Topology from Fermionic Operators

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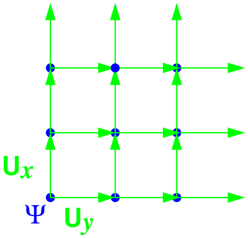
Introduction to lattice QCD

Quantum ChromoDynamics (QCD): Theory of strong interaction which governs interaction between quarks and gluons.
In contrast to Quantum Electrodynamics (QED), The effective coupling of QCD decreases in high energy, hence is calculable by hand, but not in low energy. $\rightarrow$ Nonperturbative techniques such as lattice QCD is needed for *ab initio* calculations. $(\psi(x), A_\mu(x)) \rightarrow (\psi(n), U_\mu(n) = \exp(-iA_\mu))$



$$
\begin{aligned}
Z &= \int [dU]\det(\not{D} + m)e^{-(S_g)} \\
&= \int [dU][d\bar{\psi}][d\psi]\ \exp[-(S_g + S_f)] \\
S_f &= \bar{\psi}(D^\dagger D)^{-1}\psi, \qquad S_{eff} = S_g + S_f \\
S_g &= \beta \sum [(U_\mu(x)U_\nu(x+\hat{\mu})U^\dagger_\mu(x+\hat{\nu})U^\dagger_\nu(x)]
\end{aligned}
$$

Current "typical" calculation: $V = 64^3 \times 128$, rank(D) $\sim 10^{10}$, nonzero element per row $=\sim 10^2$

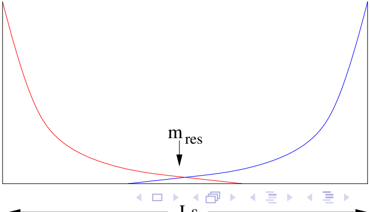
Different discretizations in Lattice QCD

Basic problem/motivation: Naive discretization

$$(\partial_\mu + iA_\mu)\psi(x) \rightarrow \frac{(U_\mu(x)\psi(x+\mu) - U_\mu^\dagger(x-\mu)\psi(x-\mu))}{2a}$$

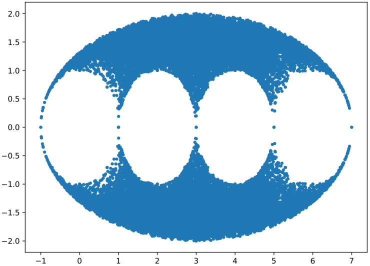
turns p into $\sin(p)$. $2^4 = 16$ particles instead of 1 (doubler).
It is impossible to have a chirally invariant, doubler-free, local, translationally invariant, real bilinear fermion action on the lattice (Nielsen-Ninomiya no-go theorem).
Wilson Fermion: Add Laplacian-like term
Additive mass renormalization $\rightarrow$ fine tuning needed.
$-\frac{a}{2}\Delta\psi(x) = -\frac{a}{2}\sum_\mu[\psi(x+a\hat{\mu}) + \psi(x-a\hat{\mu}) - 2\psi(x)]$

Domain Wall Fermion (DWF) /Mobius/Overlap fermions: Dirac operator in 5D with repeating gauge field in 4D



Spectrum of Free Field D_w, H_w, D_{ov}

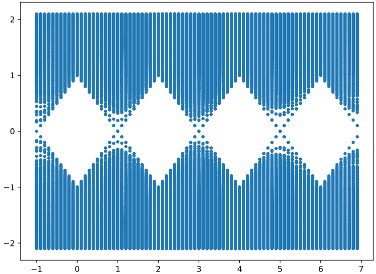
Wilson-Dirac operator $D_w(-1)$



Hermition Wilson operator

$H_w(M_5) = \gamma_5 D_w(-M_5)$

('spectral flow')

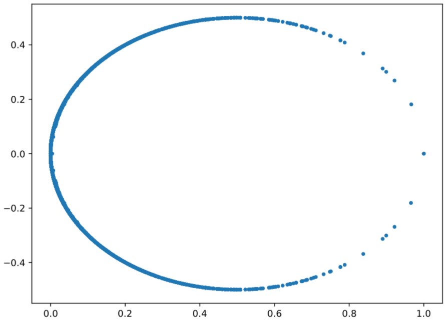


$D_w(M)_{xx'} =$

$M + 4 - \frac{1}{2}\sum_\mu \left[(1-\gamma_\mu)\mathbf{U}_\mu(x)\delta_{x+\mu,x'} + (1+\gamma_\mu)\mathbf{U}_\mu{}^\dagger(y)\delta_{x-\mu,y}\right]$

Positions of 'crossings' where $H_w(M_5)$ has zero eigenvalue coincides with the real eigenvalues of D_w

Overlap D_{ov}



$$T^{-1} = -[H_M - 1]^{-1}[H_M + 1], H_M = \gamma_5 \frac{(b+c)D_W}{2+(b-c)D_W}$$

$$D_{ov}(m_f) = \frac{1+m_f}{2} + \frac{1-m_f}{2}\gamma_5 \frac{T^{-Ls}-1}{T^{-Ls}+1}$$

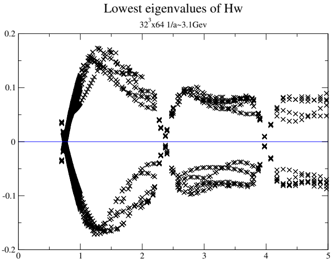
Topological index measurement on the lattice

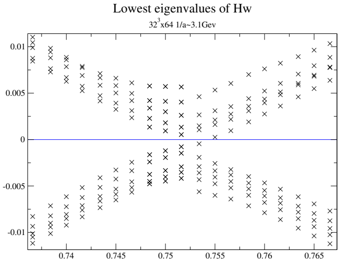
Gluonic definition: 5Li (de Forcrand et al, hep-lat/9701012) Discretization of $\int F\tilde{F}$. Combination of 5 different size loops to eliminate $\mathcal{O}(a^2, a^4)$ error. Used in combintion with cooling or smearing to control ambiguity from small instantons and produce (near-) integer number for Q.

$$\frac{dV_t(x,\mu)}{dt} = -g_0^2 \partial_{x,\mu} S_g(V(t)) V_t(x,\mu),\ V_t(x,\mu)|V_{t=0} = U(x,\mu)$$

Fermionic Definition: The topological index is the same as the 'net chirality' of exact zero modes of overlap Dirac operator. Ambiguity eliminated? No - Location of M_5 can change which mode(s) becomes 'physcial' or 'doubler' modes.

Spectral flow of 2+1f $1/a$ ~3.1Gev configuration (Q=1)

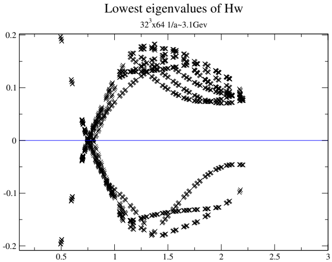


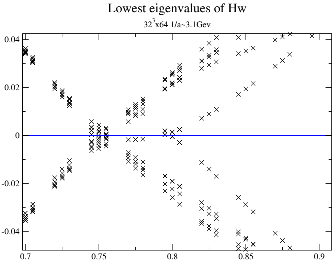


Q=1, also from Gluonic measurement

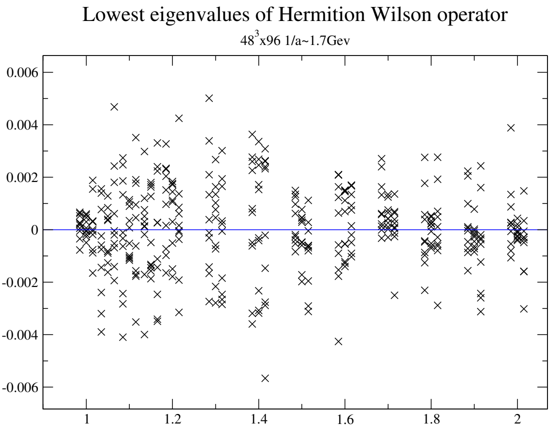
For an eigenvector $|\lambda\rangle$ of $H_w(-M_5)$, $\frac{d\lambda}{dM_5} = -\langle\lambda|\,\gamma_5\,|\lambda\rangle$.

Spectral flow of 2+1f $1/a \sim$3.1Gev configuration





Spectral flow of 2+1f $1/a \sim$1.7Gev configuration(48I)



Q$\sim$ 16

Observation

- QCD vacuum often has multiple near (anti-)instantions which generates near zero modes, and mixes heavily with exact zero modes
- Isolated (in eigenvalue space) modes are spatially better localized, consistent with 'dislocation' picture, where small (anti) instantons changes Q.
- Real eigenmodes of D_w or zero eigenmodes of H_w can be useful in tracking and improving topology evolution during ensemble generation.