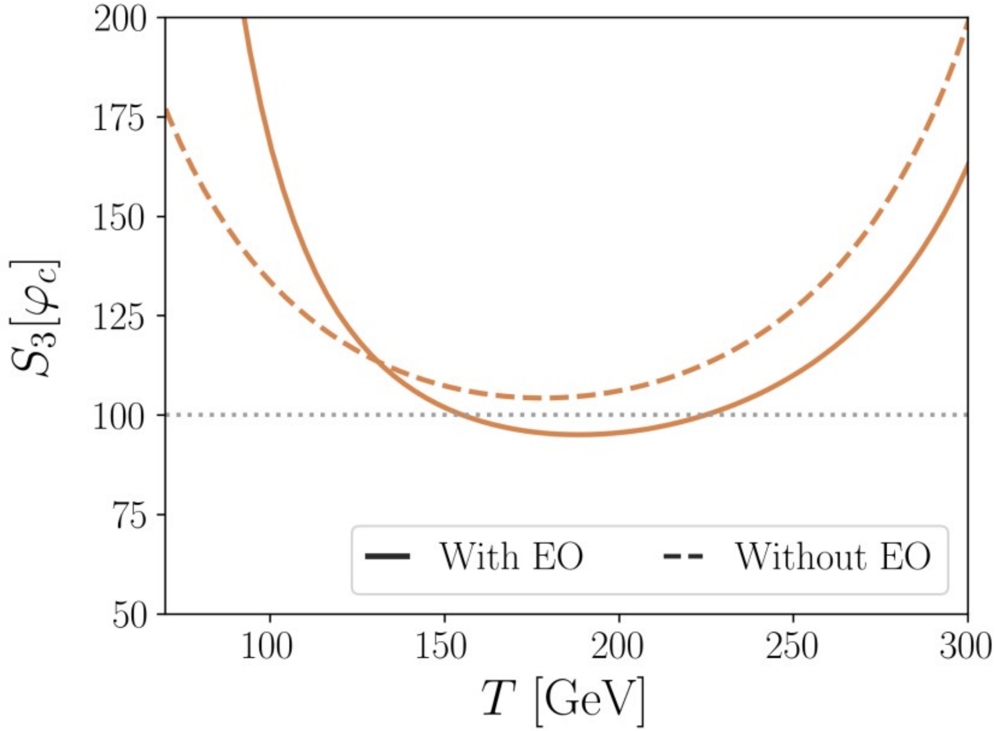
Nucleation temperature: $S_3[\varphi_c] \sim 100 - 4 \log \frac{T_*}{100\,\text{GeV}}$



Phase transition parameters

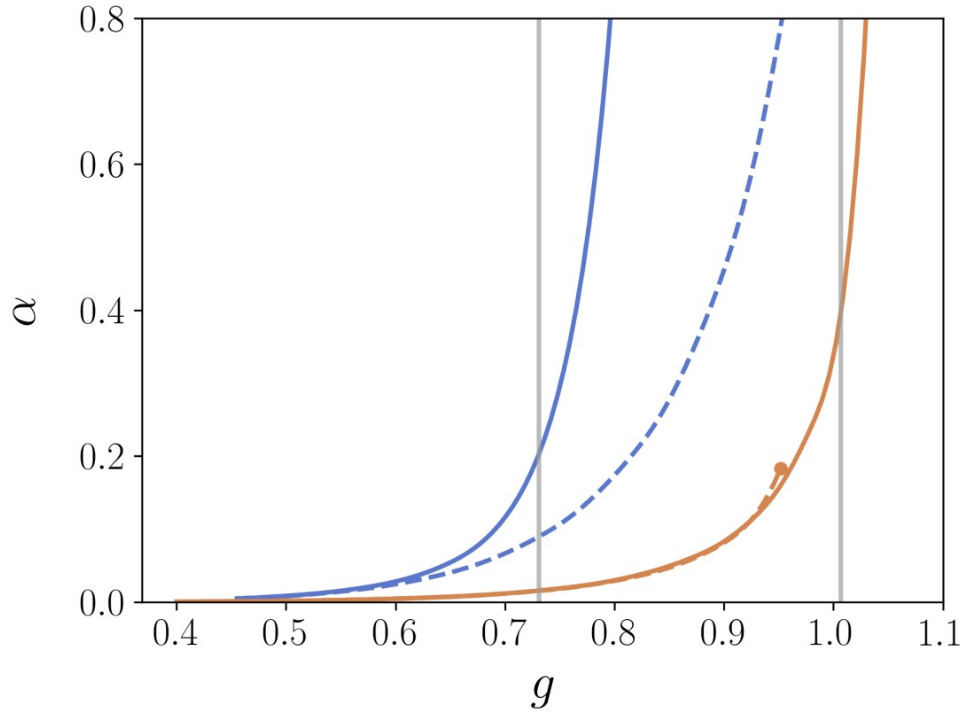
Nucleation temperature: $S_3[\varphi_c] \sim 100 - 4\log\dfrac{T_*}{100\,\mathrm{GeV}}$

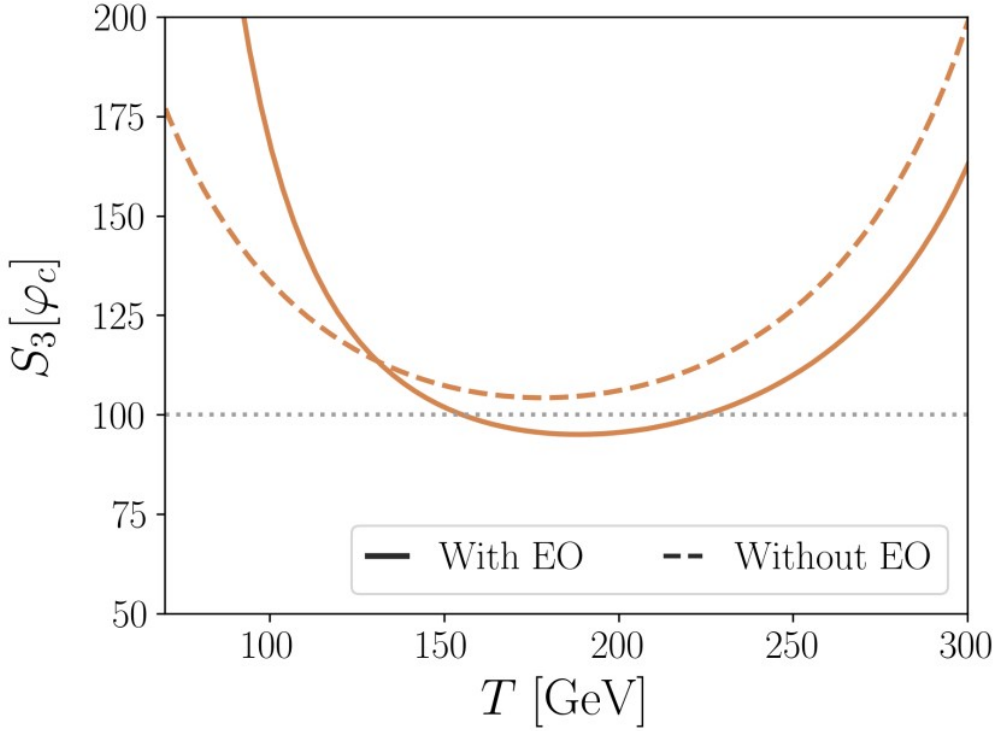




Phase transition parameters

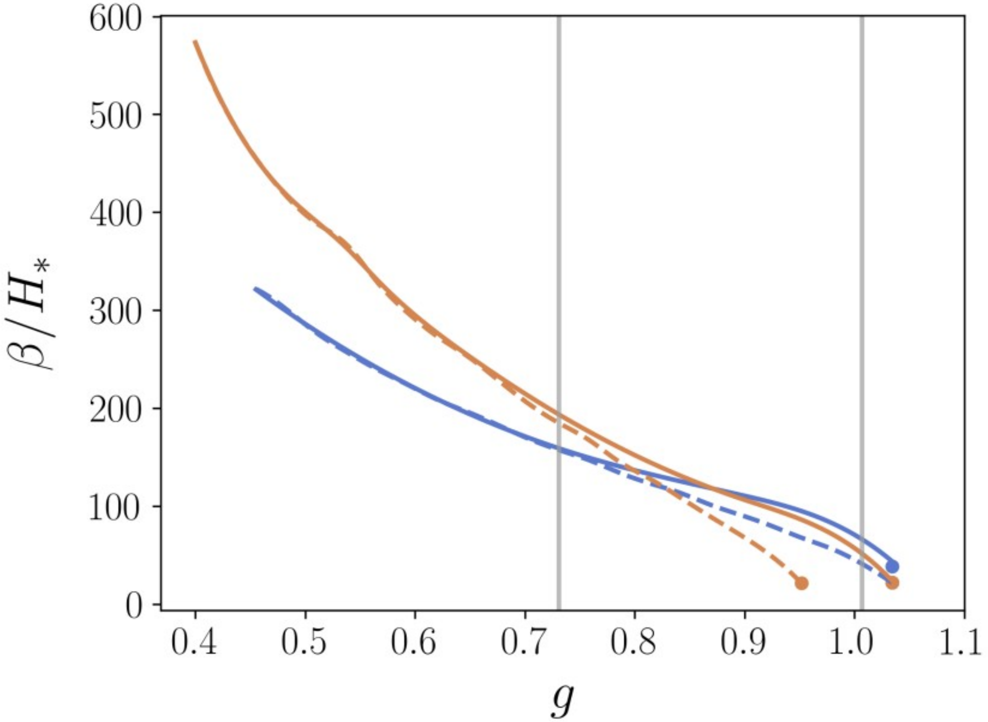
Latent heat: $\alpha = \dfrac{\Delta\left(V_3(\varphi) - \frac{T}{4}\frac{\mathrm{d}}{\mathrm{d}T}V_3(\varphi)\right)\Big|_{T_*}}{\rho_r(T_*)} \approx -0.03\dfrac{\Delta\left(V_3(\varphi) - \frac{T}{4}\frac{\mathrm{d}}{\mathrm{d}T}V_3(\varphi)\right)\Big|_{T_*}}{T_*^3}$

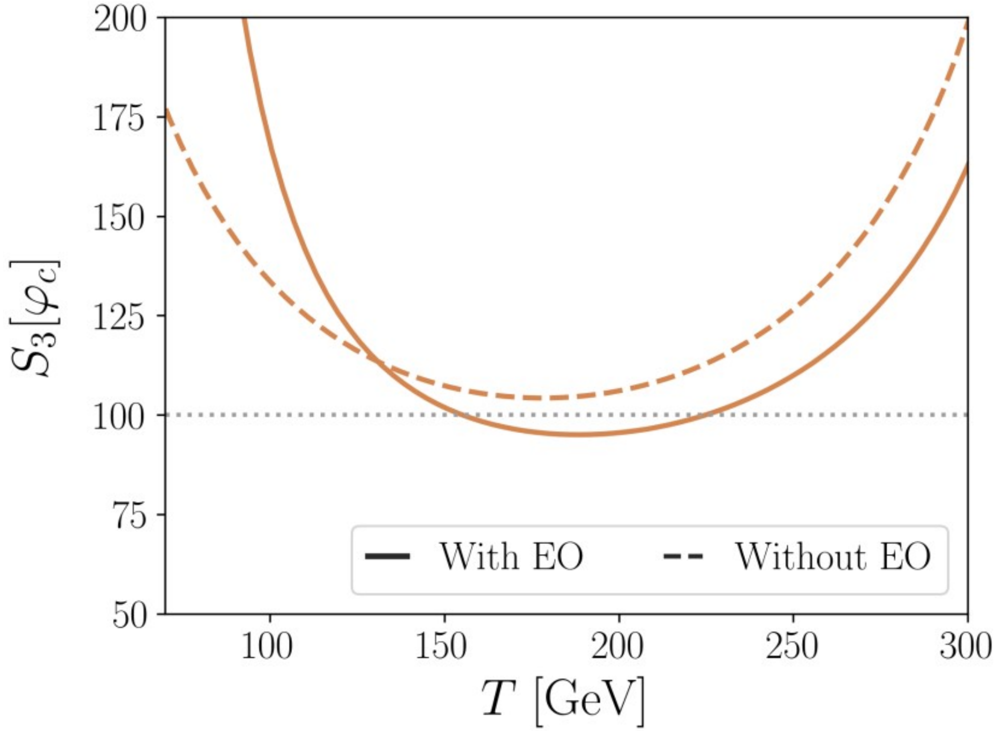




Phase transition parameters

Inverse duration: $$\frac{\beta}{H_*} = T_* \frac{\mathrm{d}S_3[\varphi_c]}{\mathrm{d}T}\bigg|_{T_*}$$





Gravitational waves

Higher-order terms open the parameter space of phase transitions and modify substantially the phase transition parameters.

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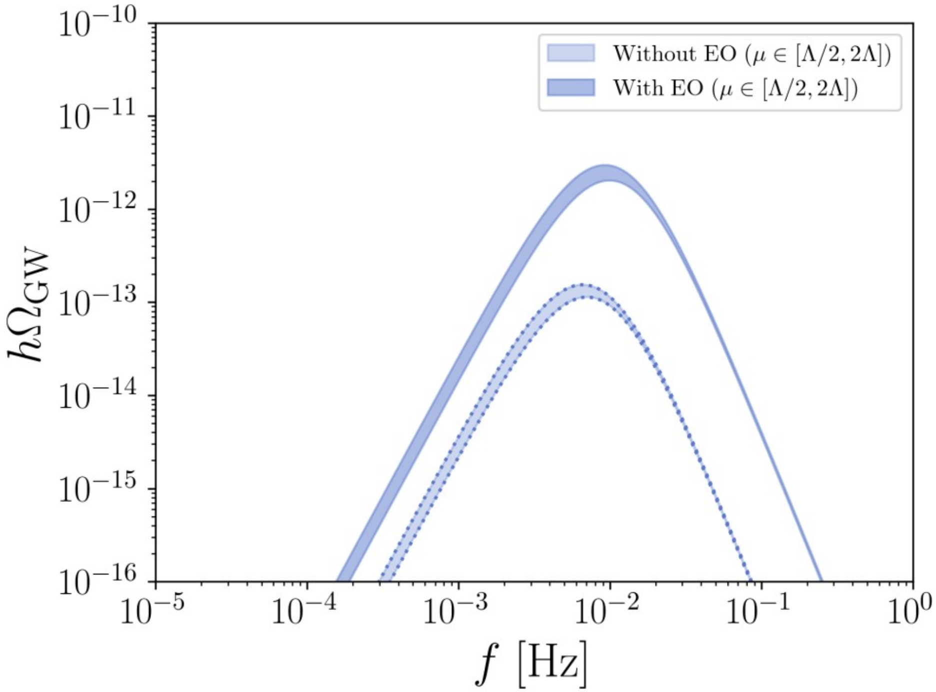
Effect complementary to renormalisation scale dependence in gravitational wave spectrum:



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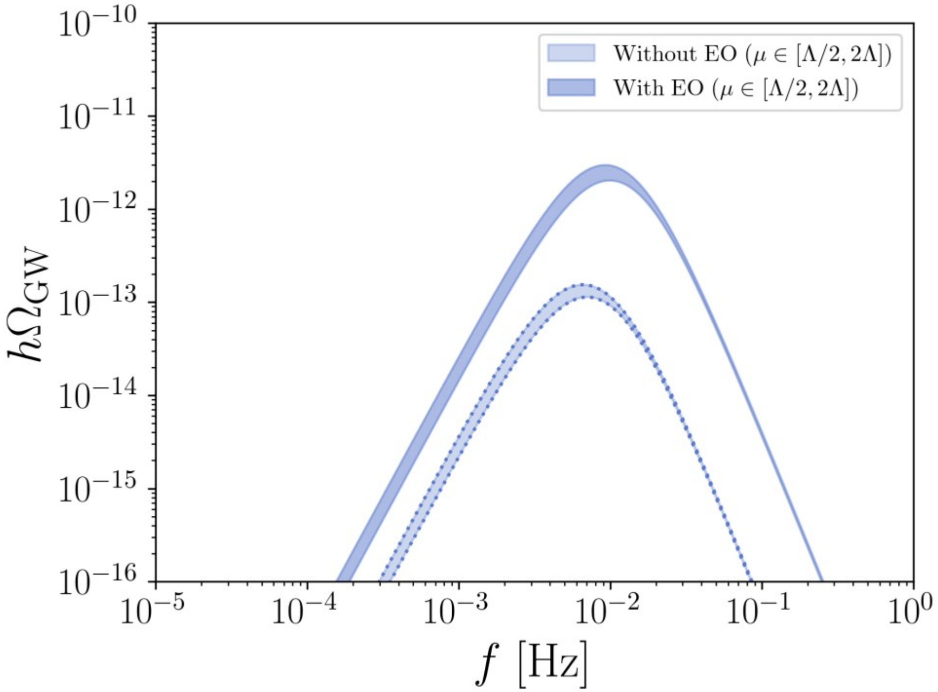
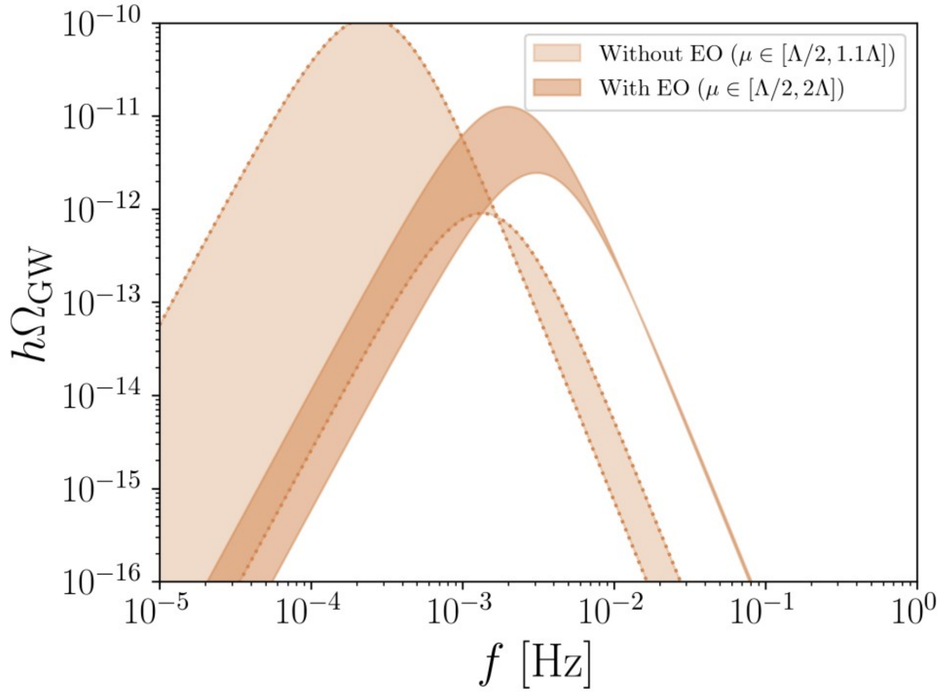




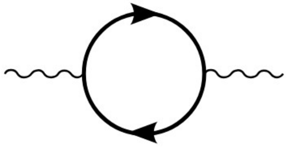
Further progress in dimensional reduction

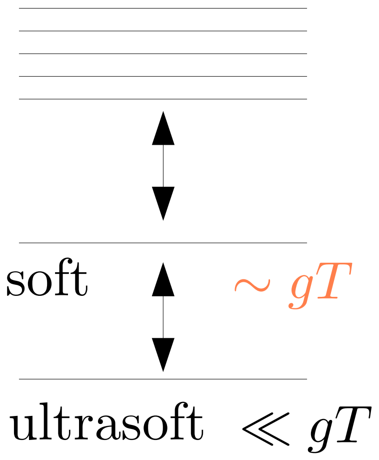
3-dimensional theory of the SM? Leading part used to determine the nature of electroweak phase transition [D'Onofrio and Rummukainen '16]

Partial results for higher-dimensional operators [Moore '95]

Complications associated to the presence of gauge bosons. First, because further scalars (temporal components) proliferate. Second, because they acquire Debye masses larger than m_{eff} [**MC**, Ekstedt and Guedes 'work in progress]

$$\mathcal{L}^{(4)}_{\text{SM3D}} = k_\phi (D_r\phi)^\dagger (D^r\phi) + \frac{k_{B_0}}{2}(D_r B_0)(D^r B_0) - \frac{k_B}{4} B_{rs}B^{rs}$$
$$+ \lambda_{\phi^4}|\phi|^4 + \lambda_{B_0^4} B_0^4 + \lambda_{\phi^2 B_0^2}|\phi|^2 B_0^2 \,,$$





Outlook

Dimensional-reduction is the most appropriate description of systems at finite temperature

Up to now, the higher-point/higher-derivative terms in the 3-dimensional theory have been mostly ignored

We derived the correct way of computing physical quantities taking these interactions into consideration

For strong phase transitions, the effects of these interactions are often larger than those ensuing from variations of renormalisation scale

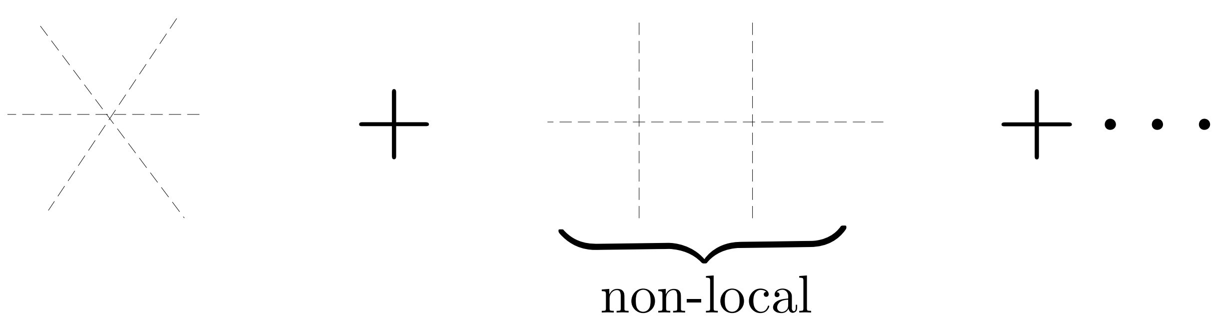
Thank you!

Field redefinitions

No systematic/automated way of removing the redundant interactions (notice that equations of motion correct only to linear order [Criado and Perez-Victoria '18])

The closest public solution implemented in Matchete [Fuentes-Martin et al '22], however the target physical basis cannot be chosen

Our solution [**MC**, Lopez-Miras, Santigo, Vilches-Bravo 'wip]: if the redundant and non-redundant bases are physically equivalent, find relation between both upon requiring they give the same S-matrix



Field redefinitions

Many restrictions on-shell: $\quad p^2 = m^2, \sum_i p_i = 0, \cdots$

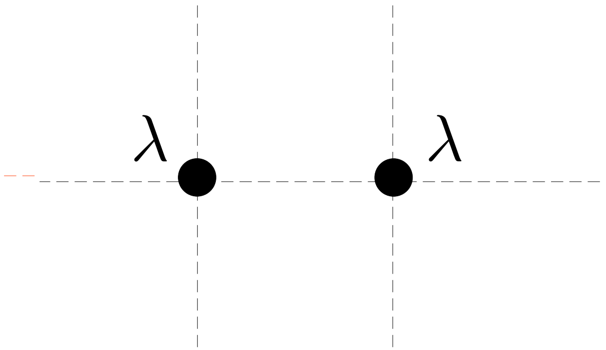
Non-localities makes the system of equations complicated (impossible?) to solved symbolically

Proposal: compute S-matrix elements in specific physical phase-space points

We use rational kinematics, based on momentum twistors [Badger '16, Angelis '22]

$$\alpha_1 \left(\begin{array}{c}\text{Complicated} \\ \text{non-local function} \\ \text{of momenta}\end{array} \right) + \alpha_2 \left(\begin{array}{c}\text{another} \\ \text{complicated} \\ \text{non-local function} \\ \text{of momenta}\end{array} \right) + \cdots = 0$$

Field redefinitions



$$m^2 \to m^2 \left(1 - \frac{2m^2\beta_{61}}{\Lambda^2} + \frac{1}{\Lambda^4}(8m^4\beta_{61}^2 + 2\beta_{81})\right),$$

$$\lambda \to \lambda + \frac{m^2}{\Lambda^2}(\beta_{62} - 8\lambda\beta_{61}) + \frac{m^4}{\Lambda^4}\left(64\lambda\beta_{61}^2 - 10\beta_{61}\beta_{62} + 12\lambda\beta_{81} - \beta_{82} - \beta_{83}\right),$$

$$\alpha_{61} \to \alpha_{61} + 16\lambda^2\beta_{61} - 4\lambda\beta_{62} -$$
$$- \frac{m^2}{\Lambda^2}\left(\frac{1728}{5}\lambda^2\beta_{61}^2 + \frac{22}{5}\beta_{62}^2 - \frac{512}{5}\lambda\beta_{61}\beta_{62} + 12\alpha_{61}\beta_{61} + \frac{304}{5}\lambda^2\beta_{81} - \frac{56}{5}\lambda\beta_{82} - 8\lambda\beta_{83} + \beta_{84}\right)$$

$$\alpha_{81} \to \alpha_{81} - \frac{3072}{5}\lambda^3\beta_{61}^2 - \frac{108}{5}\lambda\beta_{62}^2 + \frac{1248}{5}\lambda^2\beta_{61}\beta_{62} - 48\alpha_{61}\beta_{61} + 6\alpha_{61}\beta_{62} - \frac{576}{5}\lambda^3\beta_{81} +$$
$$+ \frac{144}{5}\lambda^2\beta_{82} + 16\lambda^2\beta_{83} - 4\lambda\beta_{84},$$

$$\alpha_{82} \to \alpha_{82}.$$