



Novel Directions in Dark Matter Direct Detection

CARLOS BLANCO



The Glaring Problem: Dark Matter (DM)

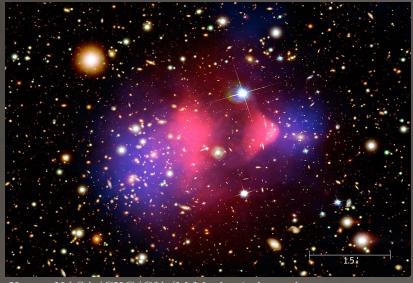
Cosmic Microwave Background

cold non-interacting fluid

ESA and the Planck Collaboration 2020

Bullet cluster

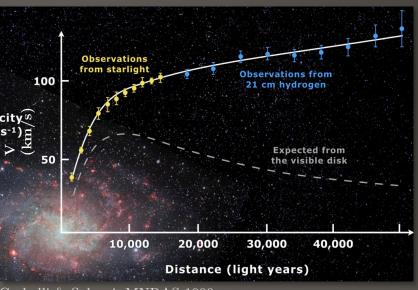
dark and visible components separate



X-ray: NASA/CXC/CfA/M.Markevitch et al.; Optical: NASA/STScI; D.Clowe et al.; Lensing Map: NASA/STScI; D.Clowe et al.

Galactic rotation curves

invisible gravitating mass



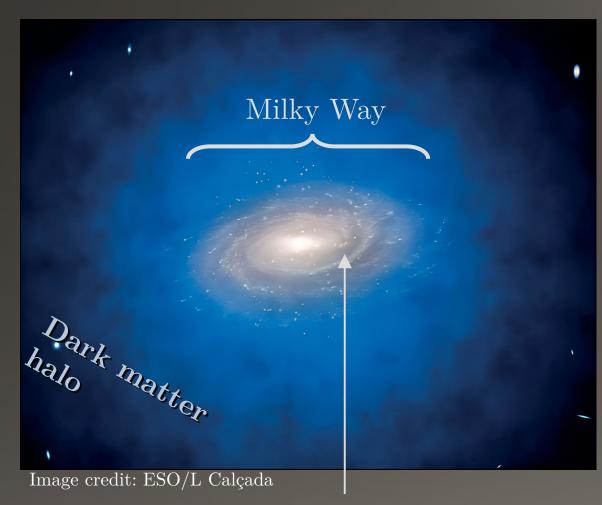
Corbelli & Salucci, MNRAS 1999 Bosma, Astronomical Journal, 1989

"...we must delve deep... and search wide, to explore as much motivated DM parameter space as possible."

Report of the Topical Group on Particle Dark Matter for Snowmass, arXiv:2209.07426.

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Our dark matter halo



We are here.

Local dark matter density

$$\rho_{\rm DM} = 0.4\,{\rm GeV/cm^3}$$

Local dark matter velocity

$$\langle v_{\rm DM} \rangle \approx 300 \, {\rm km/s}$$

Lighter masses \rightarrow more particles

$$n_{\rm DM} = \rho_{\rm DM}/m_{\rm DM}$$

More particles \rightarrow larger fluxes $\phi_{\rm DM} \sim v_{\rm DM} \, n_{\rm DM}$

What We Don't Know

Go After the Bare Minimum The Method:

Mass: How heavy and how many?

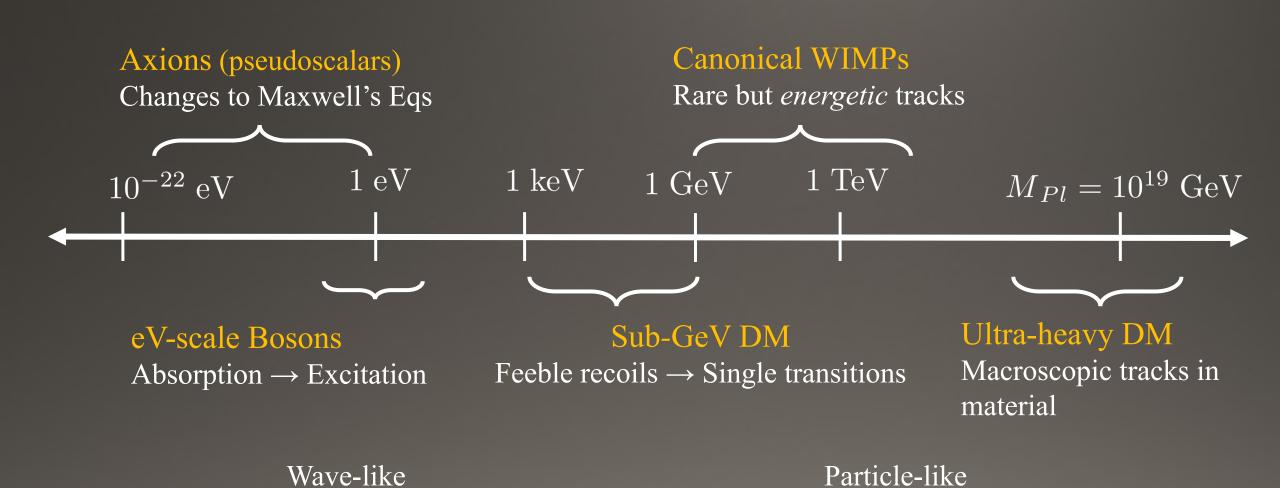
Coupling: How does it interact?

Production: How is it made?

Bring together particle physics, condensed matter, chemistry, and astrophysics.

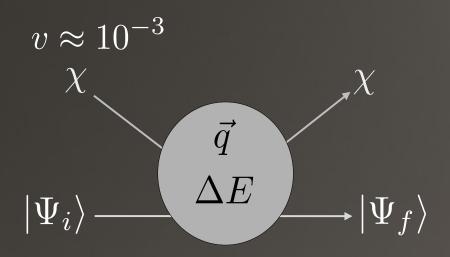
In brief, an *interdisciplinary* approach to the search for dark matter.

Mass Range & Detection Methods



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How to Read These Plots

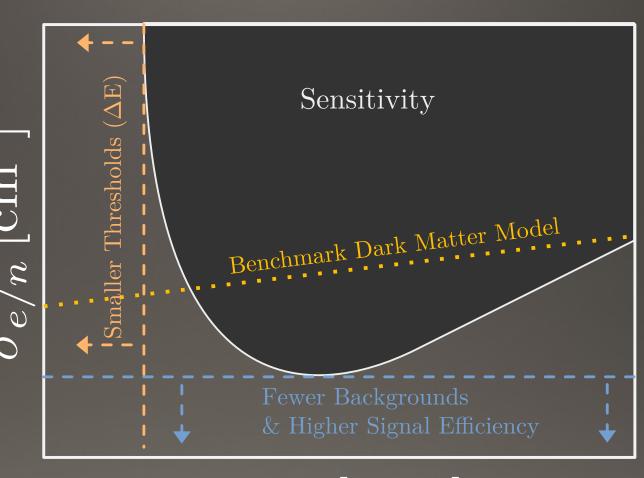


Energy Budget

$$\Delta E \sim v^2 m_{\chi} \approx \mathcal{O}(\text{eV}) \left(\frac{\text{m}_{\chi}}{1 \,\text{MeV}}\right)$$

Momentum Budget

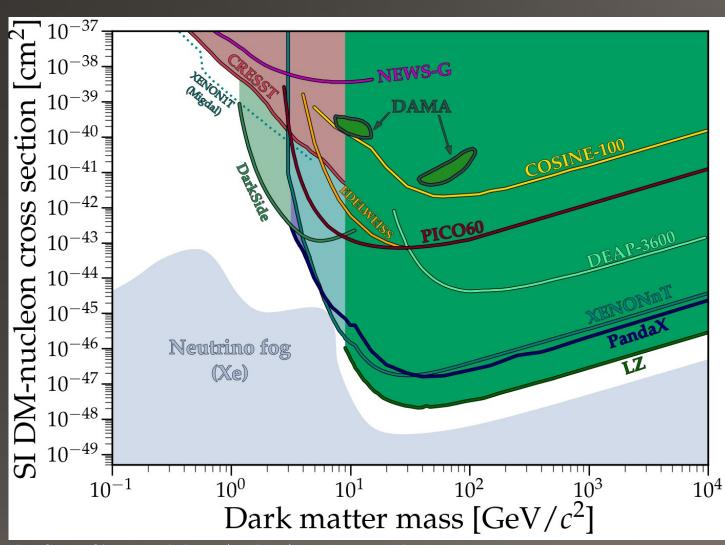
$$q \sim v m_{\chi} \approx \mathcal{O}(\text{keV}) \left(\frac{m_{\chi}}{1 \, \text{MeV}}\right)$$



 $m_{
m DM} \, [{
m MeV}]$

Nucleon Scattering

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Plot: Ciaran O'Hare - github.com/cajohare/DirectDetectionPlots See e.g. arXiv:2410.17036 & 2406.01705 and refs therein.

Noble Fluid-Based

- XENON Xe
- PandaX Xe
- LZ Xe
- DarkSide Ar
- DEAP-3600 Ar
- NEWS-G (gas) NH₄ (H, He, Ne)

Crystal Scintillator

- COSINE-100 NaI (Ti)
- $\overline{\text{CRESST}} \overline{\text{CaWO}_4}$

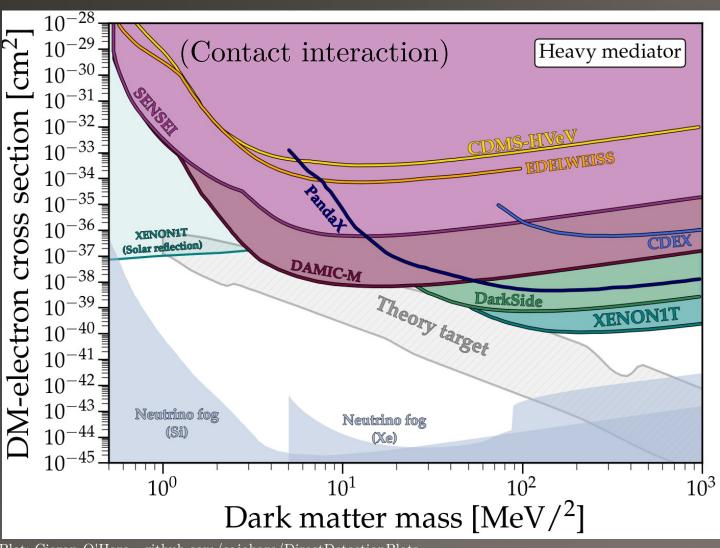
Bubble Chamber

• PICO60 - C₃F₈

Bolometers

- $CRESST CaWO_4$
- EDELWEISS Ge

Electron Recoil



Silicon CCDs

- SENSEI
- DAMIC-M

Liquid Noble Gases

- XENON Xe
- PandaX Xe
- Darkside Ar

Bolometers

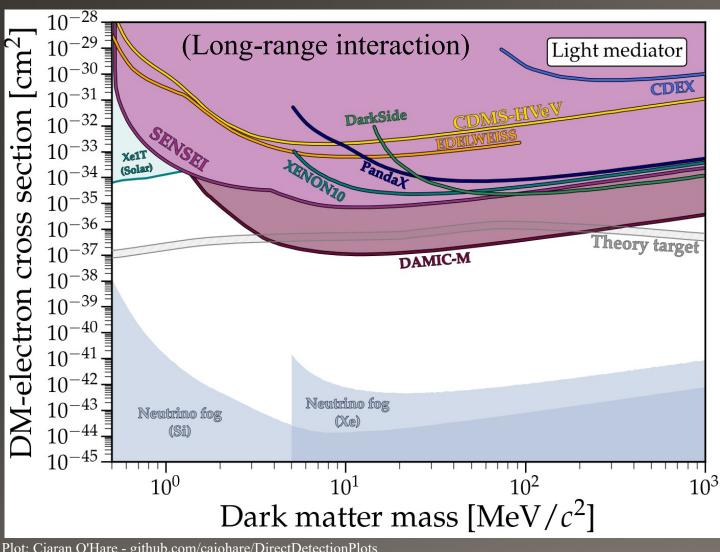
- EDELWEISS Ge
- CDMS Ge & Si

Theory Targets

ELDER/SIMP – Freeze out

Plot: Ciaran O'Hare - github.com/cajohare/DirectDetectionPlots See e.g. arXiv:2410.17036 & 2406.01705 and refs therein.

Electron Recoil



Silicon CCDs

- SENSEI
- DAMIC-M

Liquid Noble Gases

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Bolometers

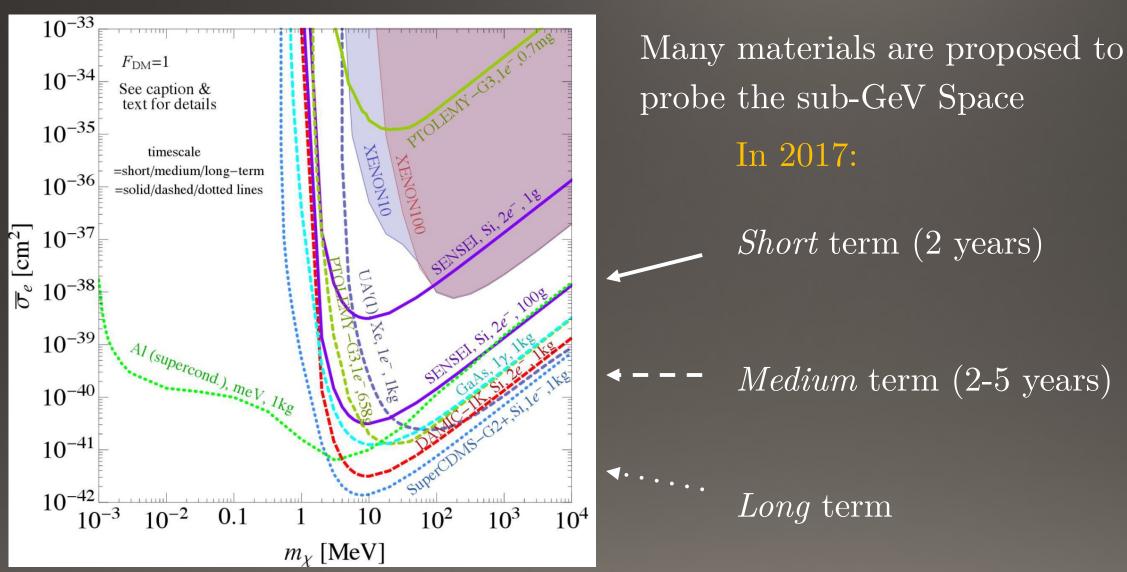
- EDELWEISS Ge
- CDMS Ge & Si

Theory Targets

• Freeze-in

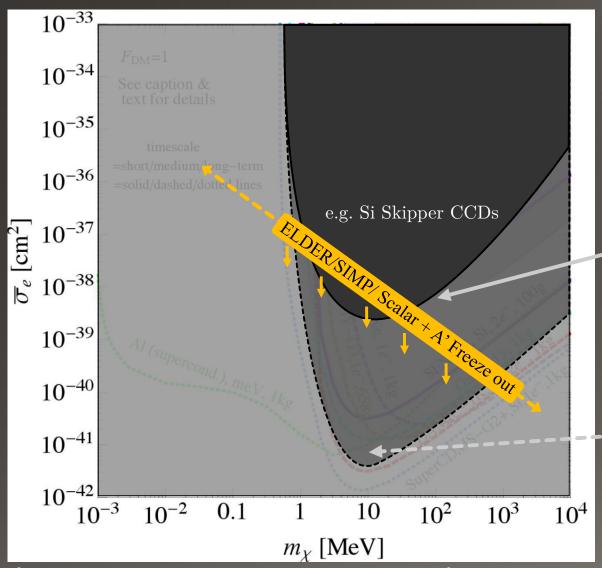
Plot: Ciaran O'Hare - github.com/cajohare/DirectDetectionPlots See e.g. arXiv:2410.17036 & 2406.01705 and refs therein.

The Field in Context



[US Cosmic Visions: Community Report: 1707.04591]

Outlook and Potential Reach



Present day:

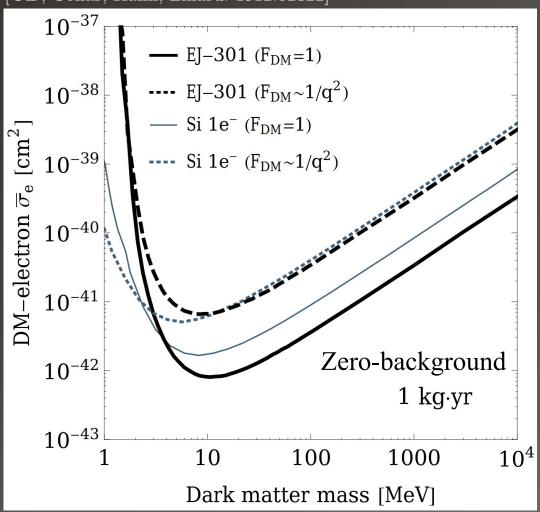
Experimental results

---- Zero-background projection

[US Cosmic Visions: Community Report: 1707.04591]

Outlook and Potential Reach

[CB, Collar, Kahn, Lillard: 1912.02822]



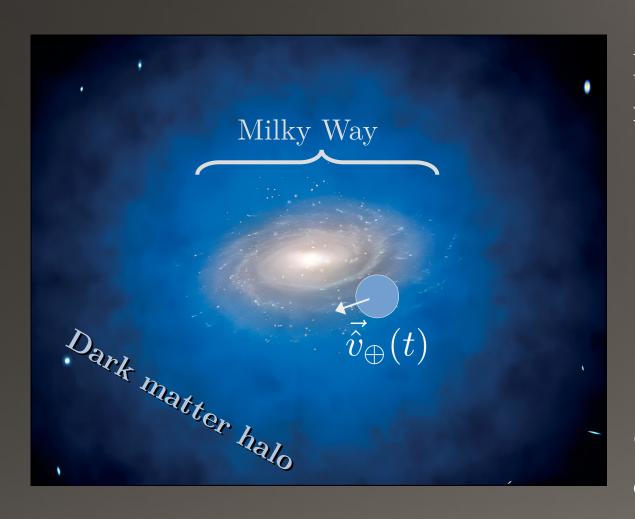
The obstacle

Backgrounds

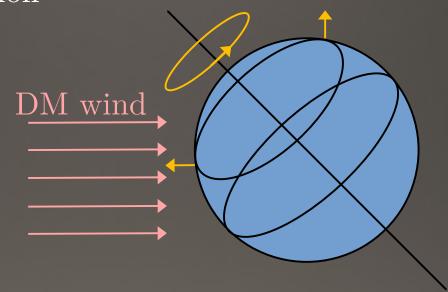
4 orders of magnitude of potential reach awaiting

Pound (kg) for Pound (kg) molecules produce about as much signal as e.g. Si.

Directional Detection

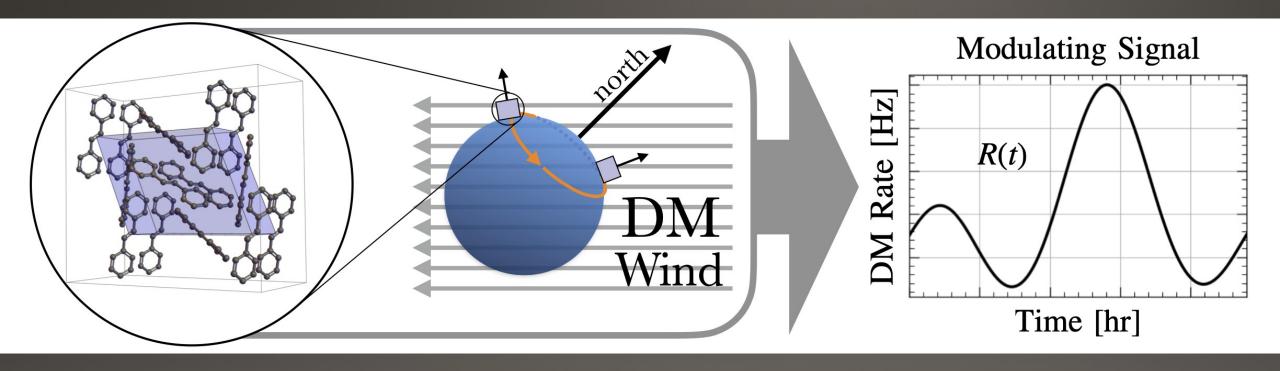


Effective dark matter "wind" from relative motion



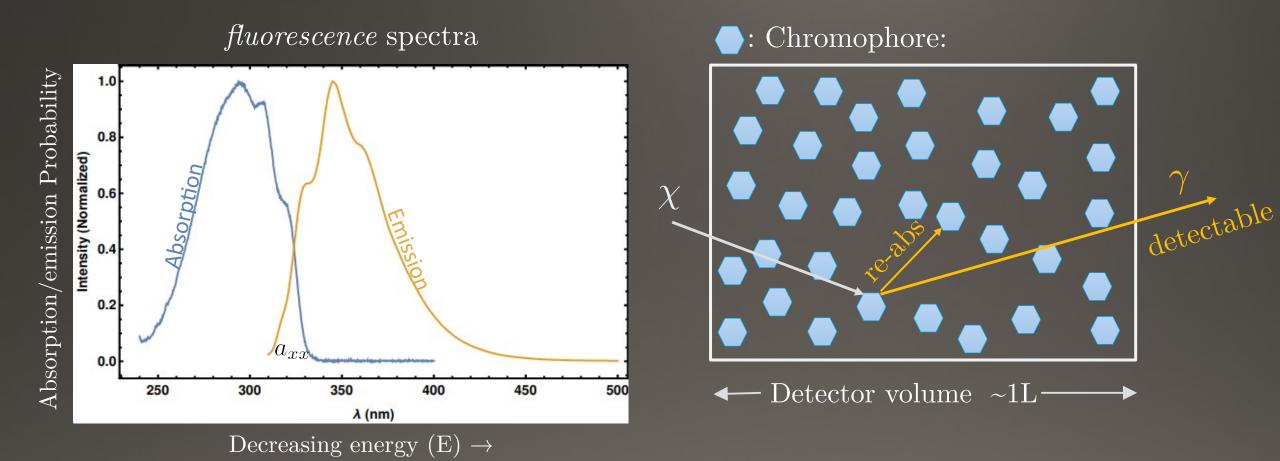
Change in relative orientation between detector and dark matter wind leads to daily modulation

Towards a Directional Detector



Modulating signals improve sensitivity by about two orders of magnitude and provide the potential for discovery.

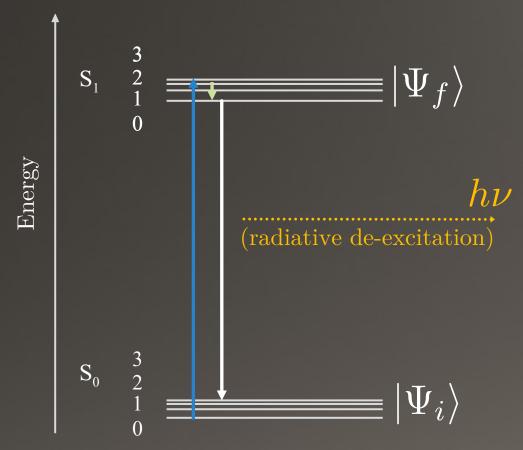
Fluorescence with DM



Probability for the photon to free stream

 $\Phi_{\mathrm{FB}} \sim (1 - a_{xx})$ e.g. molecular crystals: $\Phi_{\mathrm{FB}} \approx 65\%$

Luminescent Detectors



Fluorescence (Singlet transitions) Energy Scale

Momentum Scale

 $\overline{\Delta E \approx \mathcal{O}(\text{eV})} \quad q \sim \alpha m_e \approx \mathcal{O}(\text{keV})$

Disambiguation*

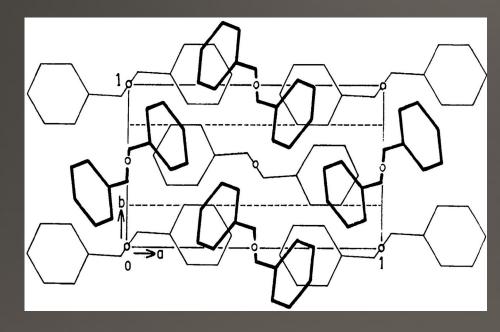
Scintillation: Initial O(keV) recoil produces *many* ionized/excited states. These quickly deexcite, producing photons.

Fluorescence: Initial O(eV) recoil produces single excited state, which deexcites, producing single photon.

First Target Candidate: Trans-Stilbene



De-localized and planar network of double bonds



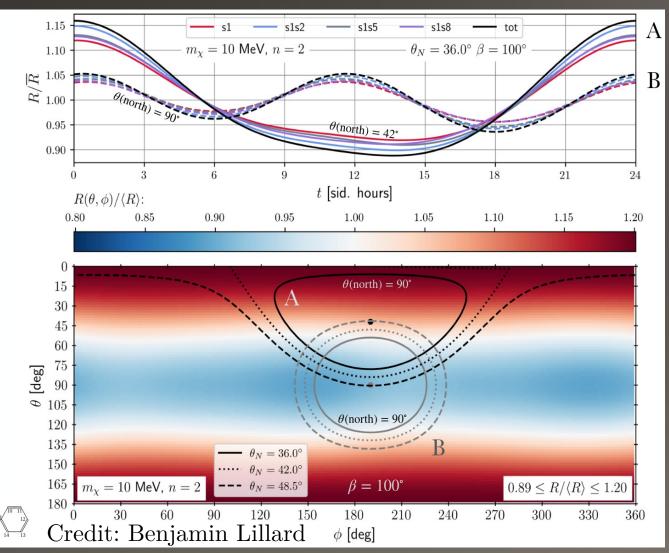
Molecular planes oriented in crystal lattice

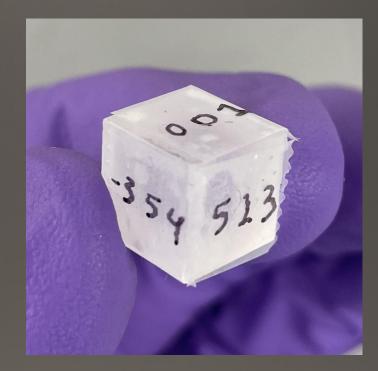


Carman, et.al. '18 (J. of Crystal Growth)

Large optical-quality crystals

Looking for Daily Modulation

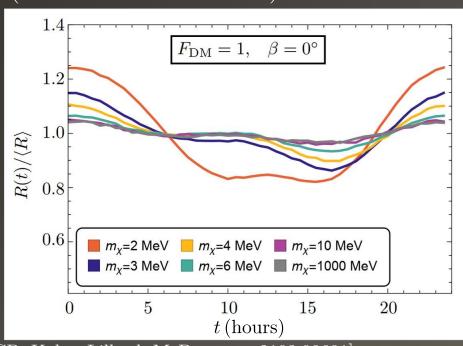




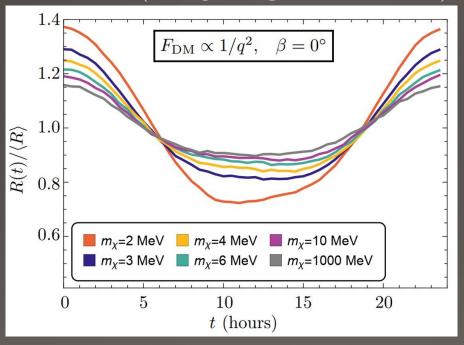
Different trajectories through rotation angles give distinct waveforms and phases. We can use the same crystal in many orientations or many different crystals.

Daily Modulation

(Contact interaction)



(Long-range interaction)



Predicted rate changes by up to 70% throughout the day.

That's a verifiable signal!

[CB, Kahn, Lillard, McDermott: 2103.08601]

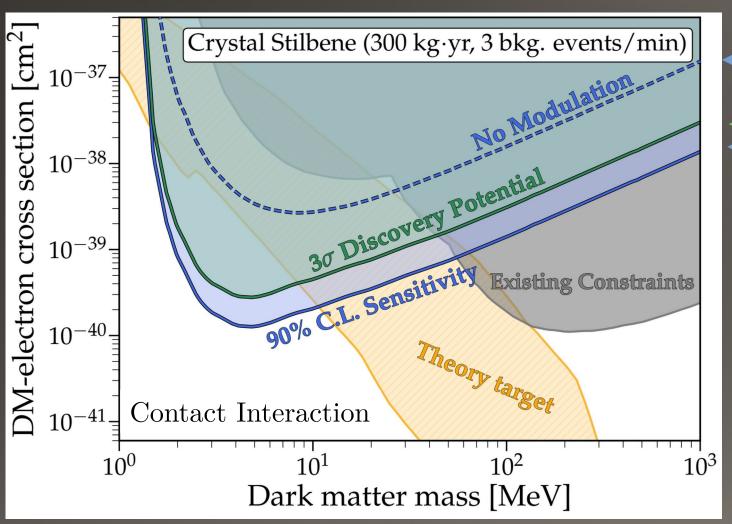
 $Modulation \ amplitude$ remains as high as 10% even at the highest masses. This is due

to the fundamental anisotropy of the molecular form factor.

$$f_{\rm RMS} = [5\% - 25\%]$$

$$N_{\sigma} \sim f_{\rm RMS} \sqrt{N_{\rm events}}$$

Sensitivity & Reach



Assuming realistic backgrounds

Exclusion w/o modulation

Discovery potential w/ mod.

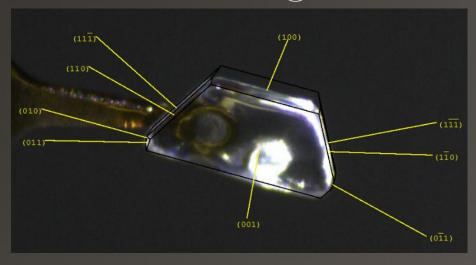
Exclusion w/ modulation

100 kg t-stilbene crystals exposed for 3 years w/ background rate of 3 events/min dominated by radioactivity of ¹⁴C, Th, & U.

Consistent w/ Borexino $^{14}\text{C}/^{12}\text{C} \sim 10^{-18}$ (0.01/min/kg)

Characterizing t-stilbene

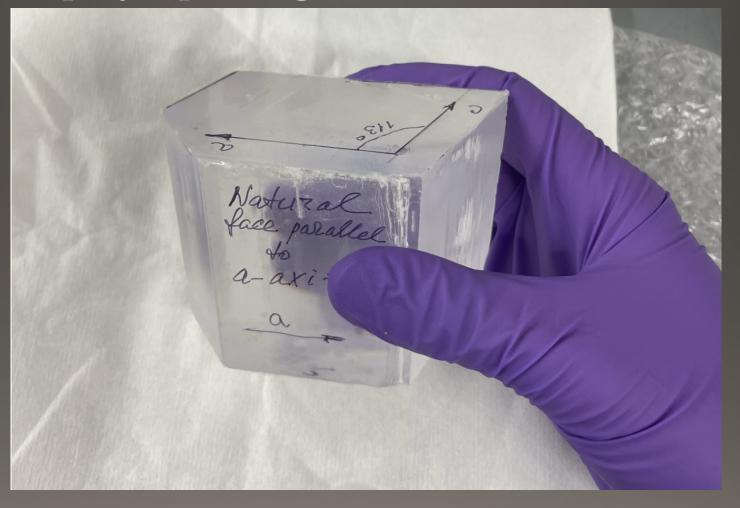
Calibrate at mg-scale



Credit: Dane Johnson Freedman Group (MIT)

Natalia Zaitseva (LLNL)

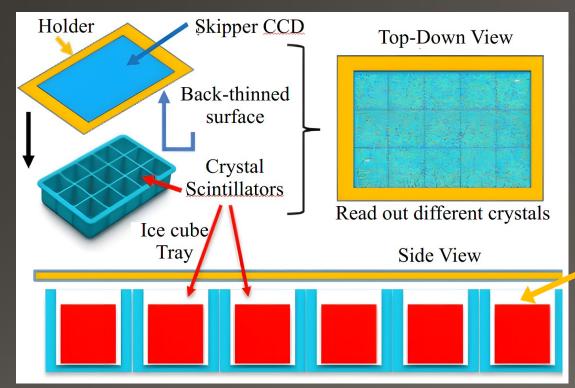
Deploy up to kg-scale



Experimental Deployment

DIANA*

Daily modulation in an Intrinsically ANisotropic Array





Many crystal samples can be read out by a single skipper CCD.

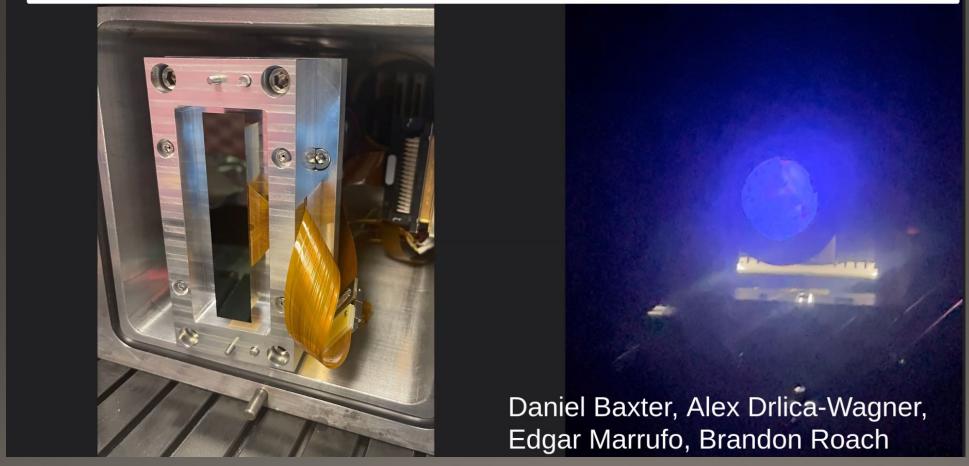
Collaboration: FermiLab, U. Toronto, MIT, UIUC, U. Oregon, Penn State

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Astro-Skipper CCDs as Readout

Taking pictures of crystals with a very sensitive CCD

Skipper CCD Measurements of trans-Stilbene



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Calibrating Anisotropic Materials

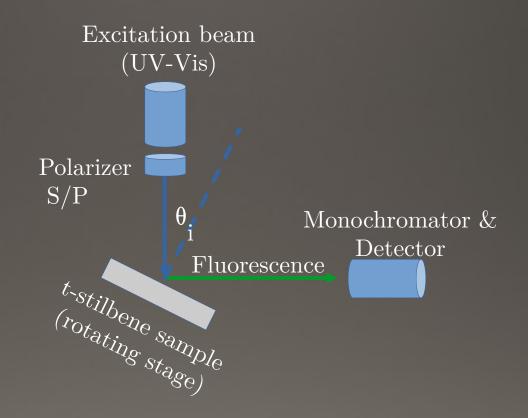
Calibration of Anisotropy via Photoabsorption

Dark Matter Scattering

$$\sigma_e \sim |f_{i \to f}(\mathbf{q})|^2 = |\langle \Psi_f | e^{i\mathbf{q} \cdot \mathbf{r}} | \Psi_i \rangle|^2$$
$$\approx |\mathbf{q} \cdot \langle \Psi_f | \mathbf{r} | \Psi_i \rangle|^2$$

UV photon absorption

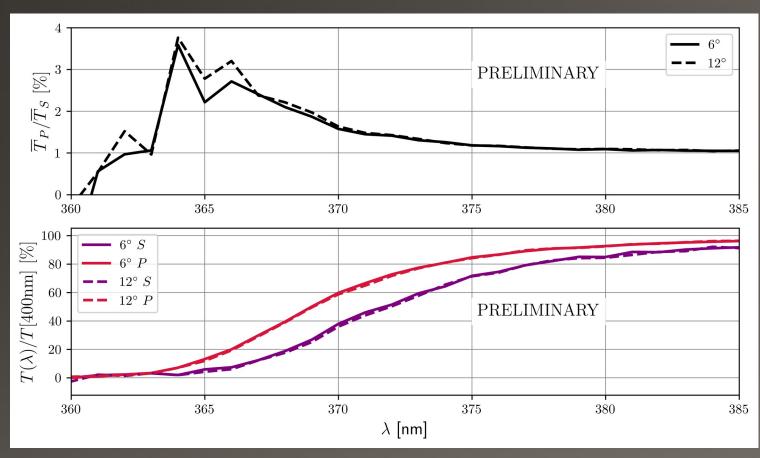
$$\sigma_{\text{UV}} \sim (E_f - E_i) |\vec{\epsilon} \cdot \langle \Psi_f | \mathbf{r} | \Psi_i \rangle|^2$$
$$A_{\text{UV}} \sim n_{\text{mol}} |\vec{\epsilon} \cdot \langle \Psi_f | \mathbf{r} | \Psi_i \rangle|^2$$

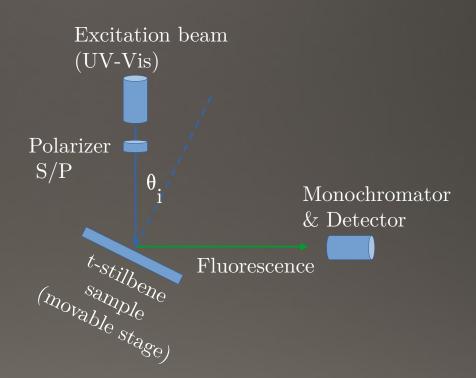


Polarization is a proxy for DM momentum

Calibrating Anisotropic Materials

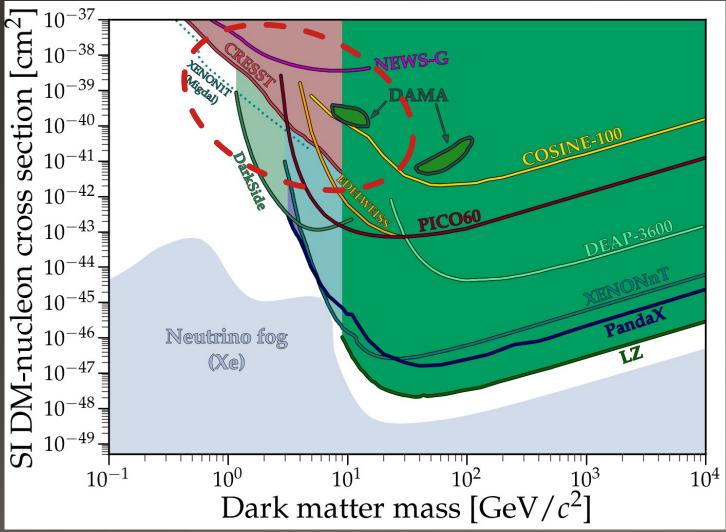
Calibration of Anisotropy via Photoabsorption





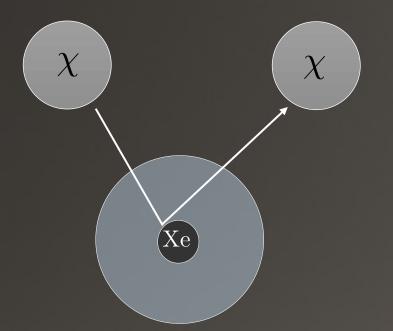
$$A_{\rm UV} \sim n_{\rm mol} |\vec{\epsilon} \cdot \langle \Psi_f | \mathbf{r} | \Psi_i \rangle |^2 \qquad A_{\rm UV} \approx {\rm const} - T_{\rm UV}$$

Migdal in Liquid Noble Gas TPCs

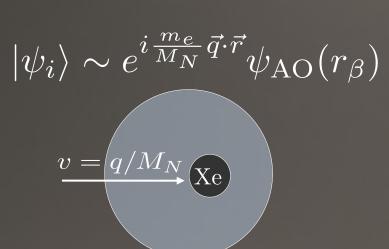


Migdal effect extends sensitivity from few GeV to 100s MeV

Ionizing Atoms Through Nuclear Recoil: Migdal Effect

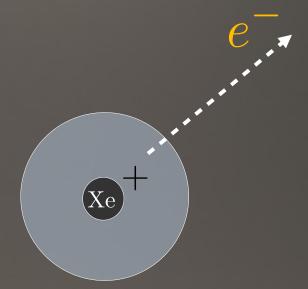


Initial nuclear recoil



Nucleus moves faster than <u>electrons</u>

The Migdal effect Inelasticity extends the sensitivity of detectors to lower masses. e.g. Xenon



Electronic transition to ionized state

$$f_{i \to f} pprox \frac{m_e}{M_N} \vec{q} \cdot \langle \vec{r} \rangle_{i \to f}$$

Kinematic penalty

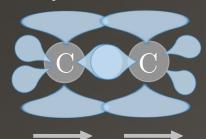
The Molecular Migdal Effect(s)

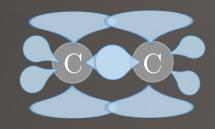
Sensitivity to Nuclear Recoils

Center of mass recoil (CMR) effect

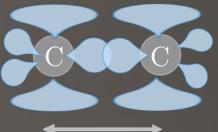
Non-adiabatic coupling (NAC) effect

Caused by center of mass motion





Caused by relative motion



COM recoil effect is the molecular analog of the semiclassical Migdal effect

$$P_{CMR} \sim \frac{m_e}{M_{mol}}$$

Suppressed by kinematic factor due to moving the whole molecule.

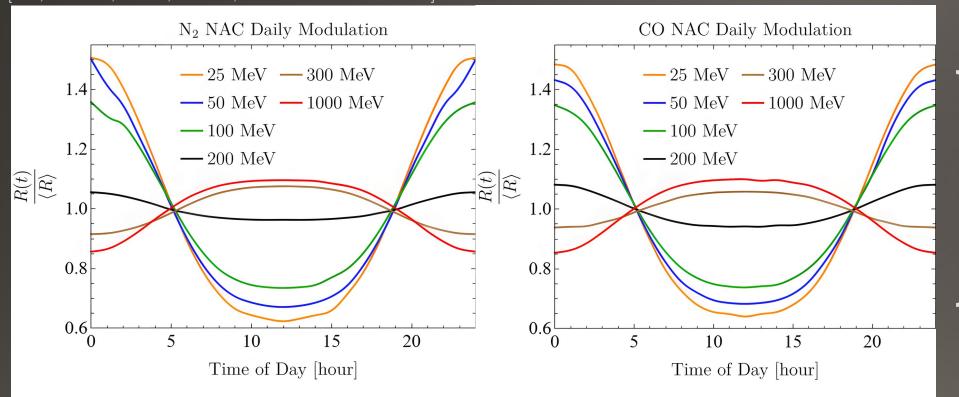
NAC caused by effects beyond Born-Oppenheimer approximation

$$P_{NAC} \sim \frac{m_e}{M_N}$$

Suppressed by kinematic factor due to moving a single atom.

Directional Molecular Migdal Effect

[CB, Harris*, Kahn, Lillard, Perez-Rios: 2208.09002]



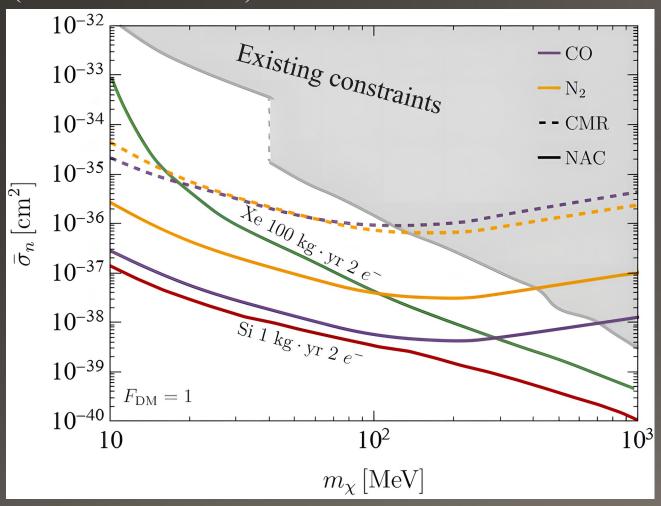
Predicted rate changes by up to 80% throughout the day.

The daily modulation phase is mass dependent.

Persistent daily modulation at large dark matter mass is generically predicted for the molecular Migdal effects. We predict that the same class of molecules that make good directional detectors for electron scattering will also be ideal for nuclear scattering because of the directional molecular Migdal effects.

The Molecular Migdal Effect(s) Sensitivity to Nuclear Recoils

(Contact interaction)



Si rate is entirely due to non-adiabatic processes. [Esposito & Rocchi 2505.08864]

Center of mass recoil effect is predicted to be subdominant at all masses.

Non-adiabatic coupling effect is predicted to dominate due to favorable kinematic factor.

Simplest molecular models already competitive.

Is there an optimal molecular target?

[CB, Harris*, Kahn, Lillard, Perez-Rios: 2208.09002]

Anisotropic Materials

2D Targets: e.g. graphene, MoS₂, GaN

Polar Semiconductors: e.g. Al₂O₃, SiC, GaAs, Eu₅In₂Sb₆ (SPLENDOR)

Dirac materials: e.g. ZrTe5

Crystalline scintillators:

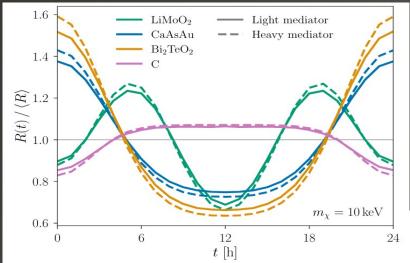
e.g. NaI, GaAs, trans-stilbene, etc...

Need

- > 10 kg low-background material
- Large anisotropy, $N_{\sigma} \sim f_{RMS} (m T_{exp})^{1/2}$

- Y. Hochberg, Y. Kahn, M. Lisanti, C. G. Tully and K. M. Zurek, *Directional detection of dark matter with two-dimensional targets*, Phys. Lett. B **772**, 239 (2017), doi:10.1016/j.physletb.2017.06.051, 1606.08849.
- R. Budnik, O. Chesnovsky, O. Slone and T. Volansky, *Direct Detection of Light Dark Matter and Solar Neutrinos via Color Center Production in Crystals*, Phys. Lett. B **782**, 242 (2018), doi:10.1016/j.physletb.2018.04.063, 1705.03016.
- Y. Hochberg, Y. Kahn, M. Lisanti, K. M. Zurek, A. G. Grushin, R. Ilan, S. M. Griffin, Z.-F. Liu, S. F. Weber and J. B. Neaton, *Detection of sub-MeV Dark Matter with Three-Dimensional Dirac Materials*, Phys. Rev. D **97**(1), 015004 (2018), doi:10.1103/PhysRevD.97.015004, 1708.08929.
- S. Griffin, S. Knapen, T. Lin and K. M. Zurek, Directional Detection of Light Dark Matter with Polar Materials, Phys. Rev. D 98(11), 115034 (2018), doi:10.1103/PhysRevD.98.115034, 1807.10291.
- A. Coskuner, A. Mitridate, A. Olivares and K. M. Zurek, *Directional Dark Matter Detection in Anisotropic Dirac Materials*, Phys. Rev. D **103**(1), 016006 (2021), doi:10.1103/PhysRevD.103.016006, 1909.09170.
- R. M. Geilhufe, F. Kahlhoefer and M. W. Winkler, *Dirac Materials for Sub-MeV Dark Matter Detection: New Targets and Improved Formalism*, Phys. Rev. D **101**(5), 055005 (2020), doi:10.1103/PhysRevD.101.055005, 1910.02091.
- S. M. Griffin, Y. Hochberg, K. Inzani, N. Kurinsky, T. Lin and T. Chin, Silicon carbide detectors for sub-GeV dark matter, Phys. Rev. D **103**(7), 075002 (2021), doi:10.1103/PhysRevD.103.075002, 2008.08560.
- C. Blanco, Y. Kahn, B. Lillard and S. D. McDermott, Dark Matter Daily Modulation With Anisotropic Organic Crystals, Phys. Rev. D 104, 036011 (2021), doi:10.1103/PhysRevD.104.036011, 2103.08601.
- A. Coskuner, T. Trickle, Z. Zhang and K. M. Zurek, *Directional detectability of dark matter with single phonon excitations: Target comparison*, Phys. Rev. D **105**(1), 015010 (2022), doi:10.1103/PhysRevD.105.015010, 2102.09567.
- C. Blanco, I. Harris, Y. Kahn, B. Lillard and J. Pérez-Ríos, Molecular Migdal effect, Phys. Rev. D 106(11), 115015 (2022), doi:10.1103/PhysRevD.106.115015, 2208.09002.
- C. Boyd, Y. Hochberg, Y. Kahn, E. D. Kramer, N. Kurinsky, B. V. Lehmann and T. C. Yu, Directional detection of dark matter with anisotropic response functions (2022), 2212. 04505.

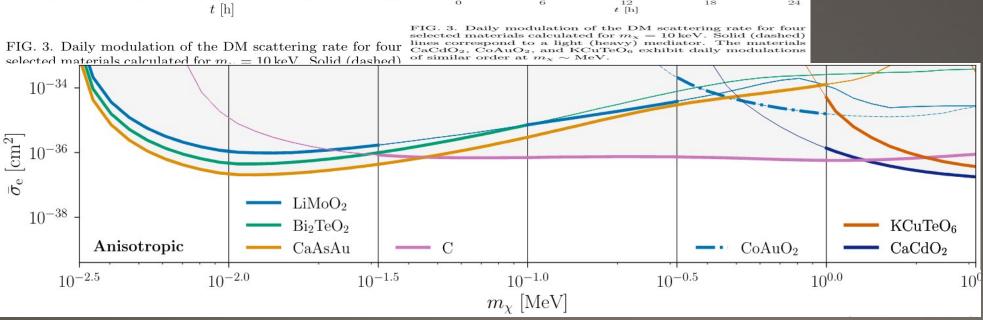
Mining for Crystals in Data



First High-throughput evaluation of Dark Matter Detector Materials

- Used the crystals in the Material Project
- Fit anisotropic Lindhard dielectric tensor to data

 $m_{\chi} = 10 \, \text{keV}$



0.6

[Griffin, Hochberg, Lehmann, Ovadia, Persson, Suter, Yang: 2506.19905]

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Inventing Optimal Targets

Problem: Chemical space is unreasonably large

How many molecules possible with C, O, N, F, H?

< 9 atoms: 100s of Thousands (DFT Computable)

< 30 atoms: 100s of Billions (Intractable)

...toluene has 15, xylene has 18, t-stilbene has 26

Method

- 1. Look for known favorable properties cheminformatics
- 2. Extra(intra)polate onto new molecules machine learning

ML for DM Direct Detection

Property prediction Molecular Generation

Energies & Matrix elements Sample latent space \rightarrow new molecules

Small molecules: Using exhaustive database (< 9 atoms)

Characterize neural nets

→ Prove it's possible to learn from small subsample

Large molecules: Sparse dataset up to 10s of atoms

Scale architecture

 \rightarrow Generate candidate molecule shortlist

Dataset of energies + oscillator strengths:

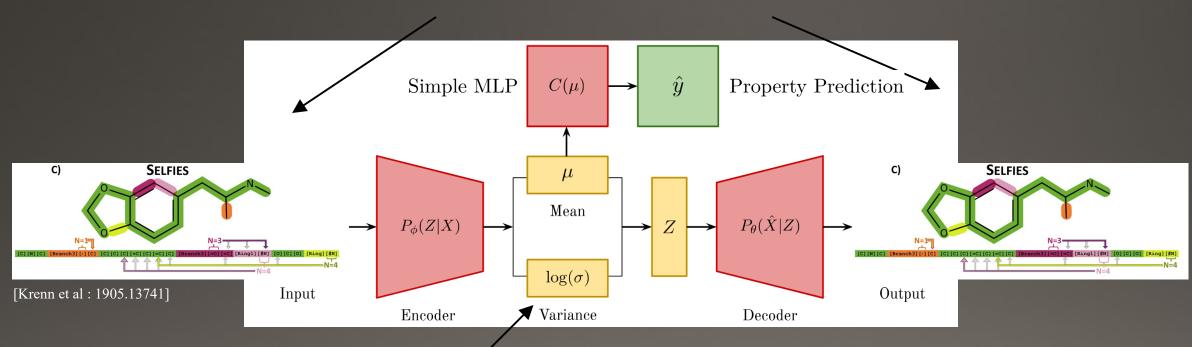
PubChemQ: a dataset of ~3M organic molecules computed using Density Functional Theory

ML for DM Direct Detection

Molecular Space: Variational Autoencoders

Energies & Matrix elements Sample latent space \rightarrow new molecules

Discrete and sparse data space



Continuous latent space

Regression Model in Action

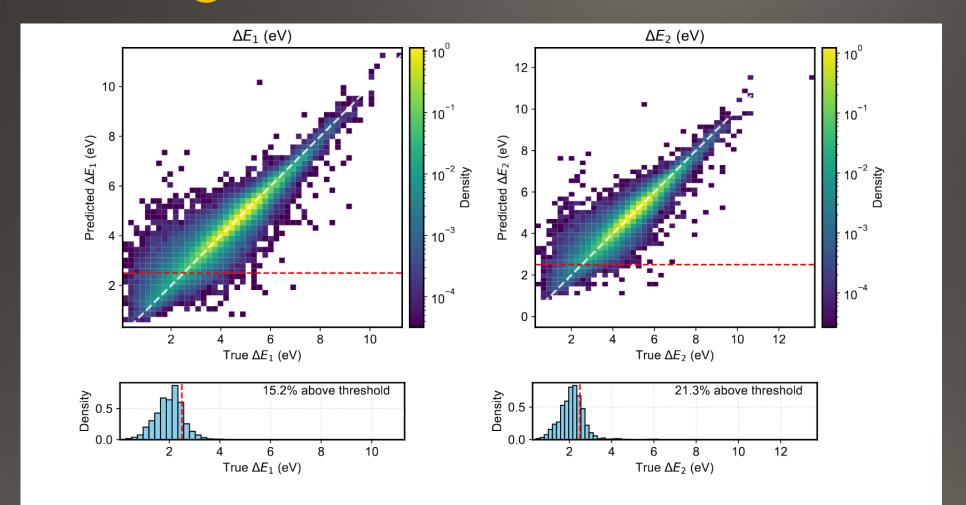


FIG. 19. We show the density of predicted values vs true values for the transition energies of our validation set, with a red line that corresponds to the threshold placed on the predicted value. The Histogram shows the distribution of true values for molecules whose predicted value is above the threshold.

Regression Model in Action

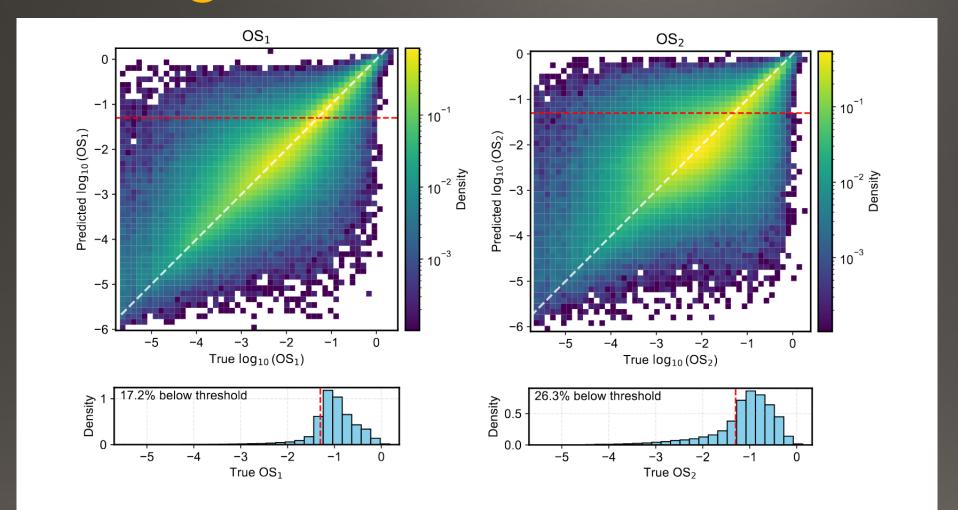


FIG. 20. We show the density of predicted values vs true values for the oscillator strength of our validation set, with a red line that corresponds to the threshold placed on the predicted value. The Histogram shows the distribution of true values for molecules whose predicted value is above the threshold.

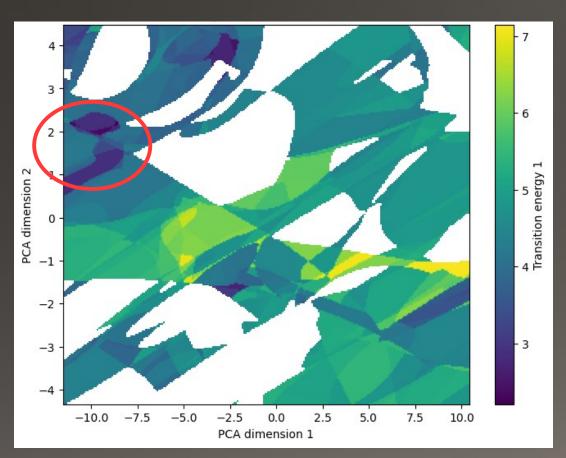
ML for DM Direct Detection

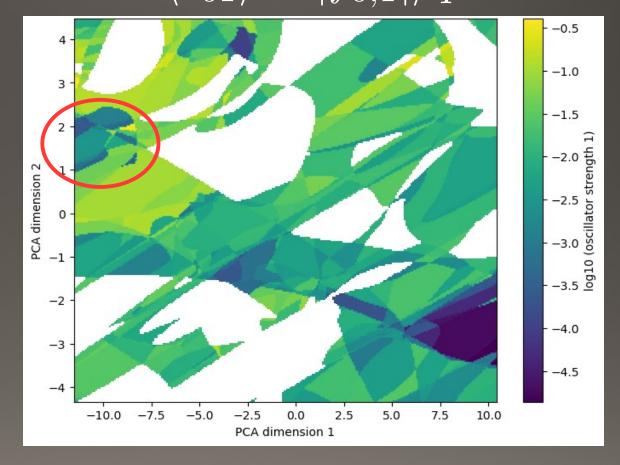
Property prediction: Molecular latent space

Energies & Matrix elements Sample latent space \rightarrow new molecules

 ΔE

 $\langle r_{01}
angle pprox |f_{0,1}|/q$





[C. Cook*, CB, J. Smirnov 2501.00091]

Clustering: Results

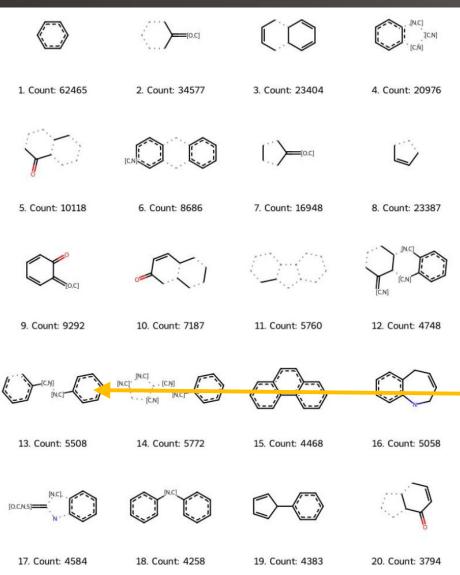
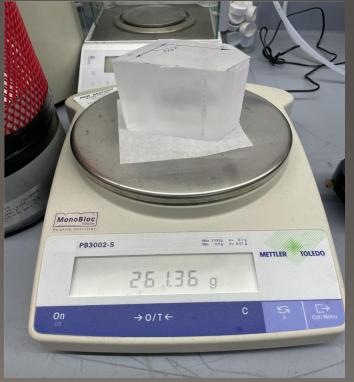


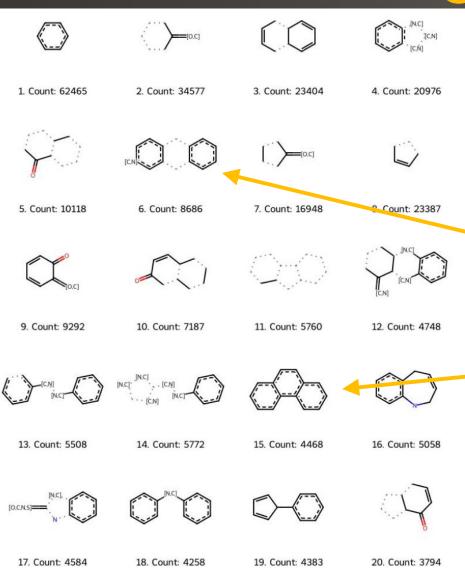
FIG. 7. Top 20 molecular motifs ranked by score. Note that certain atoms can be substituted by [C,N,O,S] as long as the structure remains isoelectronic. Dotted lines indicate that the bond could be a double or single bond.

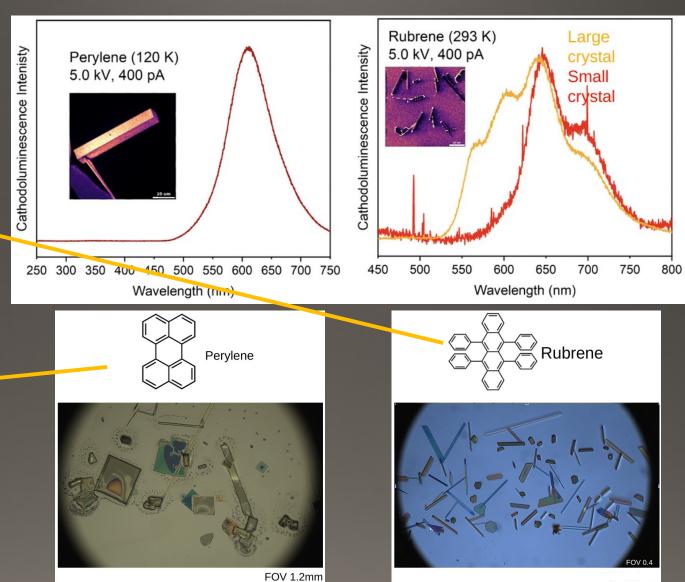
[C. Cook*, CB, J. Smirnov 2501.00091]

In hindsight, we're probably on the right track.



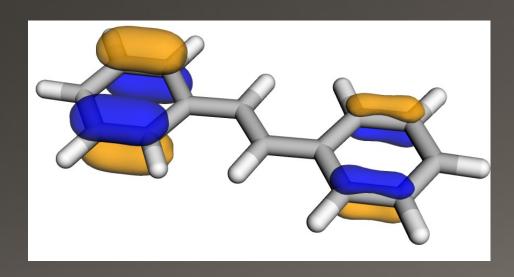
New Target Materials

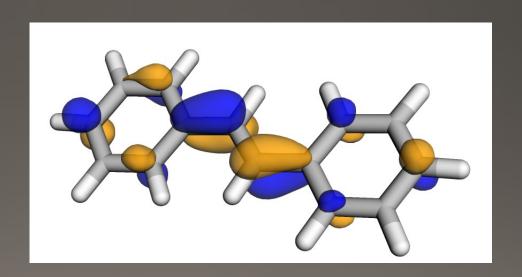




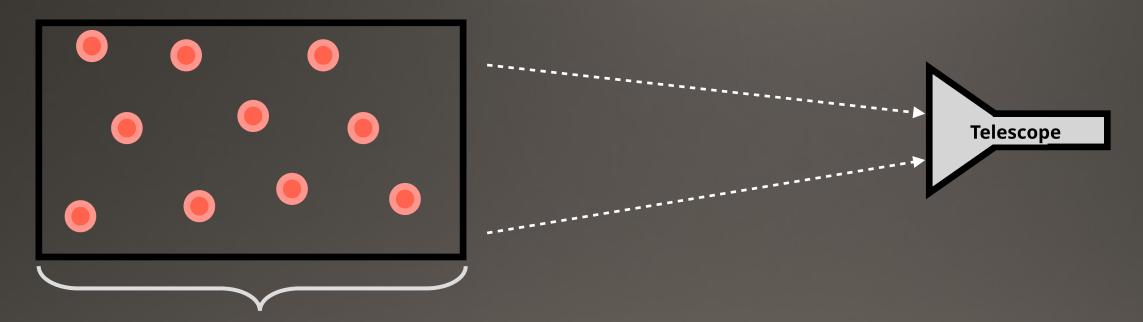
Conclusions

- 1) We have done an extremely effective job looking for WIMPs above a GeV, now we must look beyond.
- 2) By developing the formalism that describes the interaction between dark matter and molecules, we can develop detection strategies capable of *delving deep* and *searching wide* across the dark matter parameter space.





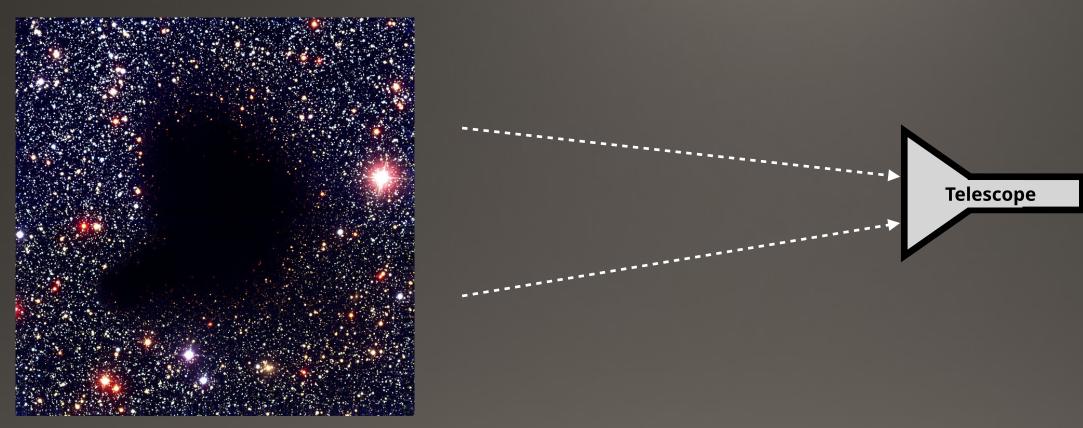
Beyond direct detection



Astrophysical volume of molecules

We can use the same theoretical techniques that we've developed to predict rates in *astrophysical* objects

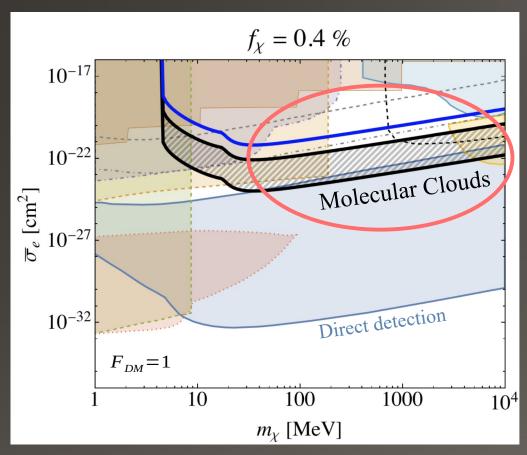
Beyond direct detection



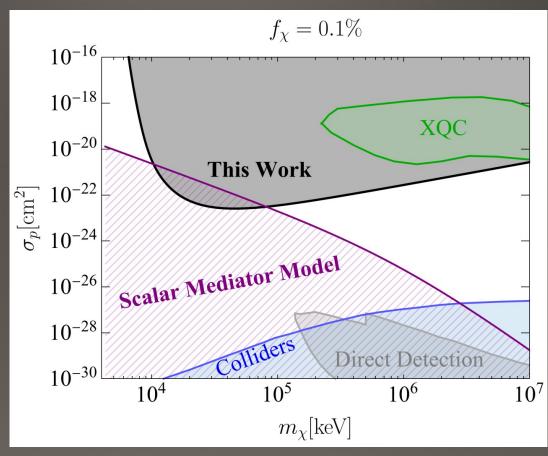
Cold molecular cloud

We can use the same theoretical techniques that I've developed to predict rates in detectors to predict rates in astrophysical objects

Dark matter in Molecular Clouds



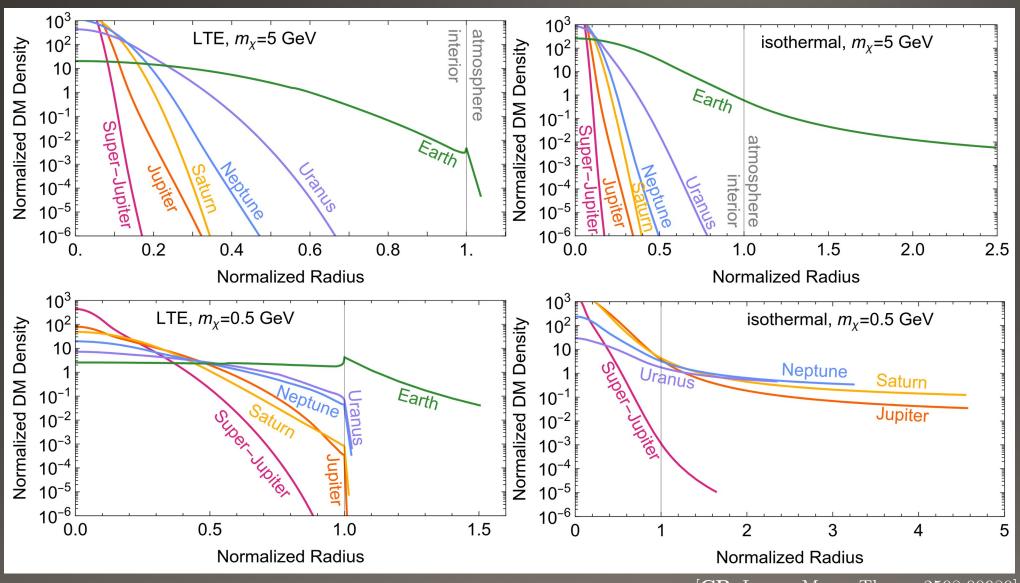
[Prabhu, **CB** : 2211.05787]



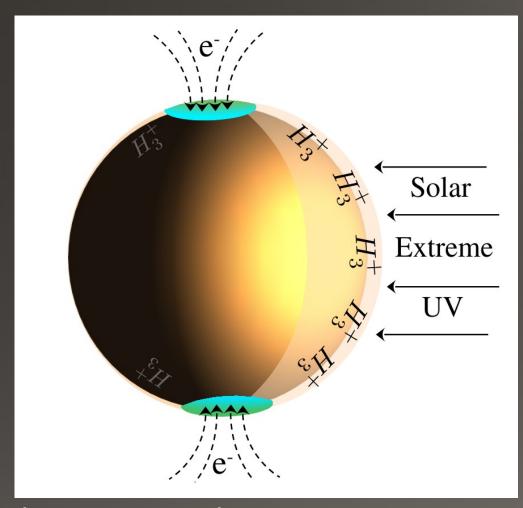
[CB, Harris, Kahn, Prabhu: 2310.00740]

The Molecular Migdal effect(s) in space

Equilibrium Radial Distribution of Captured DM



IR Signatures of DM in Jupiter



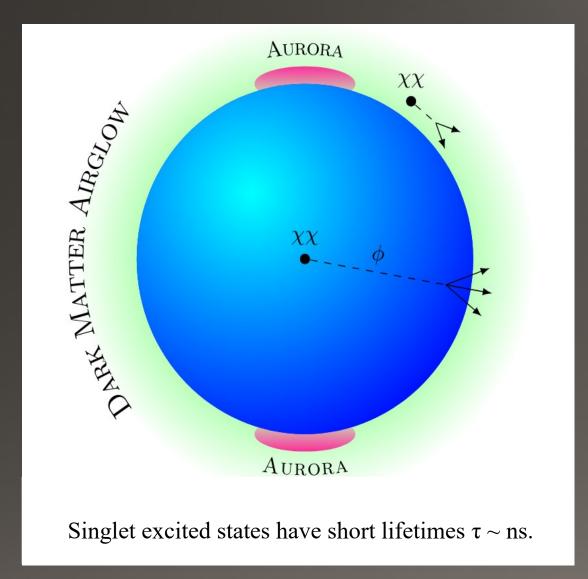
[**CB**, Leane: 2312.06758]

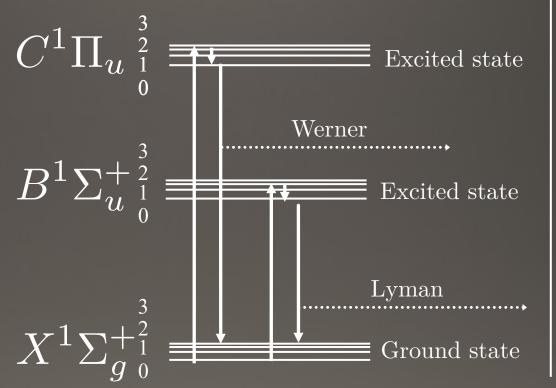
Ionizing radiation generates H₃⁺

H₃⁺ radiates extremely efficiently in the IR. It is Jupiter's *thermostat*.

DM Annihilation \rightarrow Brighter H_3^+ emission

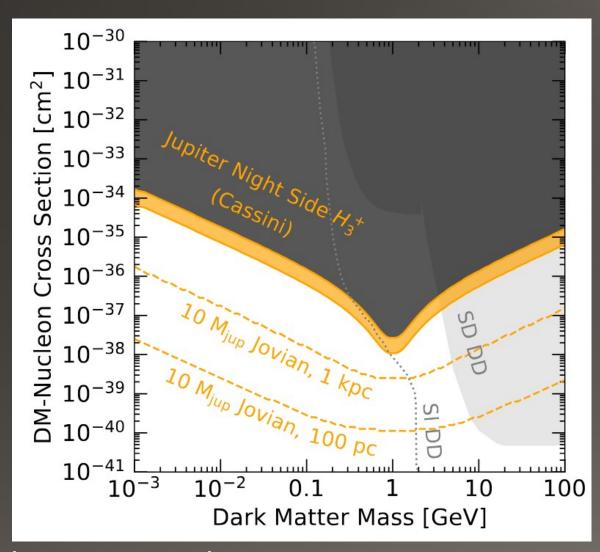
UV Signatures of DM in Jupiter

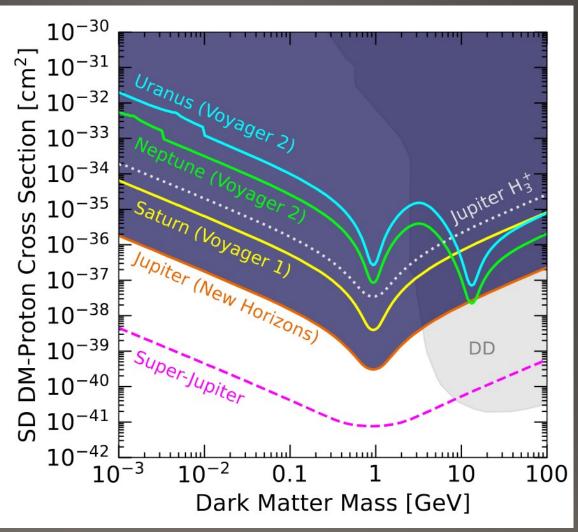




Radiative UV deexcitation (Following collisional excitation)

Constraints from IR and UV Molecular Emission





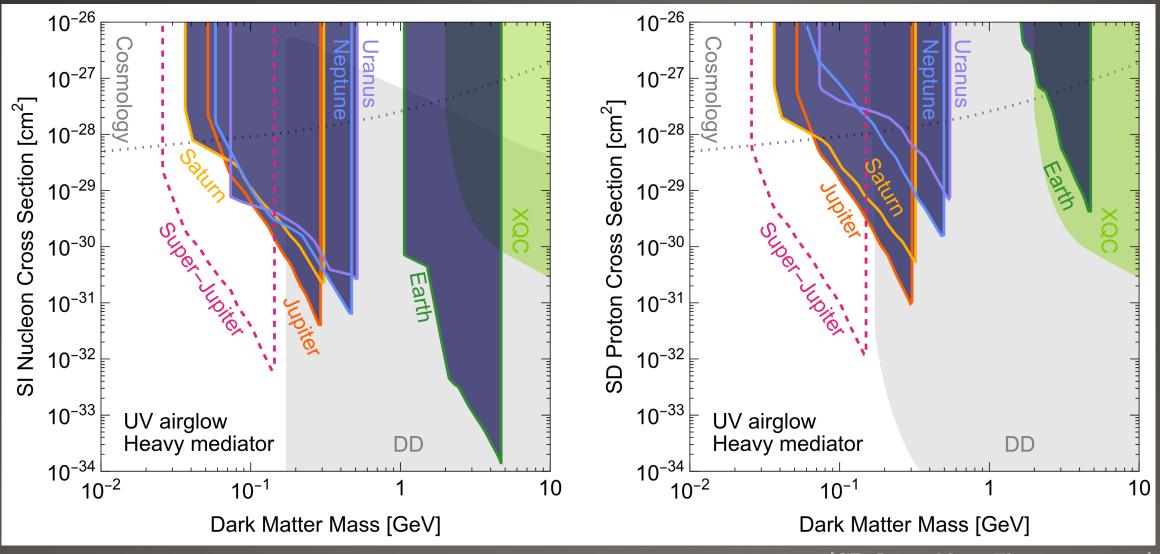
[CB, Leane: 2312.06758]

[CB, Leane, Moore, Thong: 2408.15318]

H₃⁺ IR night-side emission

UV night-side airglow

Evaporation



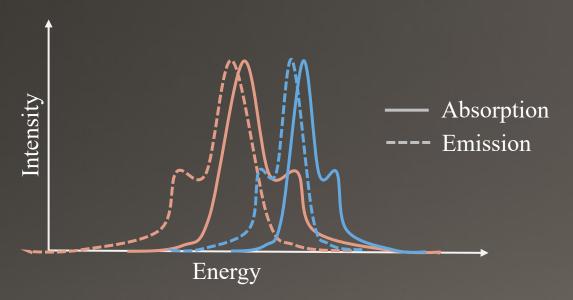
[CB, Leane, Moore, Thong: 2508.00980]

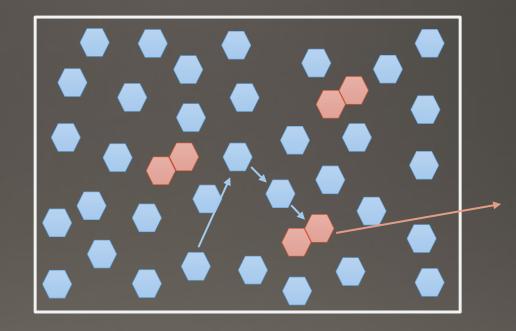
arXiv: 1912.02822

Fluorescence: Binary Scintillators

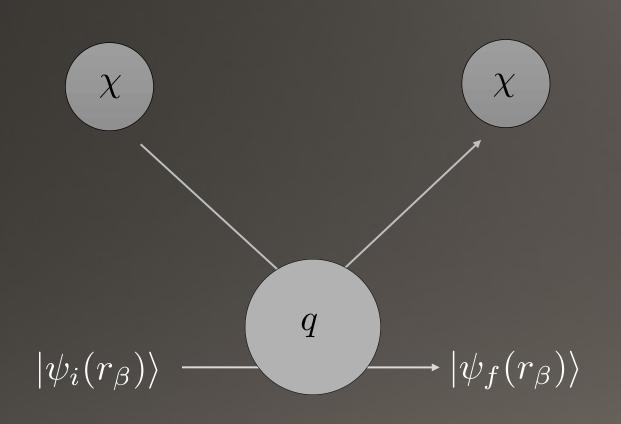
Solvent: Primary target starts the signal

Solute: Dilute fluor gets the signal out of the bulk





Electron Recoil: Charge Signal



Electron scattering
$$\Delta E_r = (m_\chi^2/m_\mathrm{T}) \times 10^{-6}$$

$$\Delta E \sim \mathcal{O}(\text{few eV}) \left(\frac{m_{\chi}}{1 \,\text{MeV}}\right)^2$$

What has such transition energies?

- Semiconductor band gaps
- Maybe atomic ionization

Electrons in crystals (exciton generation)

$$|\psi_i\rangle \sim u_v(r)e^{ik'\cdot r}$$

$$|\psi_f\rangle \sim u_c(r)e^{ik\cdot r}$$

Electrons in atoms (ionization)

$$|\psi_i\rangle \sim \psi_{\rm STO}(r_\beta) \ |\psi_f\rangle \sim e^{ik\cdot r}, \ r \gg a_0$$

Trans-Stilbene

s	Platt Symbol	Symmetry	$\Delta E \left[\mathrm{eV} \right]$	Configuration amplitudes			
$\overline{s_1}$	^{1}B	B_u	4.240	$d_{7,8} = 0.94,$	$d_{4,11} = -0.24$		
s_2	G^-	B_u	4.788	$d_{7,10} = 0.53,$	$d_{5,8} = 0.53,$	$d_{6,11} = 0.37,$	$d_{4,9} = -0.37$
s_3	G^-	A_g	4.800	$d_{7,9} = 0.53,$	$d_{6,8} = 0.53,$	$d_{5,11} = 0.37,$	$d_{4,10} = -0.37$
s_4	$(C, H)^+$	A_g	5.137	$d_{7,11} = 0.41,$	$d_{5,9} = -0.41,$	$d_{6,10} = -0.41,$	$d_{4,8} = -0.59$
s_5	$1H^+$	B_u	5.791	$d_{5,10} = 0.54,$	$d_{6,9} = 0.54,$	$d_{7,12} = 0.33,$	$d_{3,8} = 0.33$
s_6	G^+	A_g	6.264	$d_{7,9} = 0.68,$	$d_{6,8} = -0.68$		
s_7	C^-	A_g	6.013	$d_{7,11} = 0.66,$	$d_{4,8} = 0.54,$		
s_8	$1G^+$	B_u	6.439	$d_{7,10} = 0.65,$	$d_{5,8} = -0.65$		

Table 1: The first eight excited states $s_{n=1...8}$, with their energy eigenvalues

 $\Delta E(s_n)$ with respect to the ground state and coefficients $d_{ij}^{(n)}$ as calculated by

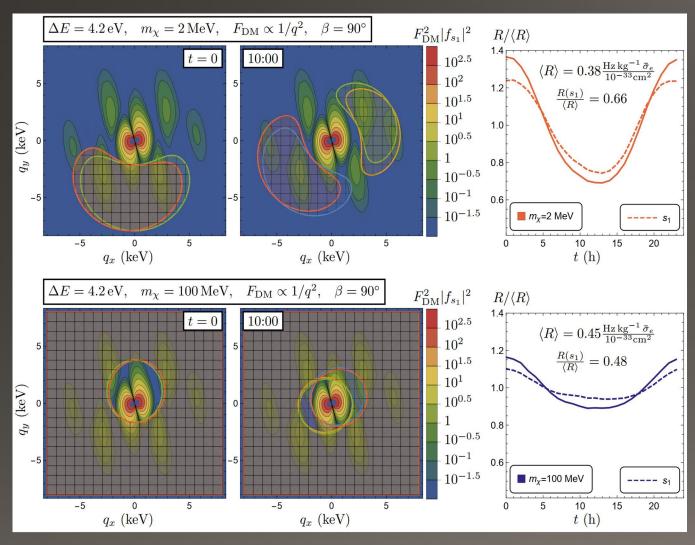
Ting and McClure.

$$|s_n\rangle = \sum_{i,j>i} d_{ij}^{(n)} |\psi_i^j\rangle,$$
$$\sum_{ij} |d_{ij}^{(n)}|^2 = 1.$$

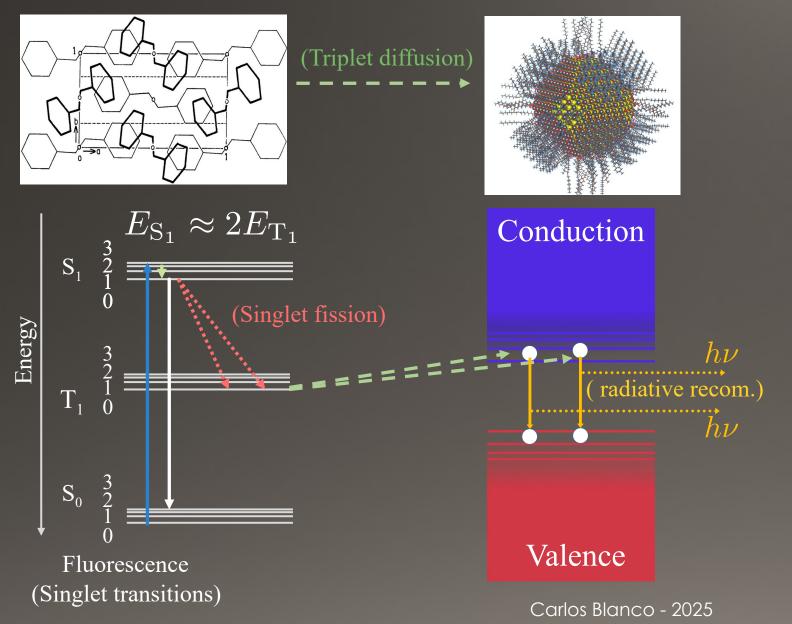
as calculated by
$$f_{g \to s_n}(\vec{q}) = \left\langle \psi_{s_n}(\vec{r}_1 \dots \vec{r}_{14}) \left| \sum_{m=1}^{14} e^{i\vec{q} \cdot \vec{r}_m} \right| \psi_G(\vec{r}_1 \dots \vec{r}_{14}) \right\rangle$$
$$= \sum_{ij} d_{ij}^{(n)} \left\langle \psi_i^j \left| e^{i\vec{q} \cdot \vec{r}} \right| \psi_G \right\rangle$$
$$= \sqrt{2} \sum_{ij} d_{ij}^{(n)} \left\langle \Psi_j(\vec{r}) | e^{i\vec{q} \cdot \vec{r}} | \Psi_i(\vec{r}) \right\rangle.$$

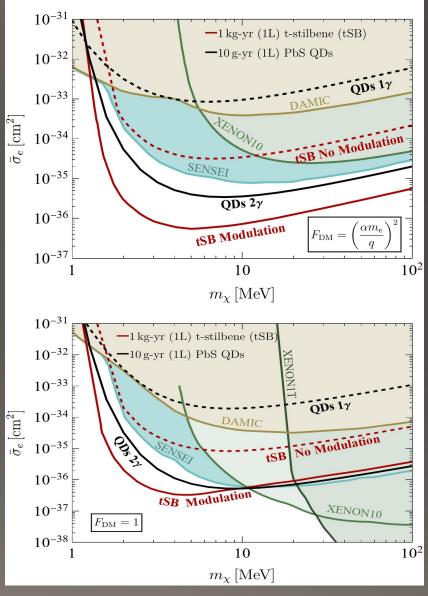
Carlos Blanco - 2025

Daily Modulation: Light Mediator

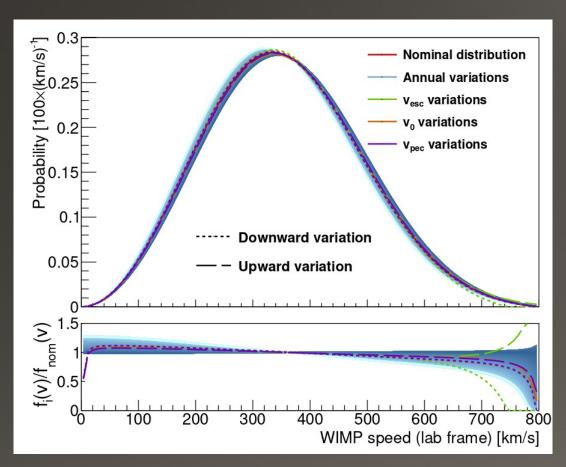


Same as previous figures (top) for a light mediator DM form factor $F_{\rm DM} = (\alpha m_e/q)^2$. Here, the contour plots show $F_{\rm DM}^2 |f(s_1)|^2$ which appears in the rate integrand; the scattering is dominated by the smallest kinematically-allowed q. **Top:** Molecular form factors with $q_z = 0$ and rate modulations for $m_\chi = 2$ MeV. **Bottom:** Molecular form factors with $q_z = 0$ and rate modulations for $m_\chi = 100$ MeV.

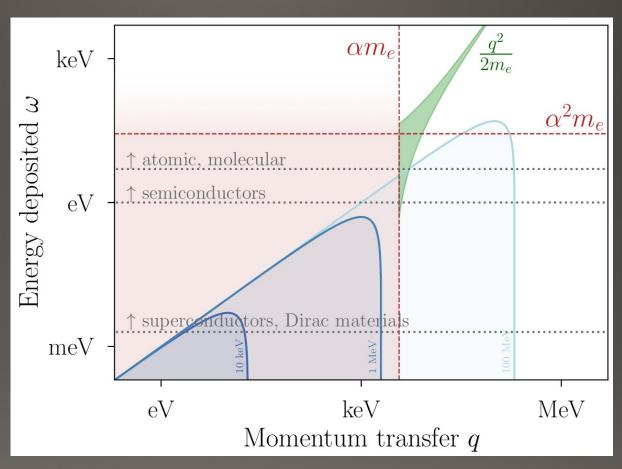




Local DM Phase Space



Baxter, D., et al. "Recommended conventions for reporting results from direct dark matter searches." The European Physical Journal C 81.10 (2021): 1-19.



Lin, Tongyan. "Sub-GeV dark matter models and direct detection." SciPost Physics Lecture Notes (2022): 043.