Chiral Anomalies and Wilson Fermions

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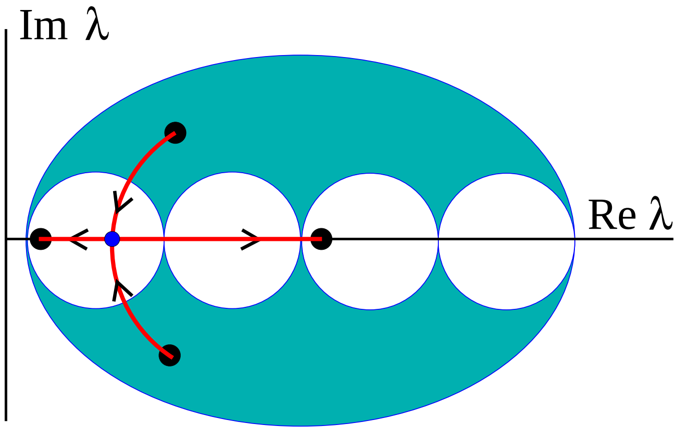


Anomalies central to understanding gauge field theories

- QED: can't conserve both axial and vector currents
  - Steinberger, Adler, Bell, Jackiw
- QCD:
  - η' not a Goldstone boson
  - proton has mass even if $m_q = 0$
- standard model: t'Hooft baryon decay

All tied to chiral anomaly

- change of variables in path integral
  - $$\psi \longrightarrow e^{i\phi\gamma_5}\psi$$
    $$\overline{\psi} \longrightarrow \overline{\psi}e^{i\phi\gamma_5}$$
- classical symmetry of $\overline{\psi}\not{D}\psi$ if $m_q = 0$
- broken in regularization process
- $\psi_L = (1-\gamma_5)\psi/2$ can mix with $\psi_R = (1+\gamma_5)\psi/2$

Wilson lattice is a regularization

- mathematically well defined path integral
- must show effects of anomalies

2024 MC: lattice standard model

- combine Wilson fermions with Higgs field

Where is the baryon decay?

Wilson Fermion Review

Classical fermion action $i\not{D} + m$

Replace derivative by nearest neighbor difference

- $i\not{\partial}\psi \longrightarrow \sum_\mu \frac{i\gamma_\mu}{a}(\psi_{i+e_\mu} - \psi_i)$
  - site i, direction μ, spacing a

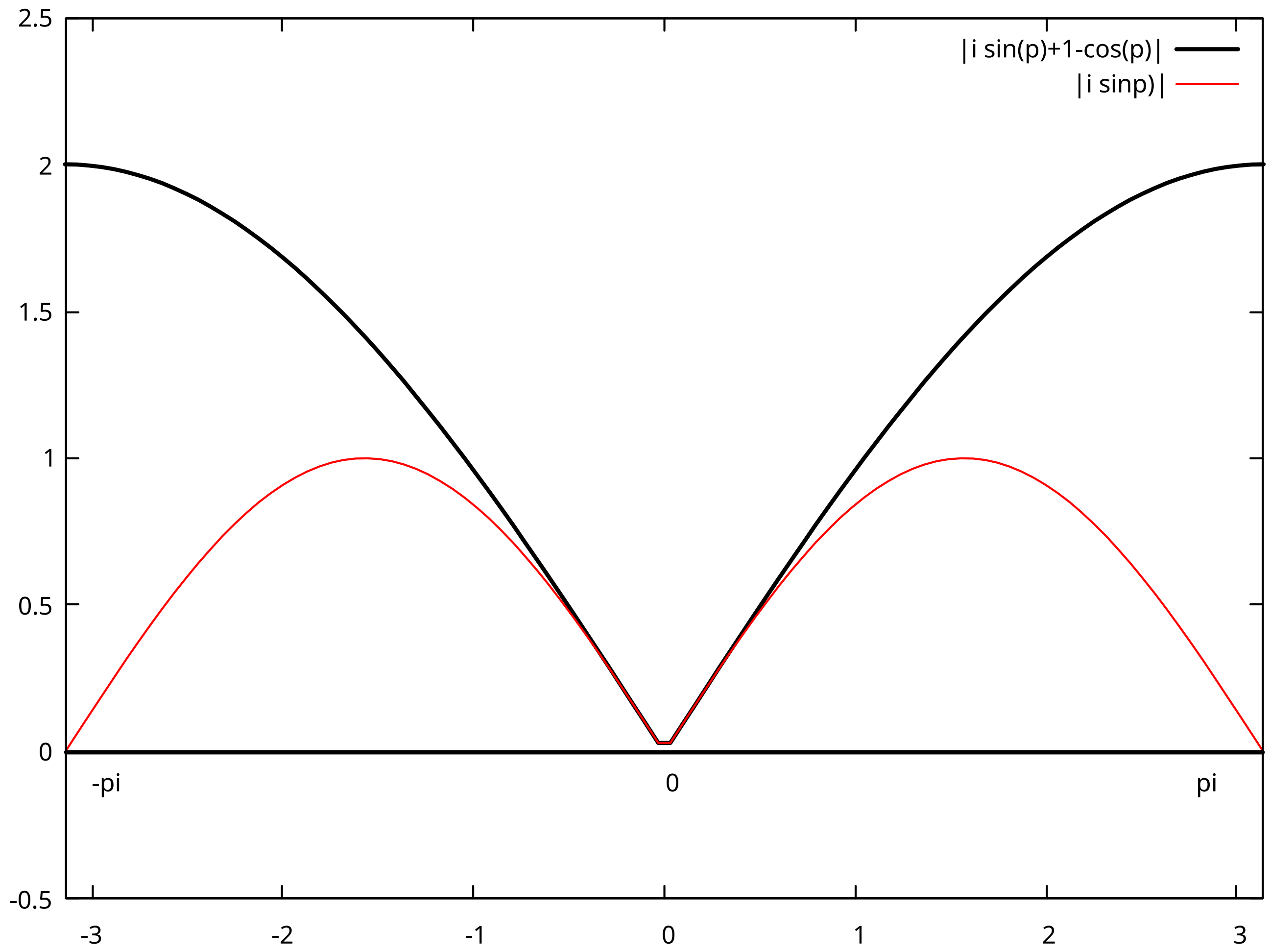
Momentum space: $p_\mu \longrightarrow \frac{1}{a}\sin(ap_\mu) = p_\mu + O(a)$

Doubling

- natural range of lattice momentum $-\frac{\pi}{a} < p_\mu \leq \frac{\pi}{a}$
- extra pole in propagator at $p_\mu \sim \frac{\pi}{a}$

Remove by giving doubler large mass

- $i\gamma_\mu p_\mu \longrightarrow \frac{1}{a}(i\gamma_\mu \sin(ap_\mu) + 1 - \cos(ap_\mu))$
- $O(ap^2)$ correction
  - formally irrelevant operator $\sim \overline{\psi}\partial^2\psi$



Heavy states break chiral symmetry

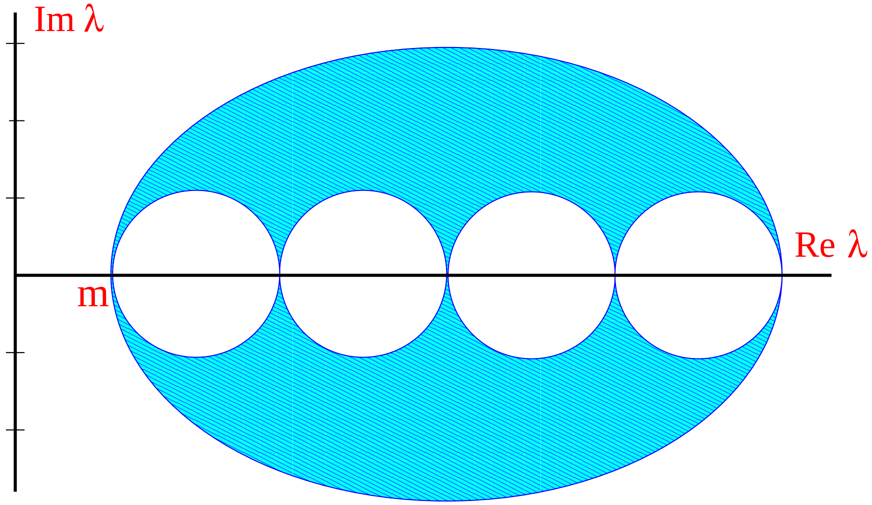
- think Pauli-Villars

Continuum: $i\not{D}+m$ is antihermitian plus a real mass term

- eigenvalues lie on a line at fixed real part m

Lattice: Wilson smears real part at high momentum

- sin's and cosine's give nested circles for free theory



Crossings count momentum components at π

When fields turn on, eigenvalues can move around

- $[D, D^\dagger] \neq 0$ non-normal matrix
  - different gauge factors
- eigenvectors are not orthogonal

Gamma-five hermiticity

- $[\gamma_5, \not{p}]_+ = 0 \qquad [\gamma_5, m]_- = 0$

$$D^\dagger = \gamma_5 D \gamma_5$$

Consequences for eigenvalue structure

- $0 = |D - \lambda| = |D^\dagger - \lambda^*| = |D - \lambda^*| = |D^\dagger - \lambda|$

Eigenvalues of D come in complex conjugate pairs

- same eigenvalues for D and $D^\dagger$
- λ, λ^* have non-orthogonal eigenvectors

Eigenvector pairs span a two dimensional space

- $D = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^* \end{pmatrix}$
- $D^\dagger = \begin{pmatrix} \lambda^* & 0 \\ 0 & \lambda \end{pmatrix}$

D block diagonal, 2 by 2 blocks

- each complex pair in a separate block

Given an eigenvalue $D\psi = \lambda\psi$ and its complex partner

- look at the spanned two dimensional space

- convenient orthogonal basis

$$\psi_R = \frac{1+\gamma_5}{2}\psi$$
$$\psi_L = \frac{1-\gamma_5}{2}\psi$$

- $\gamma_5 \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3$

General 2 by 2 matrix satisfying $D^\dagger = \sigma_3 D \sigma_3$

$$\boxed{D \to a_0 + ia_1\sigma_1 + ia_2\sigma_2 + a_3\sigma_3}$$

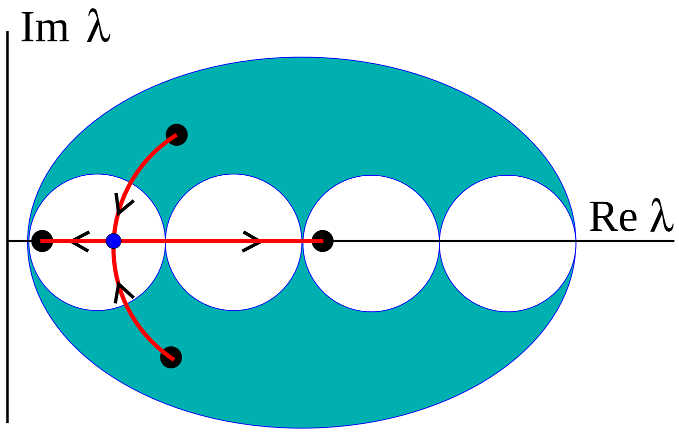
- eigenvalues $\lambda = a_0 \pm i\sqrt{a_1^2 + a_2^2 - a_3^2}$

Contributes to fermion determinant

- $|D| = a_0^2 + a_1^2 + a_2^2 - a_3^2 = \lambda\lambda^*$

Two domains

- $a_1^2 + a_2^2 > a_3^2$: two complex eigenvalues
  - $\lambda = a_0 \pm i\sqrt{a_1^2 + a_2^2 - a_3^2}$
- $a_1^2 + a_2^2 < a_3^2$: eigenvalues become real
  - $\lambda = a_0 \pm \sqrt{a_3^2 - a_1^2 - a_2^2}$



Large a_3 can give small real modes

- cooling to classical fields $\longrightarrow$ “instantons”
  - zero modes of index theorem
- γ_5 commutes with set of real modes
  - modes are chiral $\gamma_5\psi = \pm\psi$
  - $\overline{\psi}D\psi$ mixes left and right fields

Large a_3 requires γ_5 terms in $|D|$

- $a_3 = 0$ in free theory
- where do they come from
  - fermion loop that includes all four dimensions
  - fill out loop with Wilson term hoppings

Eigenvalue collision non-perturbative

- does not occur until γ_5 effects large enough

Anomaly effects appear after real modes separate

- zero mode limit not required

Classical instanton limit proves modes must exist

Peter Boyle and Chulwoo Jung: recent Friday talks

- eigenvalues of Hermitean $\gamma_5 D$
  - $\pm|\lambda| = \pm\sqrt{a_0^2 + a_1^2 + a_2^2 - a_3^2}$ (complex case)
  - $a_3 \pm \sqrt{a_0^2 + a_1^2 + a_2^2}$ (real case)

Small eigenvalues suppressed in path integral

- $Z = \int (dA)(d\overline{\psi} d\psi)\ e^{-S_g + \overline{\psi} D \psi} = \int (dA)\ e^{-S_g(A)}\ \prod \lambda_i$

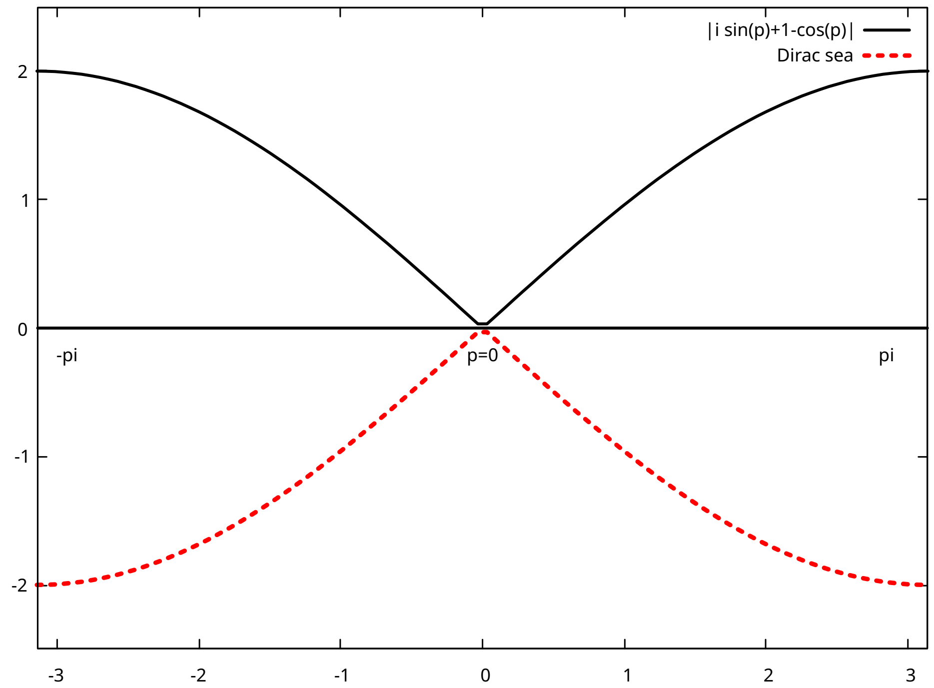
Enhanced in observables via source terms ('t Hooft)

- $$Z(\eta, \overline{\eta}) = \int (dA)\ (d\overline{\psi} d\psi)\ e^{-S_g + \overline{\psi} D \psi + \overline{\psi} \eta + \overline{\eta} \psi}$$
$$= \int (dA)\ e^{-S_g + \overline{\eta}\ D^{-1}\ \eta/4}\ \prod \lambda_i$$

Zero modes remain relevant to anomalies

Hamiltonian view

- gauge fields drive left modes into dirac sea
- while right modes rise out of the sea



Physical consequences

D depends on gauge

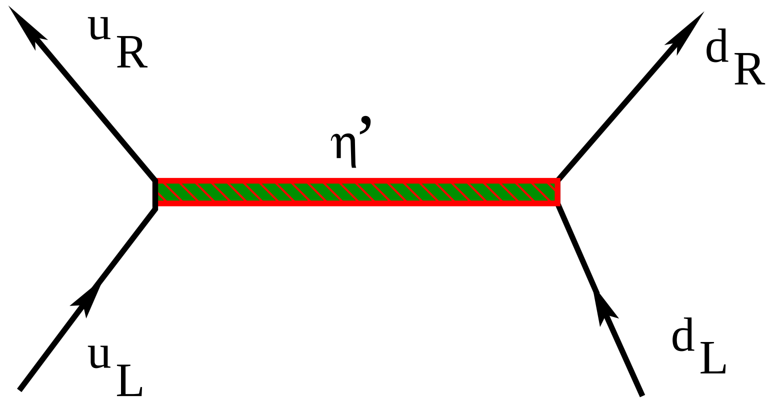
- find gauge invariant observables to see chiral modes

Electrodynamics $U(1)$: axial current not conserved

- modes break symmetry $\begin{array}{l} \psi \longrightarrow e^{i\phi\gamma_5}\psi \\ \overline{\psi} \longrightarrow \overline{\psi}e^{i\phi\gamma_5} \end{array}$
- Steinberger: $\pi \longrightarrow 2\gamma$
- doublers play role of Pauli-Villars field

QCD: one generation

- u, d two SU(3) triplets, both flip spin



- effective η' mass term

Proton $\psi = \epsilon_{rbg}\psi_{u,r}\psi_{u,b}\psi_{d,g}$

- color antisymmetrization makes gauge singlet
- $\overline{\psi}\psi$ sees the mode for each color
- effective mass even if quarks massless

Mass scale from asymptotic freedom

- $m_{\eta'}, m_p \propto \Lambda_{qcd} \propto \frac{1}{a}\; g^{-\beta_1/\beta_0^2}\;\; \exp\left(\frac{-1}{2\beta_0 g^2}\right)$

Not from instantons alone

- $\exp(-S_I) = \exp(\frac{-8\pi^2}{g^2})$
  - versus
- $\exp(\frac{-1}{2\beta_0 g^2}) = \exp(\frac{-8\pi^2}{g^2(11-2N_f/3)})$
- fermions and quantum fluctuations enhance effect

Weak $SU(2)$

First generation has four $SU(2)$ left handed doublets

- $\begin{pmatrix} u^r \\ d^r \end{pmatrix}_L \begin{pmatrix} u^g \\ d^g \end{pmatrix}_L \begin{pmatrix} u^b \\ d^b \end{pmatrix}_L \begin{pmatrix} \nu \\ e^- \end{pmatrix}_L$

Antiparticles are right handed $\psi_R^c = \tau_2 \gamma_2 \psi_L^*$

- $\begin{pmatrix} d^{r,c} \\ r^{r,c} \end{pmatrix}_R \begin{pmatrix} d^{g,c} \\ r^{g,c} \end{pmatrix}_R \begin{pmatrix} d^{b,c} \\ r^{b,c} \end{pmatrix}_R \begin{pmatrix} e^+ \\ \nu^c \end{pmatrix}_R$

Combine into vectorlike combinations

- $\begin{pmatrix} \nu \\ e^- \\ \overline{d^g} \\ \overline{u^g} \end{pmatrix} \qquad \begin{pmatrix} u^r \\ d^r \\ \overline{d^b} \\ \overline{u^b} \end{pmatrix}$

Anomaly requres spin flip in every multiplet

- $\overline{q} + \overline{q} \to l + q$

Combinations that preserve all symmetries

$$p \leftrightarrow e^+, \qquad n \leftrightarrow \overline{\nu}$$

Main points

- Dirac operator reducible to 2 by 2 blocks
- colliding modes become real and chiral
- common mechanism $SU_3 \otimes SU_2 \otimes U_1$
- outside perturbative region
- zero mode limit not required
- staggered fermions X

Question: how many theta parameters?

- relative phase between mass and Wilson term
  - Seiler and Stamatescu

QCD: can move phase between flavors, i.e. m_u or m_t

Does $SU(2)$ introduce another possible phase?

QED?