Some topics of

Operator renormalization on lattice

towards precise calculation of K→ππ

– K → ππ endeavor part 2 –

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BNL HET Lunch Seminar
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K → ππ & CP violation

CP odd CP even

$$|K_L\rangle = |K_2\rangle + \varepsilon |K_1\rangle$$

ε' direct CPV

ε indirect CPV

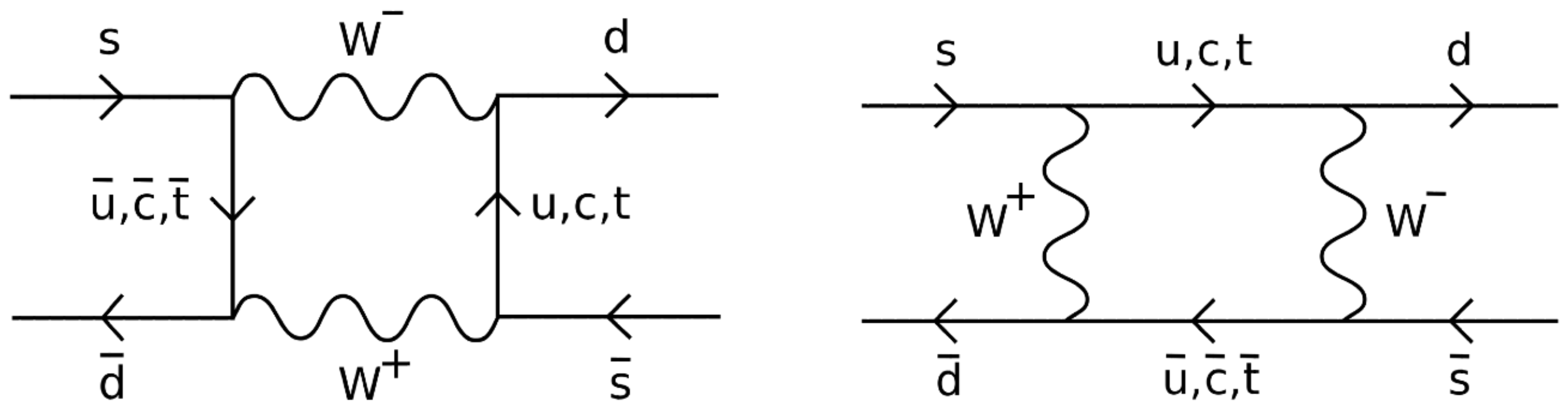
$|\pi\pi\rangle$

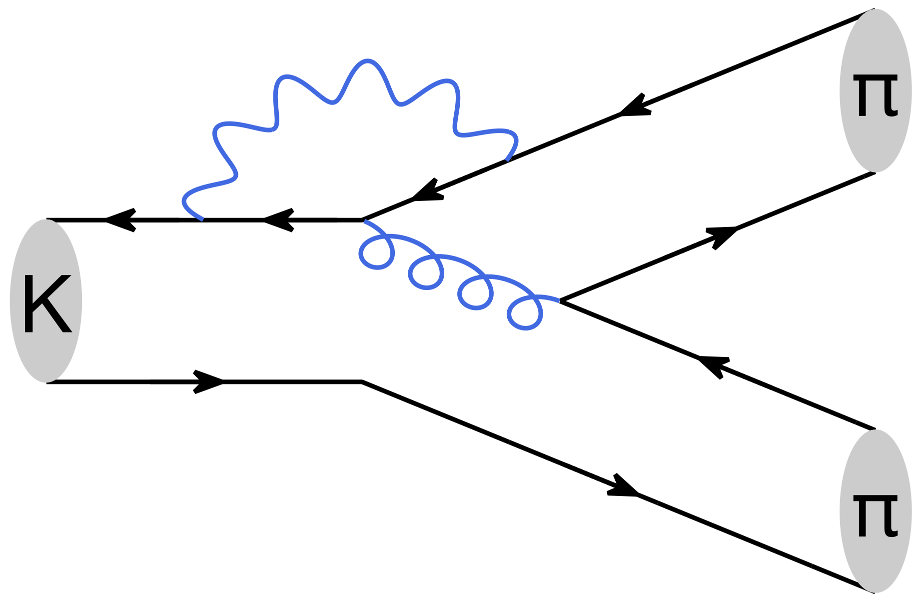
CP even

Discovered in 1964

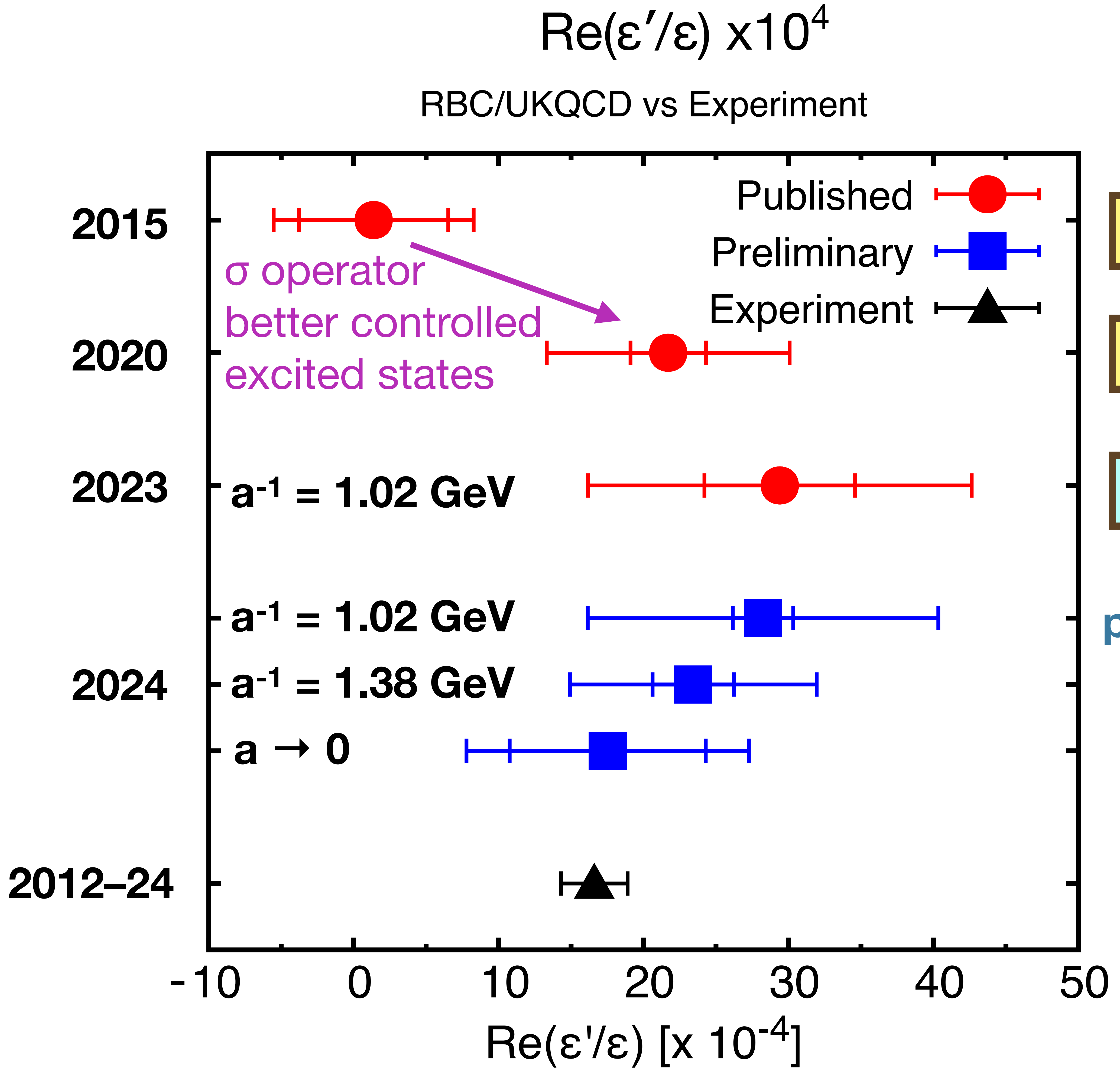
$$\frac{\Gamma(K_L \to \pi^0\pi^0)}{\Gamma(K_S \to \pi^0\pi^0)} \Big/ \frac{\Gamma(K_L \to \pi^+\pi^-)}{\Gamma(K_S \to \pi^+\pi^-)} = 1 - 6\mathrm{Re}(\varepsilon'/\varepsilon)$$

- $|\varepsilon| = 2.228(11) \times 10^{-3}$ from "odd" mixing b/w K^0 & $\overline{K}^0$
- ε' only found in decays **Discovered in 1999**
  - $\mathrm{Re}(\varepsilon'/\varepsilon)_{exp} = 1.66(23) \times 10^{-3}$ (KTeV & NA48)
  - Consistent with SM?





RBC/UKQCD achievements



PRL 115,212001

PRD 102,054509

G-parity Boundary Conditions (GPBC)
$a^{-1} \approx 1.38$ GeV

PRD 102,094517

preliminary results with multiple lattice spacings

Periodic Boundary Conditions (PBC)
$a^{-1} \approx 1.38$ GeV & 1.02 GeV

Challenges

- On-shell kinematics in euclidean space
  - final two-pion state with $E = m_K \approx 500$ MeV needed

- Operator mixing
  - 10 four-quark operators
  - desired to prevent extra mixing
  - chiral symmetry with domain wall fermions preferable

- Computational cost/ Statistics
  - disconnected diagrams

- Charm-loop effects
  - expected significant
  - lattices still not fine enough to introduce charm with physical m_π

- Overall complexity
  - requires significant human time & effort

Introduction to renormalization on lattice and continuum limit

Renormalization in lattice calculation

Divergent quantity

↓ **Regularization**

Finite quantity

Calculate the quantity with chosen lattice action
- particular regularization parameter: lattice spacing a

↓ **Renormalization**

Renormalized quantity with finite regularization parameters left

O → O' = O – O (additive) &
O' → O_R = Z O' (multiplicative)
- Condition unique to scheme & scale
- Still dependent on lattice spacing → discretization error

↓ **Removal of regularization parameter**

Renormalized quantity unique to scheme & scale

Continuum limit: a → 0
- Repeat the above steps with multiple lattice spacings and fit with some ansatz
- e.g. $O_R(a) = O_R(0) + c_2\, a^2 + c_4\, a^4$

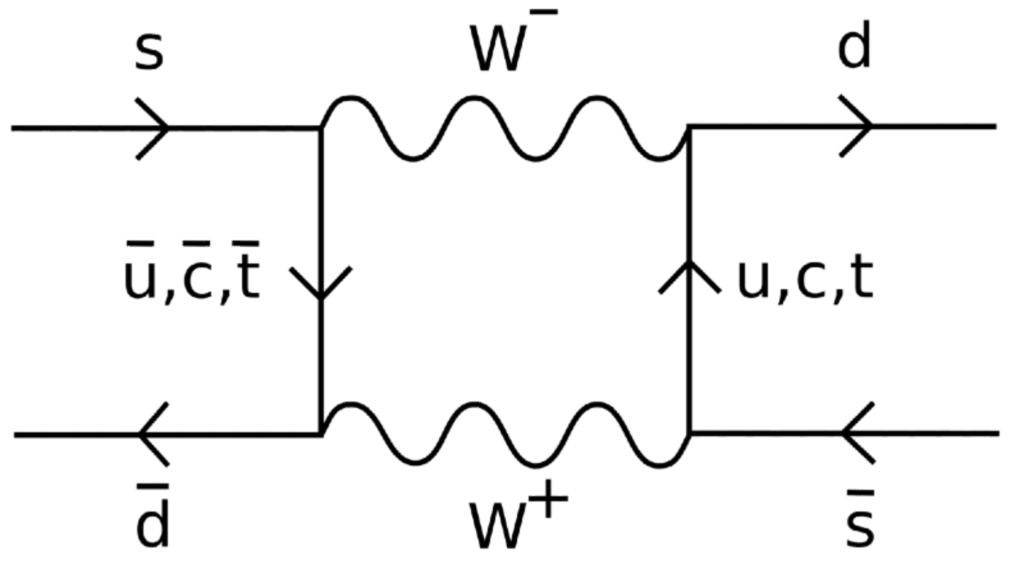
Example 1: B_K

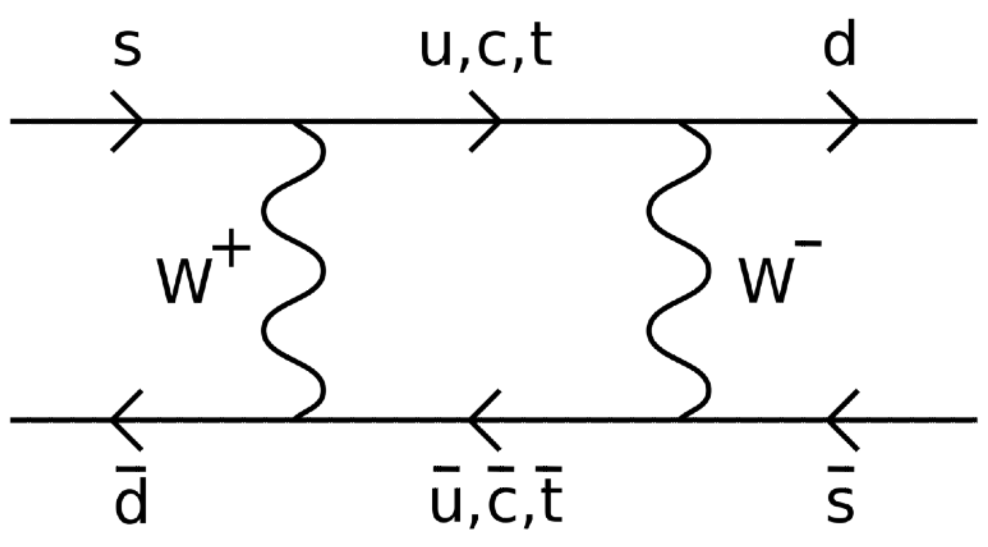
4-quark operator to be renormalized

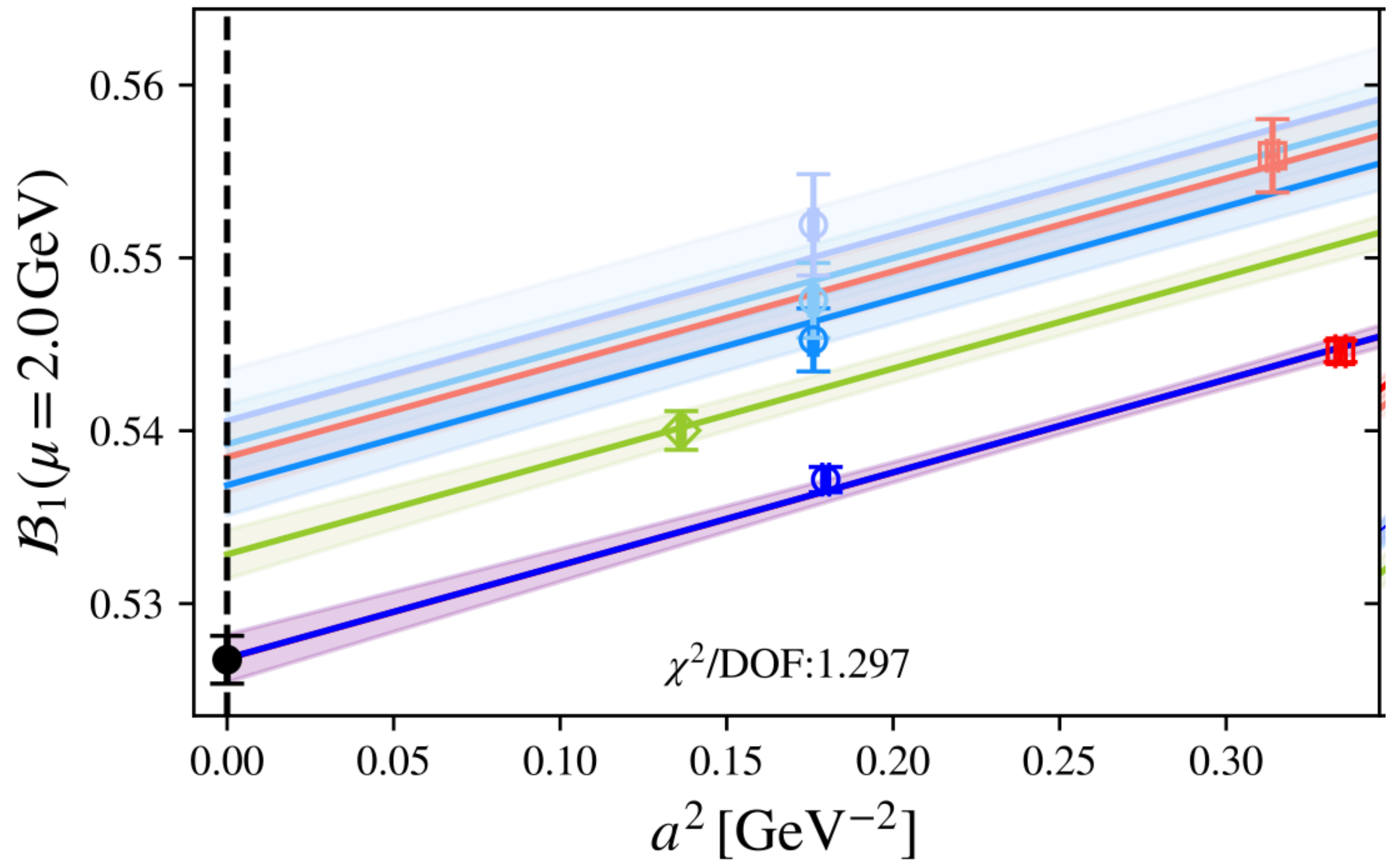
$$B_K = \frac{\left\langle \overline{K}^0 \middle| \underbrace{\bar{s}\gamma_\mu(1-\gamma_5)d \cdot \bar{s}\gamma_\mu(1-\gamma_5)d}_{} \middle| K^0 \right\rangle}{\frac{8}{3} m_K^2 f_K^2}$$

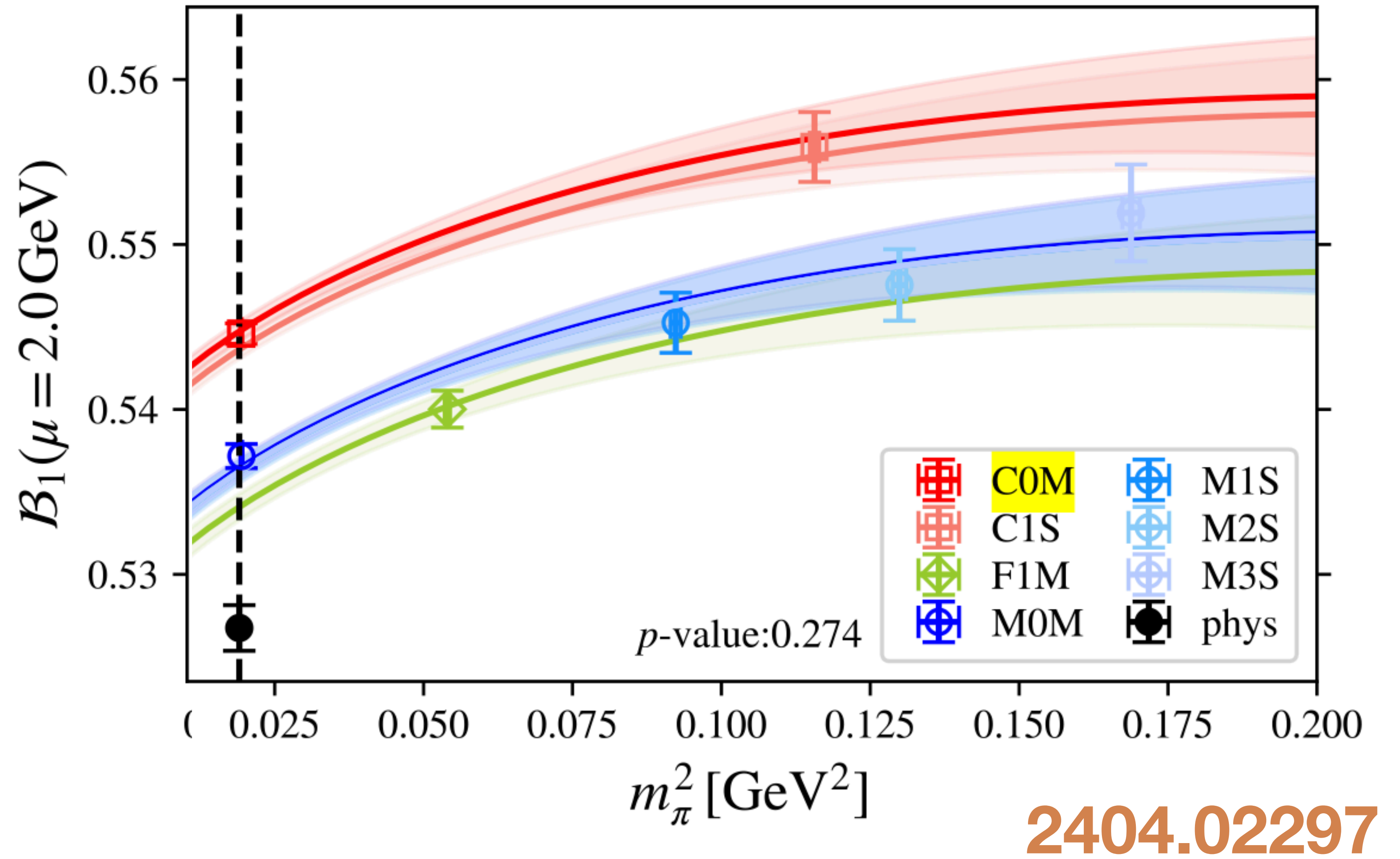
$\sim \langle 0|\bar{s}\gamma_\mu\gamma_5 d|K\rangle$

to be renormalized by Z_A









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*Included simulations at unphysical m_π

Continuum limit of renormalized quantity

- Operator renormalization

$$O^R(\mu,a) = Z(\mu,a)O^{lat}(a)$$

- Renormalized quantity

$$\left\langle O^R(\mu,a)\right\rangle = Z(\mu,a)\left\langle O^{lat}(a)\right\rangle$$
$$= \left\langle O^R(\mu,0)\right\rangle + \underline{c_2^R(\mu)}a^2 + \underline{c_4^R(\mu)}a^4 + \ldots$$

dependent on computational details

- lattice action
- renormalization scheme, scale, …
- …

Continuum limit of renormalized quantity

- Operator renormalization

$$O^R(\mu, a) = Z(\mu, a) O^{lat}(a)$$

- Renormalized quantity

$$\left\langle O^R(\mu, a) \right\rangle = Z(\mu, a) \left\langle O^{lat}(a) \right\rangle$$
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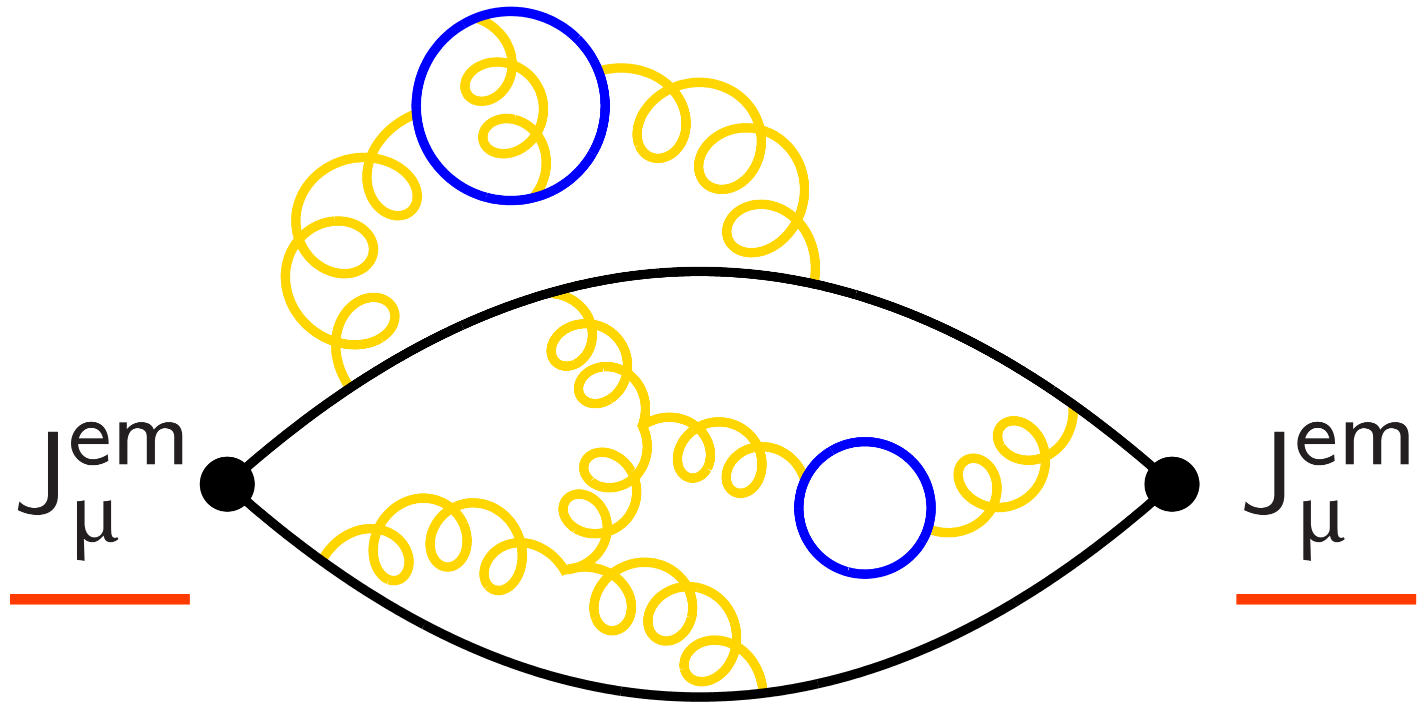
Independent of computational details

dependent on computational details

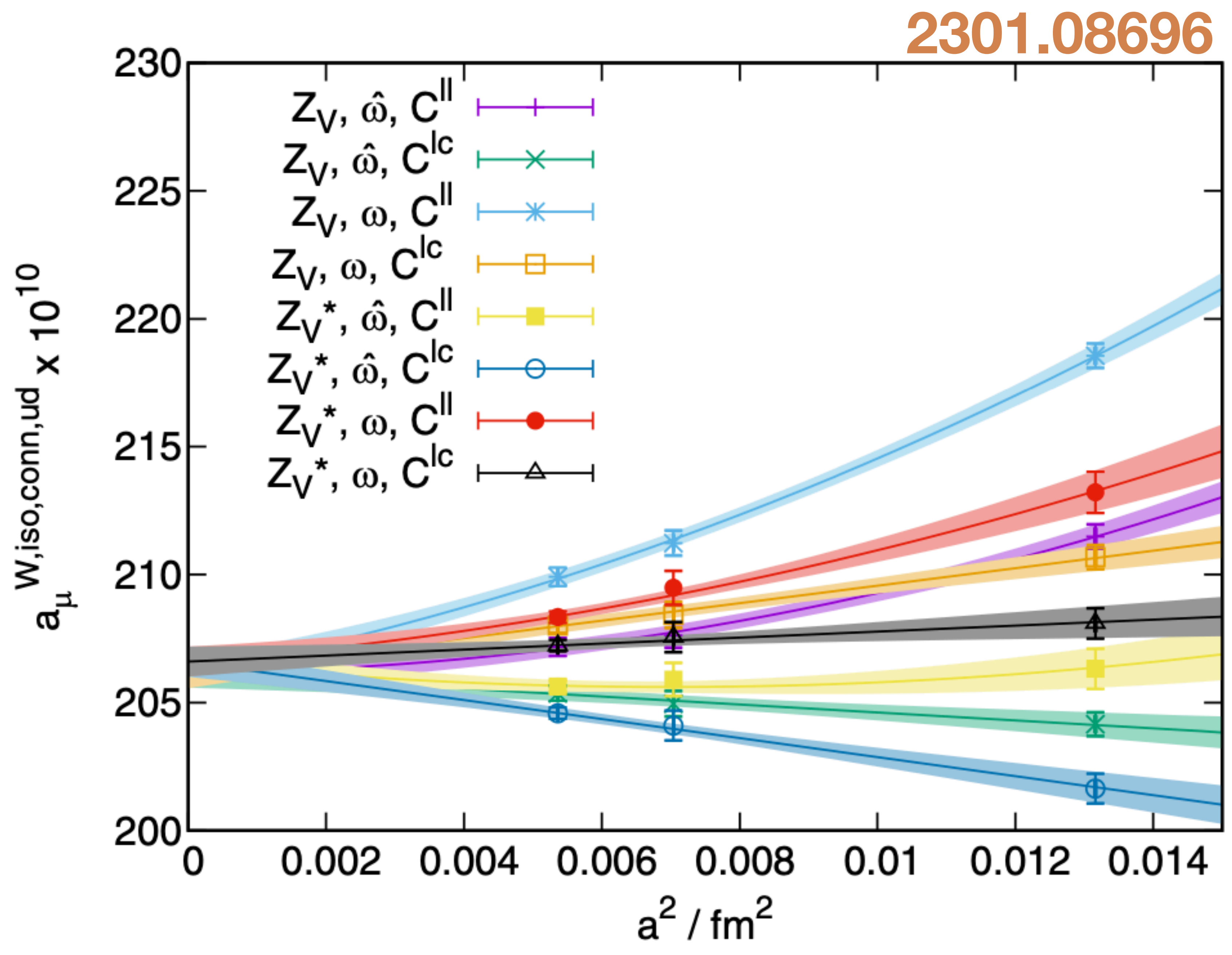
- lattice action
- renormalization scheme, scale, …
- …

Example 2: HVP



needs to be renormalized by Z_V^2

- Vector currents scale independent and not divergent, but needs renormalization b/c $\lim_{a\to 0} Z_V \neq 1$
- still some variety in how to determine → Z_V & Z_V^* in the plot



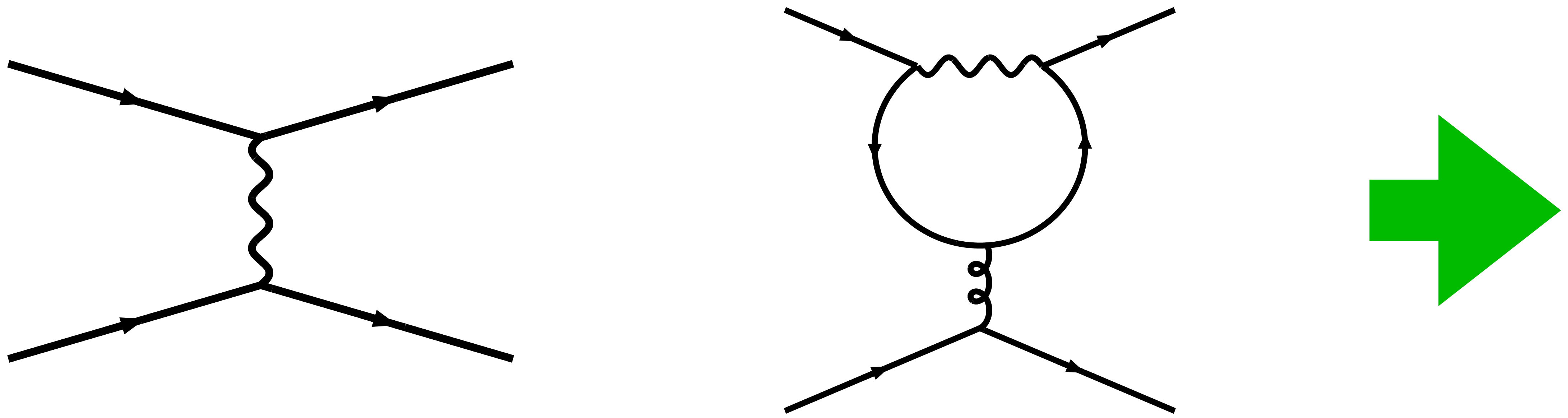
Various ways to calculate give different results at finite lattice spacing, but become consistent in a → 0

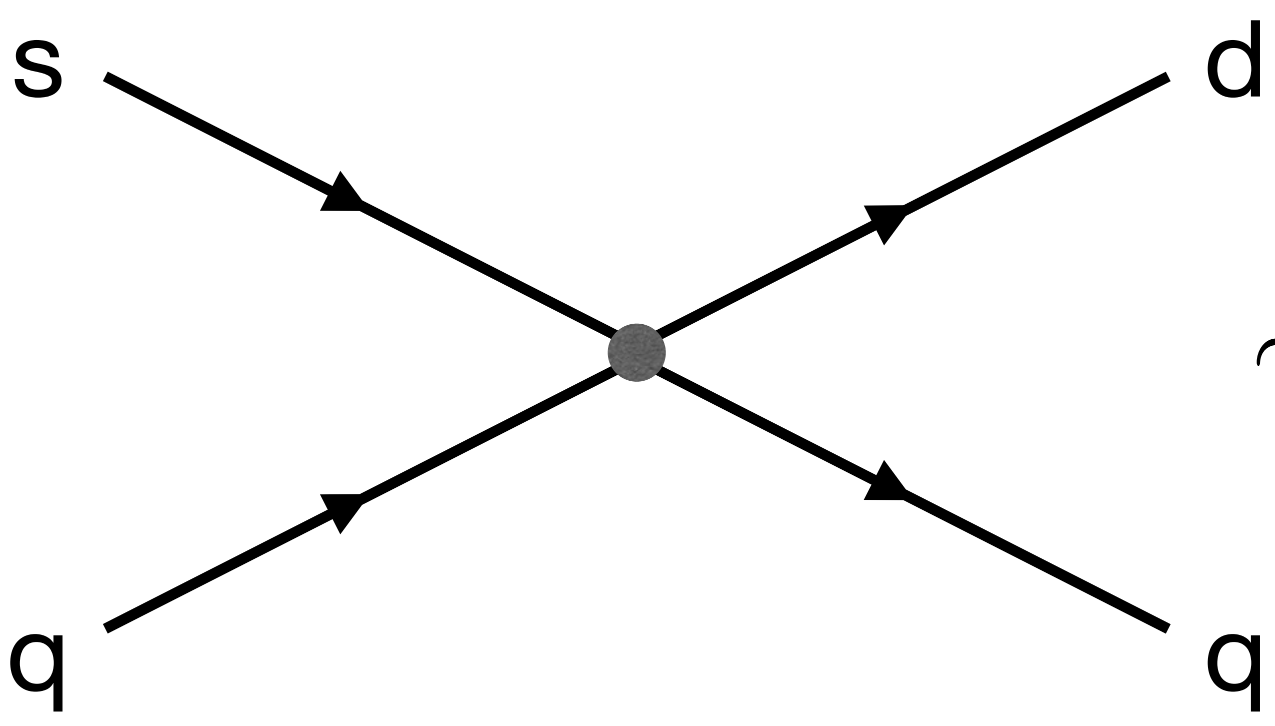
$$Z_V \langle \pi(\vec p = 0) | V_i | \pi(\vec q = 0) \rangle = 1 \qquad Z_V^* = \frac{\sum_{i=1,2,3} \langle V_i^c(t^*) V_i^\ell(0)^\dagger \rangle}{\sum_{j=1,2,3} \langle V_j^\ell(t^*) V_j^\ell(0)^\dagger \rangle}$$

Renormalization & continuum limit of $K \to \pi\pi$ matrix elements

Approach to weak matrix elements

- Two typical scales
  - Electroweak scale: m_W = 80 GeV, m_Z = 91 GeV
  - QCD scale: Λ_{QCD} ~ 300 MeV
- Low-energy effective interactions @ QCD scale





$$\sim G_F \propto \frac{g^2}{m_W^2}$$

  - $H_W = \Sigma_i \underbrace{c_i(\mu)}_{\text{Wilson coefficients}} \underbrace{O_i(\mu)}_{\text{Effective operators}}$

ΔS = 1 effective operators

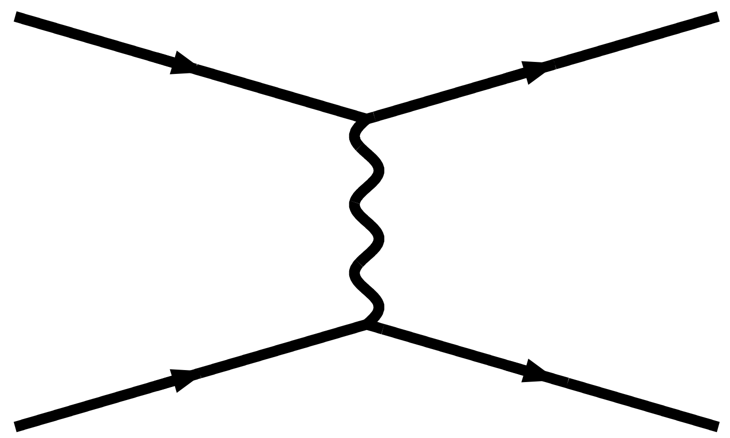
- $(\bar{s}q)_{V-A}(\bar{q}'q'')_{V\pm A} = \bar{s}\gamma_\mu(1-\gamma_5)q' \cdot \bar{q}'\gamma_\mu(1\pm\gamma_5)q''$
- α, β: color indices

$$
\begin{aligned}
Q_1 &= (\bar{s}_\alpha u_\beta)_{\mathrm{V-A}}\,(\bar{u}_\beta d_\alpha)_{\mathrm{V-A}}\,, \\
Q_2 &= (\bar{s}u)_{\mathrm{V-A}}\,(\bar{u}d)_{\mathrm{V-A}}\,, \\
Q_3 &= (\bar{s}d)_{\mathrm{V-A}}\sum_q(\bar{q}q)_{\mathrm{V-A}}\,, \\
Q_4 &= (\bar{s}_\alpha d_\beta)_{\mathrm{V-A}}\sum_q(\bar{q}_\beta q_\alpha)_{\mathrm{V-A}}\,, \\
Q_5 &= (\bar{s}d)_{\mathrm{V-A}}\sum_q(\bar{q}q)_{\mathrm{V+A}}\,, \\
Q_6 &= (\bar{s}_\alpha d_\beta)_{\mathrm{V-A}}\sum_q(\bar{q}_\beta q_\alpha)_{\mathrm{V+A}}\,, \\
Q_7 &= \frac{3}{2}(\bar{s}d)_{\mathrm{V-A}}\sum_q e_q\,(\bar{q}q)_{\mathrm{V+A}}\,, \\
Q_8 &= \frac{3}{2}(\bar{s}_\alpha d_\beta)_{\mathrm{V-A}}\sum_q e_q\,(\bar{q}_\beta q_\alpha)_{\mathrm{V+A}}\,, \\
Q_9 &= \frac{3}{2}(\bar{s}d)_{\mathrm{V-A}}\sum_q e_q\,(\bar{q}q)_{\mathrm{V-A}}\,, \\
Q_{10} &= \frac{3}{2}(\bar{s}_\alpha d_\beta)_{\mathrm{V-A}}\sum_q e_q\,(\bar{q}_\beta q_\alpha)_{\mathrm{V-A}}\,,
\end{aligned}
$$

Current-current operators

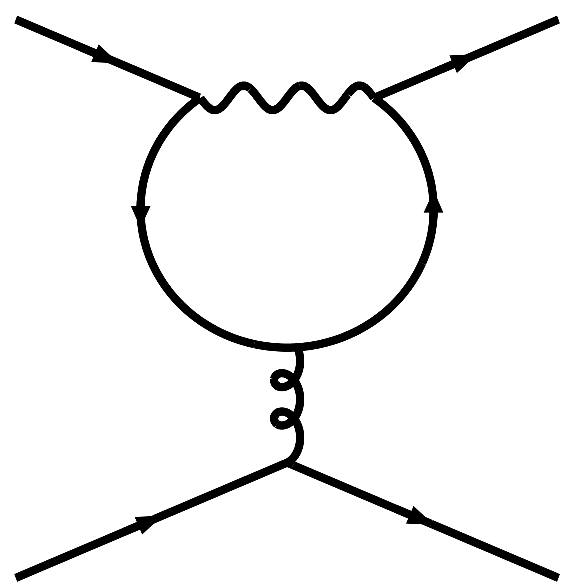
- $Q_1^c = (\bar{s}_\alpha c_\beta)_{\mathrm{V-A}}(\bar{c}_\beta d_\alpha)_{\mathrm{V-A}}$ & $Q_2^c = (\bar{s}c)_{\mathrm{V-A}}(\bar{c}d)_{\mathrm{V-A}}$

  enter when $n_f \geq 4$



QCD penguin operators

- sum over q runs for all active quarks



EW penguin operators

$K \to \pi\pi$ Amplitude and ε'

ππ phase shifts at m_K

$$\varepsilon' = \frac{i\omega e^{i\underline{(\delta_2-\delta_0)}}}{\sqrt{2}}\left[\frac{\mathrm{Im}A_2}{\mathrm{Re}A_2} - \frac{\mathrm{Im}A_0}{\mathrm{Re}A_0}\right] \qquad (\omega = \mathrm{Re}A_2/\mathrm{Re}A_0)$$

$$A_I = \frac{G}{\sqrt{2}} V_{us}^* V_{ud} \sum_i \underbrace{[z_i(\mu) + \tau y_i(\mu)]}_{\text{Wilson coefs. pQCD}} \underbrace{M_{I,i}(\mu)}_{\text{Matrix elements LQCD}}$$

$$M_{I,i}(\mu) = \lim_{a\to 0} \sum_j Z_{ij}(\mu, a) \left\langle (\pi\pi)_I \middle| Q_j^{lat}(a) \middle| K \right\rangle$$

K → ππ Amplitude and ε′

ππ phase shifts at m_K

$$\varepsilon' = \frac{i\omega e^{i\underline{(\delta_2-\delta_0)}}}{\sqrt{2}}\left[\frac{\mathrm{Im}A_2}{\mathrm{Re}A_2} - \frac{\mathrm{Im}A_0}{\mathrm{Re}A_0}\right] \qquad (\omega = \mathrm{Re}A_2/\mathrm{Re}A_0)$$

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Wilson coefs.
pQCD

Matrix elements
LQCD

$$M_{I,i}(\mu) = \lim_{a\to 0} \sum_j \underline{Z_{ij}(\mu, a)} \left\langle (\pi\pi)_I \middle| Q_j^{lat}(a) \middle| K \right\rangle$$

Q_j can mix
Need 10x10 = 100 conditions?

Q_i not linearly independent in 4D

- Fierz identity

$$
\begin{aligned}
(\bar{q}_1 q_2)_L (\bar{q}_3 q_4)_L &= \bar{q}_1 \gamma_\mu (1-\gamma_5) q_2 \cdot \bar{q}_3 \gamma_\mu (1-\gamma_5) q_4 \\
&= (\bar{q}_1 q_4)_L (\bar{q}_3 q_2)_L + (4-d) E
\end{aligned}
$$

evanescent operators vanish in 4D

- Linear dependence in 4D

$$
\begin{aligned}
Q_4 &= -Q_1 + Q_2 + Q_3 \\
Q_9 &= \frac{3}{2} Q_1 - \frac{1}{2} Q_3 \\
Q_{10} &= \frac{1}{2} Q_1 + Q_2 - \frac{1}{2} Q_3
\end{aligned}
$$

$\longrightarrow$ 7 independent operators

Chiral representations

- Renormalization schemes usually defined in the chiral limit, m → 0
    - Further constraints based on chiral symmetry
- Irreps of 4-quark operators in 3-flavor case: $SU(3)_L \times SU(3)_R$
    - (n_L, n_R): $n_{L/R}$ indicates irreducible representation of $SU(3)_{L/R}$
    - Left-right operators: $(\bar{s}d)_L(\bar{q}q)_R$ → (8,1) & (8,8) representations possible
    - Left-left operators: $\bar{3}\times 3\times \bar{3}\times 3$ only yields 27, 8 and 1 but no singlet in $(\bar{s}d)_L(\bar{q}q)_L$ → (27,1) & (8,1)

Chiral basis

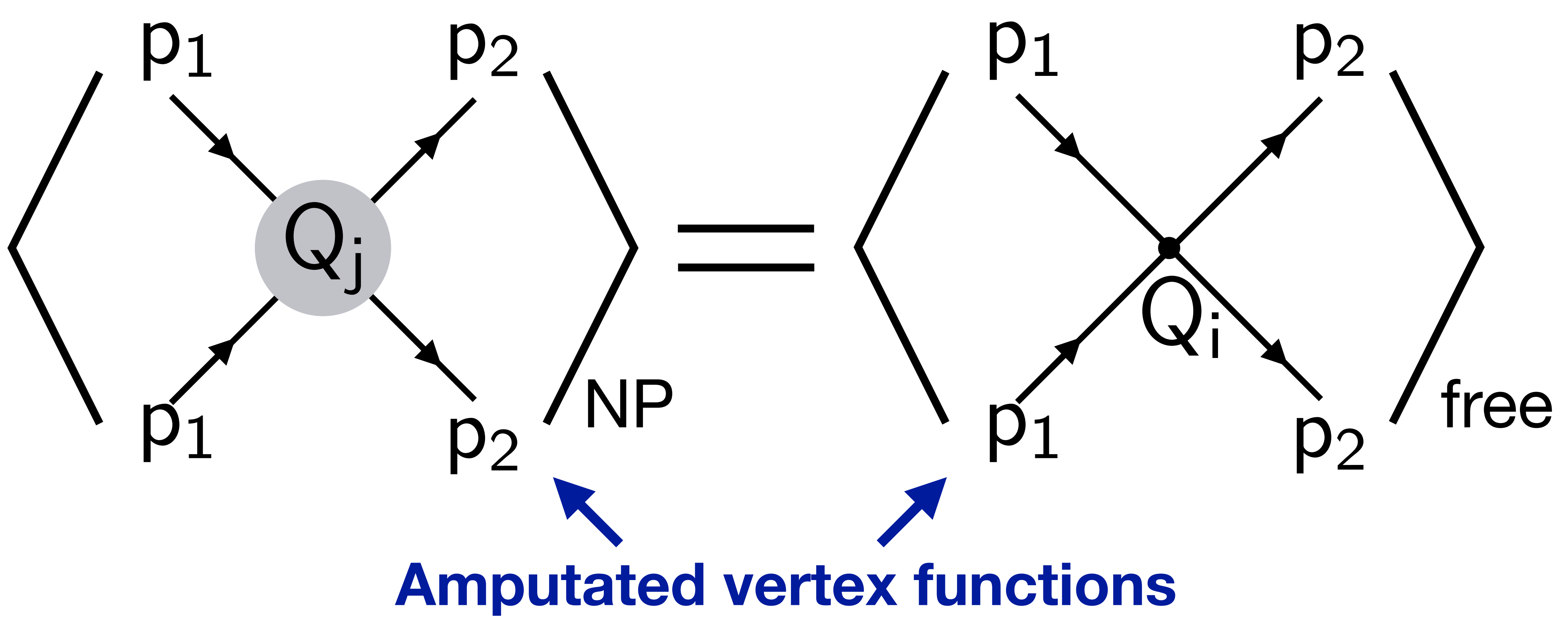
$Q'_1 = 3Q_1 + 2Q_2 - Q_3$	(27,1)	ΔI = 1/2 & 3/2
$Q'_2 = \frac{2}{5}Q_1 - \frac{2}{5}Q_2 + \frac{1}{5}Q_3$		
$Q'_3 = -\frac{3}{5}Q_1 + \frac{3}{5}Q_2 + \frac{1}{5}Q_3$	(8,1)	ΔI = 1/2
$Q'_{5,6} = Q_{5,6}$		
$Q'_{7,8} = Q_{7,8}$	(8,8)	ΔI = 1/2 & 3/2

- Mixing only with operators in the same chiral multiplet allowed

RI/SMOM schemes 0712.1061

- Renormalization condition

$$\frac{Z_{ij}^{\mathrm{RI/SMOM}}(\mu)}{Z_q(\mu)^2}$$



- SMOM momenta & scale definition: $\mu^2 = p_1^2 = p_2^2 = (p_1 - p_2)^2$
- Landau gauge chosen
- 12^4 (= $(n_c\, n_s)^4$) elements projected to only several equations (labeled by m)

$$\sum_j \frac{Z_{ij}^{s}(\mu)}{Z_q(\mu)^2} P_m^{\alpha\beta\gamma\delta} \Lambda_j^{\alpha\beta\gamma\delta} = P_m^{\alpha\beta\gamma\delta} \Lambda_j^{\alpha\beta\gamma\delta}\Big|_{\mathrm{free}}$$

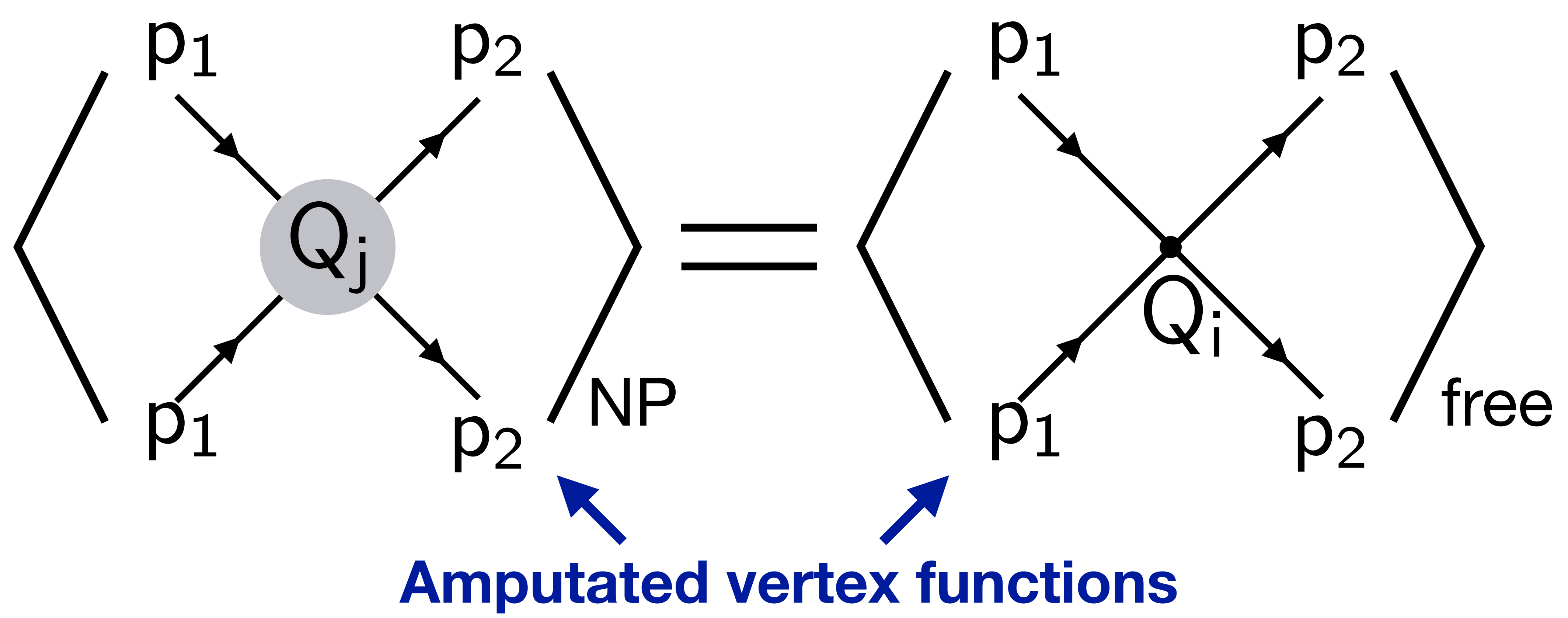
*Projectors specifically define the scheme

RI/SMOM schemes 0712.1061

symmetric

- Renormalization condition

$$\frac{Z_{ij}^{\mathrm{RI/SMOM}}(\mu)}{Z_q(\mu)^2}$$

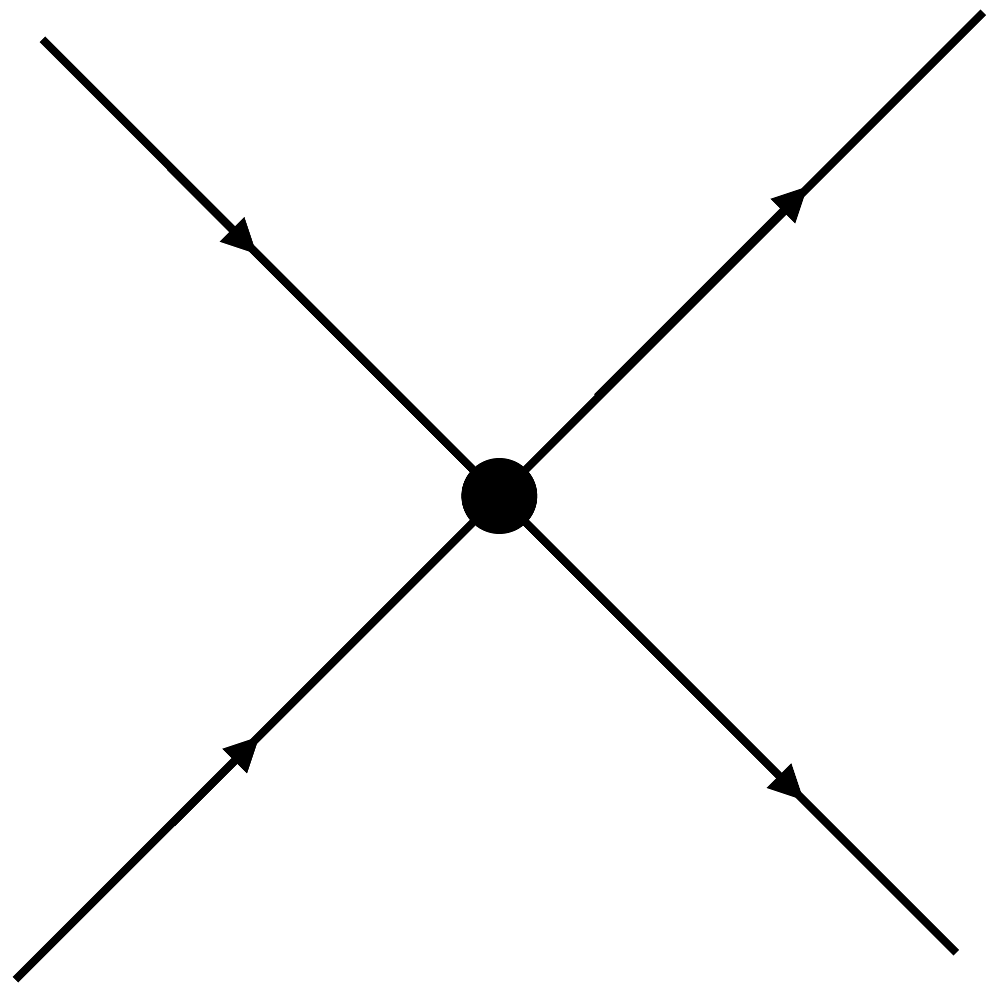


- SMOM momenta & scale definition: $\mu^2 = p_1^2 = p_2^2 = (p_1 - p_2)^2$ operator carries momentum with "symmetric" magnitude
- Landau gauge chosen
- 12^4 (= $(n_c\, n_s)^4$) elements projected to only several equations (labeled by m)

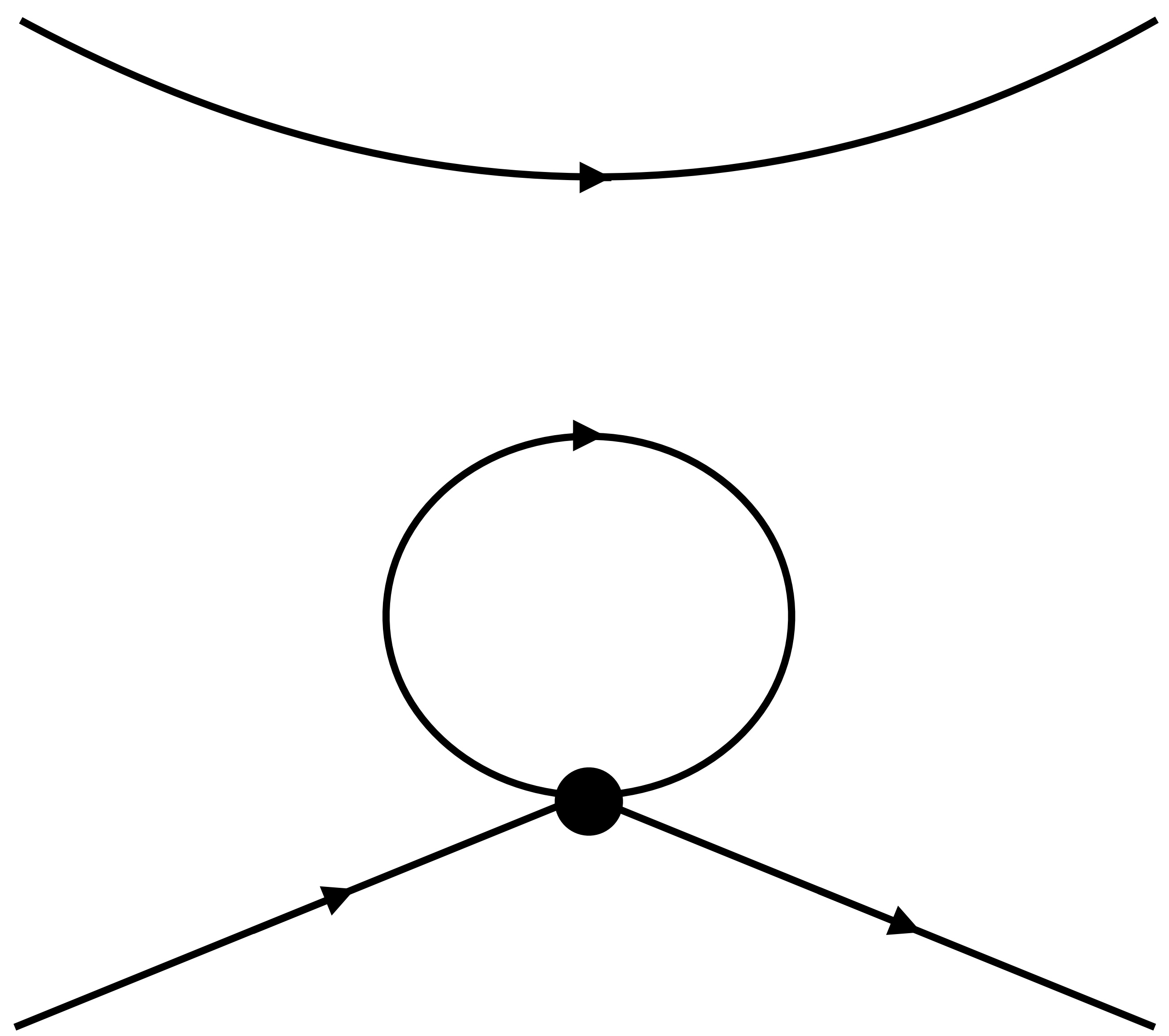
$$\sum_j \frac{Z_{ij}^s(\mu)}{Z_q(\mu)^2} P_m^{\alpha\beta\gamma\delta} \Lambda_j^{\alpha\beta\gamma\delta} = P_m^{\alpha\beta\gamma\delta} \Lambda_j^{\alpha\beta\gamma\delta}\Big|_{\mathrm{free}}$$

*Projectors specifically define the scheme

Diagrams



connected



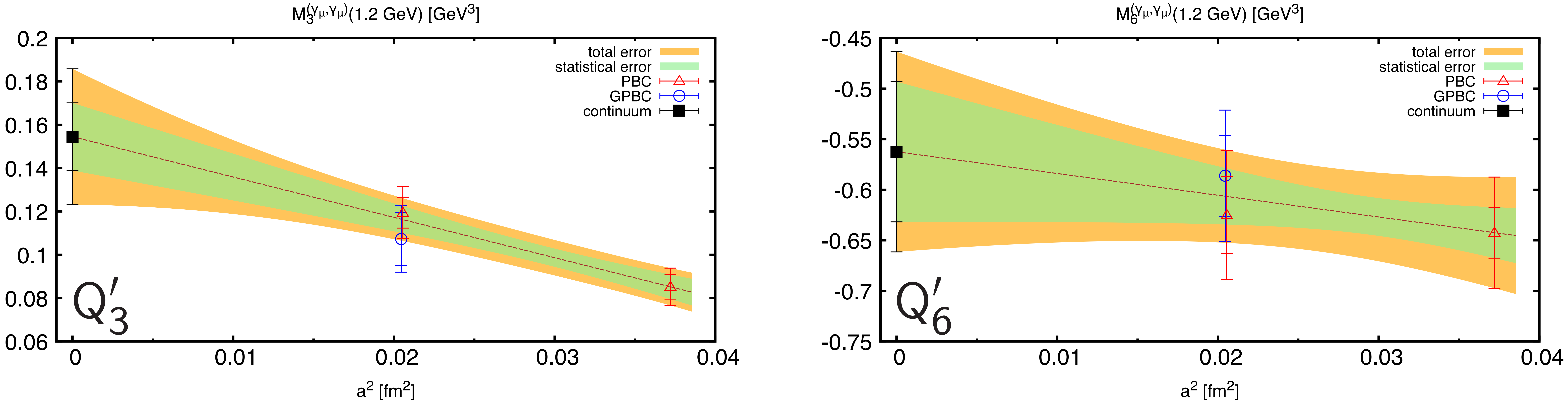
disconnected

Discretization error on Z

- Roughly $O((a\mu)^2)$ but …
- O(4) violation on lattice → Z dependent on direction of p_1 & p_2
  - example at $O(a^2)$: $\sum_\mu (ap_i^\mu)^4/(a\mu)^2$
- We have to make the $O(a^2)$ coefficient mutual for all lattices
  - Simply choosing the same p_1 & p_2 is enough
  - Not a problem if the physical volume L is mutual for these ensembles

possible momenta: $p = \frac{2\pi n}{L}$ $\quad n = -\frac{L}{2a} + 1, \ldots, \frac{L}{2a}$

Continuum extrapolations of MEs

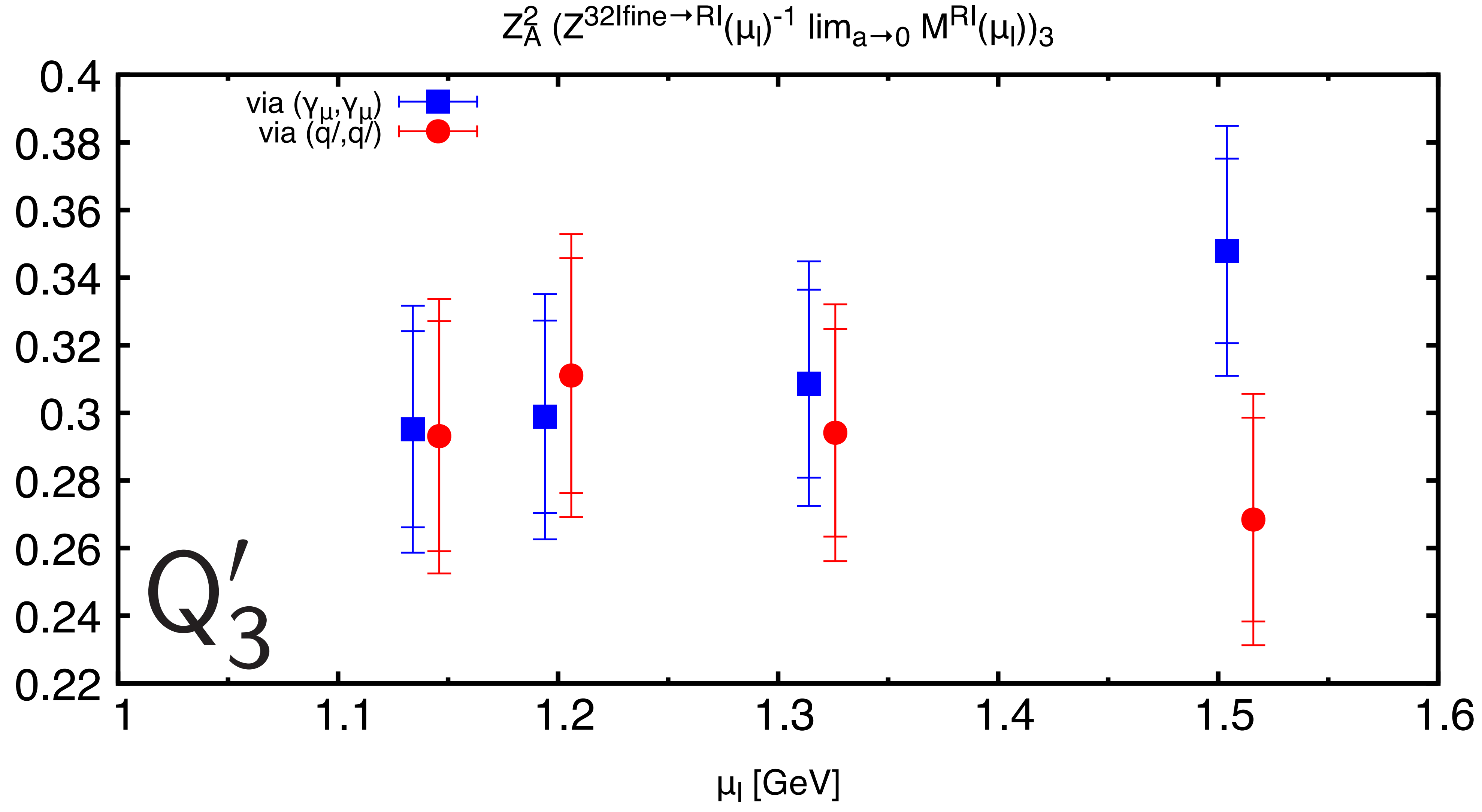


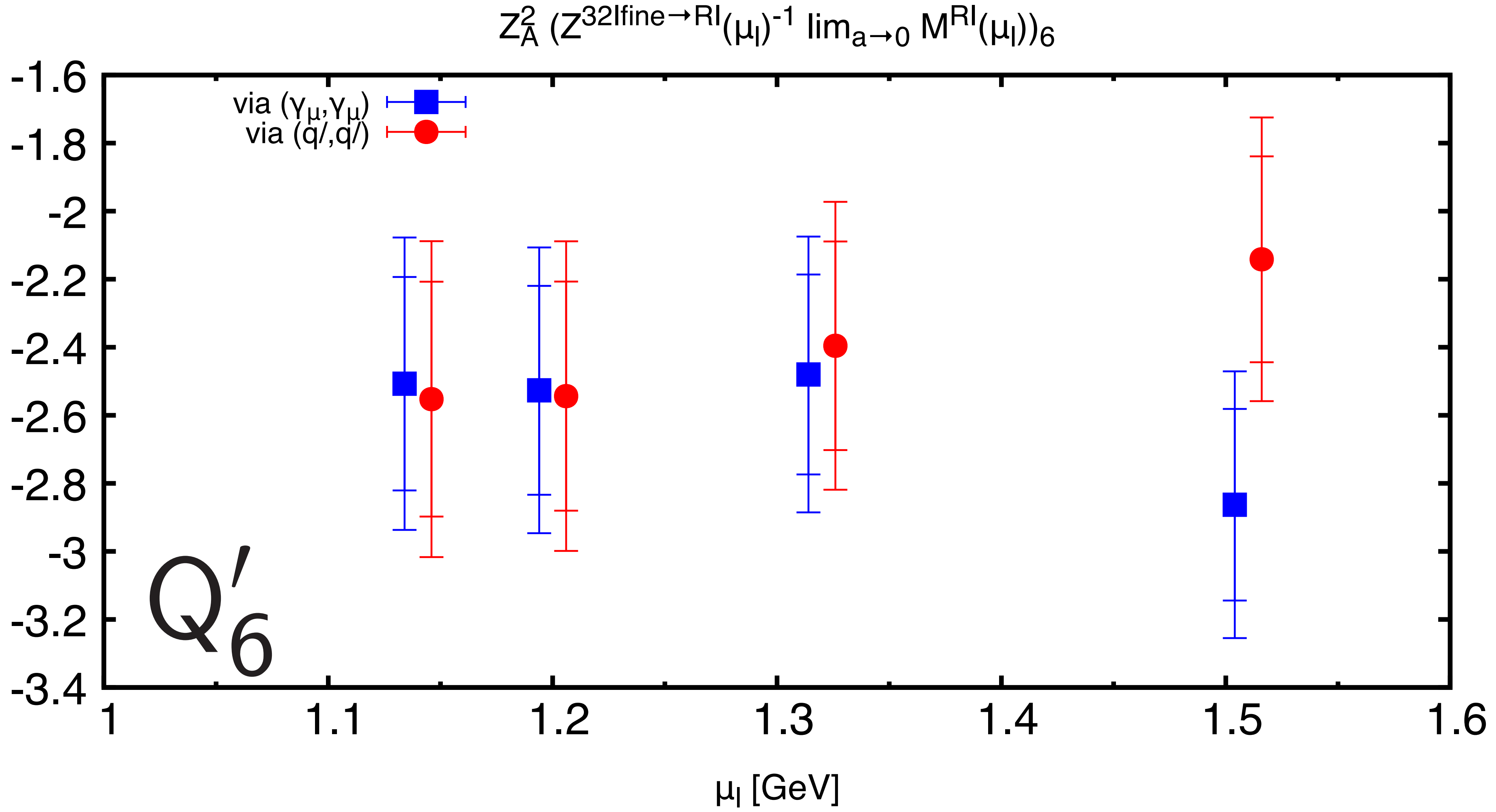
- SMOM(γ_μ,γ_μ) scheme at 1.2 GeV
- Only 2 lattice spacings → potential effect of $O(a^4)$?

Intermediate scheme/scale-dependence

$$Z^{\mathrm{lat}\to\mathrm{RI}}(\mu, a_{\mathrm{fine}})^{-1} \lim_{a\to 0} Z^{\mathrm{lat}\to\mathrm{RI}}(\mu, a)\vec{M}_I(a)$$

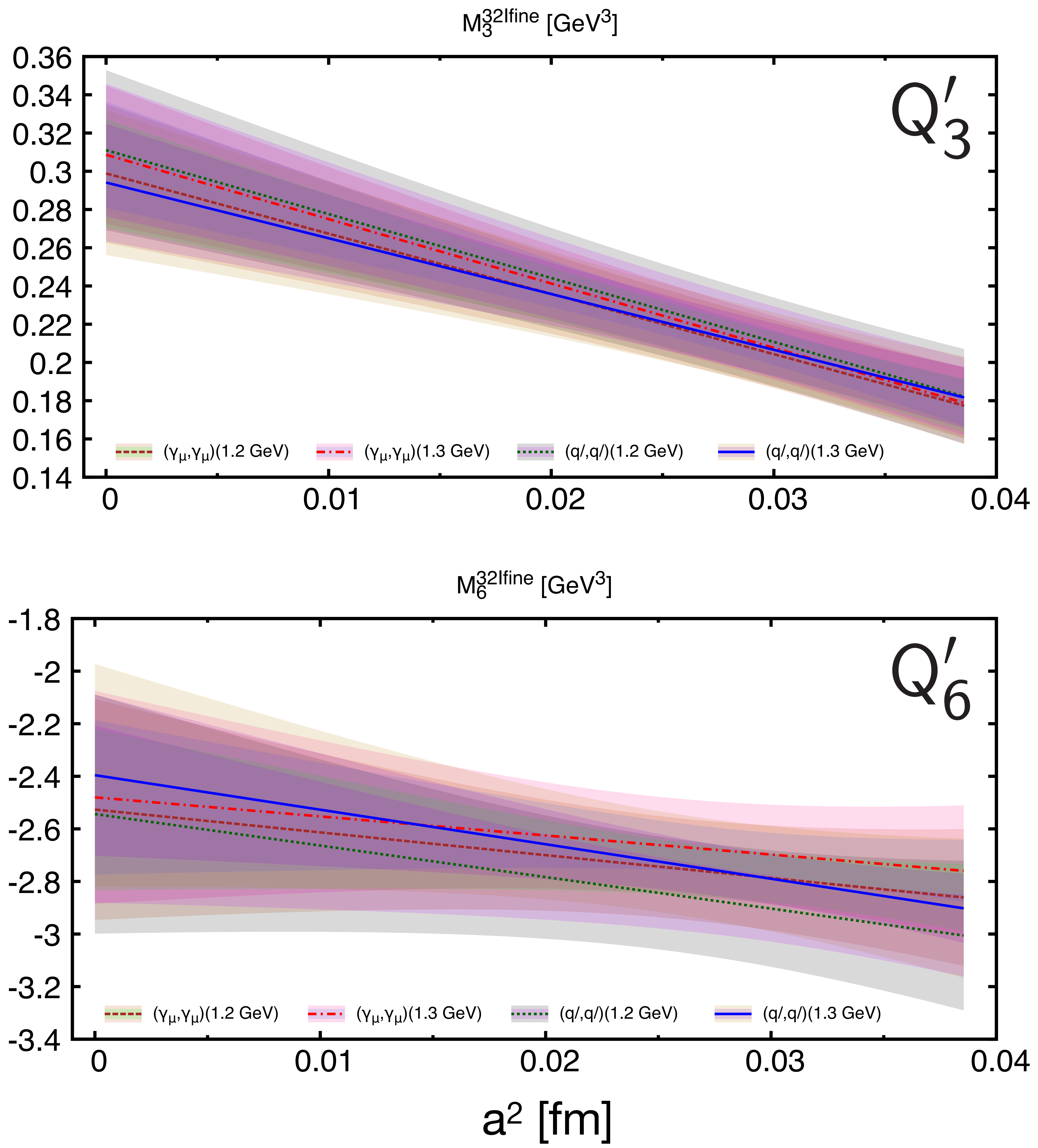
- a_{fine} on a very fine lattice ensemble
- This should be the bare matrix elements on that fine lattice
- should be independent of intermediate scheme/scale
- Difference should be from
  - failure of the continuum extrapolation with $O(a^2)$ assumption
  - mixing with excluded operators



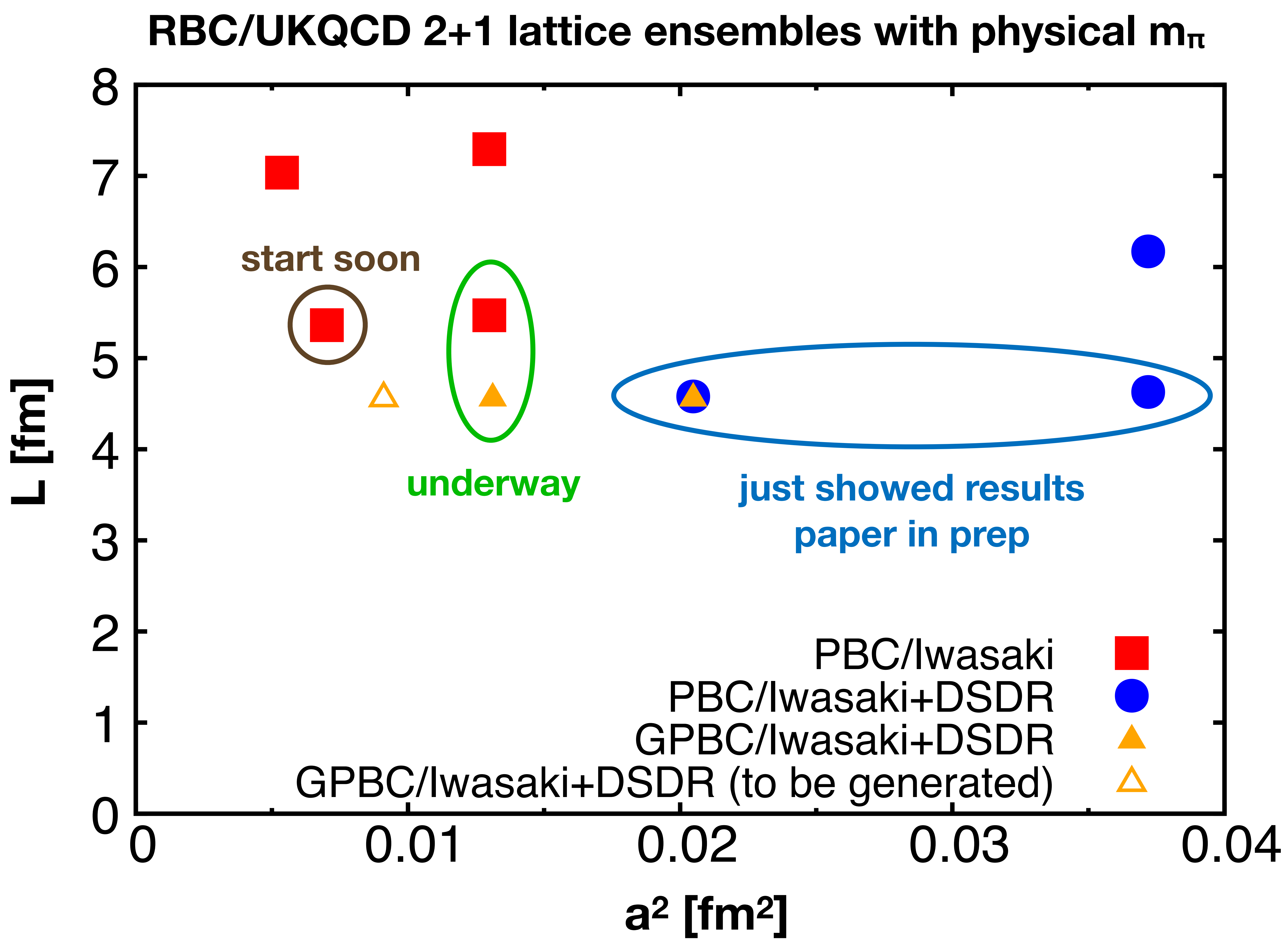


a → 0 trajectories in the “fine lattice scheme” via 4 different sets of intermediate scheme/scale

- Trajectories similar for μ < 1.5 GeV
  - data points at finite a not significantly affected by discretization error on Z
  - a-dependence appears rather coming from calculation of bare matrix elements
- But the trajectory and the value in continuum limit via μ ≈ 1.5 GeV quite different
- More study on $O(a^4)$ effect underway



Towards more accurate continuum limit



Nonperturbative scale evolution and matching to $\overline{\mathrm{MS}}$

Perturbative matching & window problem

$$A_I = \frac{G}{\sqrt{2}} V_{us}^* V_{ud} \sum_i \underbrace{[z_i(\mu) + \tau y_i(\mu)]}_{\substack{\text{Wilson coefs.} \\ \text{pQCD}}} \underbrace{M_{I,i}(\mu)}_{\substack{\text{Matrix elements} \\ \text{LQCD}}}$$

- Continuum pQCD works
  - Wilson coefficients computed to NLO in $\overline{\mathrm{MS}}$: NPB 408,209 (1993)
  - Perturbative matching between RI/SMOM & $\overline{\mathrm{MS}}$ in 3-flavor done to NLO: PRD 84,014001 (2011)

- Window problem
  - Perturbative matching requires $\mu \gg \Lambda_{QCD}$
  - Renormalization on finite lattices requires $\mu \ll \pi/a$

Step scaling

$$\vec{M}_I^{RI}(\mu_h) = \Sigma^{RI}(\mu_h, \mu_\ell)\vec{M}_I^{RI}(\mu_\ell)$$

$$\Sigma^{RI}(\mu_h, \mu_\ell) = \lim_{a_{fine}\to 0} Z^{lat\to RI}(\mu_h, a_{fine}) Z^{lat\to RI}(\mu_\ell, a_{fine})^{-1}$$

- Nonperturbative scale evolution
- $\mu_\ell < \mu_h \ll \pi/a_{fine}$
- Sometimes (and in this work) done at single fine lattice a_{fine} when required precision is not high
- These fine lattice ensembles often have unphysical m_π and smaller L
  - still useful to understand short-distance behavior of QCD such as $\Sigma(\mu_h,\mu_\ell)$

Intermediate scheme-dependence on high-energy side

$$\vec{M}_I^{RI}(\mu_h) = Z^{lat\to RI}(\mu_h, a_{fine}) \underline{Z^{lat\to RI}(\mu_\ell, a_{fine})^{-1} \vec{M}_I^{RI}(\mu_\ell)}$$

Matrix elements in the "fine lattice scheme"

- Found insensitive to the intermediate scheme/scale for $\mu_\ell < 1.5$ GeV

$$= Z^{lat\to RI}(\mu_h, a_{fine}) \vec{M}_I^{lat}(a_{fine})$$

$$\vec{M}_I^{\overline{MS}}(\mu_h) = \underline{R^{RI\to\overline{MS}}(\mu_h)} Z^{lat\to RI}(\mu_h, a_{fine}) \vec{M}_I^{lat}(a_{fine})$$

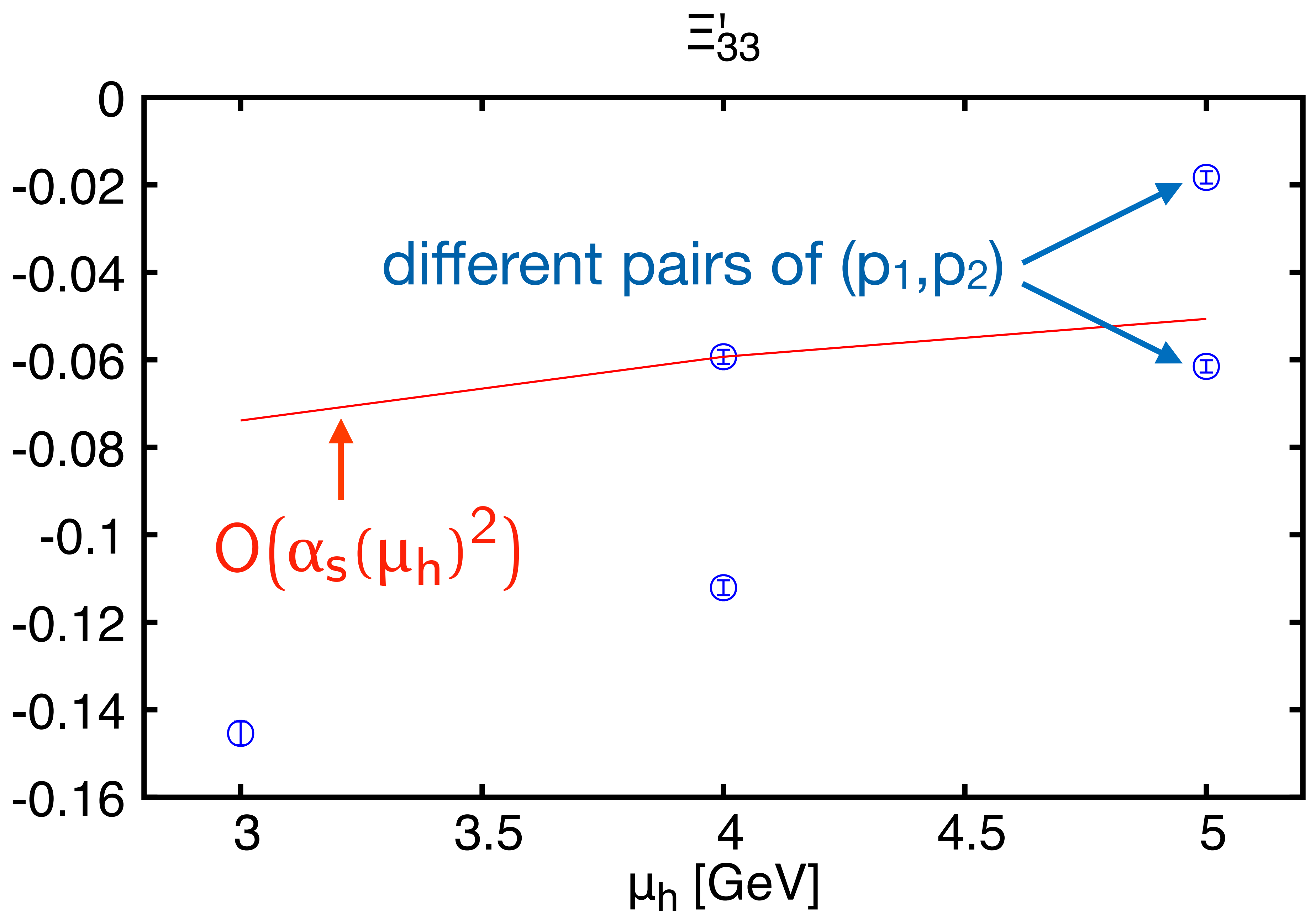
Perturbative matching to NLO by Lehner-Sturm (2011)

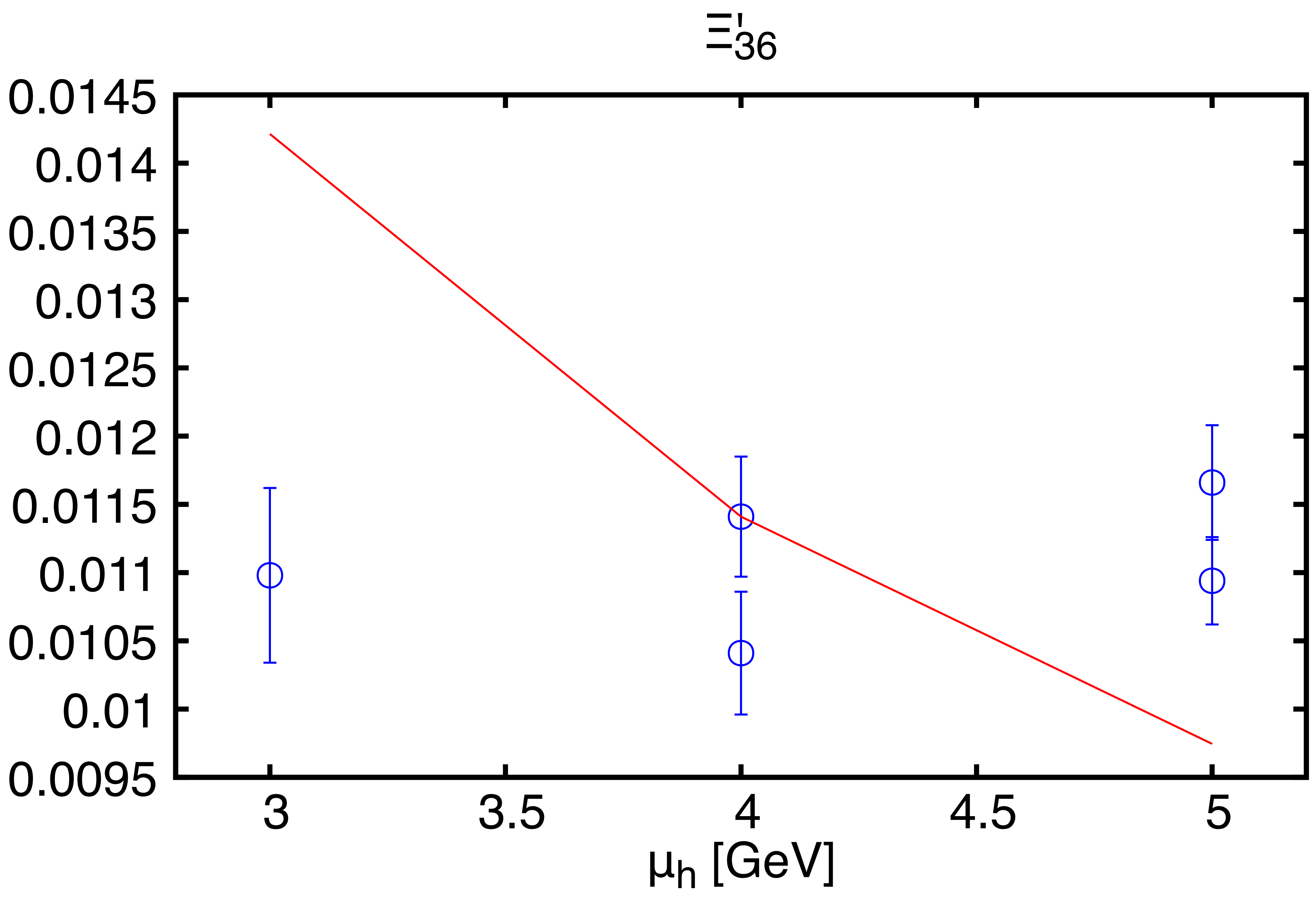
- Difference between the results through (γ_μ, γ_μ) & $(\not{q}, \not{q})$ schemes should be from
  - truncation of perturbative matching
  - $O(a_{fine}^2)$ error

Intermediate scheme-dependence on high-energy side

$$\Xi'(\mu_h) = \left(R^{(\not{q},\not{q})\to\overline{\mathrm{MS}}}(\mu_h)\, Z^{\mathrm{lat}\to(\not{q},\not{q})}(\mu_h, a_{\mathrm{fine}})\right)^{-1}$$

$$\times R^{(\gamma_\mu,\gamma_\mu)\to\overline{\mathrm{MS}}}(\mu_h)\, Z^{\mathrm{lat}\to(\gamma_\mu,\gamma_\mu)}(\mu_h, a_{\mathrm{fine}}) - \mathbb{1}_{7\times 7},$$ should be 0 if no systematic error





Intermediate scheme-dependence on high-energy side

$$\Delta Z^{\overline{\mathrm{MS}}\leftarrow 32\mathrm{Ifine}}(\mu_h) = R^{(\not q,\not q)\to\overline{\mathrm{MS}}}(\mu_h) Z^{\mathrm{lat}\to(\not q,\not q)}(\mu_h, a_{\mathrm{fine}}) - R^{(\gamma_\mu,\gamma_\mu)\to\overline{\mathrm{MS}}}(\mu_h) Z^{\mathrm{lat}\to(\gamma_\mu,\gamma_\mu)}(\mu_h, a_{\mathrm{fine}})$$

via $(\not q,\not q)$

$\mu_h \approx 3.0$ GeV

i	$\Delta Z^{\overline{\mathrm{MS}}\leftarrow 32\mathrm{Ifine}}_{(8,1)}(\mu_h)$				$M^{32\mathrm{Ifine}}_{i,(8,1)}$	$\Delta M^{\overline{\mathrm{MS}}}_{i,(8,1)}(\mu_h)$	$M^{\overline{\mathrm{MS}}}_{i,(8,1)}(\mu_h)$
2	-0.0415(51)	-0.0711(41)	0.0000(17)	0.00099(88)	-0.160	-0.0132	-0.0995[89]
3	0.0544(27)	0.0781(21)	0.00364(88)	-0.00812(50)	0.243	0.0279	0.1409(81)
5	-0.003(17)	-0.010(13)	0.0028(59)	0.0016(28)	-0.887	-0.0085	-0.333[31]
6	-0.0491(44)	-0.0998(43)	0.0012(17)	0.0048(13)	-2.563	-0.0298	-1.231[74]

Table 7: $\mu_h = 3.0$ GeV. Last three columns given in physical units GeV^3.

$\mu_h \approx 5.0$ GeV

i	$\Delta Z^{\overline{\mathrm{MS}}\leftarrow 32\mathrm{Ifine}}_{(8,1)}(\mu_h)$				$M^{32\mathrm{Ifine}}_{i,(8,1)}$	$\Delta M^{\overline{\mathrm{MS}}}_{i,(8,1)}(\mu_h)$	$M^{\overline{\mathrm{MS}}}_{i,(8,1)}(\mu_h)$
2	-0.0068(30)	-0.0138(22)	-0.0001(11)	0.00141(42)	-0.160	-0.00579	-0.01096[94]
3	0.01142(96)	0.01089(83)	0.00332(34)	-0.00716(20)	0.243	0.01623	0.1498[84]
5	0.0070(95)	0.0125(70)	0.0020(33)	0.0005(12)	-0.887	-0.00114	-0.383[32]
6	-0.0238(24)	-0.0607(22)	0.00486(85)	-0.01116(66)	-2.563	0.01334	-1.463[88]

Table 10: $\mu_h = 5.0$ GeV. Last three columns given in physical units GeV^3.

Intermediate scheme-dependence on high-energy side

$$\Delta Z^{\overline{\text{MS}}\leftarrow 32\text{Ifine}}(\mu_h) = R^{(\not q,\not q)\to\overline{\text{MS}}}(\mu_h) Z^{\text{lat}\to(\not q,\not q)}(\mu_h, a_{\text{fine}}) - R^{(\gamma_\mu,\gamma_\mu)\to\overline{\text{MS}}}(\mu_h) Z^{\text{lat}\to(\gamma_\mu,\gamma_\mu)}(\mu_h, a_{\text{fine}})$$

via $(\not q, \not q)$

$\mu_h \approx 3.0$ GeV

i	$\Delta Z^{\overline{\text{MS}}\leftarrow 32\text{Ifine}}_{(8,1)}(\mu_h)$				$M^{32\text{Ifine}}_{i,(8,1)}$	$\Delta M^{\overline{\text{MS}}}_{i,(8,1)}(\mu_h)$	$M^{\overline{\text{MS}}}_{i,(8,1)}(\mu_h)$
2	-0.0415(51)	-0.0711(41)	0.0000(17)	0.00099(88)	-0.160	-0.0132	-0.0995[89]
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20%

Table 7: $\mu_h = 3.0$ GeV. Last three columns given in physical units GeV^3.

$\mu_h \approx 5.0$ GeV

i	$\Delta Z^{\overline{\text{MS}}\leftarrow 32\text{Ifine}}_{(8,1)}(\mu_h)$				$M^{32\text{Ifine}}_{i,(8,1)}$	$\Delta M^{\overline{\text{MS}}}_{i,(8,1)}(\mu_h)$	$M^{\overline{\text{MS}}}_{i,(8,1)}(\mu_h)$
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6	-0.0238(24)	-0.0607(22)	0.00486(85)	-0.01116(66)	-2.563	0.01334	-1.463[88]

11%

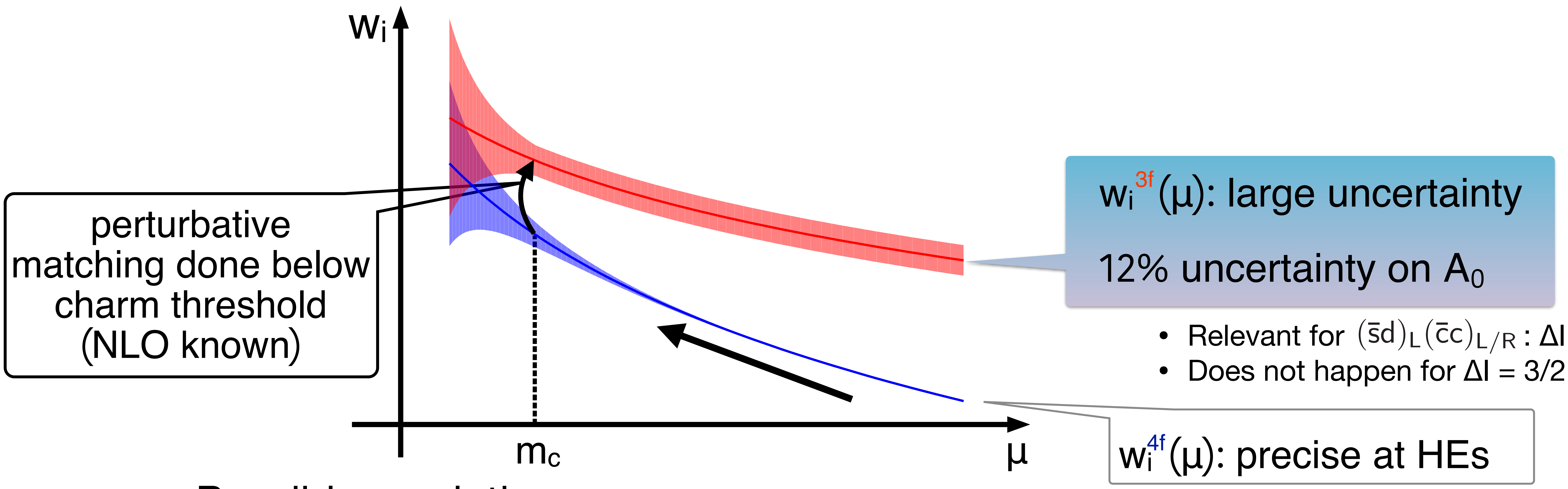
Table 10: $\mu_h = 5.0$ GeV. Last three columns given in physical units GeV^3.

Q_3' significant for ReA_0 → Need to better understand

Nonperturbative matching of $\Delta S=1$ operators b/w 3 & 4-flavor theories

Wilson coefs

$$\langle f|H_W|i\rangle = \Sigma_i \underbrace{w_i^{3f}(\mu)}_{\text{pQCD}} \underbrace{\langle f|O_i^{3f}(\mu)|i\rangle}_{\text{LQCD}}$$

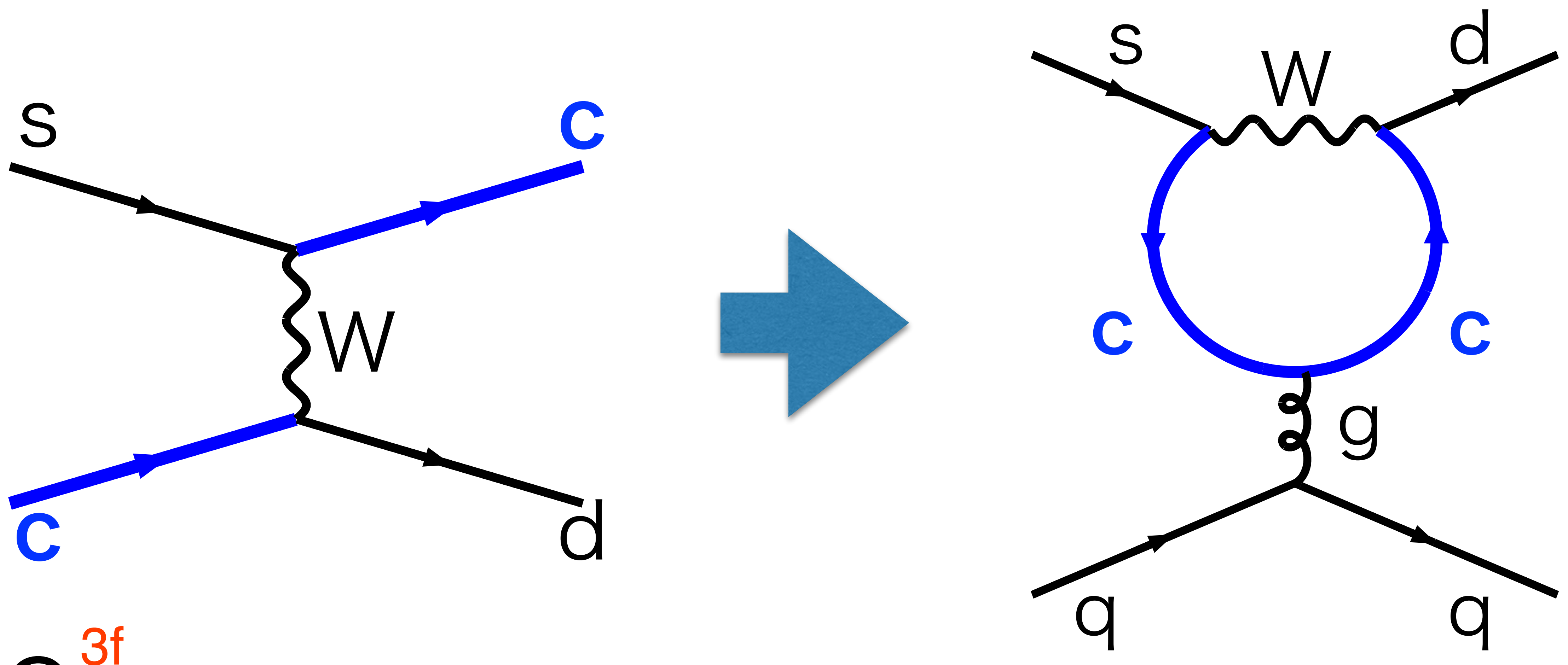


- Relevant for $(\bar{s}d)_L(\bar{c}c)_{L/R}$: $\Delta I = 1/2$
- Does not happen for $\Delta I = 3/2$

- Possible resolutions
  - NNLO perturbative matching [Cerda-Sevilla et al. Acta Phys.Polon.B 4 (2018) 1087-1096]
  - Matching nonperturbatively [MT, LATTICE2019]

How much $w_i^{3f} \neq w_i^{4f}$?

- Sea charm quark → $O(\alpha_s^2)$
- If O_i^{4f} involves charm quark…
  - WCs of charm operators will be a part of 3f WCs
  - Difference can be $O(\alpha_s)$



- w_i^{3f} necessary if MEs are calculated w O_i^{3f}
  - our fine lattice: $a^{-1} \approx 1.38$ GeV used for MEs
  - not appropriate to introduce charm on this lattice
  - $80^3 \times 160$ may be needed to have fine enough lattice maintaining m_π^{phys}

NP matching of 4-quark operators

- Basic idea

$$O_i^{4f} \to \sum_j M_{ij} O_j^{3f}$$

i.e. $\langle E_{out}|O_i^{4f}|E_{in}\rangle = \sum_j M_{ij}\langle E_{out}|O_j^{3f}|E_{in}\rangle$ for small E_{out} & E_{in} compared to m_c

- Strategy
  - Consider many 3pt functions on fine lattice

$$C_{i,ab}^{3f/4f}(t_{out}, t, t_{in}) = \langle \mathcal{O}_a(t_{out}) O_i^{3f/4f}(t) \mathcal{O}_b(t_{in})\rangle$$

  - Perform fit with many pairs of O_a & O_b at large t_{out} – t & t - t_{in} $\boxed{C_{i,ab}^{4f} = M_{ij} C_{j,ab}^{3f}}$
  - Trying with ~ 200 relevant pairs of O_a & O_b
  - Automatic Wick contractor in use

Operator basis for ΔS=1

(n_L, n_R): Representation of $SU(3)_L \times SU(3)_R$

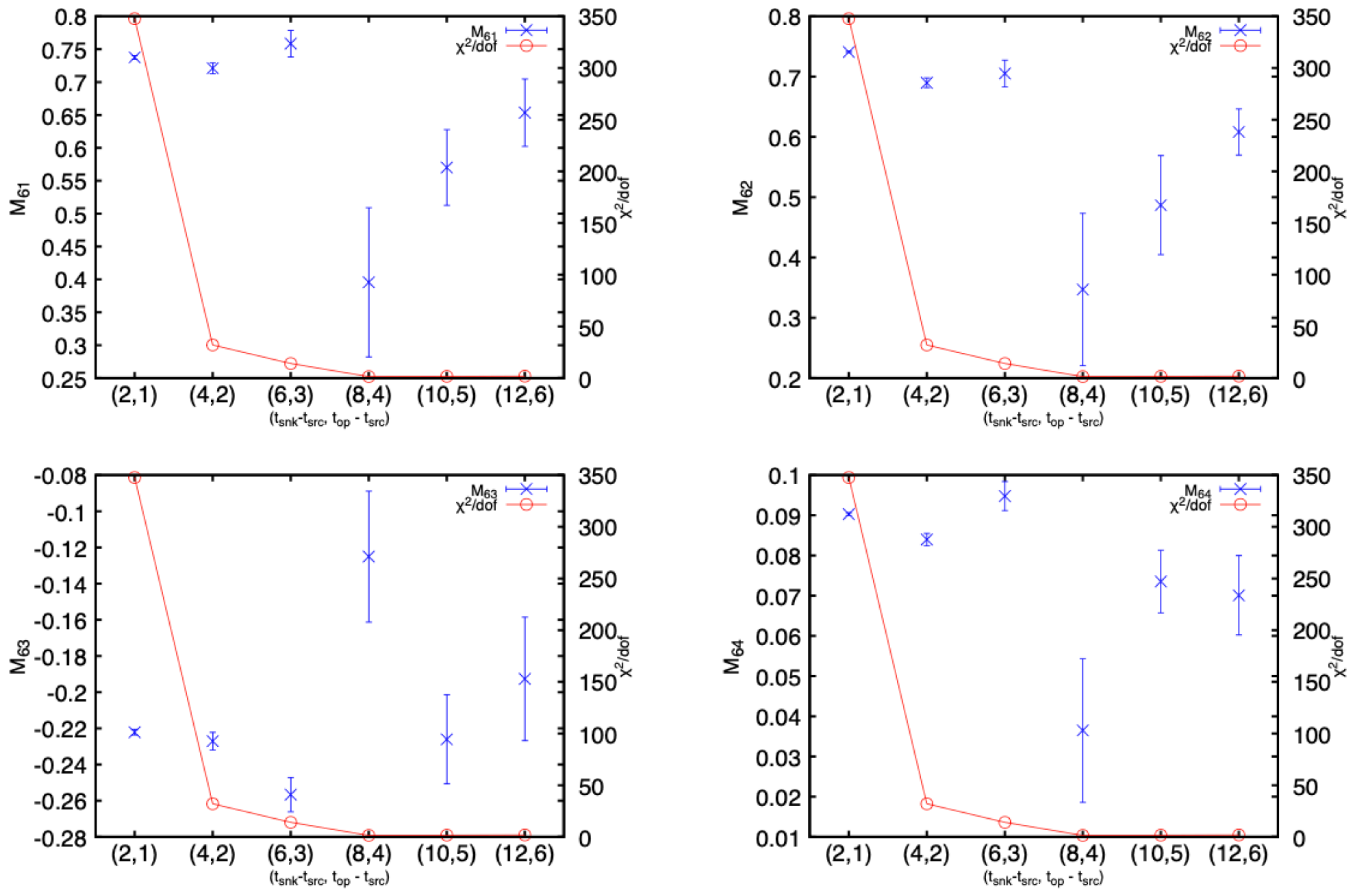
- Classification of 7 independent $n_f = 3$ operators
  - 1 in (27,1); <u>4 in (8,1)</u>; 2 in (8,8)
- 4 charm operators $\underline{(\bar{s}_\alpha d_{\alpha/\beta})_L(\bar{c}_\alpha c_{\beta/\alpha})_{L/R}}$ all in (8,1) with $SU(3)_L \times SU(3)_R$
- Only operators in (8,1) matter
  - $O_j^{3f} = (Q_1', Q_2', Q_3', Q_4')$
- $O_i^{4f} = (Q_1', Q_2', Q_3', Q_4', P_1, P_2, P_3, P_4)$

$$\Rightarrow M_{ij} = \begin{pmatrix} \mathbf{1}_{4x4} \\ \cdots\cdots\cdots \\ \begin{matrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{matrix} \end{pmatrix}$$

16 nontrivial elements

SU(3)$_L$xSU(3)$_R$ to SU(4)$_L$xSU(4)$_R$ chiral bases

- 7-operator basis w $n_f = 3$
  - (27,1) x1, (8,1) x4, (8,8) x2
- 9-operator basis w $n_f = 4$
  - (84,1) x2, (20,1) x1, (15,1) x4, (15,15) x2
- 9x7 matching matrix can be determined once the 16 elements of M_{ij} are obtained
- Instability of M_{ij} doesn't propagate to 4-flavor matrix elements
  - Instability still concerning and trying to better understand



$SU(3)_L \times SU(3)_R$ to $SU(4)_L \times SU(4)_R$ chiral bases

- 7-operator basis w $n_f = 3$
  - (27,1) x1, (8,1) x4, (8,8) x2
- 9-operator basis w $n_f = 4$
  - (84,1) x2, (20,1) x1, (15,1) x4, (15,15) x2
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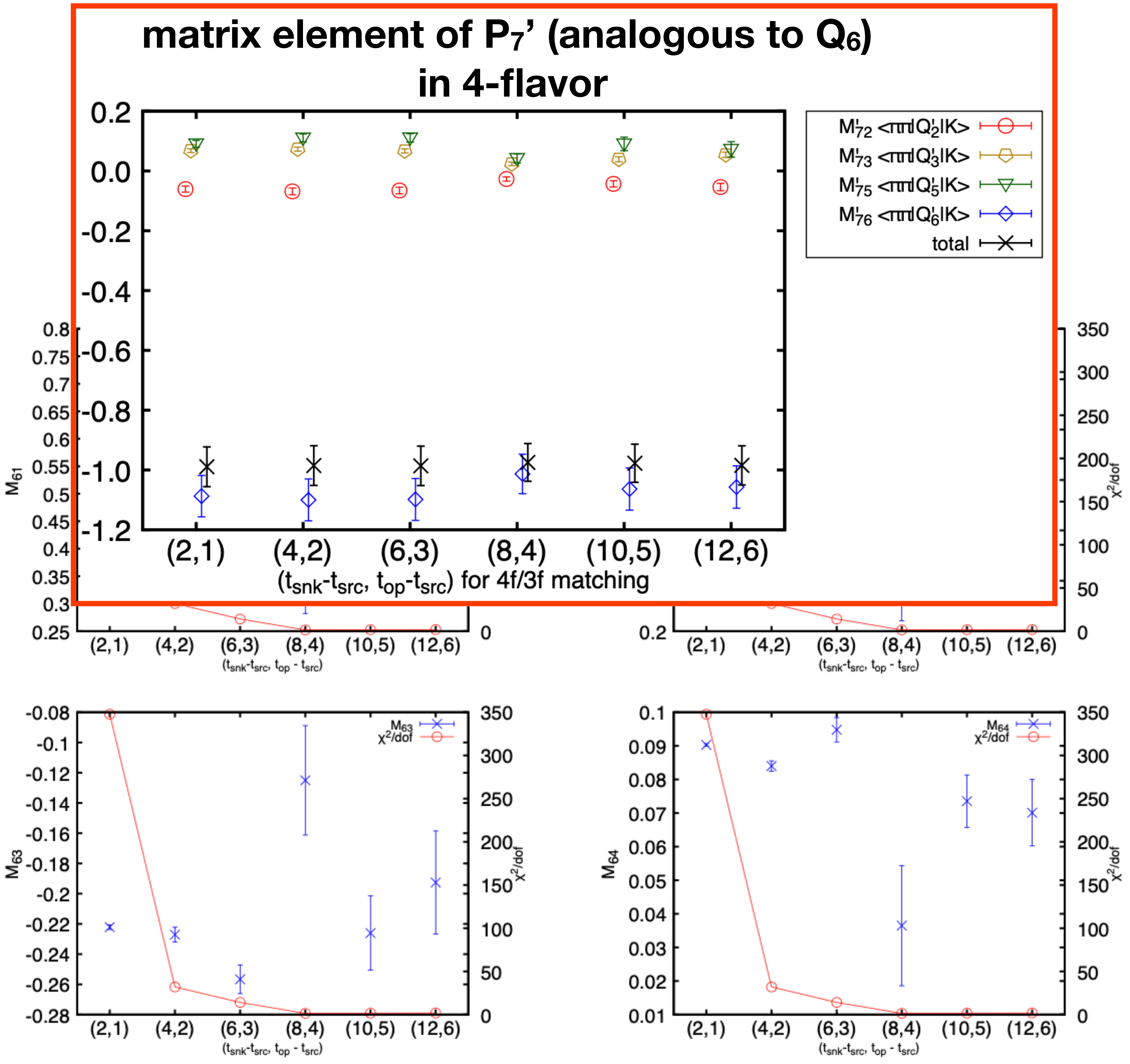


Example results on finer lattice

		value [GeV]	significant statistical errors	significant systematic errors
$\mathrm{Re}A_0$	perturbative	$2.89(23)_{stat}(35)_{sys}$ x 10^{-7}	matrix elements 8.0%	Wilson coefficients 12%
	nonperturbative	$2.97(22)_{stat}(1)_{sys}$ x 10^{-7}	• matrix elements 6.5% • 3/4-flavor matching 2.3%	
$\mathrm{Im}A_0$	perturbative	$-7.11(54)_{stat}(94)_{sys}$ x 10^{-11}	matrix elements 7.6%	• Wilson coefficients 12% • High-energy parameters 5.6%
	nonperturbative	$-6.82(94)_{stat}(37)_{sys}$ x 10^{-11}	• matrix elements 11% • 3/4-flavor matching 6.3%	High-energy parameters 5.4%

****outdated results**
****Discretization effects (12% overall) and finite volume effects (7% overall) not included above**

Summary

- So many issues to address for precise SM prediction of $K \to \pi\pi$
  - better continuum limit → calculation at $a^{-1} \approx 1.7$ GeV underway, 2.3 GeV planned
  - continuum limit of step scaling
  - dependence on intermediate scheme/scale on the high-energy side (μ_h)
    - NNLO perturbative matching would be very helpful
  - Nonperturbative 3/4-flavor matching mostly done, but maybe a bit more to understand before writing up a paper
  - NNLO Wilson coefficients should significantly reduce the systematic error

Why periodic BC?

- Already have lattice ensembles with physical pion mass
  - a^{-1} = 1 GeV, 24^3 x 64 & a^{-1} = 1.4 GeV, 32^3 x 64 & …
  - Continuum limit easier
- Hope to introduce QED/IB effects near future
  - G-parity BC violate charge conservation
  - PBC appear necessary
- Presence of $E_{\pi\pi} = 2m_\pi$ state challenging
  - S/N ratio of $E_{\pi\pi} = m_K$ state should be the same as in G-parity BC: $\sim e^{-(m_K - 2m_\pi)t}$
  - interesting to see feasibility of extracting signal of excited states

Lattice setup

- RBC/UKQCD's 2+1-flavor MDWF ensembles at physical pion & kaon masses
  - 24^3 x 64, a^{-1} = 1.0 GeV, 258 confs
  - 32^3 x 64, a^{-1} = 1.4 GeV, 107 confs
- Chiral symmetry of DWF → strong constraints on operator mixings
  - with lower-dimensional operators
  - among different representations w.r.t. chiral symmetry (8,1), (8,8) & (27,1)
- All-to-all quark propagators
  - 2,000 low modes for light quarks (no low mode for strange)
  - high-mode part: spin, color and time dilutions => 768 inversions

Matrix elements

- For extraction of ground-state ME

$$M^{\mathrm{eff}}(t_2, t_1) = C^{(3)}(t_2, t_1)\left[\frac{e^{E^{\pi\pi}t_2}e^{E^K t_1}}{C^{\pi\pi}(t_2)C^K(t_1)}\right]^{1/2} \xrightarrow{\text{large } t_1 \ \& \ t_2} M$$

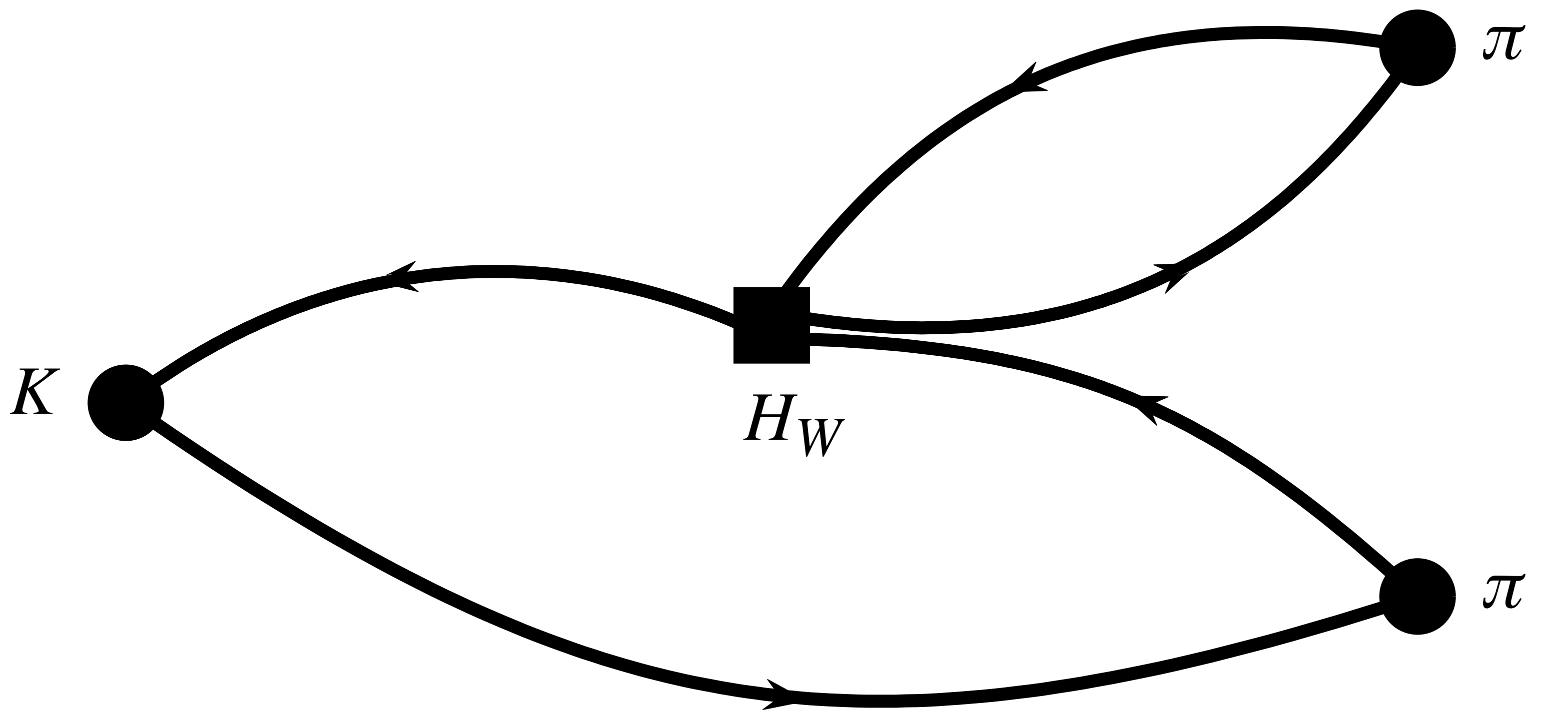
- Excited (n-th) ππ state needed for on-shell kinematics with PBC

$$\boxed{M_n^{\mathrm{eff}}(t_2, t_1) = C_n^{(3)}(t_2, t_1)\left[\frac{e^{E_n^{\pi\pi}t_2}e^{E^K t_1}}{C_n^{\pi\pi}(t_2)C^K(t_1)}\right]^{1/2}} \xrightarrow{\text{large } t_1 \ \& \ t_2} M_n$$

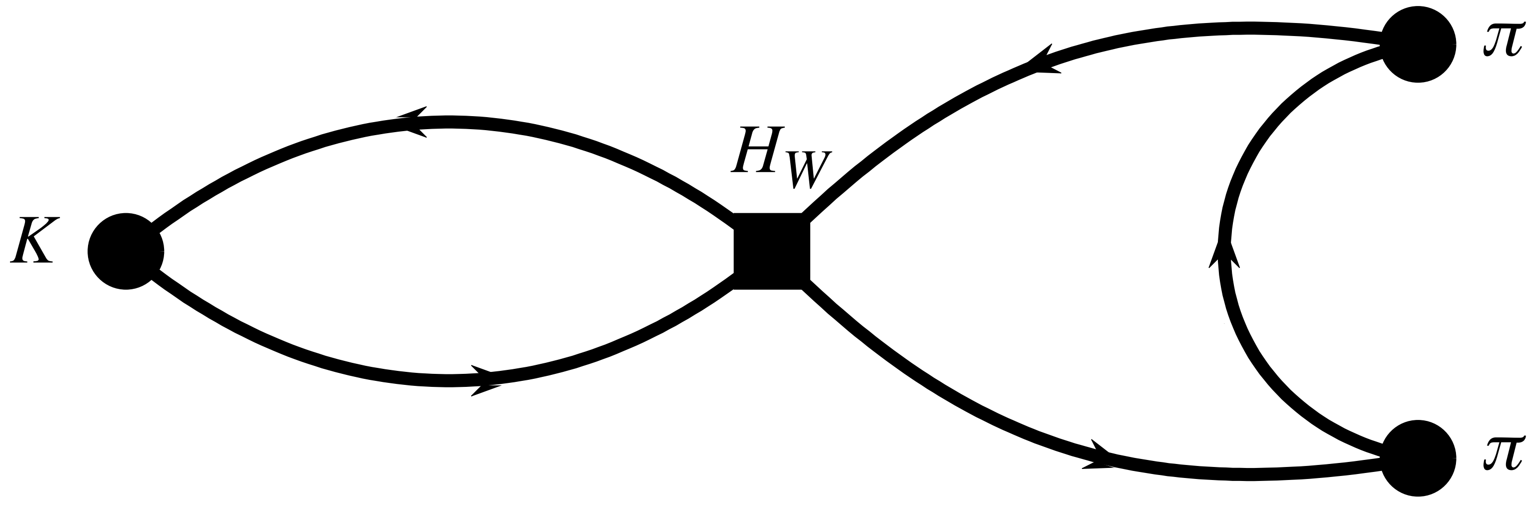
$C_n^{\pi\pi}$: 2-pt function of ππ operators diagonalized by GEVP

$C_n^{(3)}$: K → ππ 3-pt function with ππ operator used in $C_n^{\pi\pi}$

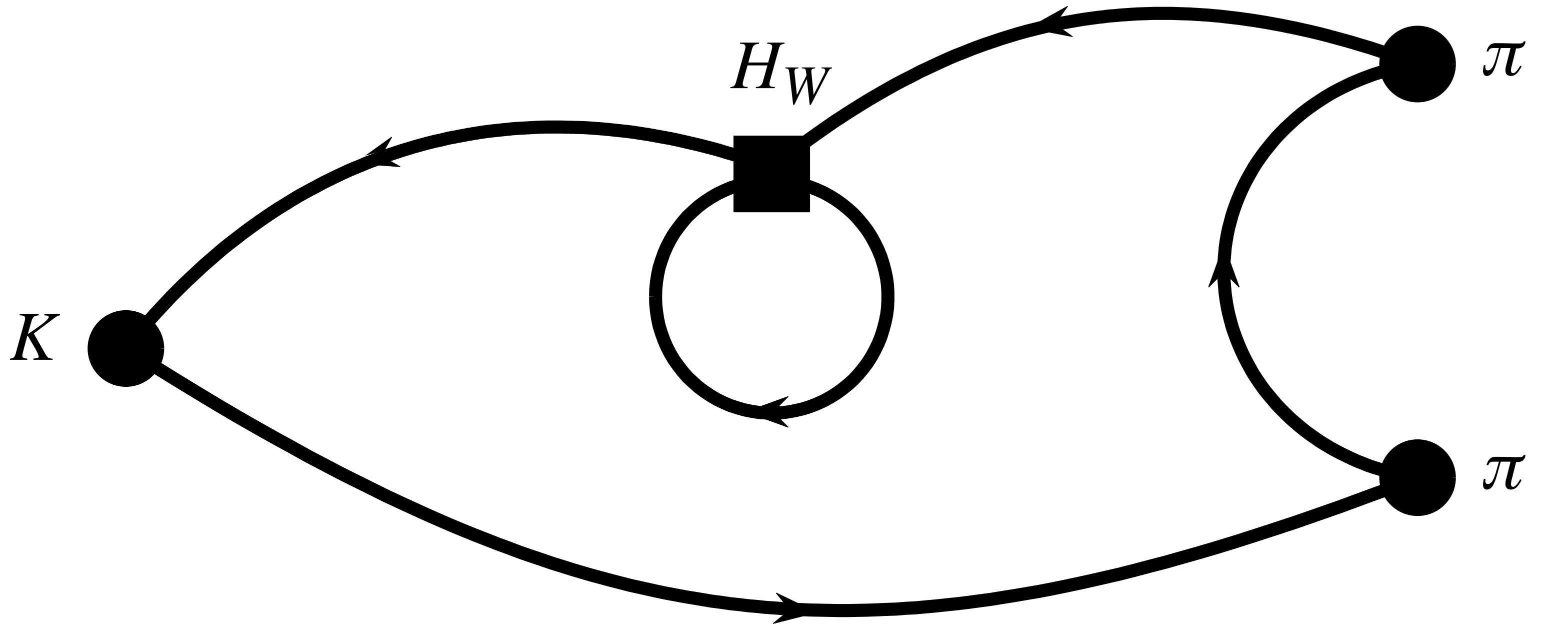
Diagrams for K → ππ 3pt functions



type 1



type 2



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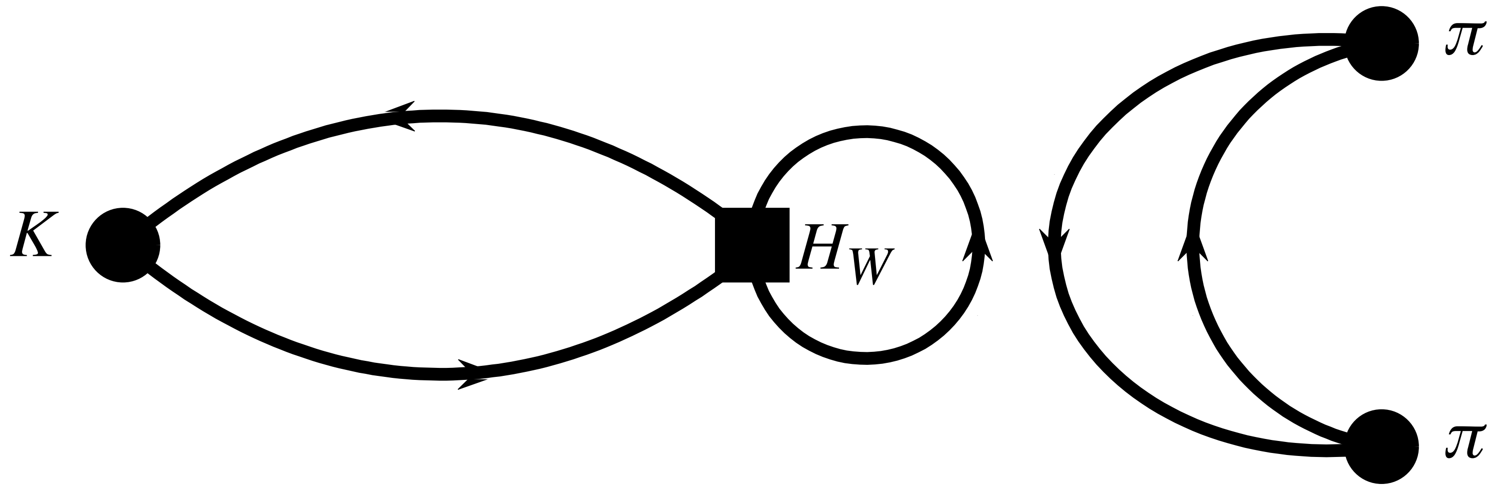


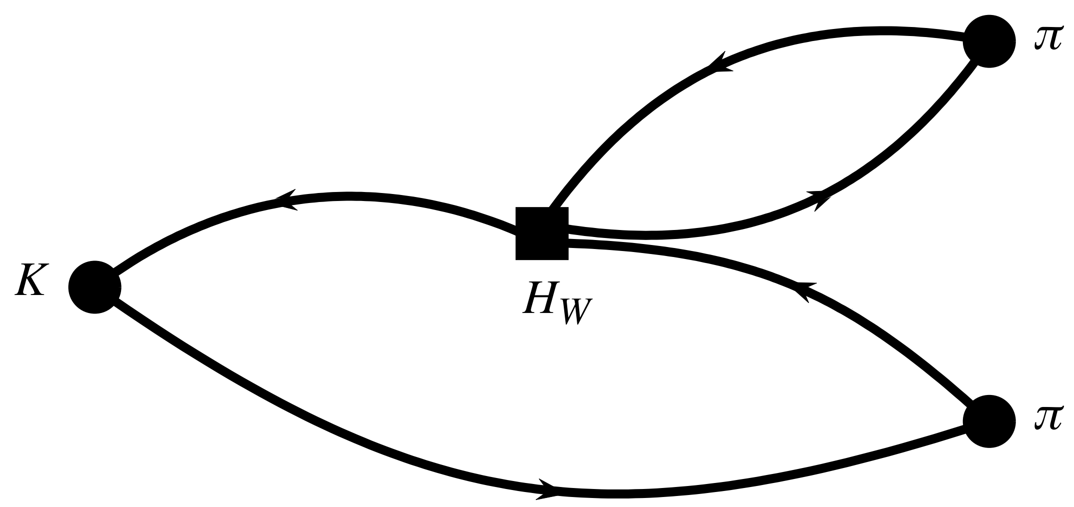
type 4

type 4 dominates stat. error

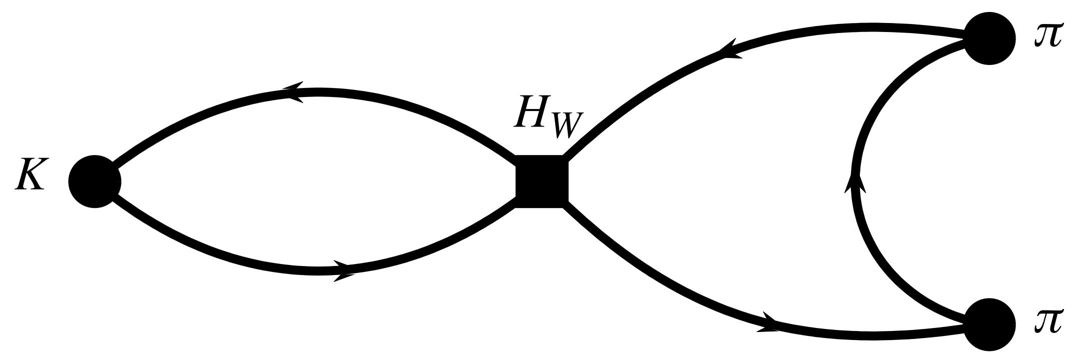
- Previous G-parity calculation
  - types 1,2: averaged over every 8 time translations
  - types 3,4: averaged over every time translation

- types 1,2 still expensive but no need of such precision

  → cost reduction?



type 1



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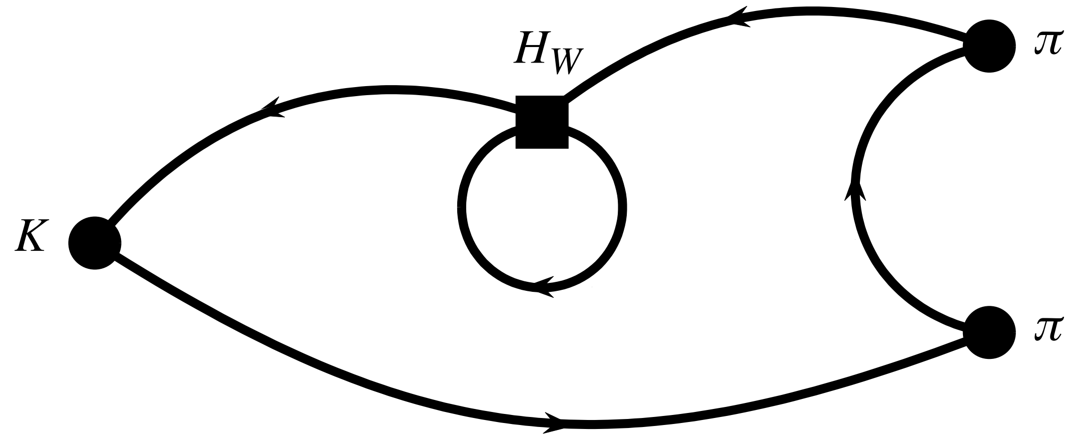

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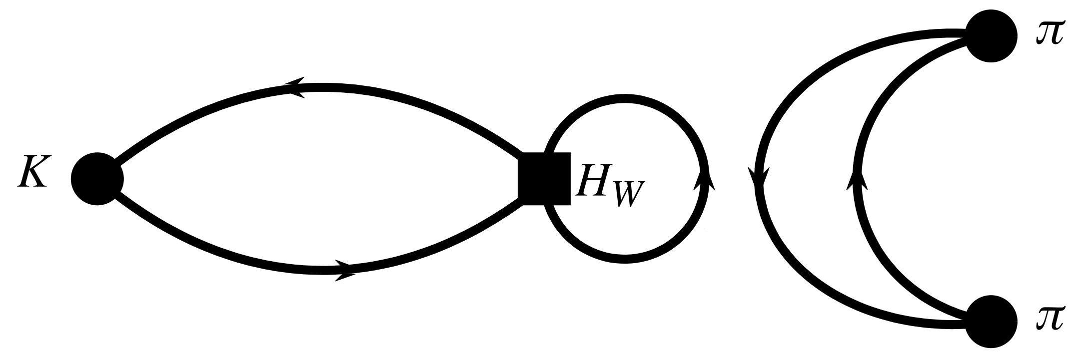
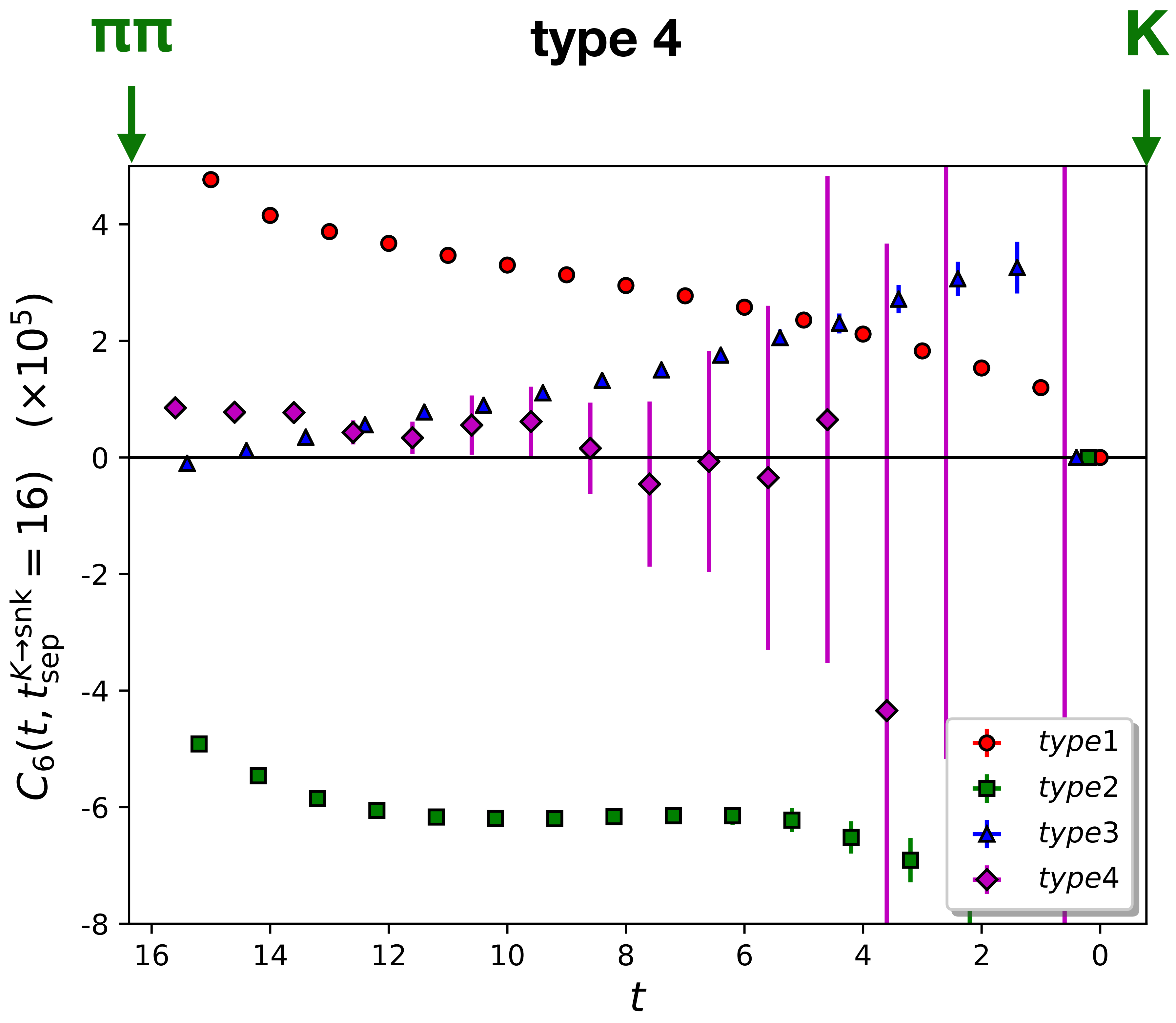


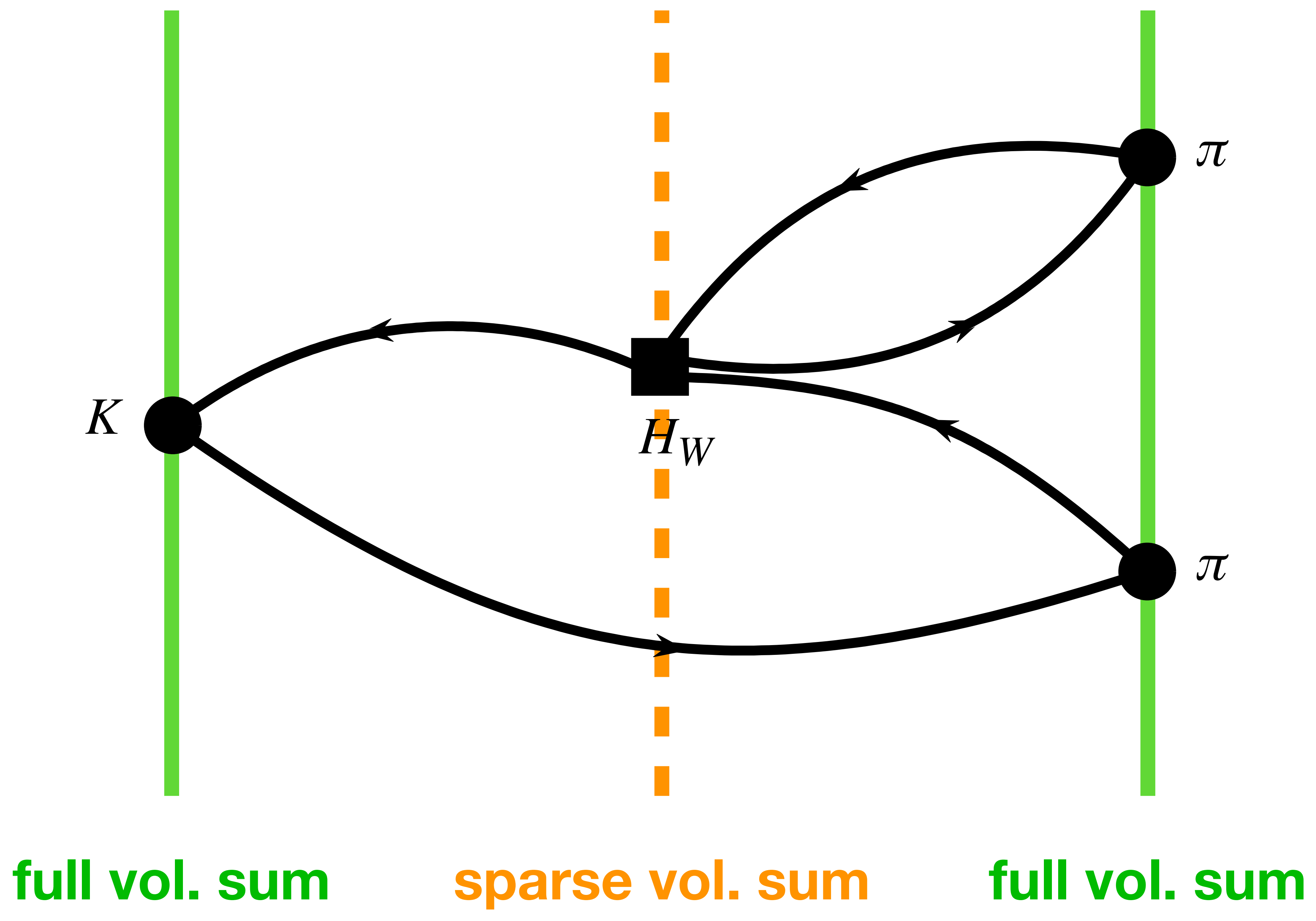
type 4



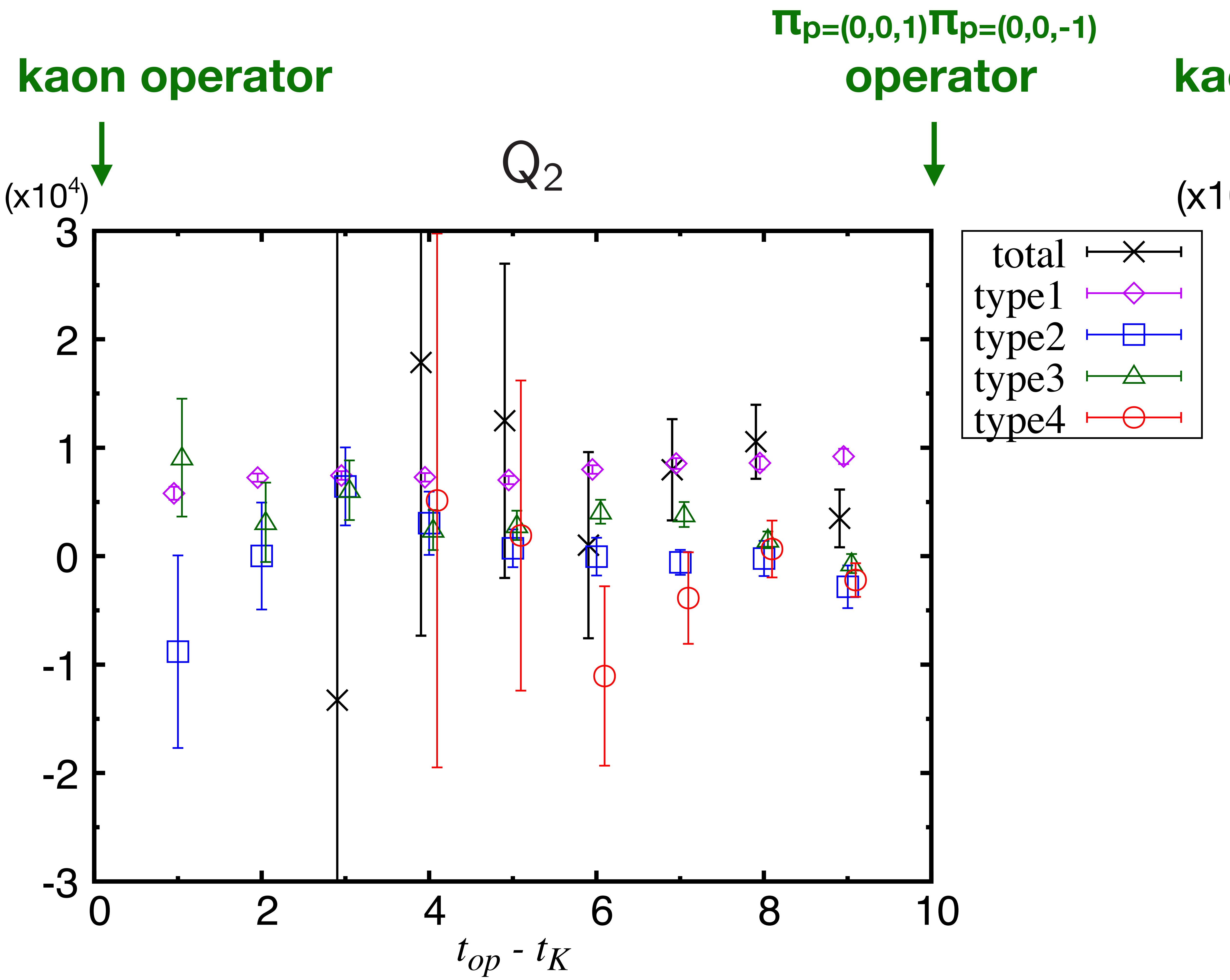
RBC/UKQCD, PRD 102,054509

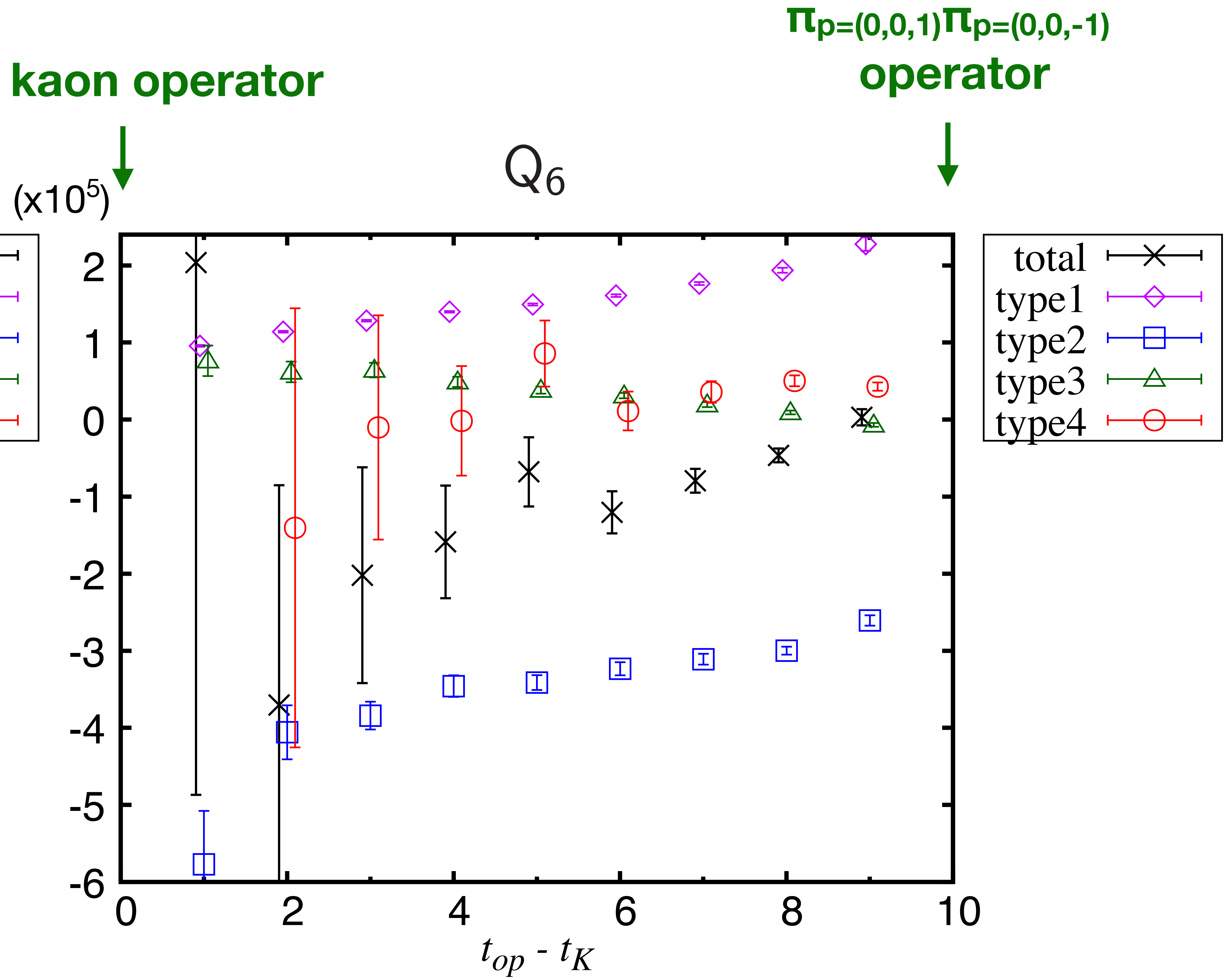
Sparsening H_W

- Contraction cost mostly proportional to volume of H_W
- G-parity calculation: summed H_W over full 3D volume
- This calculation: volume of H_W ($24^3 \to 8^3$: 27x speed up) for types 1 & 2



I = 0 correlation functions



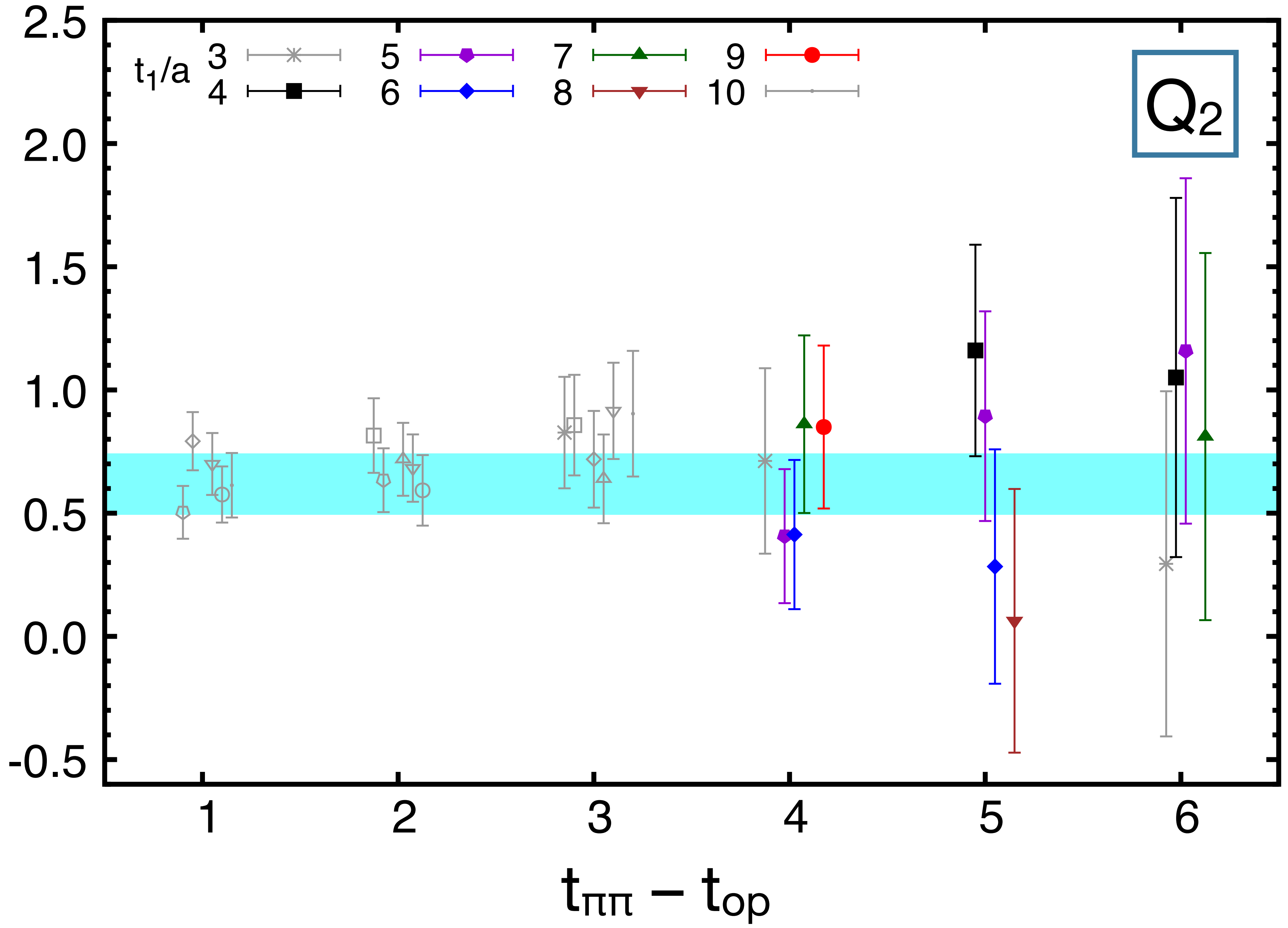


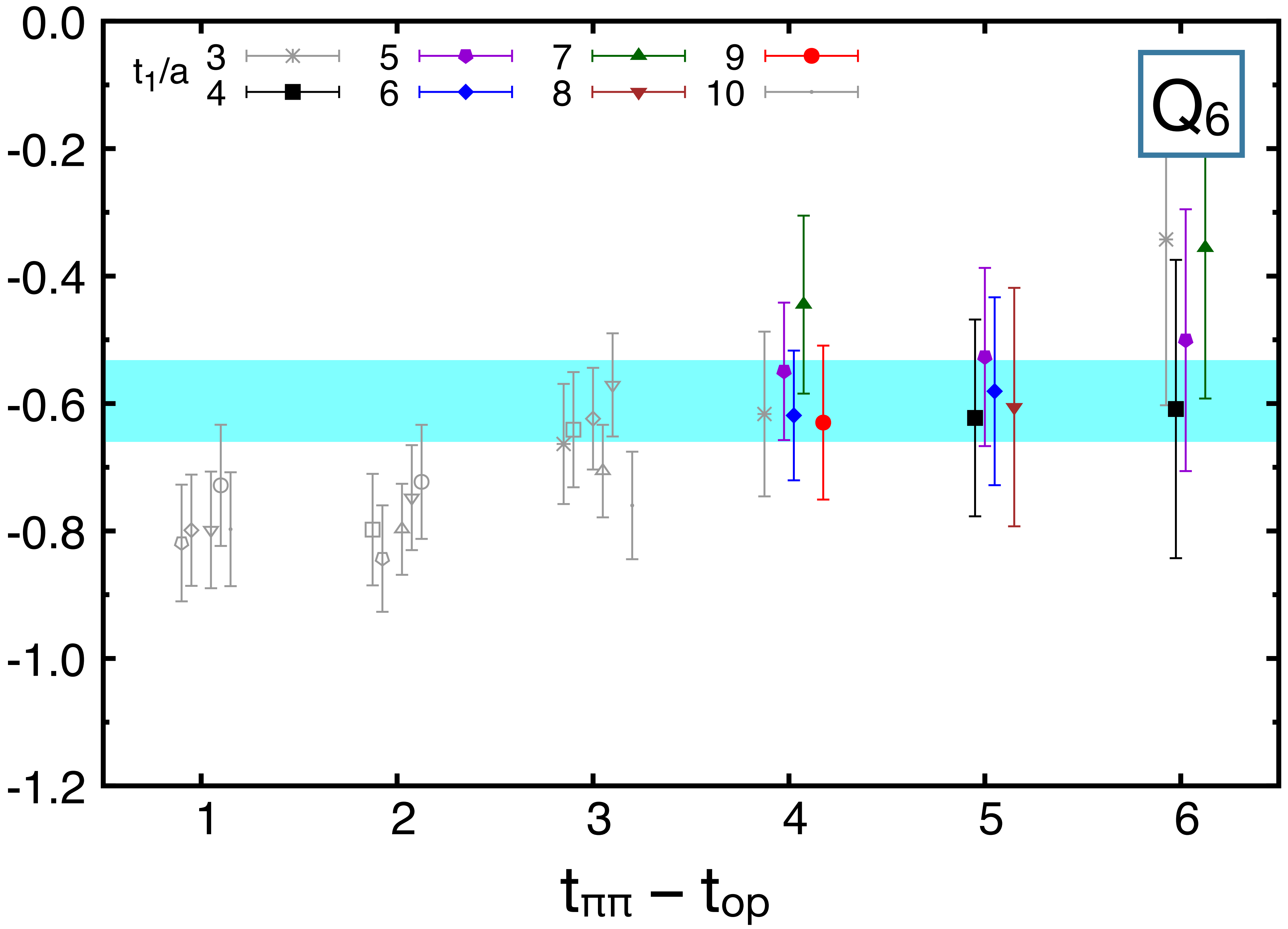
- Sparsen H_W for types1,2 – still more precise than type4
- Precision of type4 disconnected diagram

Effective matrix elements (ΔI = 1/2)

- 24^3 x 64
- Plateau appears
- : Correlated fit result with

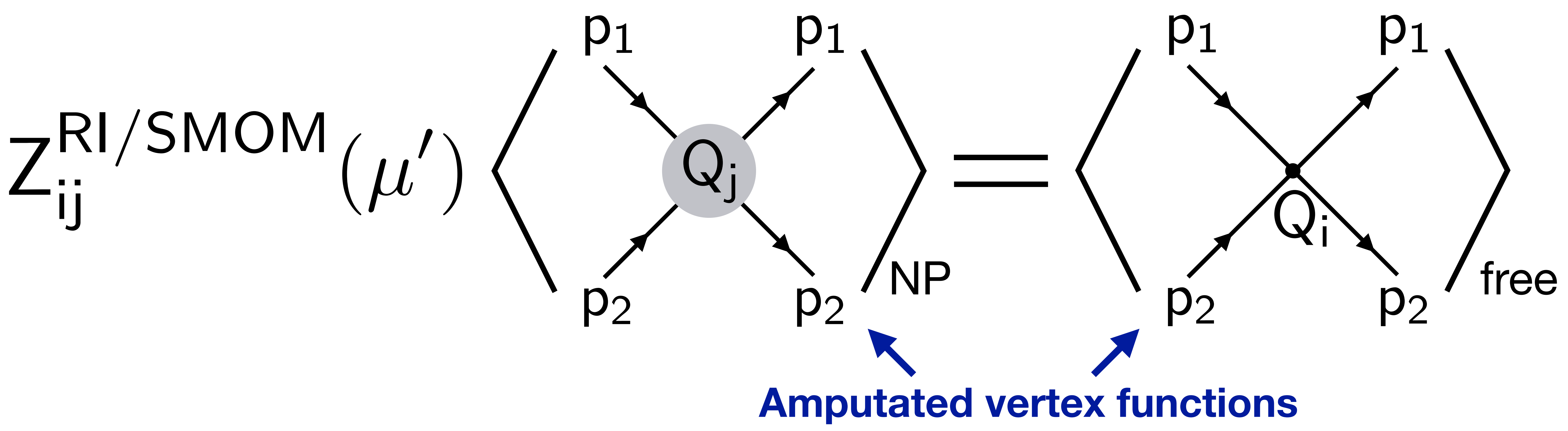
$t_1 = t_{op}-t_K \geq 4$ && $t_2 = t_{\pi\pi}-t_{op} \geq 4$ (colored filled data points)





Translating to more physical ME

- Renormalization (RI/SMOM scheme)

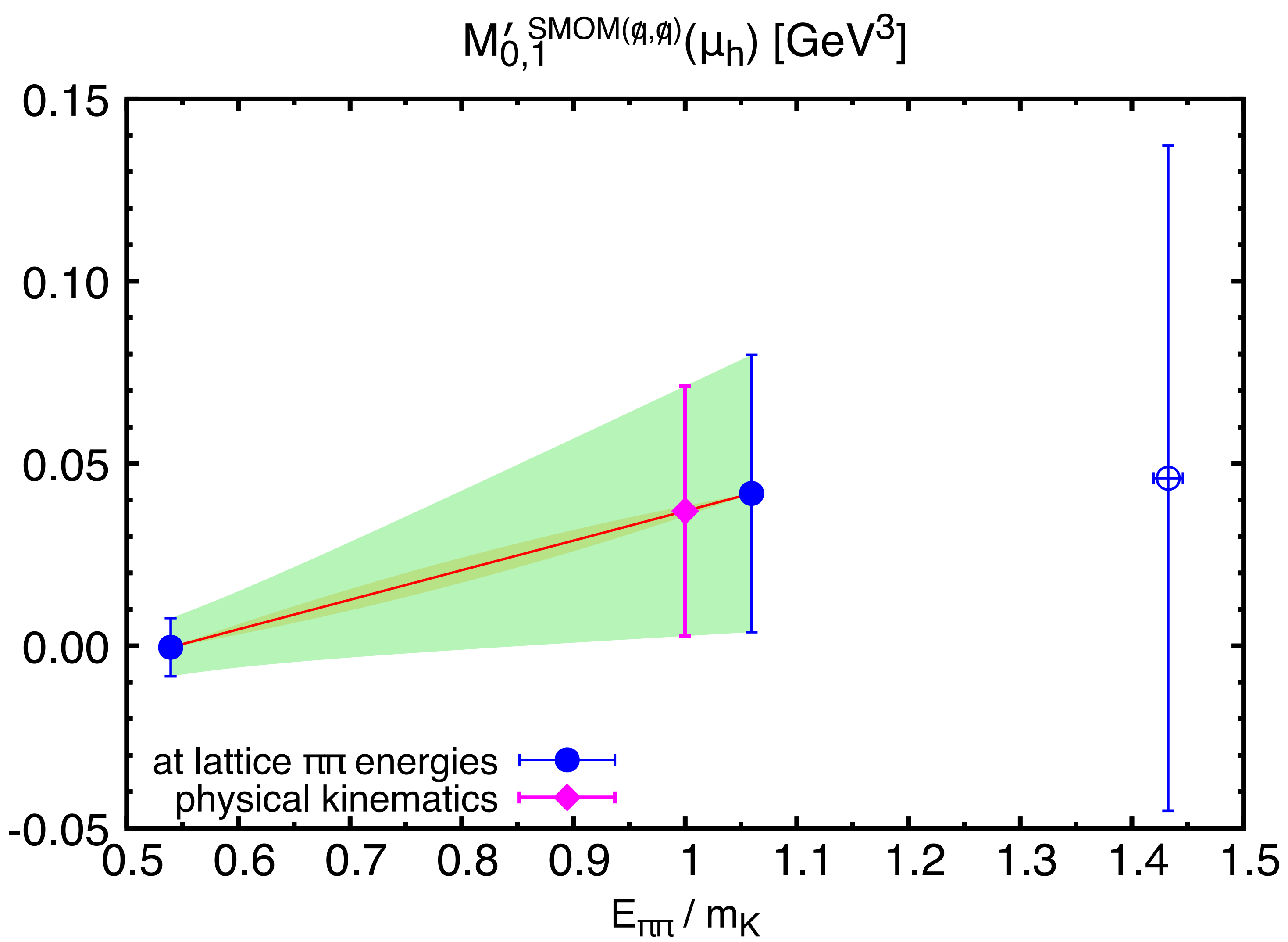


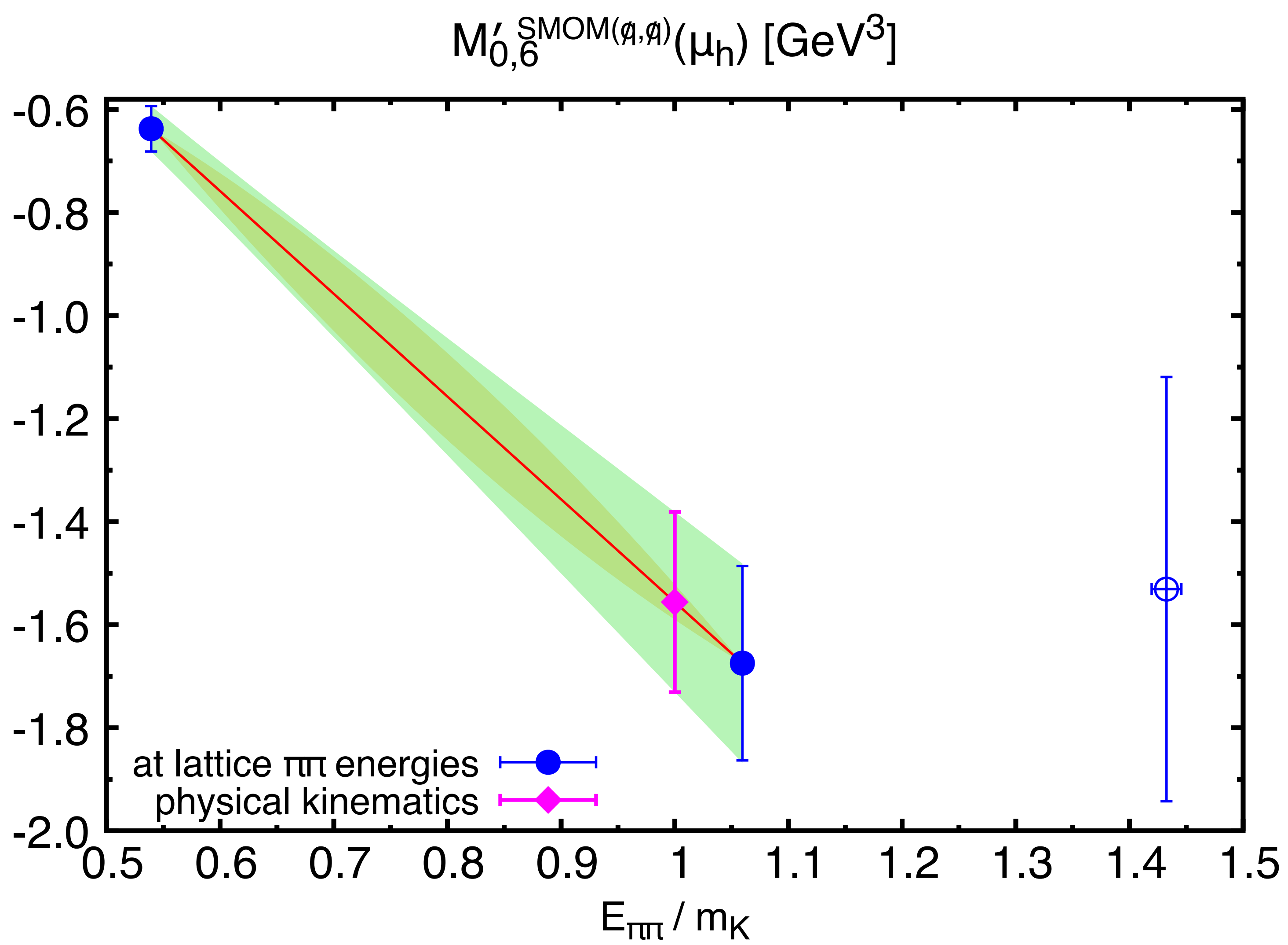
$$\mu'^2 = p_1^2 = p_2^2 = (p_1 - p_2)^2$$

- Interpolation

Examples of interpolation of renormalized ME

- Linear & quadratic in $E_{\pi\pi}/m_K$
- Systematic error estimated as lin vs quad is small as 1st excited st. close to on-shell





Precision performance

	32^3 G-parity BC (previous work)	24^3 Periodic BC	32^3 Periodic BC
# of configurations	741	258 → 440	107 → 470
$\Delta I = 1/2$ ME via Q_2^{lat}	10%	14% → (11%)	14% → (6.7%)
$\Delta I = 1/2$ ME via Q_6^{lat}	6.5%	8.9% → (6.8%)	11% → (5.3%)
Re A_0	11%	13% → (10%)	14% → (6.7%)

Error % (statistical)

expectations of upcoming analysis in ( )

- Good precision performance of PBC (ME with excited-state ππ) compared to G-parity BC calculation (ME with ground-state ππ)

Summary & Outlook

- Main sources of systematic errors at the moment
  - Finite lattice spacing - *Easier to take continuum limit with PBC as we already have lattice ensembles*
  - Wilson coefficients - *Independent study on-going*
  - QED/IB effects - *Theoretical approach being developed [Christ et al, PRD106, 014508 (2021)] with PBC*

- We are successful in
  - Extracting excited-state signals of the challenging $\Delta I = 1/2$ process
  - Good precision performance of PBC approach
- Precision will reach that of experiment in the near future
  - Could attract a lot of attention