Brookhaven Forum 2025
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Lattice study of K → ππ decays

work with RBC & UKQCD Collaborations

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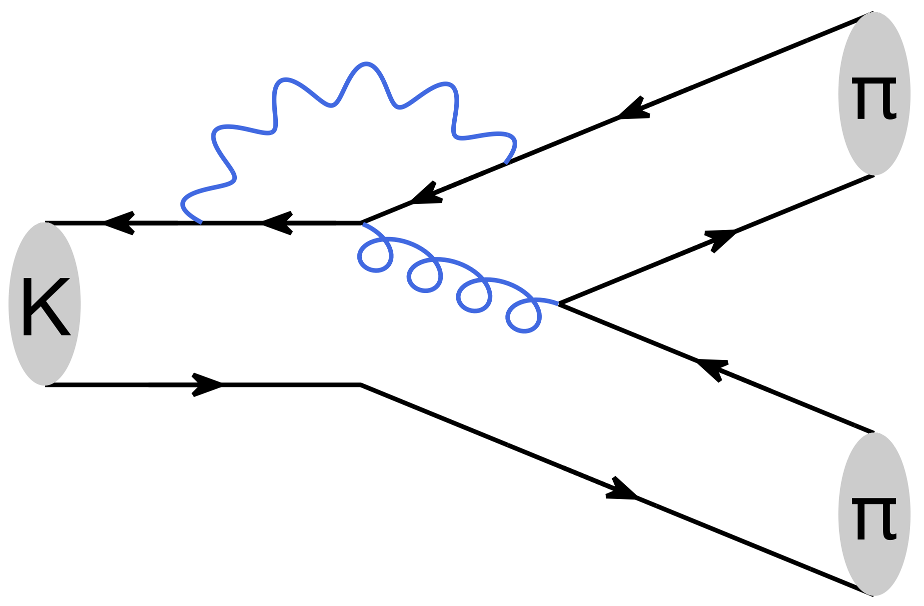

RIKEN BNL Research Center

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$K \to \pi\pi$ & CP violation

CP odd CP even

$$|K_L\rangle = |K_2\rangle + \varepsilon |K_1\rangle$$

ε' direct CPV

ε indirect CPV

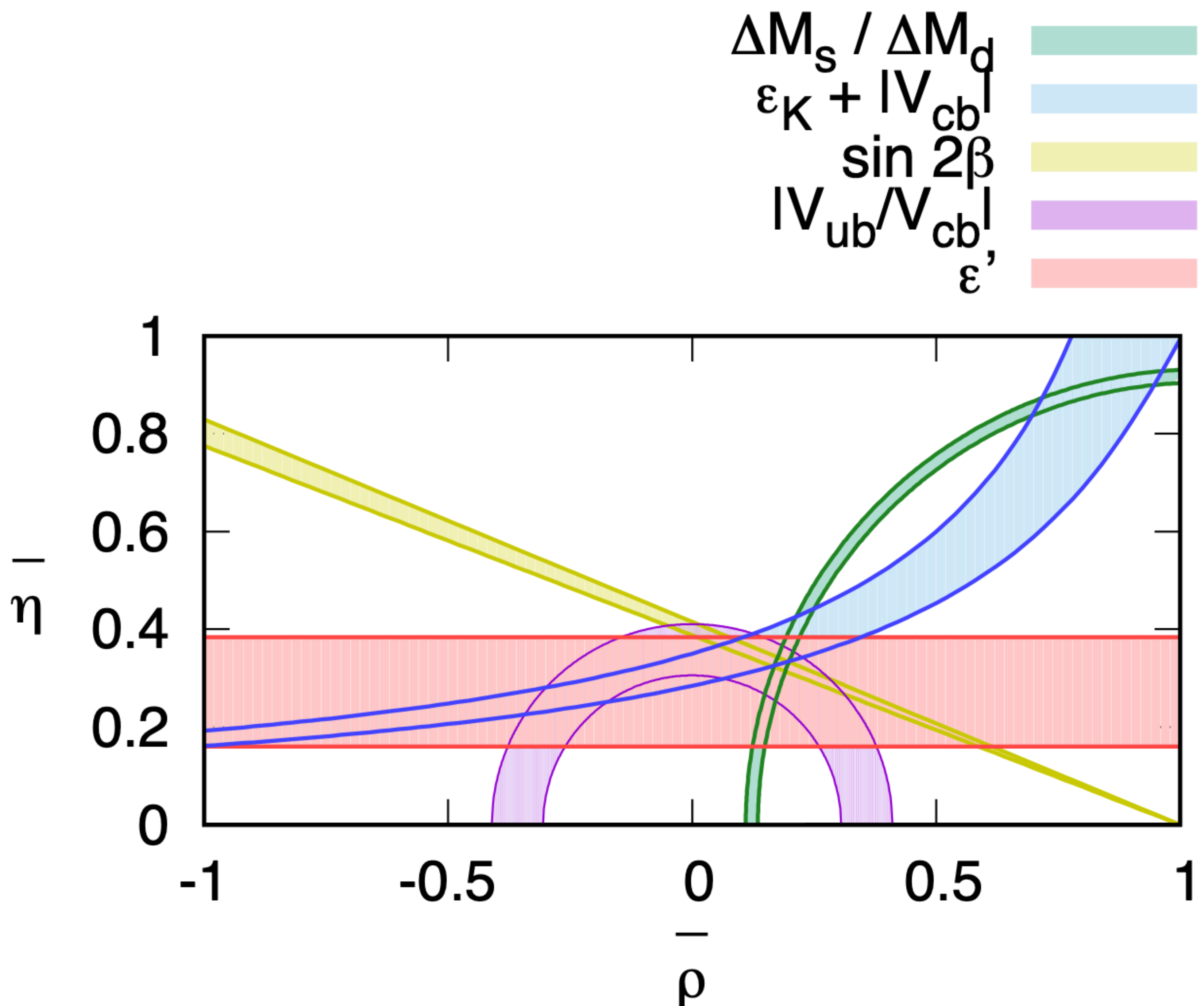
$$|\pi\pi\rangle$$

CP even

Discovered in 1964

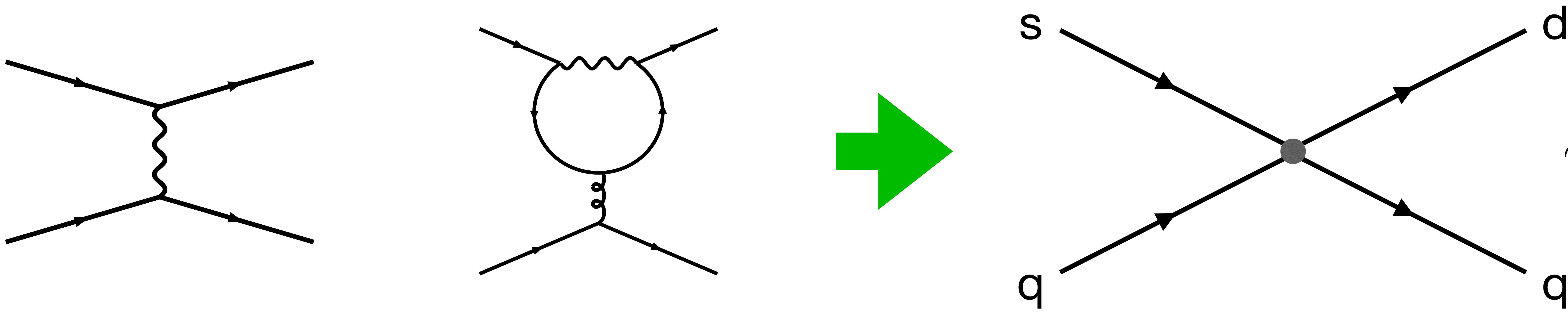
$$\frac{\Gamma(K_L \to \pi^0\pi^0)}{\Gamma(K_S \to \pi^0\pi^0)} \Big/ \frac{\Gamma(K_L \to \pi^+\pi^-)}{\Gamma(K_S \to \pi^+\pi^-)} = 1 - 6\mathrm{Re}(\varepsilon'/\varepsilon)$$

- $|\varepsilon| = 2.228(11) \times 10^{-3}$ from "odd" mixing b/w K^0 & $\overline{K}^0$
- ε' only found in decays **Discovered in 1999**
  - $\mathrm{Re}(\varepsilon'/\varepsilon)_{exp} = 1.66(23) \times 10^{-3}$ (KTeV & NA48)
  - Consistent with SM?



Approach to weak matrix elements

- Two typical scales
  - Electroweak scale: m_W = 80 GeV, m_Z = 91 GeV
  - QCD scale: Λ_{QCD} ~ 300 MeV
- Low-energy effective interactions @ QCD scale



$\sim G_F \propto \frac{g^2}{m_W^2}$

  - $H_W = \Sigma_i \underbrace{c_i(\mu)}_{\text{Wilson coefficients}} \underbrace{O_i(\mu)}_{\text{Effective operators}}$

Wilson coefficients Effective operators

ΔS = 1 effective operators

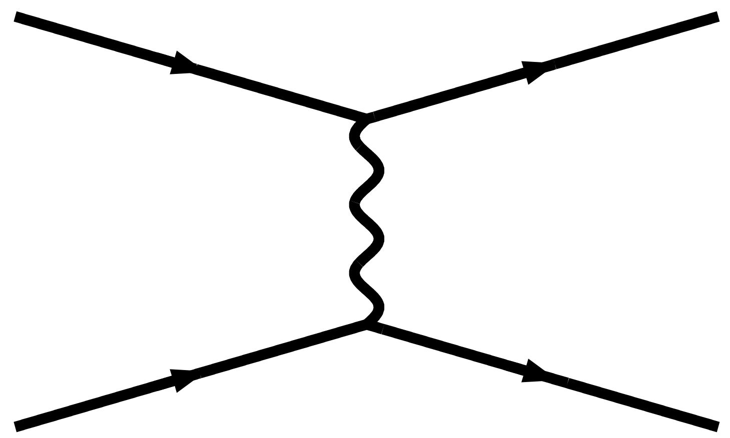
- $(\bar{s}q)_{V-A}(\bar{q}'q'')_{V\pm A} = \bar{s}\gamma_\mu(1-\gamma_5)q' \cdot \bar{q}'\gamma_\mu(1\pm\gamma_5)q''$
- α, β: color indices

$$
\begin{aligned}
Q_1 &= (\bar{s}_\alpha u_\beta)_{\mathrm{V-A}}\,(\bar{u}_\beta d_\alpha)_{\mathrm{V-A}}\,, \\
Q_2 &= (\bar{s}u)_{\mathrm{V-A}}\,(\bar{u}d)_{\mathrm{V-A}}\,, \\
Q_3 &= (\bar{s}d)_{\mathrm{V-A}}\sum_q(\bar{q}q)_{\mathrm{V-A}}\,, \\
Q_4 &= (\bar{s}_\alpha d_\beta)_{\mathrm{V-A}}\sum_q(\bar{q}_\beta q_\alpha)_{\mathrm{V-A}}\,, \\
Q_5 &= (\bar{s}d)_{\mathrm{V-A}}\sum_q(\bar{q}q)_{\mathrm{V+A}}\,, \\
Q_6 &= (\bar{s}_\alpha d_\beta)_{\mathrm{V-A}}\sum_q(\bar{q}_\beta q_\alpha)_{\mathrm{V+A}}\,, \\
Q_7 &= \frac{3}{2}(\bar{s}d)_{\mathrm{V-A}}\sum_q e_q\,(\bar{q}q)_{\mathrm{V+A}}\,, \\
Q_8 &= \frac{3}{2}(\bar{s}_\alpha d_\beta)_{\mathrm{V-A}}\sum_q e_q\,(\bar{q}_\beta q_\alpha)_{\mathrm{V+A}}\,, \\
Q_9 &= \frac{3}{2}(\bar{s}d)_{\mathrm{V-A}}\sum_q e_q\,(\bar{q}q)_{\mathrm{V-A}}\,, \\
Q_{10} &= \frac{3}{2}(\bar{s}_\alpha d_\beta)_{\mathrm{V-A}}\sum_q e_q\,(\bar{q}_\beta q_\alpha)_{\mathrm{V-A}}\,,
\end{aligned}
$$

Current-current operators

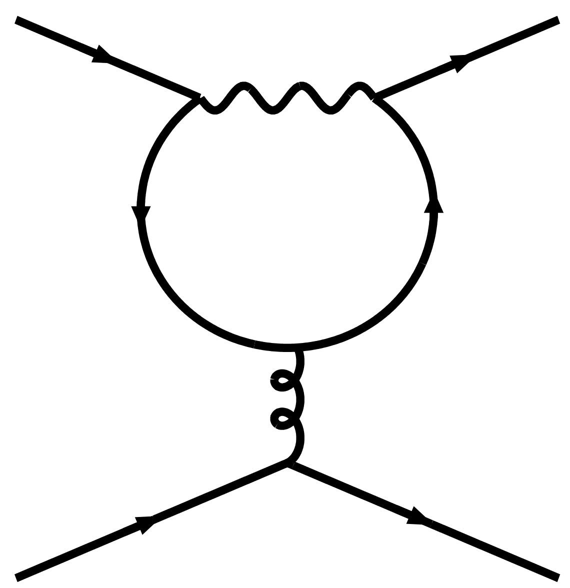
- $Q_1^c = (\bar{s}_\alpha c_\beta)_{\mathrm{V-A}}(\bar{c}_\beta d_\alpha)_{\mathrm{V-A}}$ & $Q_2^c = (\bar{s}c)_{\mathrm{V-A}}(\bar{c}d)_{\mathrm{V-A}}$

  enter when $n_f \geq 4$



QCD penguin operators

- sum over q runs for all active quarks



EW penguin operators

$K \to \pi\pi$ Amplitude and ε'

ππ phase shifts at m_K

$$\varepsilon' = \frac{i\omega e^{i(\delta_2-\delta_0)}}{\sqrt{2}}\left[\frac{\mathrm{Im}A_2}{\mathrm{Re}A_2} - \frac{\mathrm{Im}A_0}{\mathrm{Re}A_0}\right] \qquad (\omega = \mathrm{Re}A_2/\mathrm{Re}A_0)$$

$$A_I = \frac{G}{\sqrt{2}} V_{us}^* V_{ud} \sum_i \underbrace{[z_i(\mu) + \tau y_i(\mu)]}_{\text{Wilson coefs. pQCD}} \underbrace{M_{I,i}(\mu)}_{\text{Matrix elements LQCD}}$$

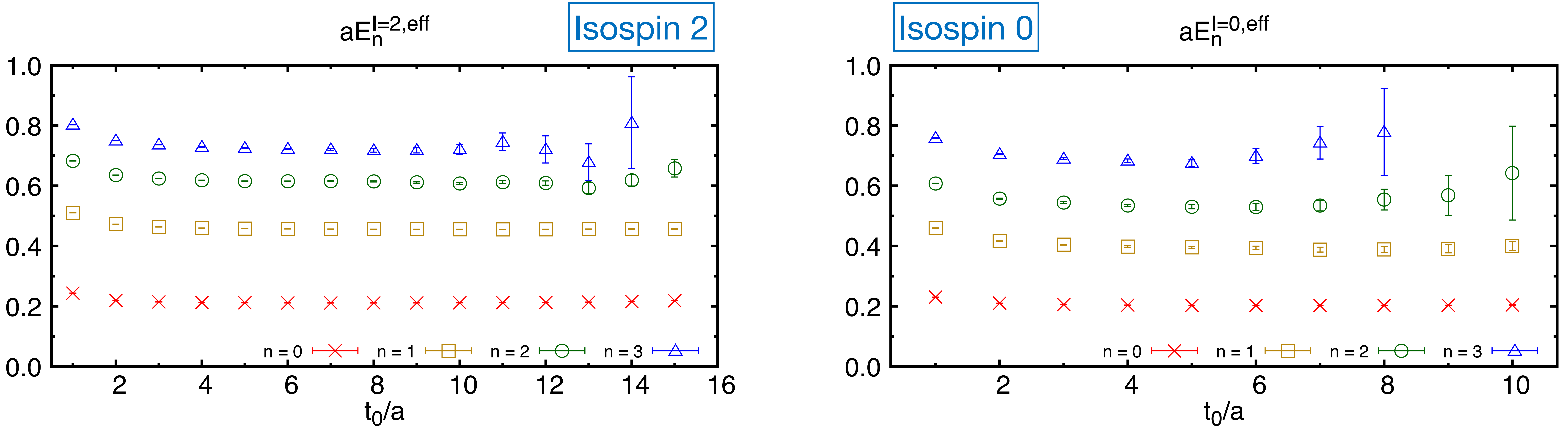
- A_2 already reached sufficient precision RBC/UKQCD PRD91 (2015) 074502
  - $\mathrm{Re}\,A_2 = 1.50(4)_{stat}(14)_{sys} \times 10^{-8}$ GeV, $\mathrm{Im}\,A_2 = -6.99(20)_{stat}(84)_{sys} \times 10^{-13}$ GeV
    cf: $(\mathrm{Re}\,A_2)_{exp} = 1.479(4) \times 10^{-8}$ GeV
- A_0 still challenging because of many difficulties

Two approaches (boundary conditions)

✳ **Euclidean correlation functions contributed from many energy states**
✳ **Need to extract the contribution from $E_{\pi\pi} \approx m_K$ (energy-conserving matrix element)**

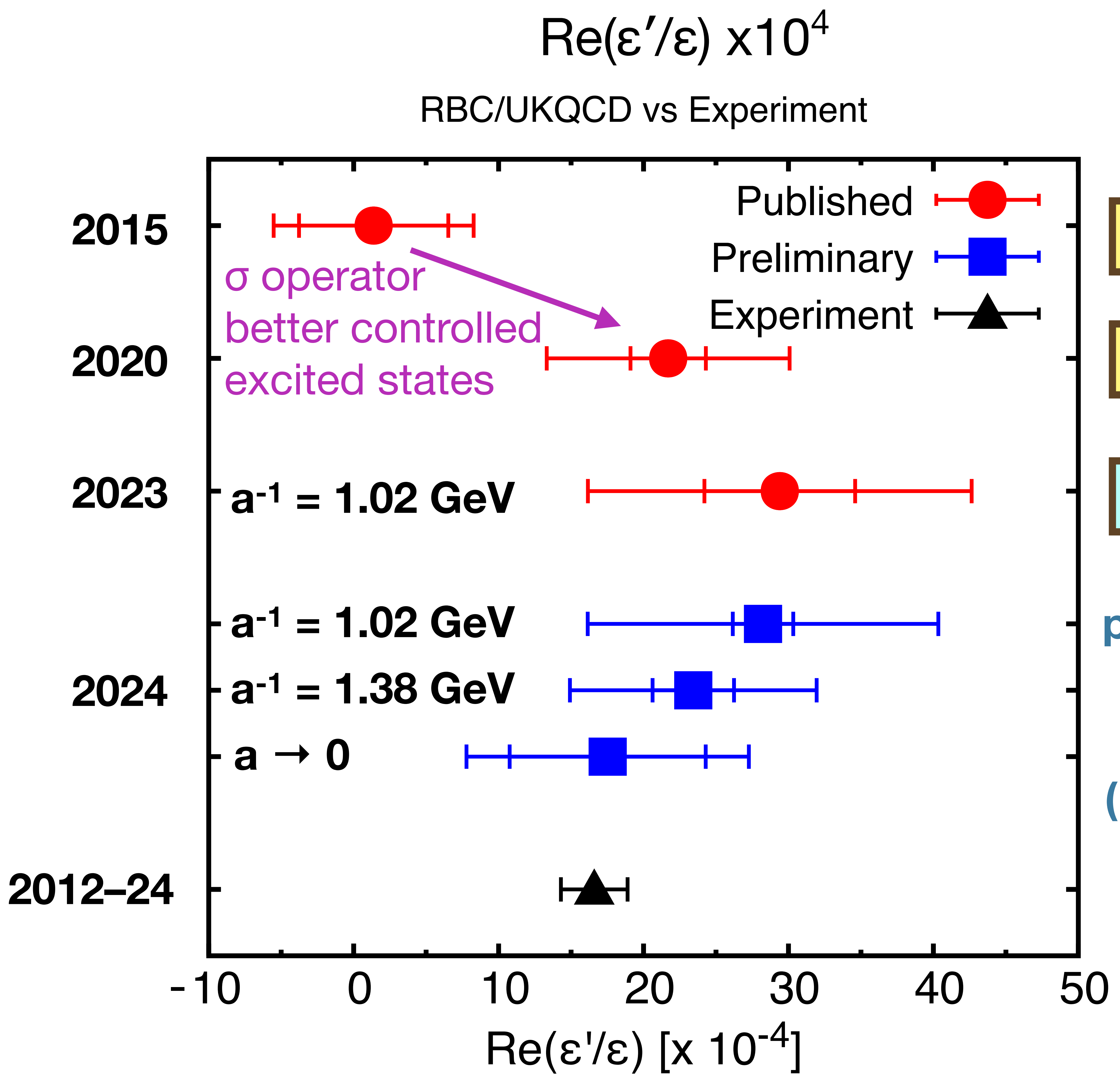
- G-parity BC (2015, 2020)
  - charge conjugation and isospin rotation for u & d
  - pions become anti-periodic → must move in finite volume
  - ground ππ state with $E_{\pi\pi} \approx m_K$ possible
  - ensemble generation just for K → ππ
  - unsuited for EM correction (charge conservation violated)
- Periodic BC (2023, …)
  - $E_{\pi\pi} \approx m_K$ only possible as an excited ππ state (in the rest frame)
  - used to be considered unrealistic to extract signals of excited states
  - GEVP (generalized eigenvalue problem) found useful for state decomposition
  - many ensembles already generated
  - suited for EM correction

ππ energies from GEVP



- 4 states up to $E_{\pi\pi} \approx 1$ GeV well resolved
- Plateau at small time slices indicate successful extraction of each state

RBC/UKQCD achievements



PRL 115,212001

PRD 102,054509

G-parity Boundary Conditions (GPBC)
$a^{-1} \approx 1.38$ GeV

PRD 102,094517

preliminary results with multiple lattice spacings

(MT LATTICE2024)

Periodic Boundary Conditions (PBC)
$a^{-1} \approx 1.38$ GeV & 1.02 GeV

- **With PBC, we already have a bunch of finer lattice ensembles**
- **PBC necessary for EM correction**

Systematic errors

- Systematic errors on Im A_0 on 1.4 GeV lattice PRD 102,054509 (2020)

Finite lattice spacing	12%
Wilson coefficients/charm-loop effects	12%
Lelloch-Lüscher FV correction	1.5%
Residual FV correction	7%
Parametric error	6%
Off-shellness	5%
Renormalization	4%
Missing G_1 operator	3%
TOTAL	21%

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- In addition
  - ε′ could be significantly affected by EM/IB effects (ΔI = 1/2 rule → 25%)

Hope to compute near future

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Finite lattice spacing → Improvement desired

Wilson coefficients/charm-loop effects → Improvement desired

Residual FV correction → based on estimate of A_2 could be more significant for A_0

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Finite lattice spacing → **Improvement desired**

Wilson coefficients/charm-loop effects → **Improvement desired**

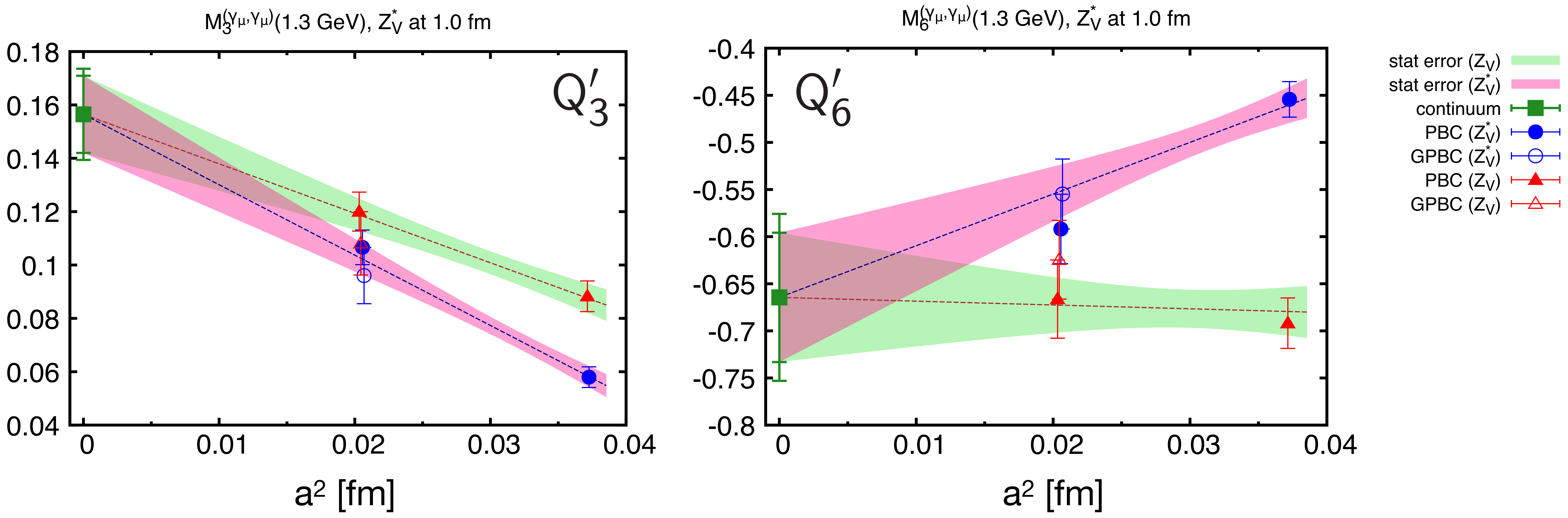
Residual FV correction → **based on estimate of A_2 could be more significant for A_0**

Renormalization → **Recently found more significant for ReA_0 (~18%)**

- In addition
  - ε' could be significantly affected by EM/IB effects (ΔI = 1/2 rule → 25%)

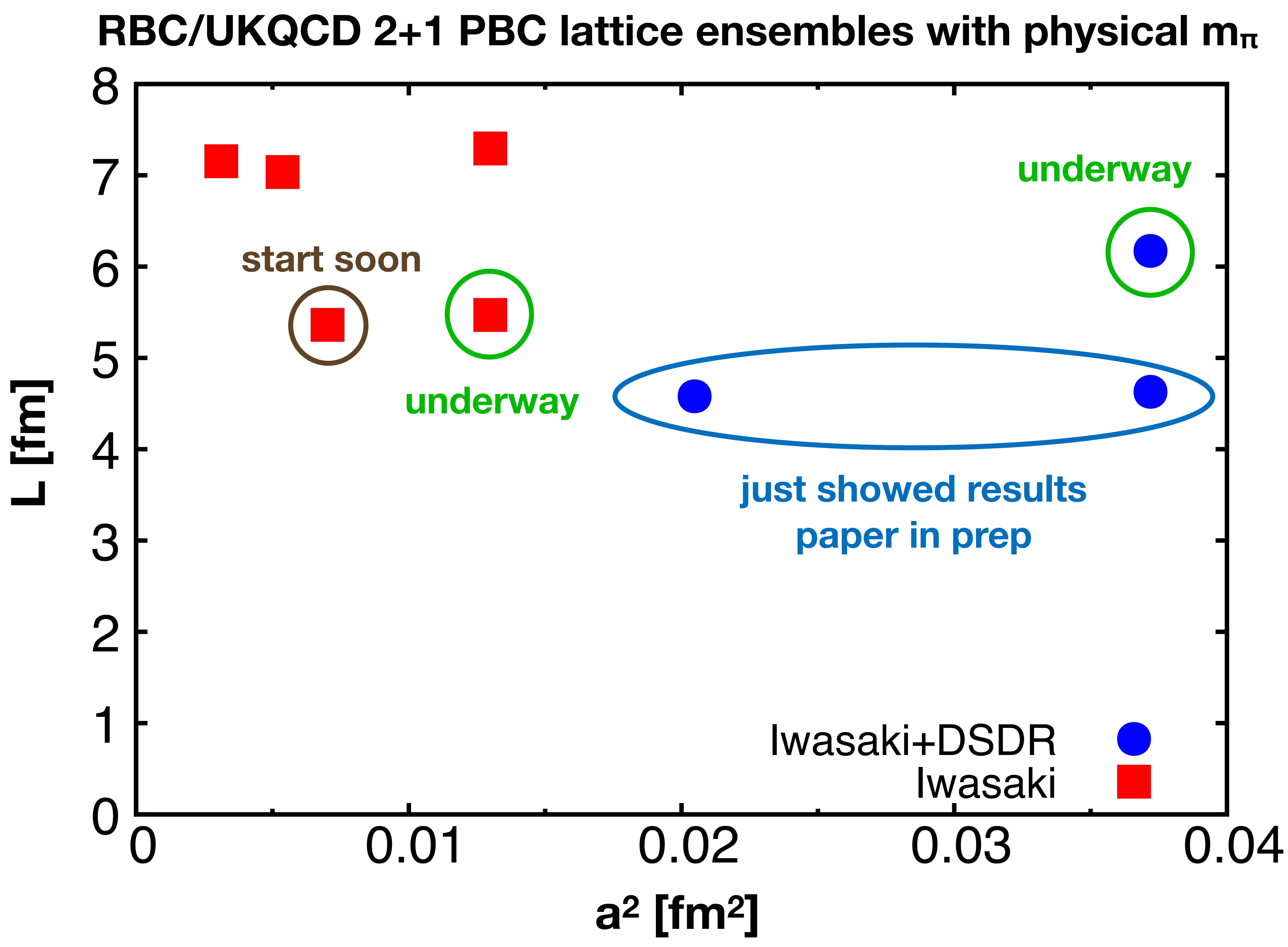
Hope to compute near future

Continuum extrapolations of MEs



- First attempt at continuum extrapolation of ΔI = 1/2 matrix elements
- SMOM(γ_μ,γ_μ) scheme at 1.3 GeV
- Only 2 lattice spacings → O(a^4) potentially significant

Towards more accurate continuum limit



Major systematic errors

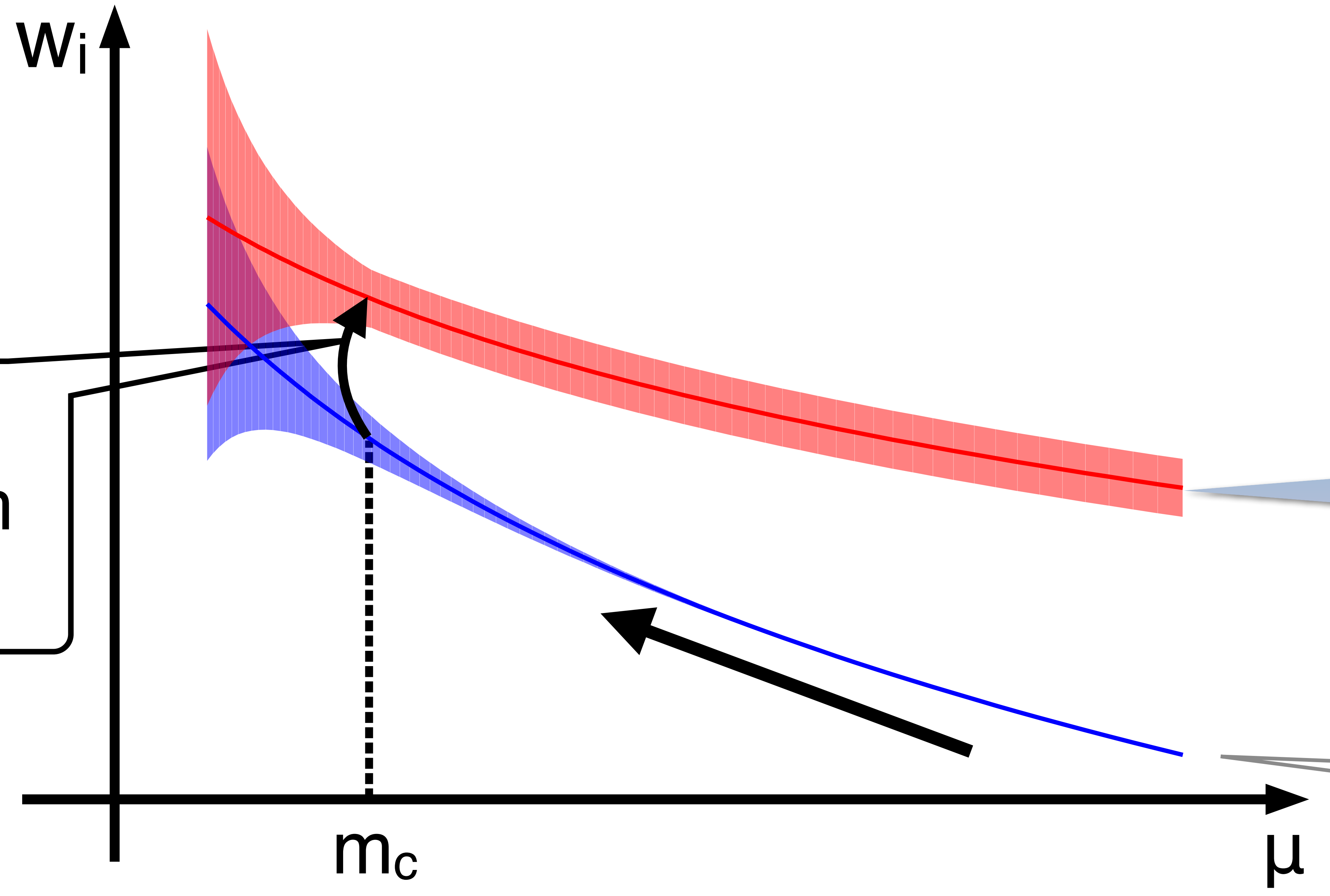
Finite lattice spacing effect **Finite volume effect**	• First attempt at a → 0 • Calculation on finer lattice & larger volume underway
Wilson coefficients	
Renormalization	
Electromagnetic & isospin-violating effects	

Wilson coefs

3-flavor calculation due to the limit on lattice cutoff

$$\langle f|H_W|i\rangle = \Sigma_i \underbrace{w_i^{3f}(\mu)}_{\text{pQCD}} \underbrace{\langle f|O_i^{3f}(\mu)|i\rangle}_{\text{LQCD}}$$

perturbative matching at charm threshold



$w_i^{3f}(\mu)$: large uncertainty

12% uncertainty on A_0 w NLO

$w_i^{4f}(\mu)$: precise at HEs

Possible approaches

- NNLO perturbation
  - Anomalous dimension with QCDP operators
    Gorbahn & Haisch, NPB713,291 (2005)
  - Threshold matching with QCDP operators
    Cerdà-Sevilla et al, 1611.08276 [Kaon 2016]
  - Matching b/w $\overline{\text{MS}}$ and RI/SMOM schemes "missing"
    a suitable scheme on lattice
- Finer lattices allow us to directly include the charm quark in lattice calculation
- Nonperturbative matching between 3 & 4-flavor operators
  MT [PoS LATTICE 2019]

Major systematic errors

Finite lattice spacing effect **Finite volume effect**	• First attempt at $a \to 0$ • Calculation on finer lattice & larger volume underway
Wilson coefficients	• Completion of NNLO would be great • Plan to include charm quark in upcoming calculation with $a^{-1} \approx 2.3$ GeV
Renormalization	
Electromagnetic & isospin-violating effects	

EM/IB effects

- O(1%) for usual quantities but …

$$\frac{\varepsilon'}{\varepsilon} = \frac{i\omega e^{i(\delta_2-\delta_0)}}{\sqrt{2}\varepsilon}\left[\frac{\mathrm{Im}A_2}{\mathrm{Re}A_2} - \frac{\mathrm{Im}A_0}{\mathrm{Re}A_0}\right] = -\frac{i\omega e^{i(\delta_2-\delta_0)}}{\sqrt{2}\varepsilon}\frac{\mathrm{Im}A_0}{\mathrm{Re}A_0}\left[1 - \frac{1}{\omega}\underline{\frac{\mathrm{Im}A_2}{\mathrm{Im}A_0}}\right]$$

$(\omega = \mathrm{Re}A_2/\mathrm{Re}A_0)$

Even small correction to this can be amplified ($1/\omega \approx 22.5$: $\Delta I = 1/2$ rule)

Ciligriano et al, JHEP 02, 032 (2020)
NLO ChPT + large N_c
(example estimation)

$$\frac{\varepsilon'}{\varepsilon} = \frac{i\omega_+ e^{i(\delta_2-\delta_0)}}{\sqrt{2}\varepsilon}\left[\frac{\mathrm{Im}A_2^{\mathrm{emp}}}{\mathrm{Re}A_2^{(0)}} - \frac{\mathrm{Im}A_0^{(0)}}{\mathrm{Re}A_0^{(0)}}(1-\hat{\Omega}_{\mathrm{eff}})\right]$$

$$\boxed{\hat{\Omega}_{\mathrm{eff}} = 0.170\left(\begin{smallmatrix}+91\\-90\end{smallmatrix}\right)}$$

- Theoretical framework to incorporate EM effects in finite volume well developed
  - [Christ et al, PRD106 (2022), 014508], vol 2 by Christ and Lundstrum soon
- Beginning test calculations on lattice!

Major systematic errors

Finite lattice spacing effect **Finite volume effect**	• First attempt at $a \to 0$ • Calculation on finer lattice & larger volume underway
Wilson coefficients	• Completion of NNLO would be great • Plan to include charm quark in upcoming calculation with $a^{-1} \approx 2.3$ GeV
Renormalization	• NNLO also important • Nonperturbative scale evolution to higher energy will be more precise
Electromagnetic & isospin-violating effects	

Renormalization

nonperturbative renormalization (& scale evolution)

perturbative matching (NLO)

$$M_i^{\mathrm{lat}}(a) \to M_i^{\mathrm{SMOM}(\not{q},\not{q})}(\mu) \xrightarrow{\text{perturbative matching (NLO)}} M_i^{\overline{\mathrm{MS}}}(\mu)$$

$$M_i^{\mathrm{lat}}(a) \to M_i^{\mathrm{SMOM}(\gamma_\mu,\gamma_\mu)}(\mu) \longrightarrow M_i^{\overline{\mathrm{MS}}}(\mu)$$

significant discrepancy for some MEs at $\mu \approx 4$ GeV

- Possible causes
  - NLO perturbative matching inaccurate
  - Discretization error in nonperturbative lattice part (improvement planned)

Major systematic errors

Finite lattice spacing effect **Finite volume effect**	• First attempt at $a \to 0$ • Calculation on finer lattice & larger volume underway
Wilson coefficients	• Completion of NNLO would be great • Plan to include charm quark in upcoming calculation with $a^{-1} \approx 2.3$ GeV
Renormalization	• NNLO also important • Nonperturbative scale evolution to higher energy will be more precise
Electromagnetic & isospin-violating effects	• Formulation to implement in finite volume well developed • Exploratory lattice calculation starting

Summary

- $K \to \pi\pi$ decay & ε' remain important in lattice QCD to reduce the theory error
- Various efforts to improve the prediction since our “best” result in 2020
  - First attempt at continuum limit (preliminary)
  - Pursuing the calculation on even finer lattices
  - Nonperturbative incorporation of charm-loop effects reduces systematic error
  - Beginning an exploratory calculation of EM & IB effects
- All significant systematic errors are set to be reduced in the near future