COLUMBIA UNIVERSITY
MEDICAL CENTER

Tune compensation and Dynamic Acceptance Studies in Imperfect Scaling FFAG

With special thanks to the international KURRI FFAG collaboration.

Malek HAJ TAHAR
September 8th , 2017
FFAG'17 workshop
Cornell university

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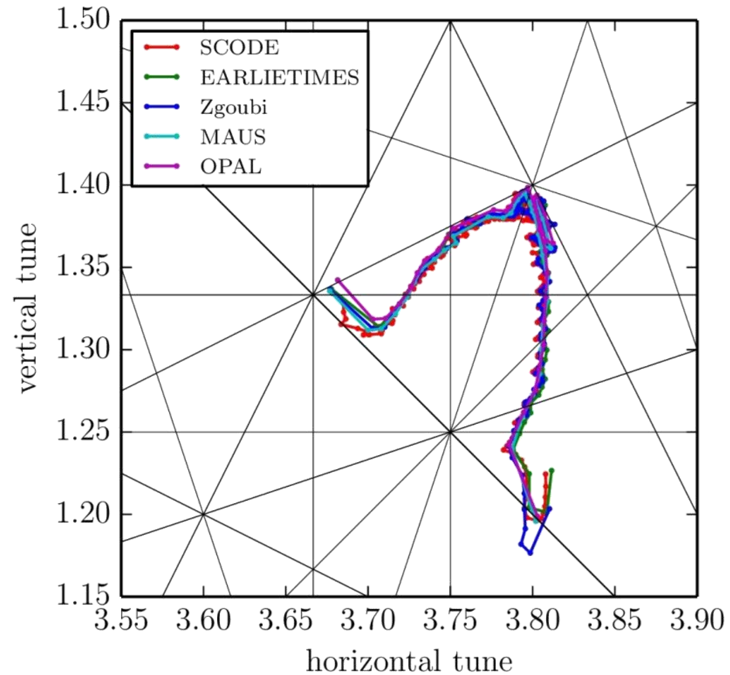
Benchmarking

- ❑ Benchmarking campaign between different codes: OPAL, SCODE, ZGOUBI, EARLIETIMES and MAUS.

- ❑ The idea is to use the same 2D or 3D field maps for the simulation.

- ❑ **Lots of work but yielded excellent results so far.**



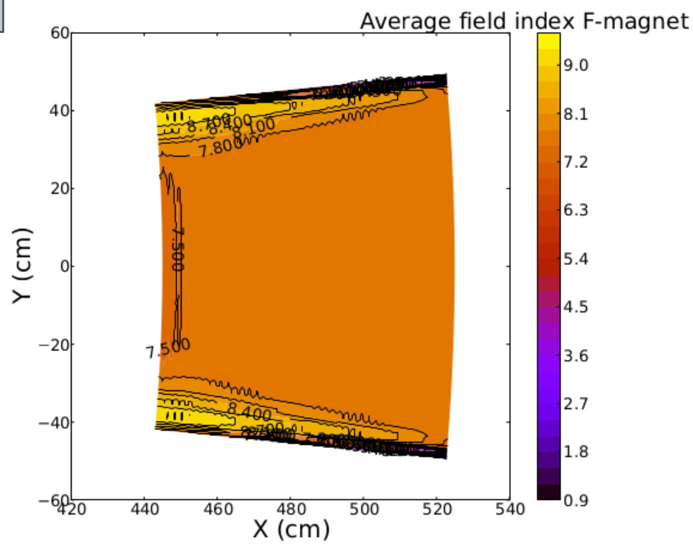
Betatron tunes from 11 to 139 MeV (left to right) calculated with several codes. Courtesy S. L. Sheehy

Mean field index of the magnets

- An extension of the mean field index k as defined by Symon consists in introducing its azimuthal variation in the following way:

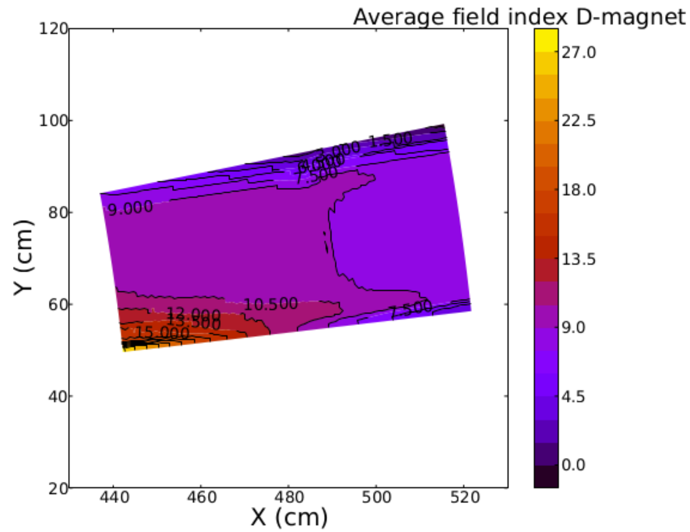
$$k_i = \frac{R}{B_i}\frac{dB_i}{dR} \qquad ; \qquad i = F, D, drift$$

F-magnet



(a) Average field index map of the focusing magnet (k_F)

D-magnet



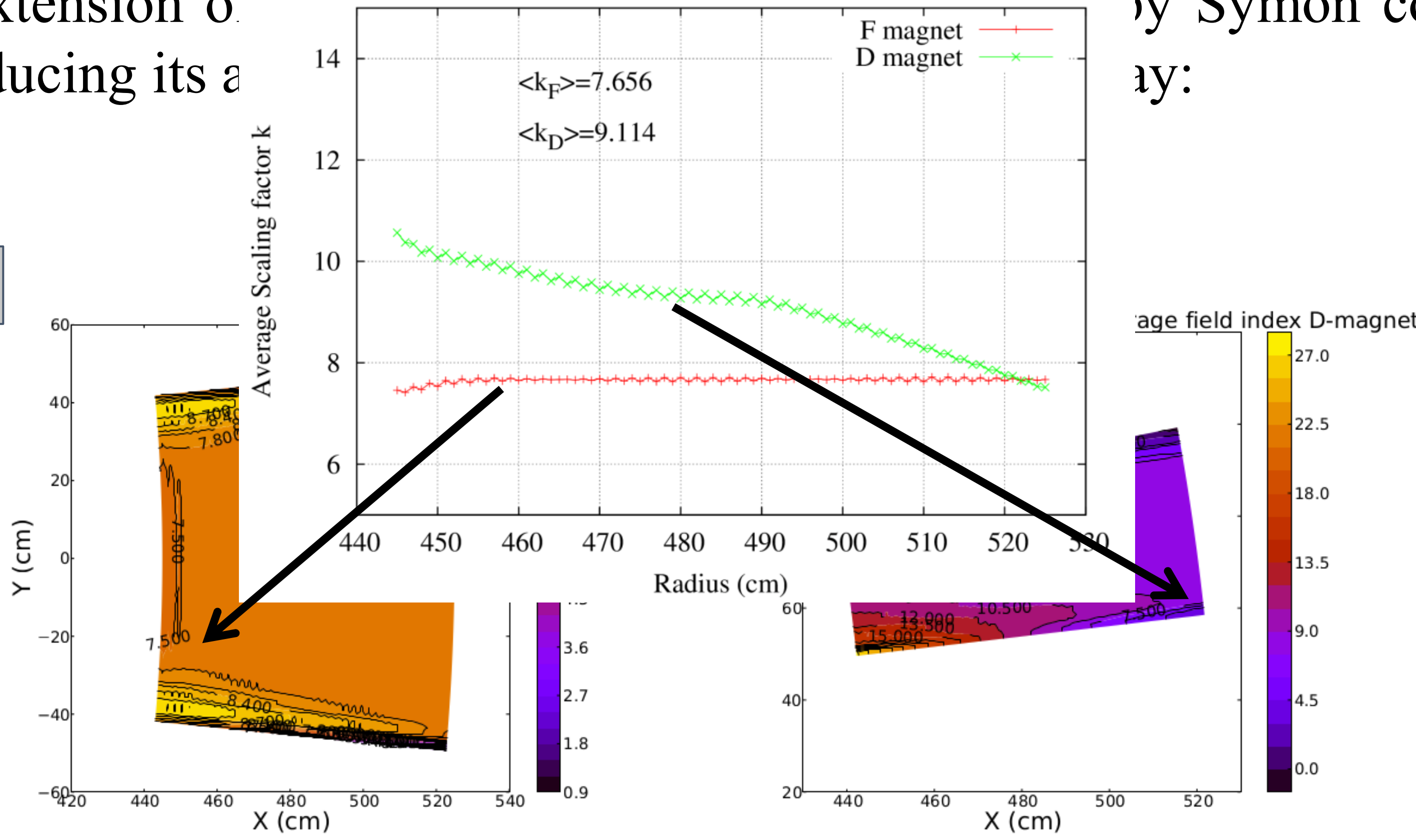
(b) Average field index map of the defocusing magnet (k_D)

Mean field index of the magnets

- An extension o[…] [b]y Symon consists in introducing its a[…] [w]ay:

F-magnet

D-magnet



(a) Average field index map of the focusing magnet (k_F)

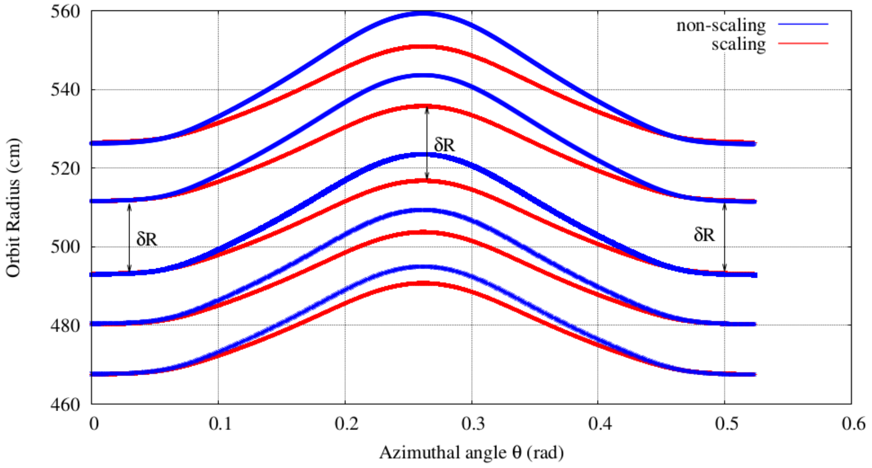
(b) Average field index map of the defocusing magnet (k_D)

Realistically, the field index of the magnet changes radially and azimuthally.

Preliminary finding

- ❑ If $\kappa = k_F - k_D \neq 0$, the tunes are energy-dependent and the orbits are not similar.

⇨ **Non scaling FFAG.**



- ❑ An accurate solution of the non-linear equation of motion is thus needed.

⇨ **Calculate an approximate solution of the tunes and compare with the numerical simulations (using the tracking code Zgoubi).**

Bogoliubov method of averages (non-scaling)

- Using the BKM's method of averages, one can compute approximately the frequencies of the betatron oscillations and their dependence on the average field index of the F and D magnet. One obtains:

$$\nu_x^2(E) = \sum_i \beta_i(E) - \sum_i \alpha_i(E) \times k_i(E) + \frac{3N^2}{(N^2-1)(N^2-4)} \mathcal{F}^2 \left[1 + \tan^2(\xi)\right]$$

$$\nu_y^2(E) = \sum_i \alpha_i(E) \times k_i(E) + \frac{N^2}{N^2-1} \mathcal{F}^2 \left[1 + 2\tan^2(\xi)\right]$$

where the subscript i denotes the F-magnet, the D-magnet and the drift and:

$$\alpha_i(E) = \frac{-1}{2\pi/N} \int_{\theta_i} \mu(R,\theta) d\theta = \frac{-1}{2\pi/N} \int_{\theta_i} \frac{R}{\rho} d\theta \quad ; \quad \beta_i(E) = \frac{1}{2\pi/N} \int_{\theta_i} \mu(R,\theta)^2 d\theta = \frac{1}{2\pi/N} \int_{\theta_i} \left(\frac{R}{\rho}\right)^2 d\theta$$

1st order index of similarity

2nd order index of similarity

Bogoliubov method of averages (non-scaling)

- Using the BKM's method of averages, one can compute approximately the frequencies of the betatron oscillations and their dependence on the average field index of the F and D magnet. One obtains:

Focusing due to the average magnetic gradient

$$\nu_x^2(E) = \boxed{\textstyle\sum_i \; \beta_i(E)} \boxed{- \sum_i \alpha_i(E) \times k_i(E)} + \frac{3N^2}{(N^2-1)(N^2-4)} \mathcal{F}^2 \left[1 + \tan^2(\xi)\right]$$

$$\nu_y^2(E) = \boxed{\textstyle\sum_i \; \alpha_i(E) \times k_i(E)} + \frac{N^2}{N^2-1} \mathcal{F}^2 \left[1 + 2\tan^2(\xi)\right]$$

Radial focusing from centrifugal forces

where the subscript i denotes the F-magnet, the D-magnet and the drift and:

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1st order index of similarity

2nd order index of similarity

x-motion focused in magnets with positive curvature and defocused in magnets with negative curvature.

Fundamental property

- Define a closed orbit, i.e. for a given energy, in a cylindrical coordinates system:

$$R(\theta) = \langle R \rangle \left[1 + f.g(N\theta)\right]$$

- Compute the first and second order index of similarity:

$$\left(\frac{R}{\rho}\right)_E = \frac{\left| R^2 + 2\left(\frac{dR}{d\theta}\right)^2 - R\frac{d^2R}{d\theta^2} \right|}{\left[R^2 + \left(\frac{dR}{d\theta}\right)^2\right]^{3/2}} R$$

- It results that, for any closed orbit:

$$\boxed{\sum_i \alpha_i(E) = -1}$$

i denotes the F-magnet, the D-magnet and the drift

Bogoliubov method of averages (non-scaling)

- Using the previous result, one obtains:

$$\nu_x^2(E) = \boxed{\left\langle \frac{R}{\rho} \right\rangle^2 (E)} \boxed{-\alpha_F(E) \times [k_F(E) - k_D(E)]} + k_D(E) + \frac{3N^2}{(N^2-1)(N^2-4)} \boxed{\mathcal{F}^2} [1 + \tan^2(\xi)]$$

$$\nu_y^2(E) = \boxed{\alpha_F(E) \times [k_F(E) - k_D(E)]} - k_D(E) + \frac{N^2}{N^2-1} \boxed{\mathcal{F}^2} [1 + 2\tan^2(\xi)]$$

- All parameters are susceptible to change when scaling imperfections introduced: the non-scaling of the orbits introduces a change of the average magnetic gradient.

- It does introduce a change of the magnetic flutter as well as the index of similarity β of the orbits.

Bogoliubov method of averages (scaling)

- Using the previous result, one obtains:

$$\nu_x^2(E) = \left\langle \frac{R}{\rho} \right\rangle^2 (E) \; -\alpha_F(E) \times \xcancel{[k_F(E) - k_D(E)]} + k_D(E)$$
$$+ \quad \frac{3N^2}{(N^2-1)(N^2-4)} \mathcal{F}^2 \left[1 + \tan^2(\xi)\right]$$

$$\nu_y^2(E) = \; \alpha_F(E) \; \times \xcancel{[k_F(E) - k_D(E)]} - k_D(E)$$
$$+ \quad \frac{N^2}{N^2-1} \mathcal{F}^2 \left[1 + 2\tan^2(\xi)\right]$$

- If $k_F = k_D$, one recovers the same expression as Symon, except for one term: $\left\langle \frac{R}{\rho} \right\rangle^2 (E)$

- This term is the contribution of the centrifugal forces and is equal to 1, only for a circular orbit.

Bogoliubov method of averages (scaling)

- Using the previous result, one obtains:

$$\nu_x^2(E) = \left\langle \frac{R}{\rho} \right\rangle^2 (E) \; -\alpha_F(E) \cancel{\times [k_F(E) -} k_D(E)] + k_D(E) + \frac{3N^2}{(N^2-1)(N^2-4)} \mathcal{F}^2 \left[1 + \tan^2(\xi)\right]$$

$$\nu_y^2(E) = \alpha_F(E) \times [\cancel{k_F(E) - k_D(E)}] - k_D(E) + \frac{N^2}{N^2-1} \mathcal{F}^2 \left[1 + 2\tan^2(\xi)\right]$$

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ZGOUBI tracking model: Approach (1/4)

Loop on x

Loop on k_F

Loop on k_D

1) Generate a median plane field map for a given (x, k_F, k_D).

2) Search for the closed orbits.

3) Compute the tunes

ZGOUBI tracking model: Approach (2/4)

1) Build the model by generating a median plane field map for a given (x, k_F, k_D). Tracking is performed using ZGOUBI: Median plane anti-symmetry is assumed and the Maxwell equations are accommodated which yields the Taylor expansions for the three components of the magnetic field.

   The field writes in the following way:

$$\boxed{B(R,\theta) = B_{F0} \times \left(\frac{R}{R_0}\right)^{k_F} \times F_F(\theta) + x \times B_{D0} \times \left(\frac{R}{R_0}\right)^{k_D} \times F_D(\theta)}$$

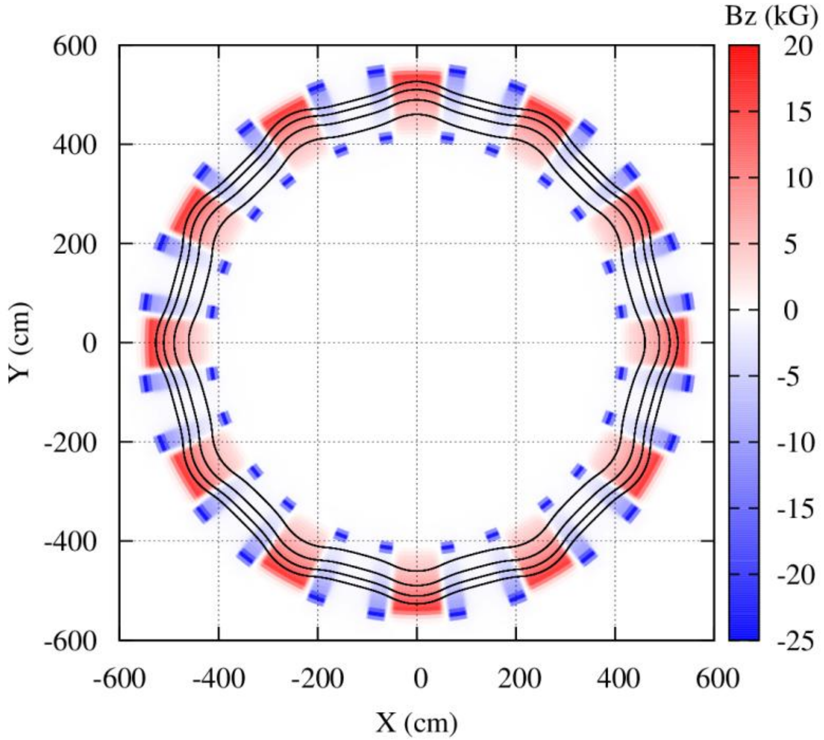
- x is a scale factor to change the FD ratio
- k_F is the average field index of the F-magnet
- k_D is the average field index of the D-magnet

Separable function:
$F_F(\theta).F_D(\theta) = 0$

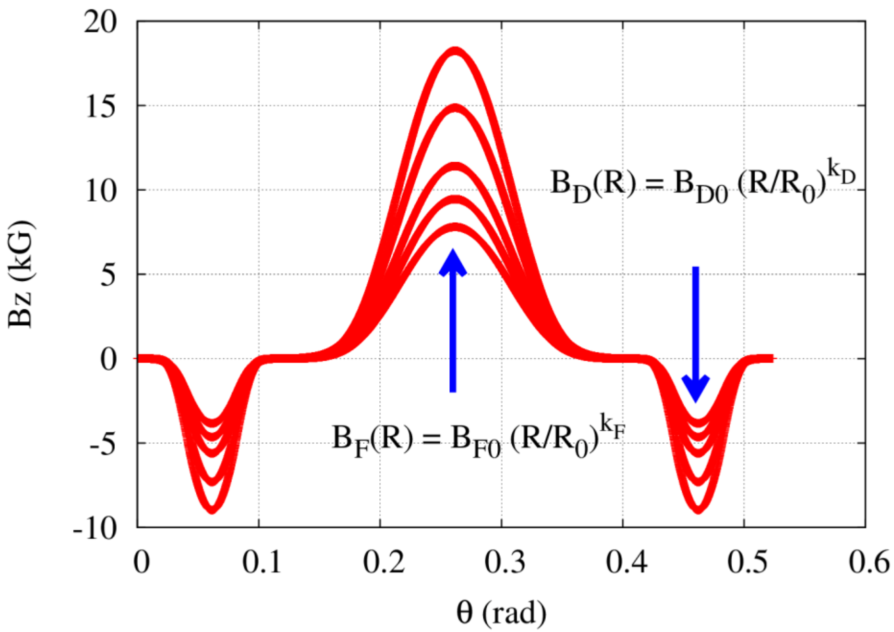
NB: if $k_F = k_D$, the field writes in the standard form of a scaling FFAG.

ZGOUBI tracking model: Approach (3/4)

2) Search for the closed orbits between injection energy and extraction energy: 30 closed orbits between $E_{inj} = 11\ MeV$ and $E_{ext} = 100\ MeV$.



Example of several closed orbits: the lattice consists of 12 DFD triplets.



Magnetic field along several closed orbits

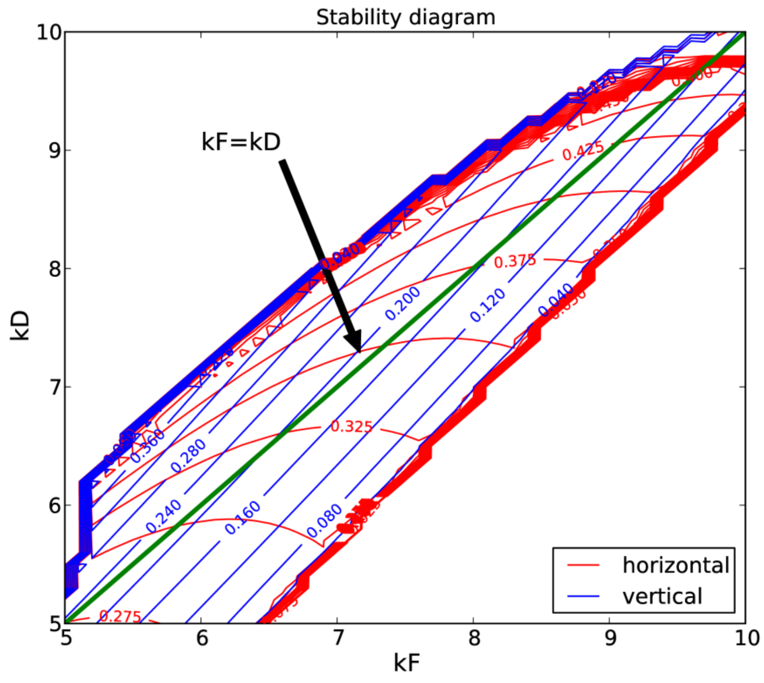
ZGOUBI tracking model: Approach (4/4)

3) For each closed orbit the number of betatron oscillations in both planes is calculated using the Zgoubi Discrete Fourier Transform (DFT).

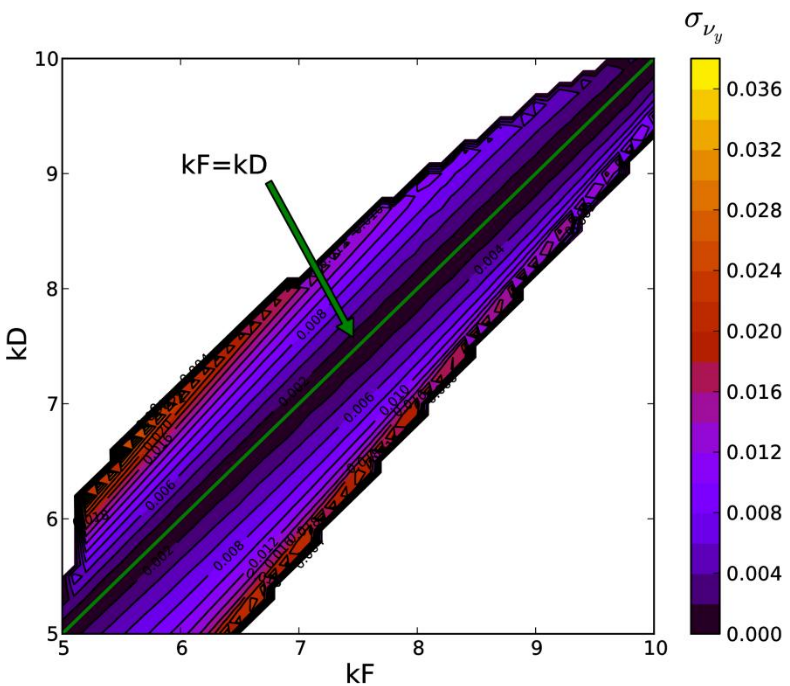
As explained earlier, the main objective of this study is to explain the origin of the tune variations in FFAGs. Therefore, we introduce two new quantities in the calculation of the tunes: the average as well as the RMS tune variations over the closed orbits (NCO):

$$\begin{cases} \nu_{x,y}^{m} = \langle \nu_{x,y} \rangle = \dfrac{1}{NCO} \displaystyle\sum_{i=1}^{NCO} \nu_{x,y,i} \\ \nu_{x,y}^{rms} = \langle \nu_{x,y}^{2} \rangle^{1/2} = \left(\dfrac{1}{NCO} \displaystyle\sum_{i=1}^{NCO} \left(\nu_{x,y,i} - \nu_{x,y}^{m} \right)^{2} \right)^{1/2} \end{cases}$$

ZGOUBI tracking model: results



Contour plot of the average cell tune.



Contour plot of the tune variations.

In the vicinity of the central line, i.e. $k_F = k_D$, the Symon formula is qualitatively verified.

Can we explain the stability diagrams? Why does it shrink linearly with k?

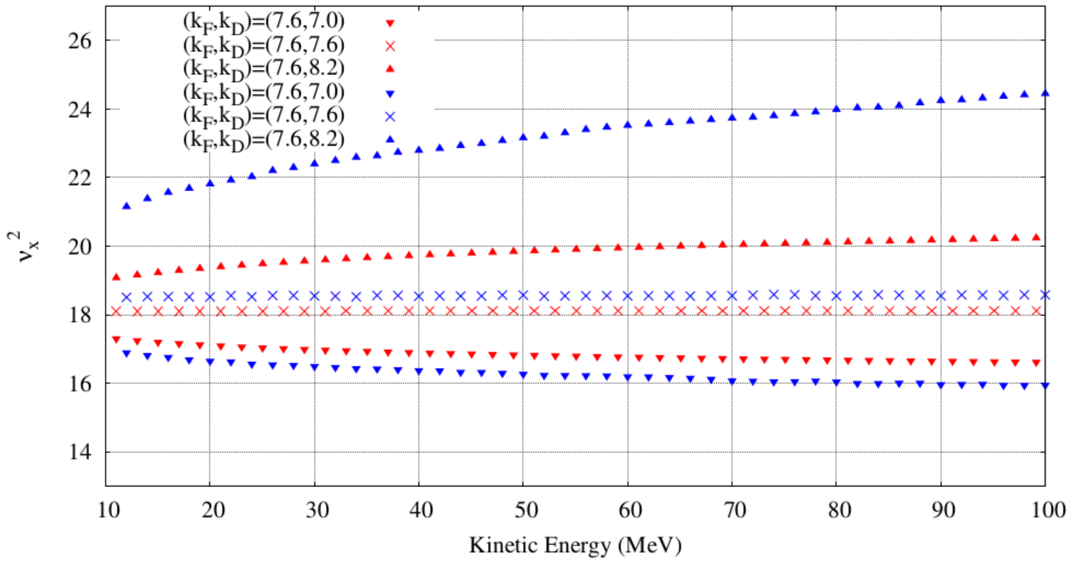
Comparison analytical vs simulation results

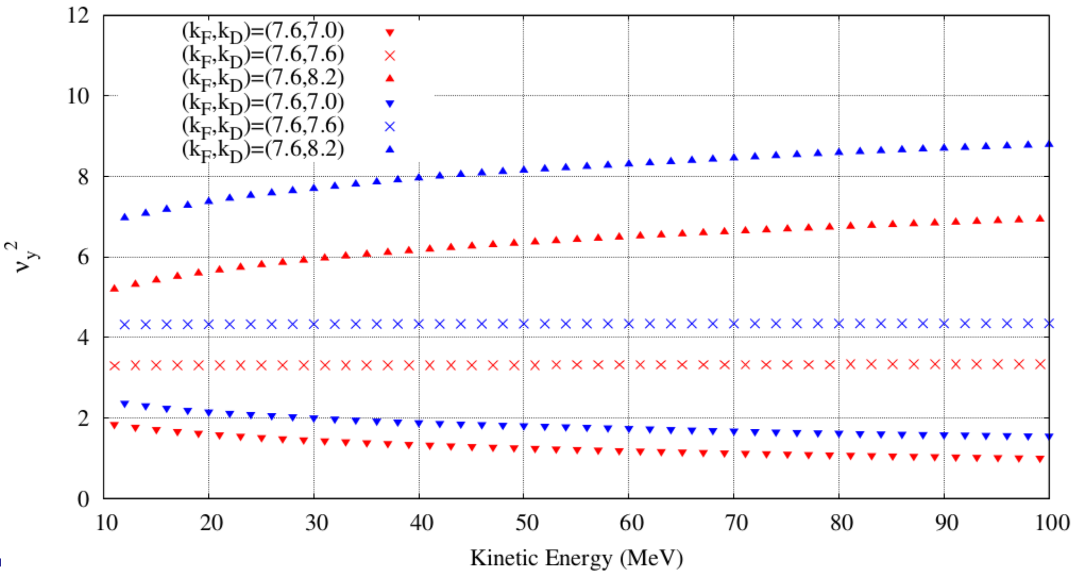
- Comparison of the Zgoubi results (blue) with the 1st order bogoliubov method of averages shows good agreement.

- Two regimes can be distinguished depending on the sign of $\boldsymbol{\kappa = k_D - k_F}$.

- The tune variations vanish at $\boldsymbol{\kappa = 0}$.

- Symon formula fails in the horizontal plane.

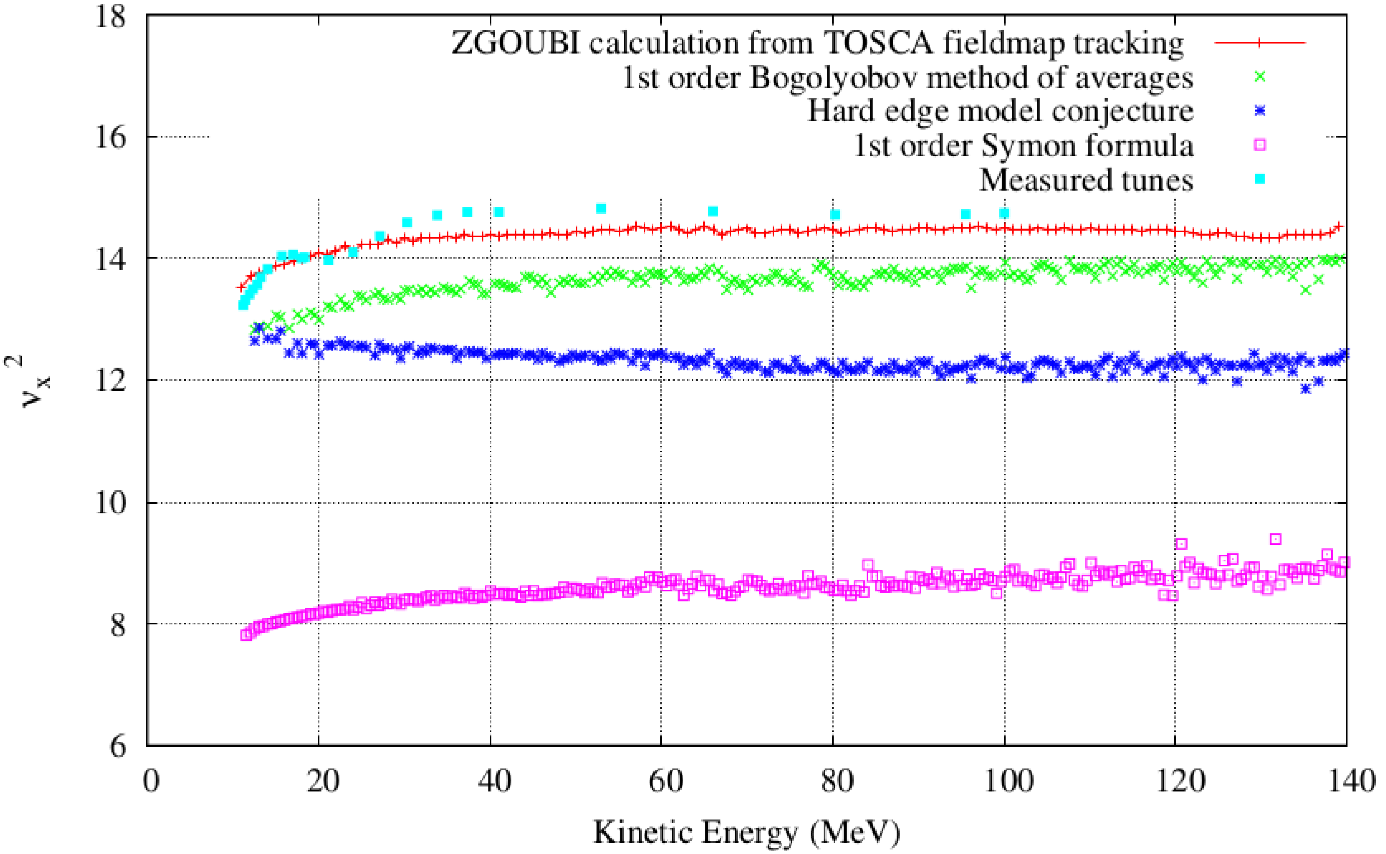




Application to the KURRI 150 MeV FFAG

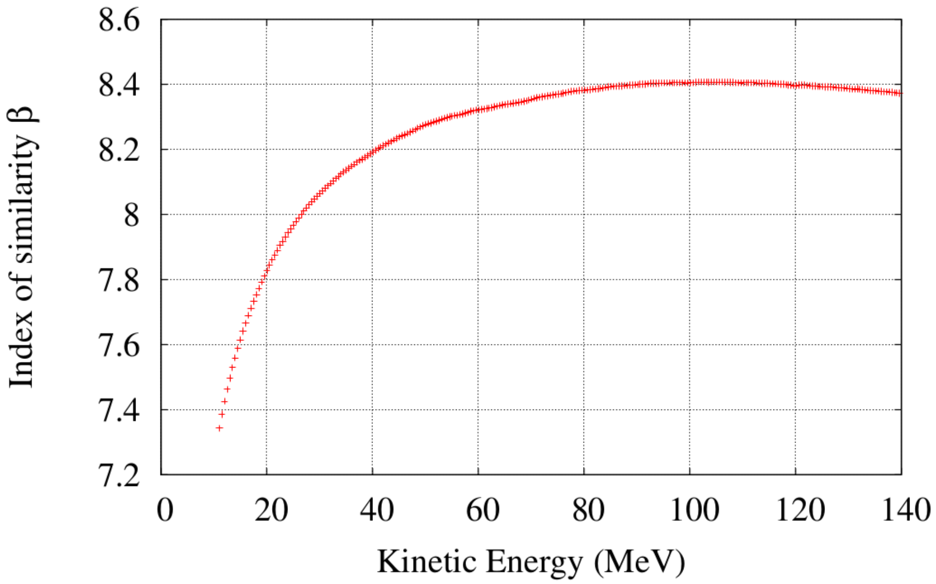
$$\nu_x^2(E) = \beta_F(E) + 2\beta_D(E) + \beta_{drift}(E) - \alpha_F(E)k_F(E) - 2\alpha_D(E)k_D(E) - \alpha_{drift}(E)k_{drift}(E)$$
$$\nu_y^2(E) = \alpha_F(E)k_F(E) + 2\alpha_D(E)k_D(E) + \alpha_{drift}(E)k_{drift}(E) + \mathcal{F}^2$$

Why the Symon formula fails?

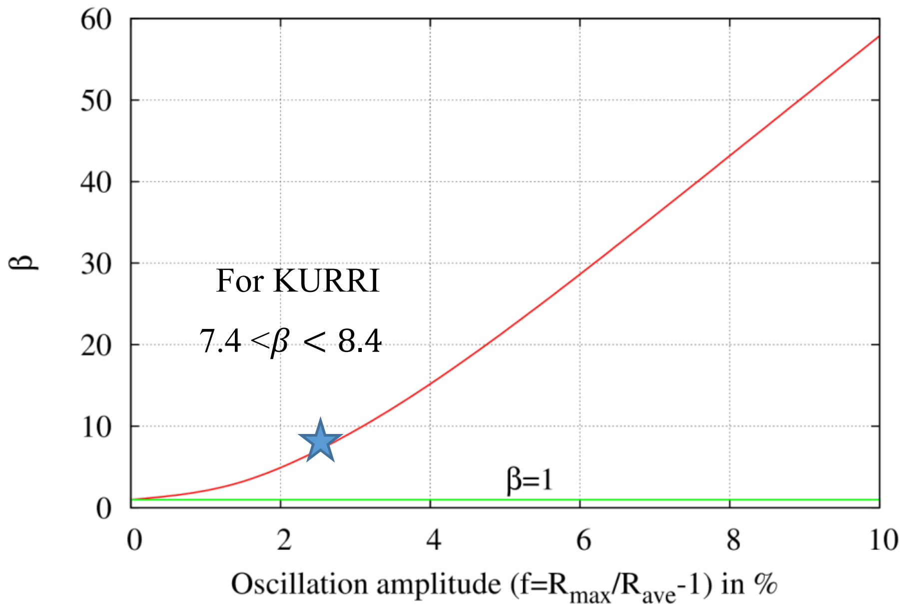


Application to the KURRI 150 MeV FFAG

$$\nu_x^2(E) = \beta_F(E) + 2\beta_D(E) + \beta_{drift}(E) - \alpha_F(E)k_F(E) - 2\alpha_D(E)k_D(E) - \alpha_{drift}(E)k_{drift}(E)$$
$$\nu_y^2(E) = \alpha_F(E)k_F(E) + 2\alpha_D(E)k_D(E) + \alpha_{drift}(E)k_{drift}(E) + \mathcal{F}^2$$



Index of similarity β computed from the TOSCA fieldmap tracking



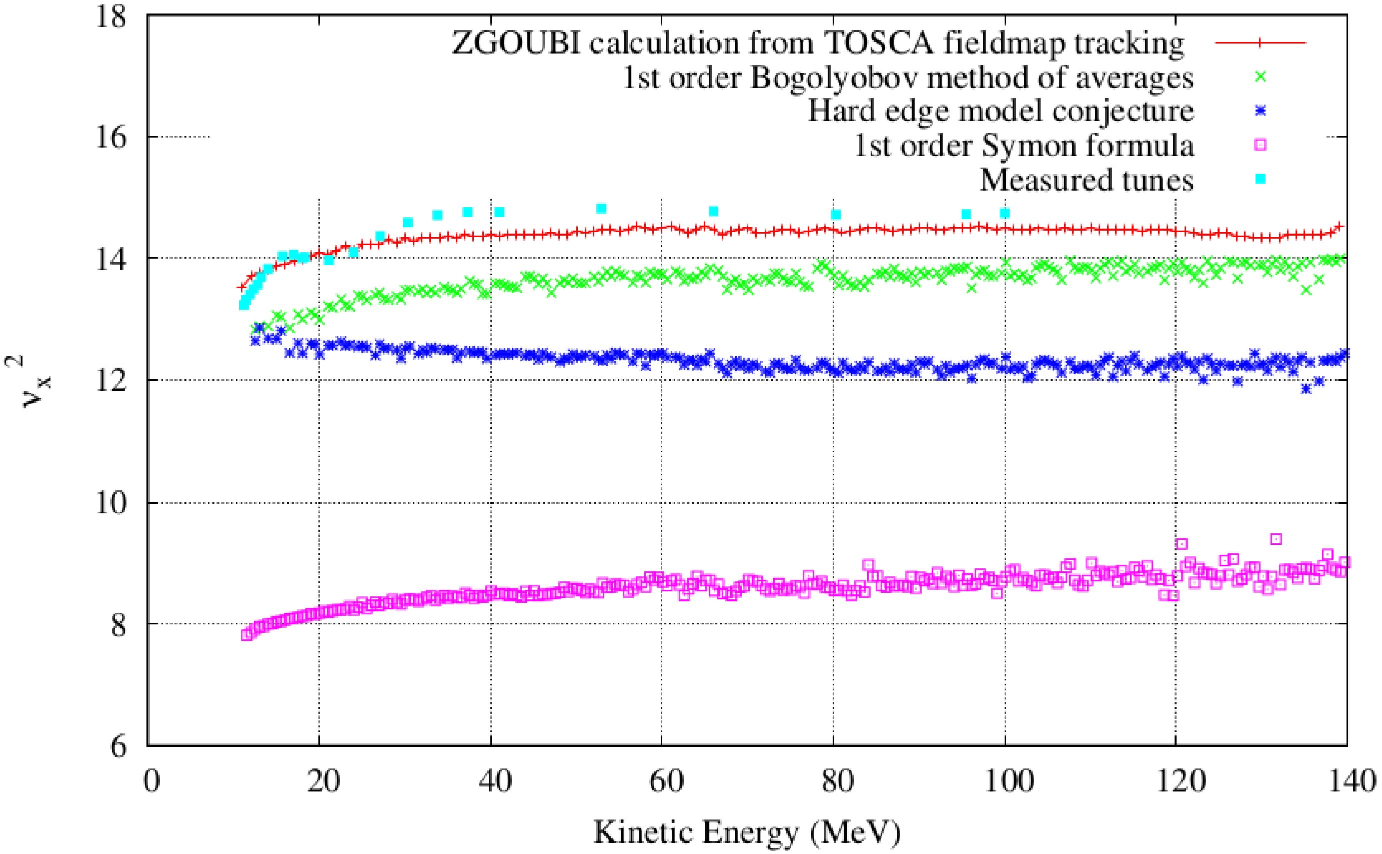
Analytical expression of β as a function of the scalloping of the orbit.

Does this explain why the tunes increase with the energy in the horizontal plane?

Application to the KURRI 150 MeV FFAG

$$\nu_x^2(E) = \beta_F(E) + 2\beta_D(E) + \beta_{drift}(E) - \alpha_F(E)k_F(E) - 2\alpha_D(E)k_D(E) - \alpha_{drift}(E)k_{drift}(E)$$
$$\nu_y^2(E) = \alpha_F(E)k_F(E) + 2\alpha_D(E)k_D(E) + \alpha_{drift}(E)k_{drift}(E) + \mathcal{F}^2$$

Yes, the monotonic behavior of the tunes in the horizontal plane is due to the increase of the horizontal restoring force due to the non-scaling of the orbits.



Key finding

I. *In presence of scaling imperfections, the number of betatron oscillations per turn* ***increases*** *(resp decreases) with the energy* ***if*** $\boldsymbol{\kappa = k_D - k_F > 0}$ *(resp* $\boldsymbol{\kappa < 0}$*).*

II. *Besides, the variations of the square of the number of betatron oscillations are, to the first order, proportional to* $|\boldsymbol{\kappa}|$.

These results are obtained from simulations using different fringe fields.
Can we prove the above results analytically?

Reminder, for KURRI, $k_D \sim 9$ while $k_F \sim 7.6$ thus $\kappa > 0$.

Proof 1/2

- It can be shown, from what preceded that the tune variations in the horizontal plane are pre-dominantly due to the change of the horizontal restoring force due to the non-scaling of the orbits:

$$\boxed{\beta_F^{ns} \approx \left(\frac{R_E^{ns}}{\rho_F^{ns}}\right)^2 = \left(\frac{A_4}{\left[1 + A_3\left(\frac{R}{R_0}\right)^{\kappa}\right]}\right)^2 \beta_F^s}$$

$$A_3 = \frac{< B_D >}{< B_F >} < 0$$

$$A_4 = 1 + A_3 > 0$$

- Similarly, in the vertical plane, the tune variations are pre-dominantly due to the change of the magnetic flutter due to the non-scaling of the orbits:

Only valid if $F_F(\theta).F_D(\theta) = 0$!

$$\boxed{\mathcal{F}^2 = \frac{\langle B^2\rangle}{\langle B\rangle^2} - 1 = \frac{\langle F_F^2(\theta)\rangle + (A_3)^2 \langle F_D^2(\theta)\rangle \left(\frac{R}{R_0}\right)^{2\kappa}}{\langle F_F(\theta)\rangle^2 \left[1 + A_3\left(\frac{R}{R_0}\right)^{\kappa}\right]^2} - 1}$$

This proves the first part of the property.

Proof 2/2

- It can also be shown, from the previous results that:

$$\Delta\left({\nu_x}^2\right) \approx \left|\beta_F^{ns}(max) - \beta_F^{ns}(min)\right| = \left| \frac{A_4^2 \beta_F^s}{\left[1 + A_3\left(\frac{R}{R_0}\right)^{\kappa}\right]^2} - \frac{A_4^2 \beta_F^s}{(1+A_3)^2} \right|$$

$$\approx \left|\frac{2A_3\beta_F^s}{1+A_3}\right| \frac{\Delta R}{R_0} |\kappa| \propto |\kappa|$$

R_0 is the injection radius

$$\Delta\left({\nu_y}^2\right) \approx \left|\mathcal{F}^2(max) - \mathcal{F}^2(min)\right| = \left| \frac{\left\langle F_F^2(\theta)\right\rangle + (A_3)^2\left\langle F_D^2(\theta)\right\rangle \left(\frac{R}{R_0}\right)^{2\kappa}}{\left[1 + A_3\left(\frac{R}{R_0}\right)^{\kappa}\right]^2} - \frac{\left\langle F_F^2(\theta)\right\rangle + (A_3)^2\left\langle F_D^2(\theta)\right\rangle}{\left[1+A_3\right]^2} \right|$$

$$\approx \frac{2\left[|A_3|\left\langle F_D^2(\theta)\right\rangle + \left\langle F_F^2(\theta)\right\rangle\right]}{(1+A_3)^3} |A_3| \frac{\Delta R}{R_0} |\kappa| \propto |\kappa|$$

This proves the second part of the property. Furthermore, one demonstrated that the tune variations increase linearly with the radial excursion of the magnets in both planes. Hence the interest of scaling FFAG with small orbit excursion (second stability region).

Vanishes when the FD ratio goes to zero. Expected since one of the two magnets is removed!

Verification tests 1/3

- Start from a scaling FFAG lattice: $k_F = k_D = 7.6$ and $A_3 = -0.55$

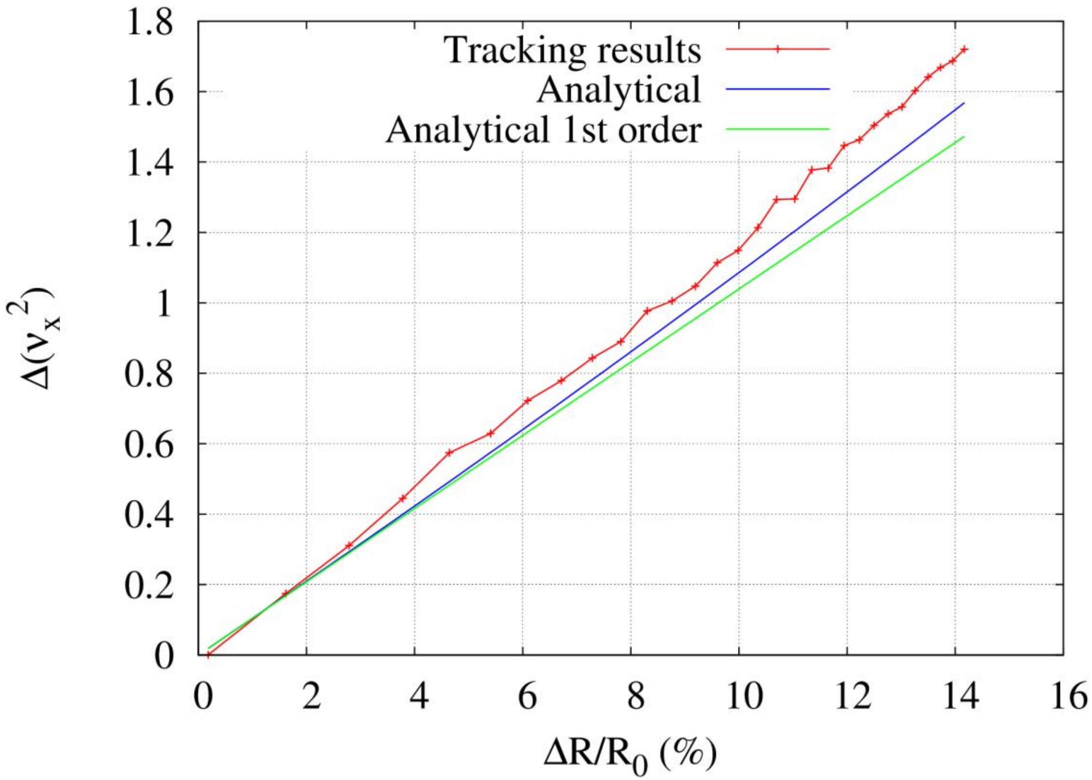
This yields ${\beta_F}^s = 10.7$

- Introduce an error such that $\boldsymbol{\kappa = 0.4}$

As shown earlier,

$$\Delta\left({\nu_x}^2\right) \approx \left|\frac{2A_3\beta_F^s}{1+A_3}\right| \frac{\Delta R}{R_0}|\kappa| \propto |\kappa|$$

Good agreement between the numerical simulations and the analytical formula.



Verification tests 2/3

- ❑ Start from a scaling FFAG lattice: $k_F = k_D = 7.6$ and $A_3 = -0.55$

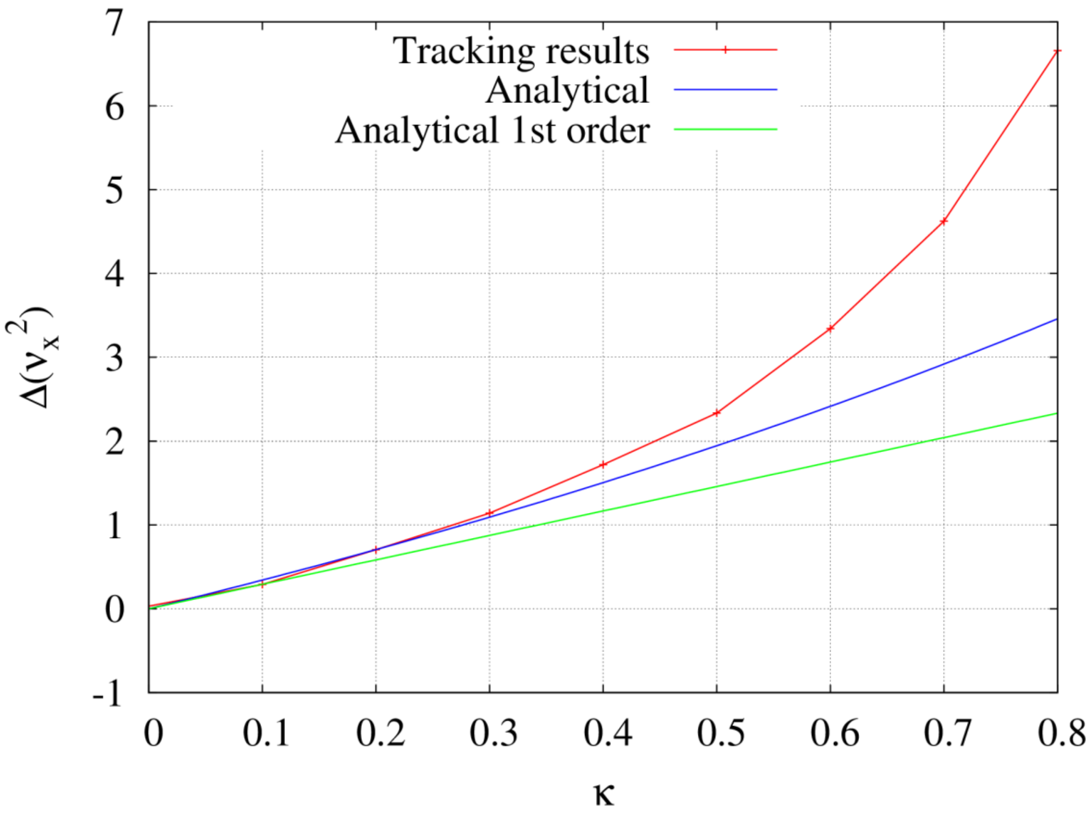
This yields ${\beta_F}^s = 10.7$

- ❑ Introduce an error such that $\boldsymbol{\kappa > 0}$

As shown earlier,

$$\Delta\left({\nu_x}^2\right) \approx \left|\frac{2A_3\beta_F^s}{1+A_3}\right|\frac{\Delta R}{R_0}|\kappa| \propto |\kappa|$$

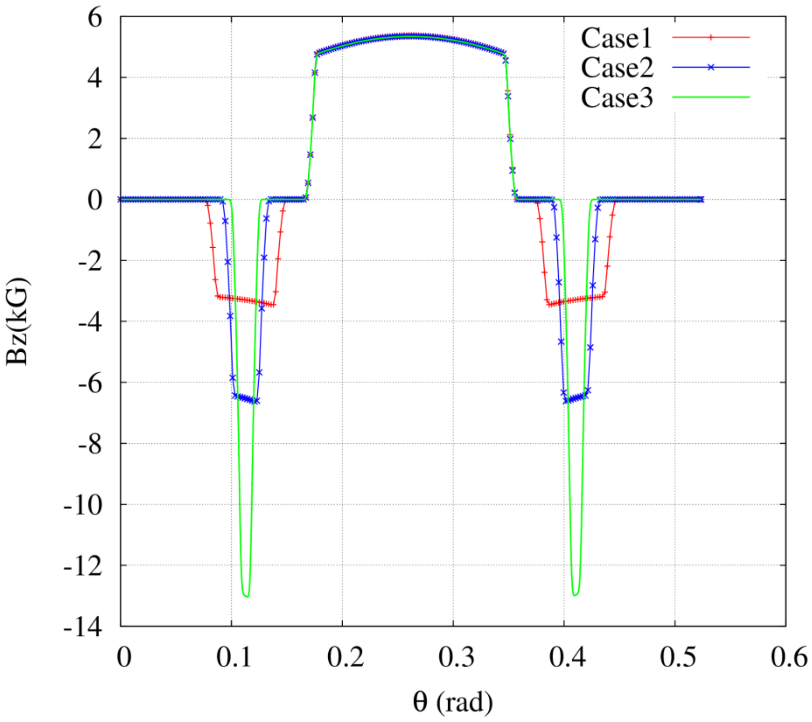
Good agreement except for large $\boldsymbol{\kappa}$-values .



Verification tests 3/3

For the same FD ratio at injection and the same average field index of the magnet (or same type of error), is there a field profile of the magnets that is less sensitive to these types of imperfections?

Keeping the same average magnetic flutter for the D-magnet means the width of the magnet can be reduced while increasing the amplitude of its field.



Field along the orbit in one cell.

Verification tests 3/3

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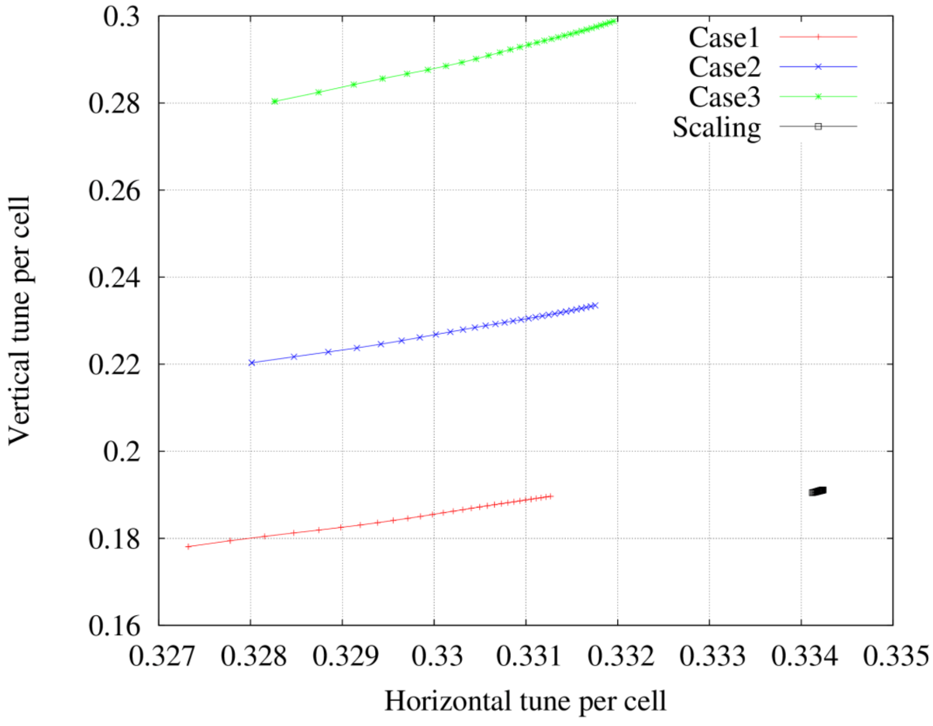
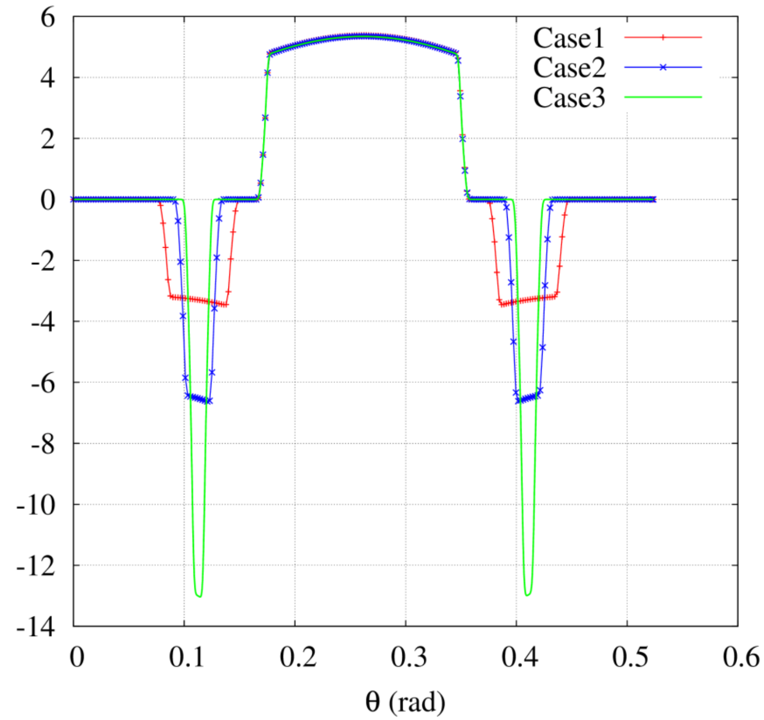


Tune diagram for various cases.



Field along the orbit in one cell.

Verification tests 3/3

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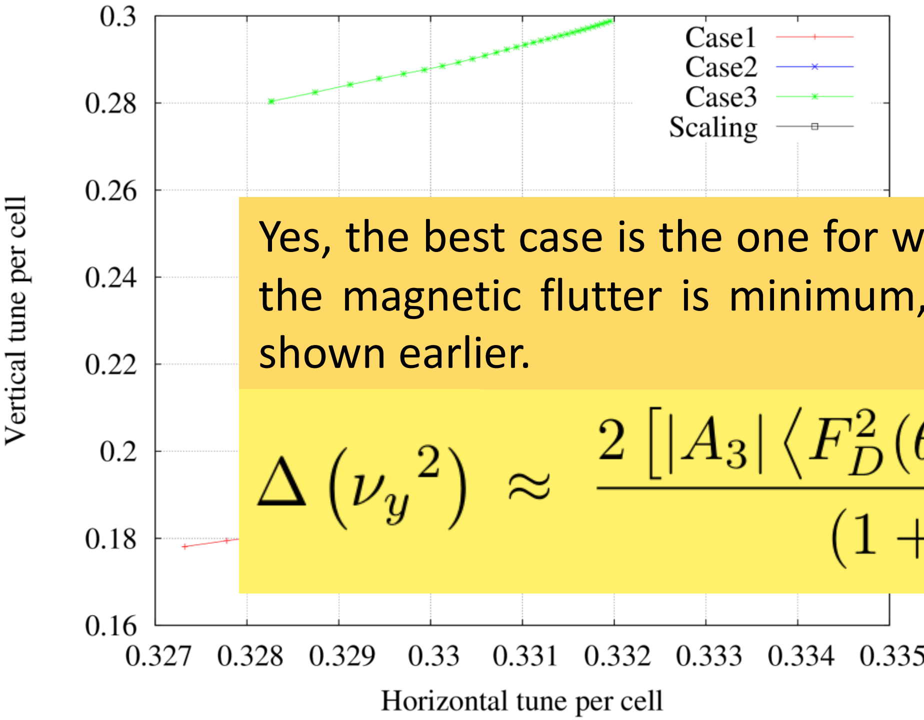
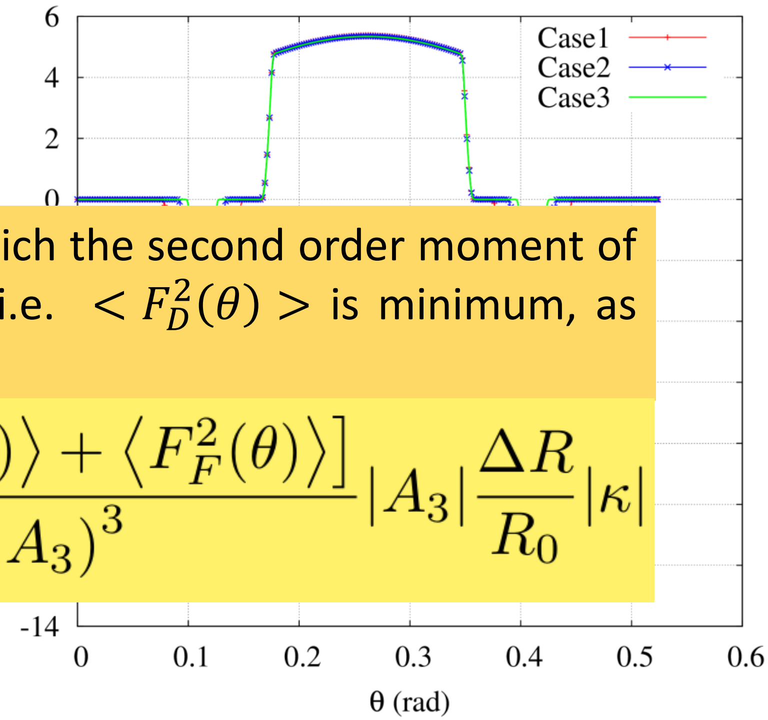




Yes, the best case is the one for which the second order moment of the magnetic flutter is minimum, i.e. $< F_D^2(\theta) >$ is minimum, as shown earlier.

$$\Delta\left(\nu_y{}^2\right) \approx \frac{2\left[|A_3|\left\langle F_D^2(\theta)\right\rangle + \left\langle F_F^2(\theta)\right\rangle\right]}{\left(1+A_3\right)^3}|A_3|\frac{\Delta R}{R_0}|\kappa|$$

Tune diagram for various cases.

Field along the orbit in one cell.

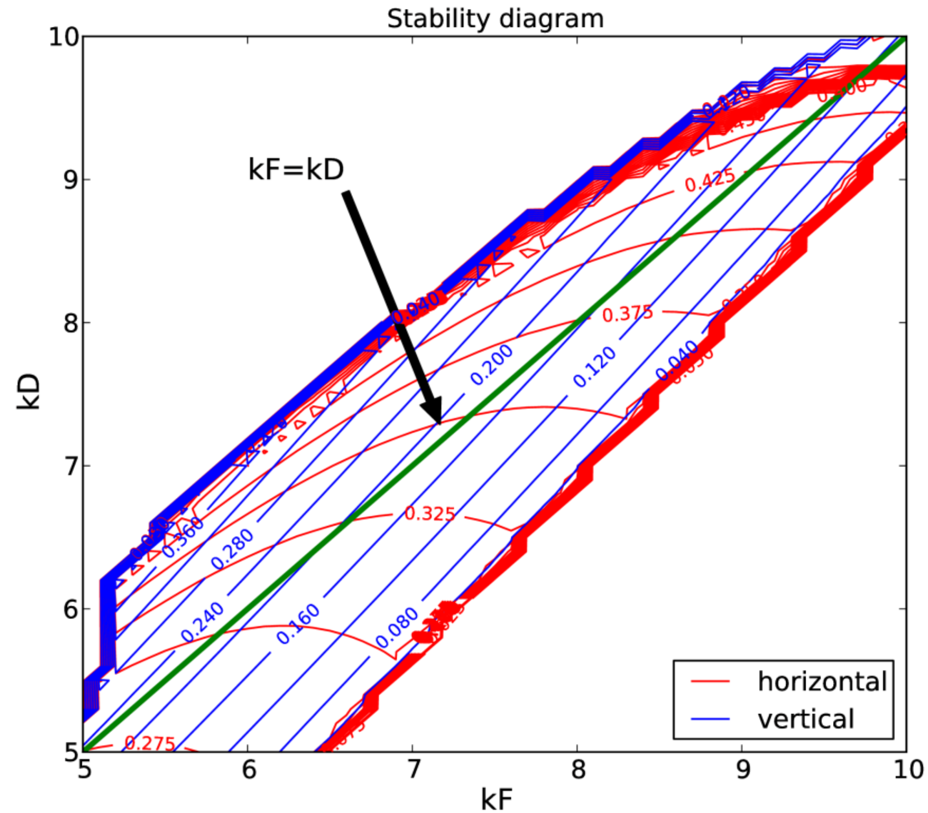
Explaining the stability diagram

Upper left side: radial pi-mode stop band resonance, or phase advance=180 deg.

Lower right side: vertical phase advance=0 deg.

Why does it shrink linearly?

$$\begin{cases} |\kappa| = \dfrac{\left(\dfrac{N}{2}\right)^2 - \nu_x^2}{B\dfrac{\Delta R}{R_0}} \approx \dfrac{\left(\dfrac{N}{2}\right)^2 - k - \beta^s}{B\dfrac{\Delta R}{R_0}} \\[2em] |\kappa| = \dfrac{\nu_y^2}{A\dfrac{\Delta R}{R_0}} \approx \dfrac{-k + \mathcal{F}^2}{A\dfrac{\Delta R}{R_0}} \end{cases}$$



Large *κ*-values are less tolerant to field imperfections. However ..

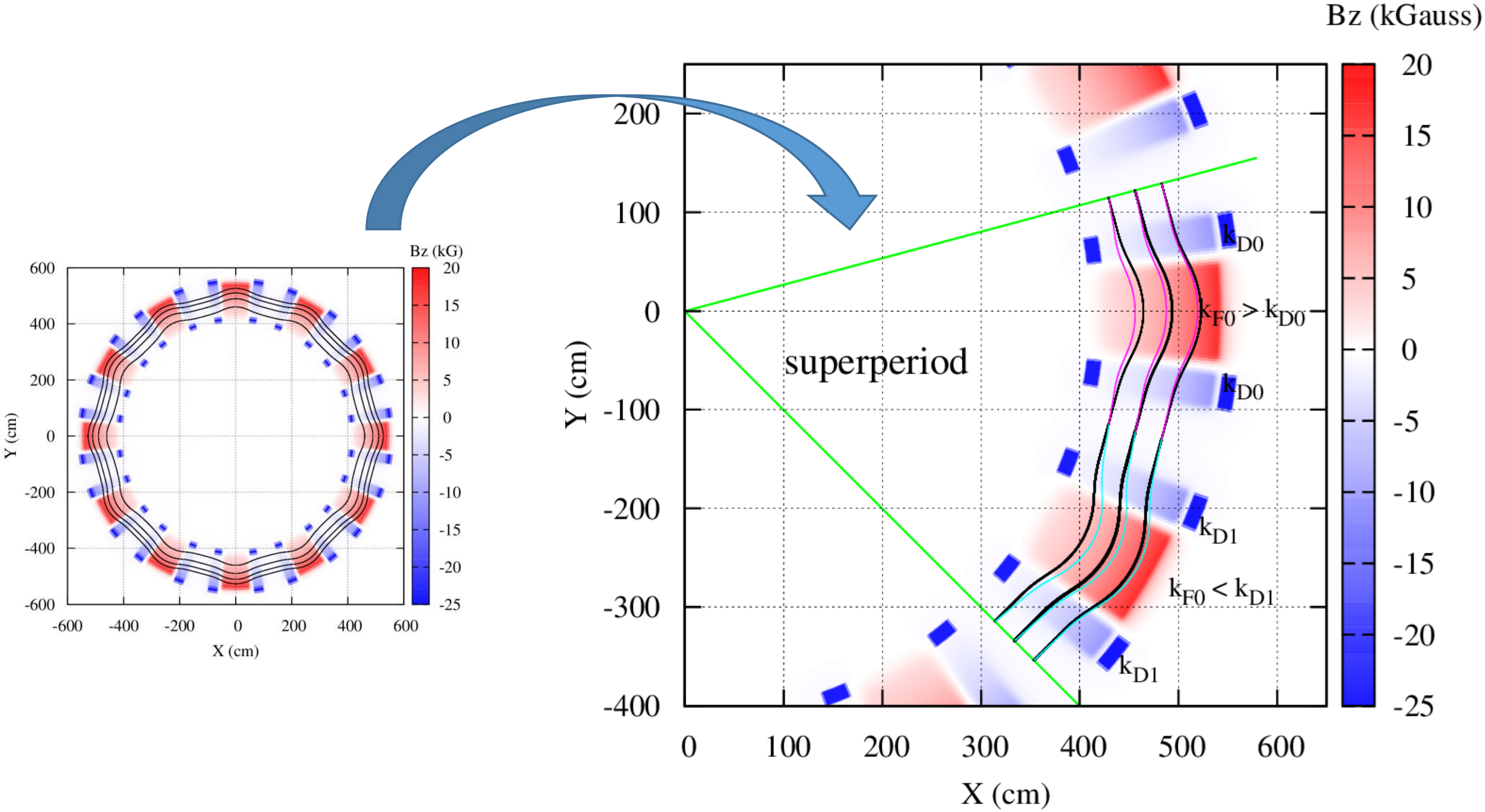
Compensation of tune variations in FFAG

I. *In presence of scaling imperfections, the number of betatron oscillations per turn* ***increases*** *(resp decreases) with the energy* ***if*** $\boldsymbol{\kappa = k_D - k_F > 0}$ *(resp* $\boldsymbol{\kappa < 0}$*).*

II. *Besides, the variations of the square of the number of betatron oscillations are, to the first order, proportional to* $|\boldsymbol{\kappa}|$*.*

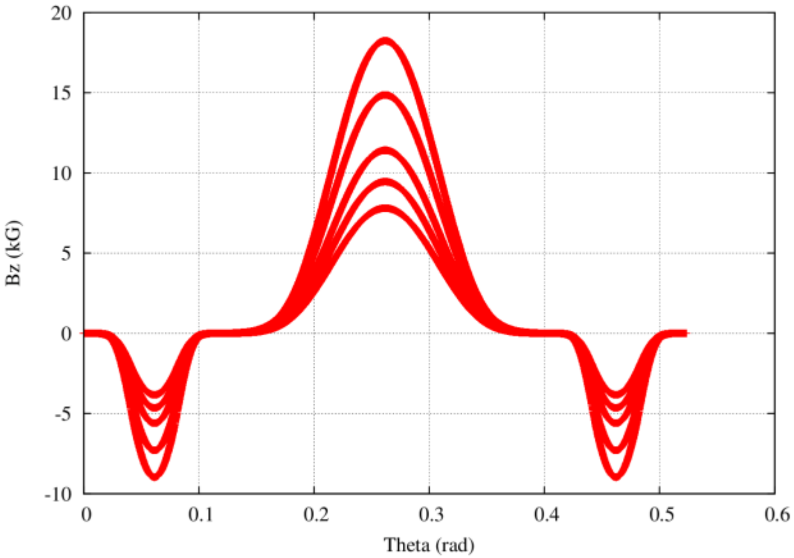
- **Main idea: if we create alternating scaling imperfections, by alternating κ, one may obtain a fixed tune machine.**

Concept of the non-scaling fixed tune FFAG

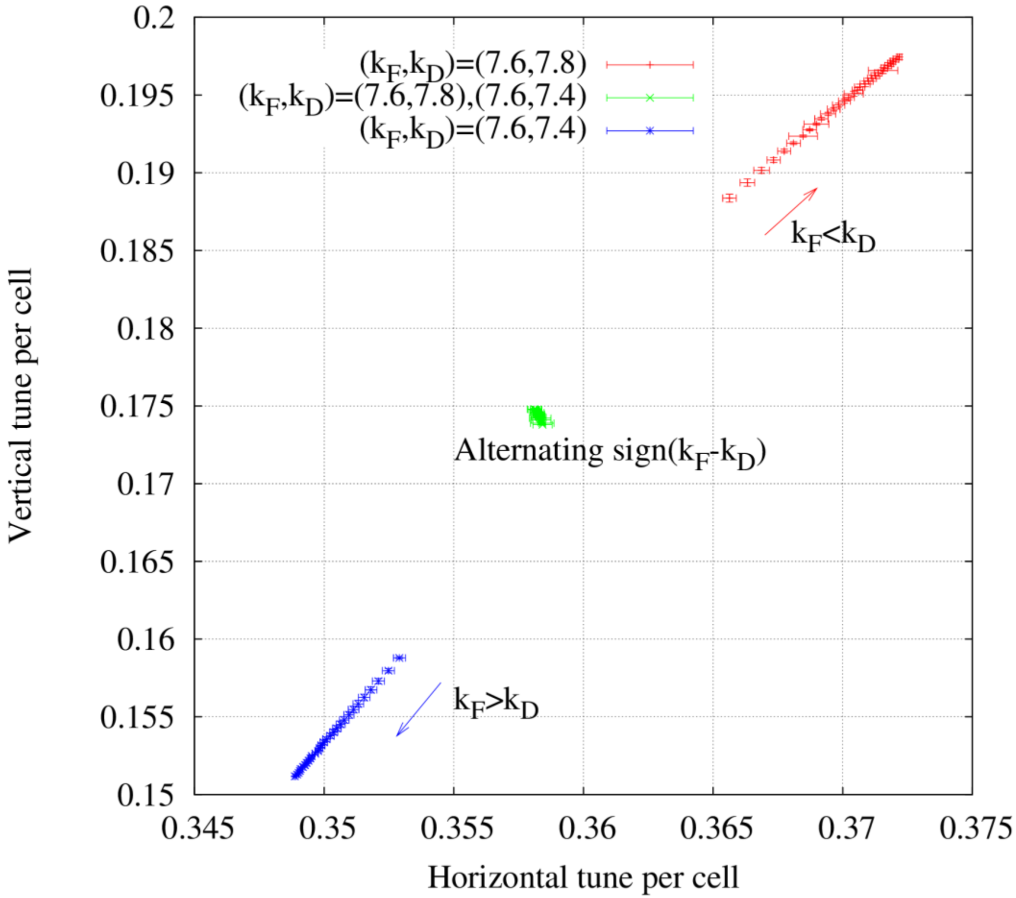


Replace the 12-fold symmetry by a 6-fold symmetry.

Concept of the non-scaling fixed tune FFAG

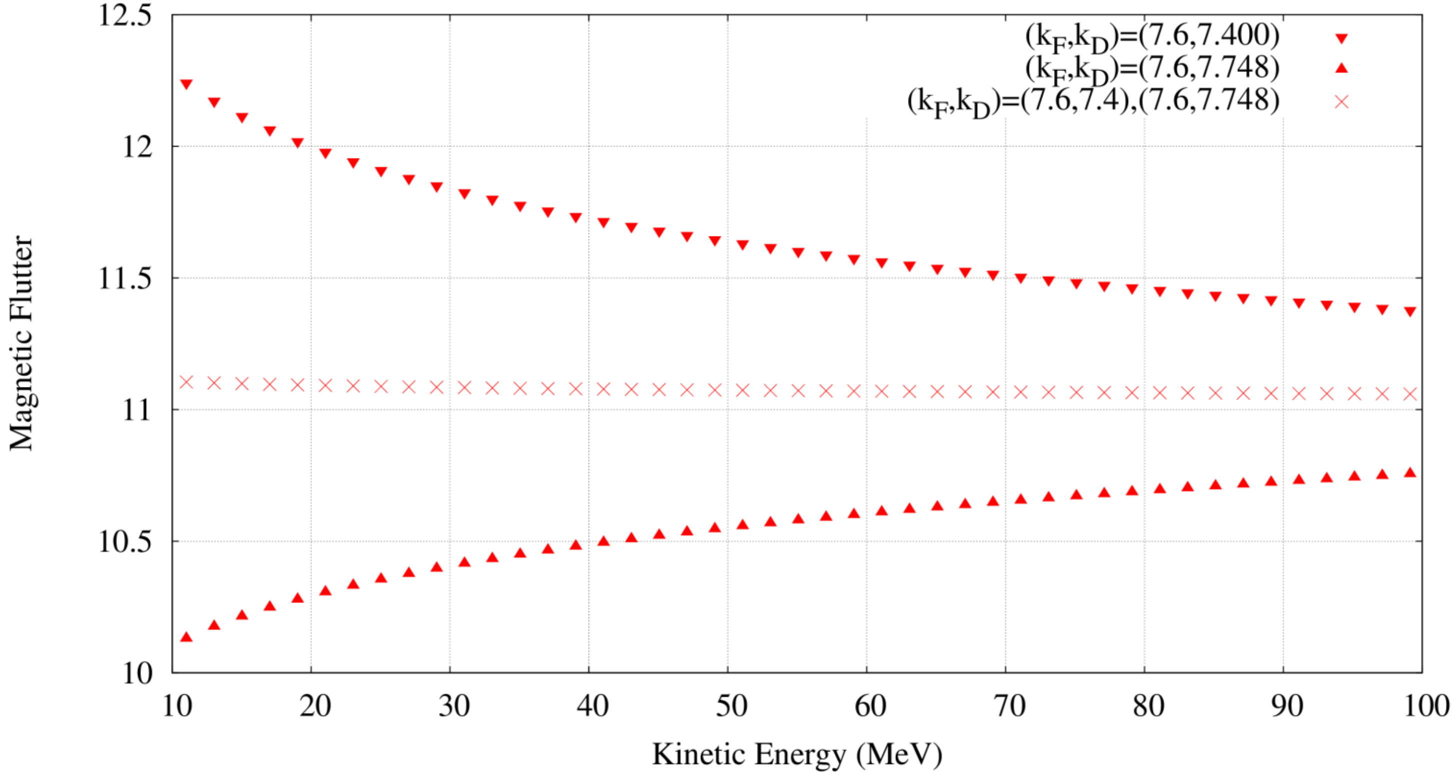


Magnetic field along several closed orbits in the ZGOUBI model.



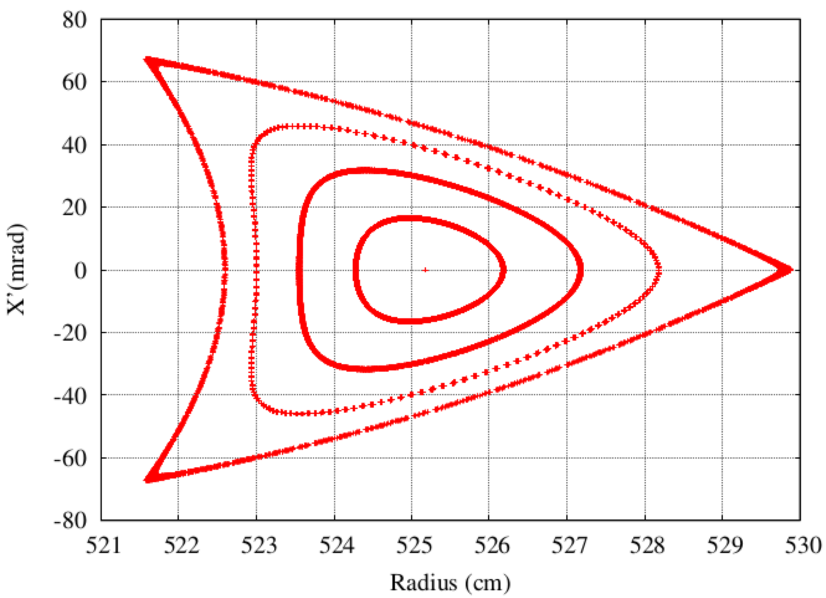
Using the mid-plane field map (or the analytical model FFAG), one can show that alternating κ allows to alternate the phase advance between the upper and lower regions so that the tune becomes constant.

Concept of the non-scaling fixed tune FFAG

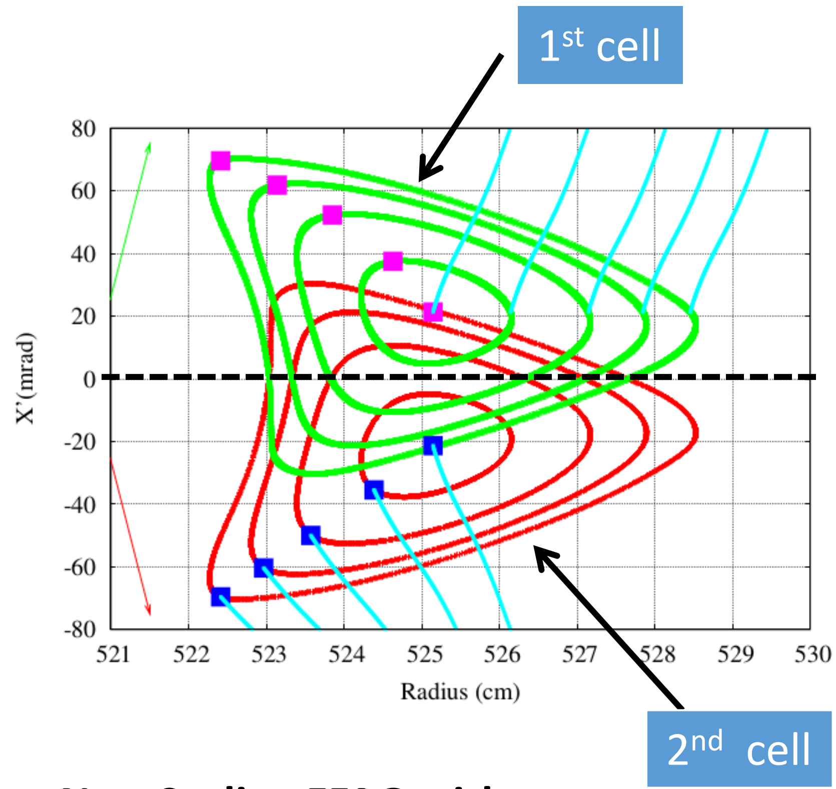


The magnetic flutter of the orbits is not energy-independent inside any magnet. However, it becomes constant when calculated over the width of the two DFD triplets combined.

Phase space



Scaling FFAG with $k_F = k_D = 7.6$

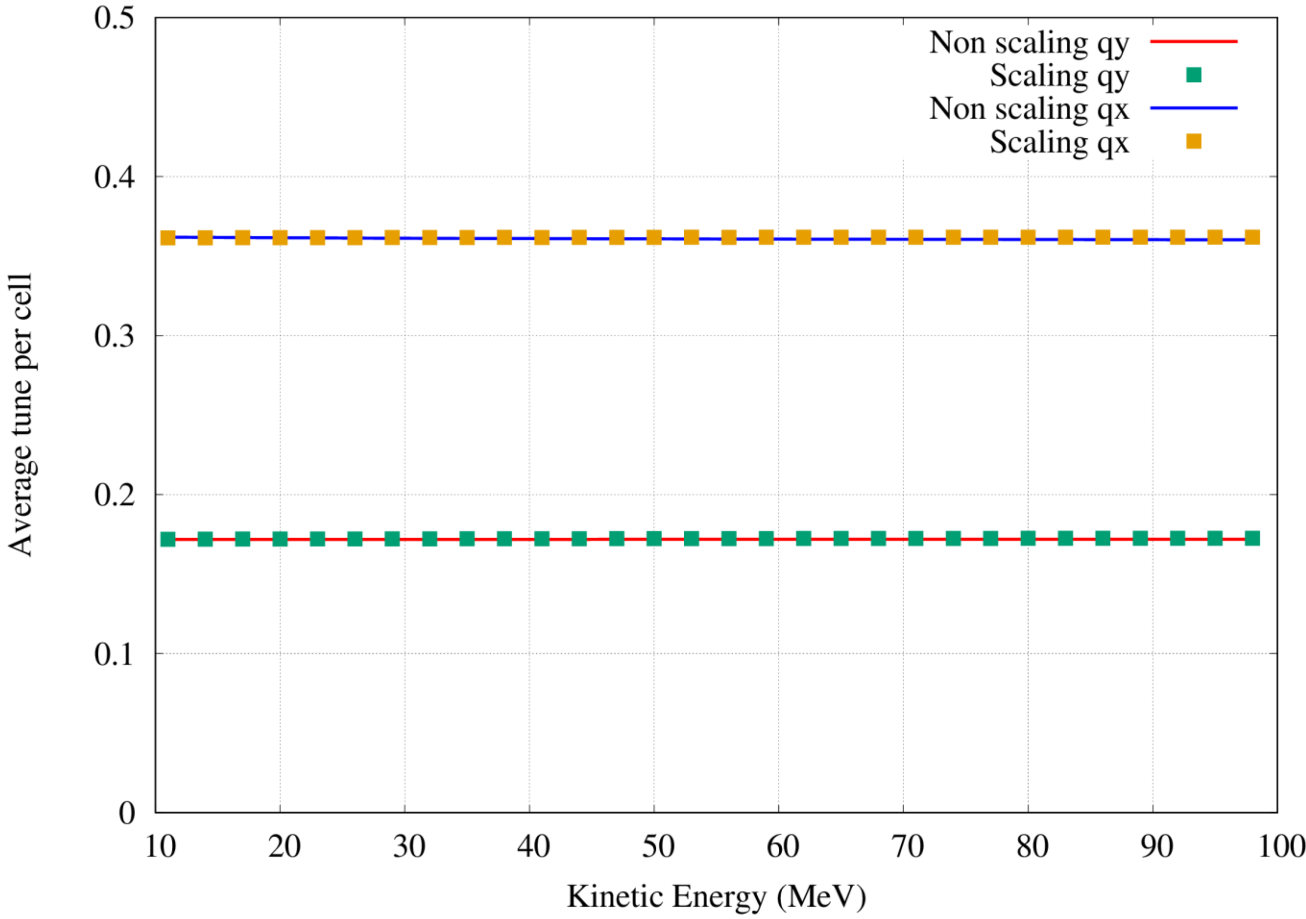


Non-Scaling FFAG with
$(k_F, k_D) = (7.6, 7.4), (7.6, 7.8)$

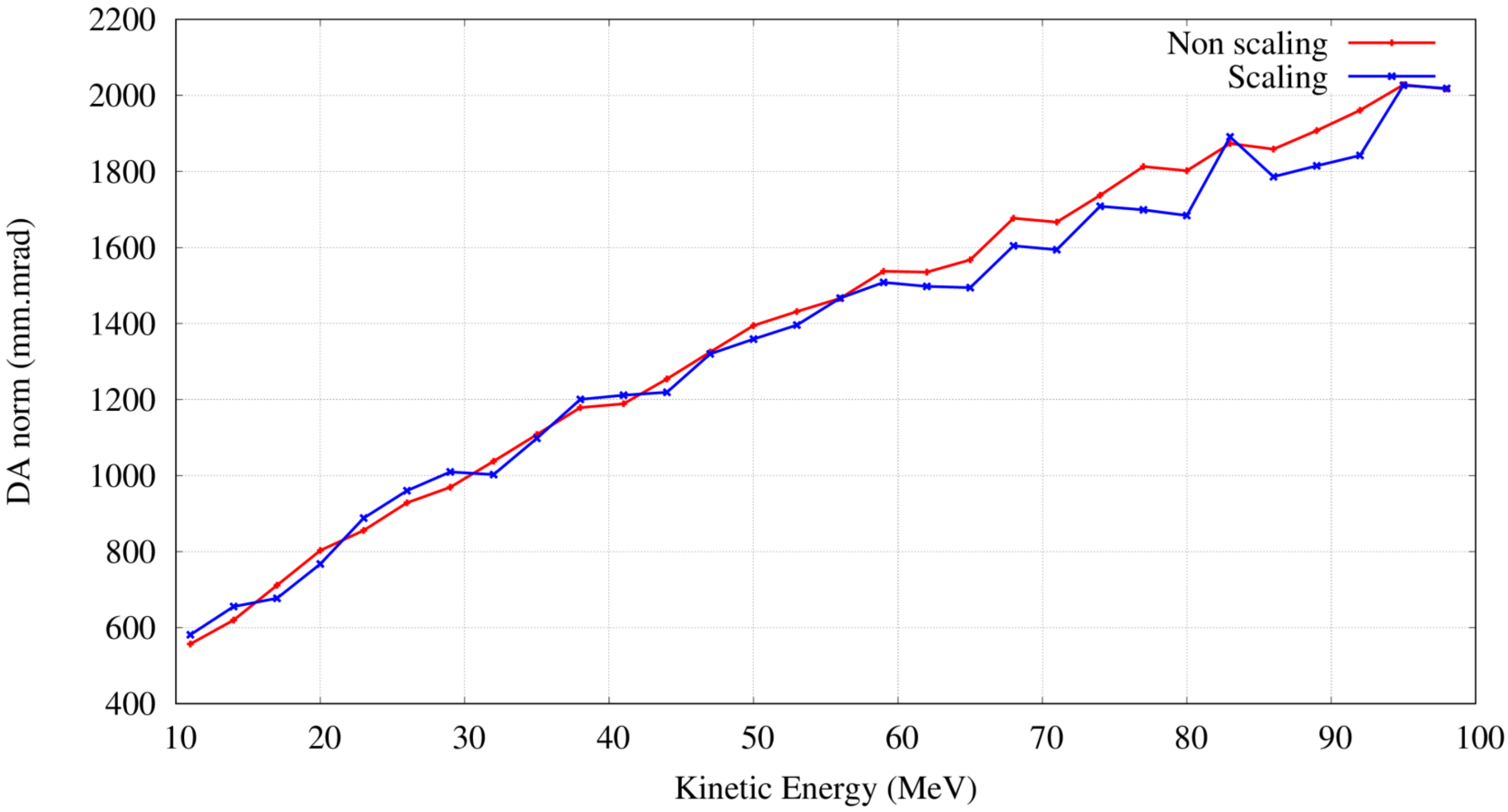
Horizontal phase space at 100 MeV including the separatrix. The limit of stable motion is obtained when the particle is lost in less than 1000 turns.

Dynamic Acceptance

Generate a scaling FFAG lattice and a non-scaling one that have the same fixed tunes.



Dynamic Acceptance



For the same tunes, the Dynamic Acceptance of the scaling and the non-scaling lattice are the same.

Next

- Reverse the problem: defining arbitrary functions of the tunes in both planes, can we obtain the expression of the magnetic field?

Thank you!

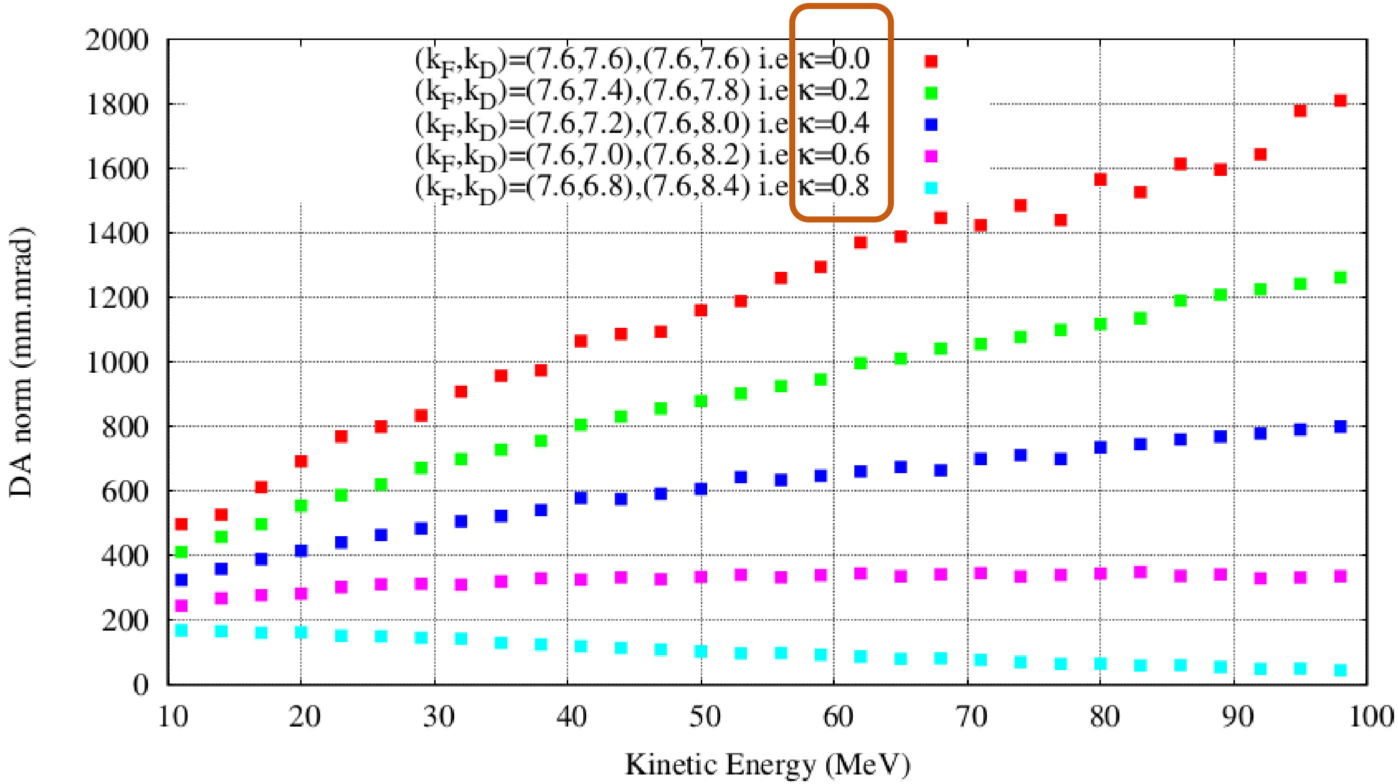
Backup slides

DA of the non-scaling FFAG

$\kappa = k_D - k_F$

$\kappa = 0 \Rightarrow$ ***scaling***

$\kappa \neq 0 \Rightarrow$ ***non-scaling***



Although large κ values can be explored and the stability limit overcome, the main finding is that the DA decreases with increasing κ.

The amplitude of κ should not exceed 10 % of the average field index of the magnets in order to maintain a reasonably large horizontal DA (> 50 π mm.mrad).