Controlling correlations and excited states in Lattice QCD analyses - a novel approach

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Outline

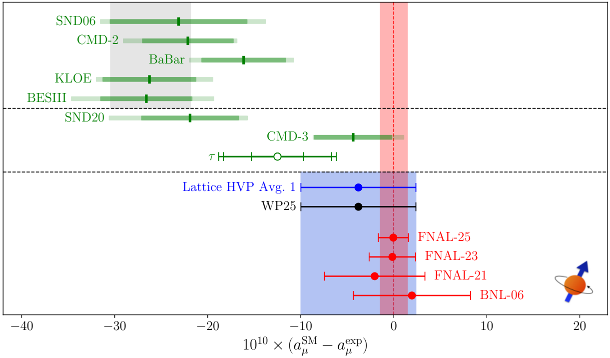
1. Introduction and motivation
2. Laplace filtering
3. Application to data
4. Annihilating correlation functions
5. Excited states
6. Conclusions and outlook

Introduction and motivation

Testing the Standard Model: Tensions?!

We know the SM is not the end of the story. Look for tensions:

- ~~$g-2$~~



[2505.21476 (Theory Initiative WP2)]

⇒ new results from BMW, Mainz, RBC/UKQCD, Fermilab/MILC/HPQCD, ETMC]

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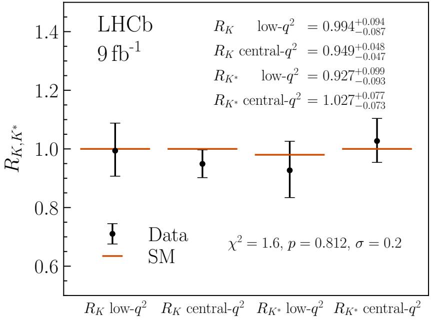
- ~~$R(K)$~~, ~~$R(K^*)$~~



LHCb PRD 108 (2023) 3, 032002

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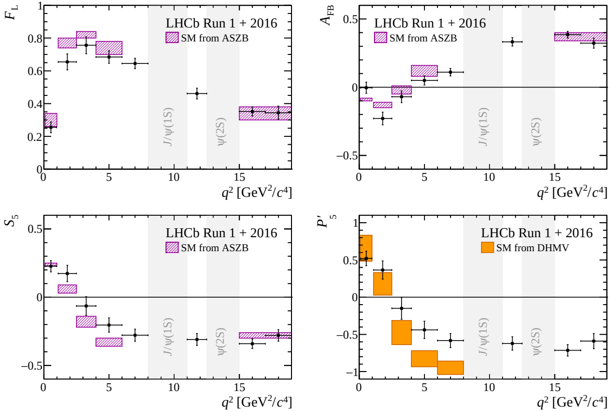

LHCb PRL 125 (2020) 1, 011802

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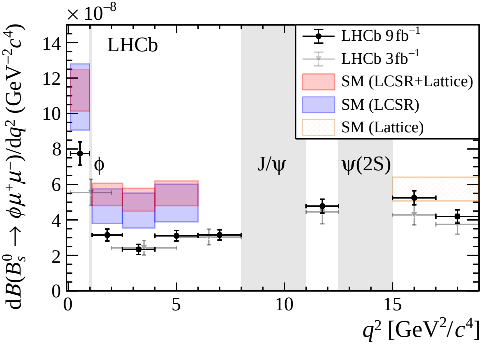
- $B_s \to \phi\ell\ell$



LHCb PRL 127 (2021) 15, 151801

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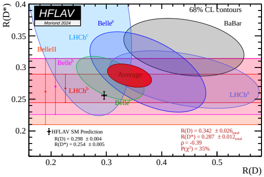
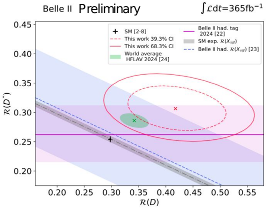
- $R(D)$, $R(D^*)$





(new result by Belle II presented by Tommy Martinov at EW Moriond)

Testing the Standard Model: Tensions?!

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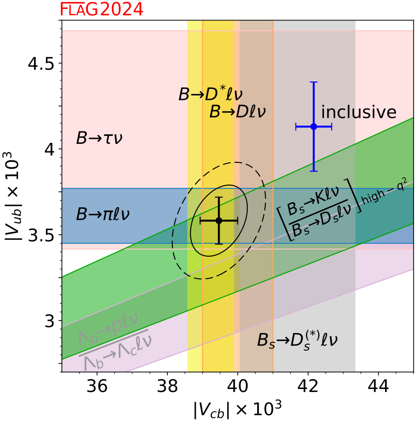
- $B \to K^* \ell\ell$
- $B_s \to \phi\ell\ell$
- $R(D)$, $R(D^*)$
- inclusive vs exclusive
- CKM unitarity?
- ...



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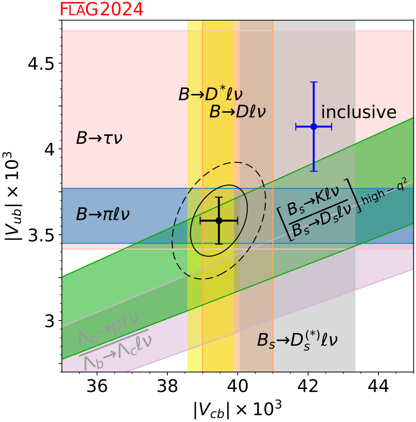


☹ Several theory-to-theory tensions ☹

Over-constraining the Standard Model: CKM

- Determine individual elements from (many) different decays $\Rightarrow$ Different determinations compatible?
- SM CKM-matrix is unitary $\Rightarrow$ Does unitarity hold?

CKM = theory $\otimes$ experiment

$$\text{experiment} \approx \left|V_{qq'}\right|^n \sum_i \text{kinematic} \times \text{non-perturbative}$$

$$\Gamma(B \to \ell\nu) \approx |V_{ub}|^2 \mathcal{K} f_B$$

$$\frac{\mathrm{d}\Gamma(B \to \pi\ell\nu)}{\mathrm{d}q^2} \approx |V_{ub}|^2 \left(\mathcal{K}_1 f_+^{B\to\pi}(q^2) + \mathcal{K}_2 f_0^{B\to\pi}(q^2)\right)$$

$$\Delta m_d \approx |V_{tb}^* V_{td}|^2 \mathcal{K} f_{B_d}^2 \hat{\mathcal{B}}_{B_d}^{(1)}$$

CKM-precision depends on knowledge of theory AND experiment.

Lattice QCD: From simulations to physics

Non-perturbative method of choice: **Lattice QCD**

Schematic lattice workflow

1. (H)MC chain
   $\Rightarrow$ Gauge ensembles
2. measurements
   $\Rightarrow$ *n*-point correlation functions
3. **correlation function fits**
   $\Rightarrow$ energies, matrix elements
4. extrapolations
   $\Rightarrow xx.xxx(yy)_{\rm stat}$
5. estimates of systematics
   $\Rightarrow xx.xxx(yy)_{\rm stat}(zz)_{\rm sys}$

Data production:
Take as given for this talk

Data analysis:
Topic of this talk.

In particular

- $\Rightarrow$ steps that use a data-estimated correlation matrix.
- $\Rightarrow$ fits to correlation functions. [required for all results presented before]

Some (well-known) nomenclature

Assume raw data on N samples for N_t time slices: $C_i(t)$

- Averaged correlation function

$$\bar{C}(t) = \frac{1}{N} \sum_{i=0}^{N-1} C_i(t)$$

- Covariance matrix

$$\mathrm{cov}(t_1, t_2) = \frac{1}{N-1} \sum_{i=0}^{N-1} \left[\bar{C} - C_i\right](t_1) \left[\bar{C} - C_i\right](t_2)$$

- Standard deviation

$$\sigma(t_1) = \sqrt{\mathrm{cov}(t_1, t_1)}$$

- Correlation matrix ("normalised covariance matrix")

$$\mathrm{corr}(t_1, t_2) = \frac{1}{\sigma(t_1)} \mathrm{cov}(t_1, t_2) \frac{1}{\sigma(t_2)}$$

Where does the correlation matrix enter?

- For a fit function f, "best" parameters $\vec{a}$ found by χ^2-fitting, i.e. minimise

$$\chi^2 = [f(\vec{a}, t_1) - C(t_1)] \, \mathrm{cov}(t_1, t_2)^{-1} \, [f(a, t_2) - C(t_2)]$$

$p(\chi^2, \mathrm{d.o.f's})$ used as criterion for which fits "work"

- Many recent works use model averaging procedures (e.g. AIC):
$\Rightarrow$ Do many different fits (vary models and included data)
$\Rightarrow$ Results get combined weighted by their χ^2

Note: Inversion of covariance matrix == inversion of correlation matrix

$$\mathrm{cov}^{-1} = \sigma^{-1} \mathrm{corr}^{-1} \sigma^{-1}$$

$\Rightarrow$ From now on we only speak about the correlation matrix!

Correlation estimation from data

- The correlation matrix is estimated from data.
- Want $N_{\mathrm{conf}} \gg N_{\mathrm{obs}}$ for accurate estimates, but often hard in practice.

$\Rightarrow$ Non-invertible or poorly conditioned covariance matrix.

$\Rightarrow$ Issues amplified since usually we require corr^{-1}!

- Several ways to "deal with" the issue [non-exhaustive list]:
  - down-sampling (a.k.a. thinning)
  - uncorrelated fits [Bruno, Sommer; "On fits to correlated and auto-correlated data", 2209.14188]
  - SVD-cuts
  - "shrinkage" [Ledoit, Wolf, https://www.zora.uzh.ch/id/eprint/139880/]
  - source-place treatment [JTT, RBC/UKQCD, "Kaon mixing beyond the standard model with physical masses", 2404.02297]
  - estimation of p-values based on bootstrapping [Christ, Ernaki, Kelly; "Bootstrap-determined p values in lattice QCD", 2409.11379]
  - Lanczos-methods [Wagman, 2406.20009, 2412.04444, . . .; Ostmeyer, Sen, Urbach, "On the equivalence of Prony and Lanczos methods for Euclidean correlation functions", 2411.14981]

$\Rightarrow$ **All not 100% satisfactory!** Can we do better?

Laplace filtering

The idea

- Find a transformation of the data to reduce correlations.
- Transform the model in the same way.

$\Rightarrow$ Fit the transformed model to the transformed data.

What transformation to choose?

- Lattice data is highly correlated in time!

$\Rightarrow$ Use a time-local transformation

Laplace filter

$$\Delta C(t) = C(t-1) - 2C(t) + C(t+1)$$

$$D_\lambda C(t) \equiv (-\Delta + \lambda^2)C(t) = -C(t-1) + (2+\lambda^2)C(t) - C(t+1)$$

Laplace filtering

$$D_\lambda C(t) = -C(t-1) + (2+\lambda^2)C(t) - C(t+1)$$

- local subtractions $\Rightarrow$ reduce local correlations
- tunable parameter λ (assume $\lambda > 0$ for definiteness)
- preserves original correlations for $\lambda \to \infty$
- invertible transformation for $\lambda > 0$, but singular as $\lambda \to 0^+$.

- "added bonus": Preserves exponential form, i.e.

$$D_\lambda \left(A e^{-Et} \right) = A'(\lambda) e^{-Et}$$

Aside: Measure of the level of correlations: CDR

- Determine eigenvalues of correlation matrix corr and define condition number: $\kappa \equiv \lambda_{\mathrm{max}}/\lambda_{\mathrm{min}}$
- Define Correlation Dynamic Range (CDR):

$$\mathrm{CDR}(\mathrm{corr}) = 10\log_{10}\kappa$$

$$\begin{aligned} \mathrm{CDR}(\mathrm{corr}) &= 0\,\mathrm{dB}: & \text{uncorrelated} \\ \mathrm{CDR}(\mathrm{corr}) &= +\infty\,\mathrm{dB}: & \text{singular} \\ \mathrm{CDR}(\mathrm{corr}) &\gtrsim 156\,\mathrm{dB}: & \text{double precision numerically singular} \end{aligned}$$

Limiting cases of the filtering:

$$\lim_{\lambda\to\infty} \mathrm{CDR}(\mathrm{corr}\,(\Delta_\lambda C(t)) = \mathrm{CDR}(\mathrm{corr}\,(C(t))$$

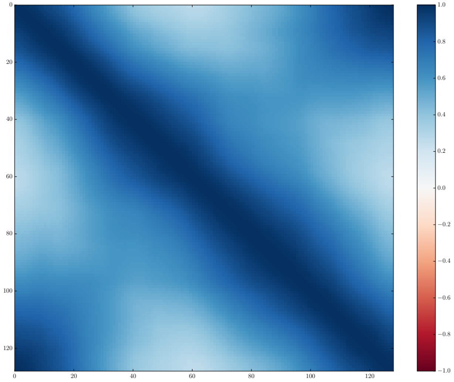
$$\lim_{\lambda\to 0^+} \mathrm{CDR}(\mathrm{corr}\,(\Delta_\lambda C(t)) = +\infty$$

Q: Is there a choice of λ that improves (lowers) the CDR?

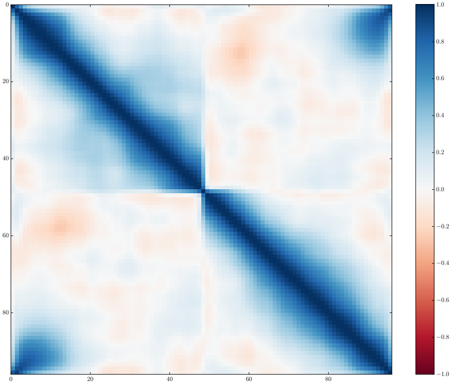
Application to data

Typical 2-point functions

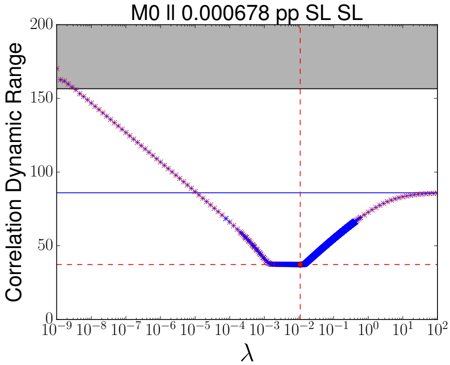
- $m_\pi^{\rm phys}$, $a^{-1} \sim 2.3\,\mathrm{GeV}$
- pion 2pt function
- Gaussian smeared $\mathbb{Z}_2$-sources + point sinks

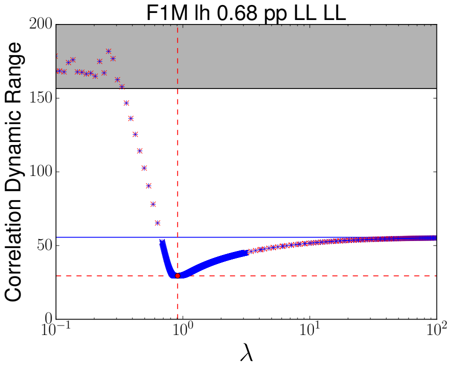


- $m_\pi \sim 230\,\mathrm{MeV}$, $a^{-1} \sim 2.7\,\mathrm{GeV}$,
- heavy-light PS meson with $am_h = 0.68$ $(m_c < m_h < m_b)$
- local $\mathbb{Z}_2$ sources + point sinks



Exploring λ: Correlations



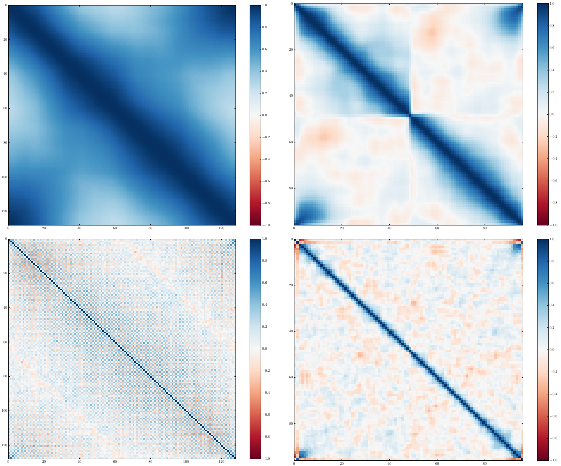


$86\,\mathrm{dB} \to 37\,\mathrm{dB}$

$56\,\mathrm{dB} \to 29\,\mathrm{dB}$

- Correct asymp. behaviour as $\lambda \to \infty$ and $\lambda \to 0^+$.
- Sizable region with significantly reduced correlations
- Similar gains when restricted to realistic subset of the time-range

Exploring λ: Correlations



- Filtered datapoints carry more independent information
- Still some "redundancy" for neighbouring time-slices
  $\Rightarrow$ can be combined with down-sampling (thinning)

Q1: Can we quantify the gain?
Q2: How do thinning and filtering interfere?

Quantifying the gains

- full range ('all t') vs representative fit range of $t \in [1/8T, 3/8T)$
- if thinned: filtering happens first

	M0M-mes-ll		**F1M-mes-lh**	
	all t	$t \in [16, 48)$	all t	$t \in [12, 36)$
raw	$86\,\mathrm{dB}$	$67\,\mathrm{dB}$	$56\,\mathrm{dB}$	$47\,\mathrm{dB}$
optimised	$37\,\mathrm{dB}$	$18\,\mathrm{dB}$	$36\,\mathrm{dB}$	$20\,\mathrm{dB}$
raw thinned	$62\,\mathrm{dB}$	$52\,\mathrm{dB}$	$37\,\mathrm{dB}$	$30\,\mathrm{dB}$
optimised thinned	$21\,\mathrm{dB}$	$11\,\mathrm{dB}$	$19\,\mathrm{dB}$	$9\,\mathrm{dB}$

- Filtering and thinning interfere constructively!
- Approximately 4 orders of magnitude improvement in all cases!

Q: How does this translate to precision & stability of correlator fits?

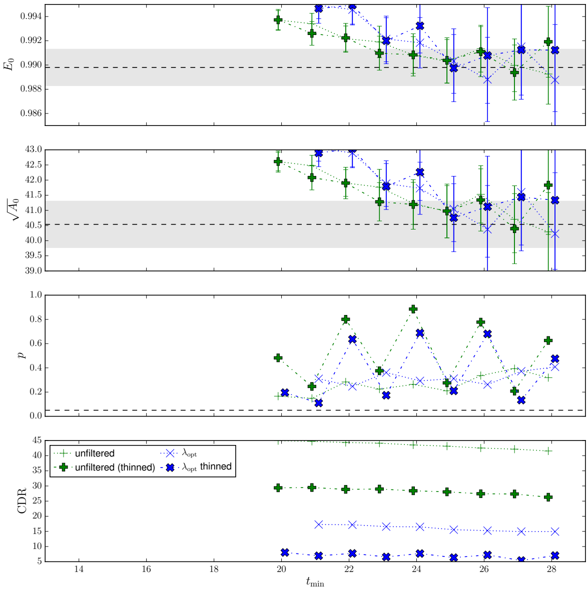
Application to two-point function fits

Q: How does this translate to precision & stability of correlator fits?

Strategy: Determine λ which minimises CDR on a given time-interval, then fit

- Test fits on the heavy-light dataset from before
- Fit to single exponential Ae^{-Et}.
- Vary $t_{\rm min}$, keep $t_{\rm max} = 3/8T = 36$ fixed
- Explore the separate and combined effects of filtering and thinning
- Only keep fits with acceptable p-value ($> 5\%$)
- Known results for E_0 and A_0 from extensive larger scale stability analysis

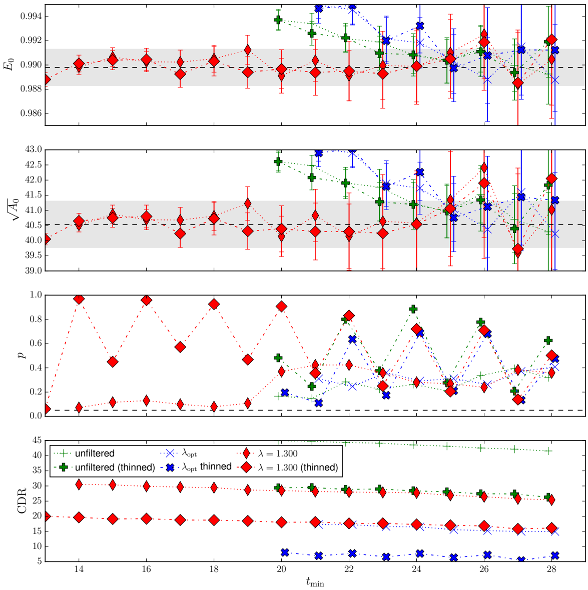
Minimising correlations



- $t_{\max}$ fixed
- single state fit
- only show fits with $p \geq 5\%$
- ☺ CDR decreases
- ☹ uncertainties increase

what about $\lambda \neq \lambda_{\mathrm{opt}}$?

~~Minimising~~ Reducing correlations



- $t_{\rm max}$ fixed
- single state fit
- only show fits with $p \geq 5\%$
- ☺ CDR decreases
- ☹ uncertainties increase

what about $\lambda \neq \lambda_{\rm opt}$?

- ☺ CDR decreases
- ☺ **largely reduced excited state**
- ☺ **competitive uncertainties at lower $t_{\rm min}$**

Annihilating correlation functions

Killing exponentials

Recall: Laplace filtering preserves exponentials, specifically

$$D_\lambda \sum_i A_i e^{-m_i t} = \sum_i \left[\tilde{E}_i^2 - \lambda^2\right] A_i e^{-m_i t} \equiv \sum_i A_i'(\lambda) e^{-m_i t}$$

with $\tilde{E}_i^2 = 2(\cosh(m_i) - 1)$

in idealised world:

- Exact correlation function $C(t)$ with exactly 2 states E_0 and E_1
- Choose $\lambda_0 = \tilde{E}_0$, $\lambda_1 = \tilde{E}_1$
- By definition we must have $D_{\lambda_1} D_{\lambda_2} C(t) = 0$ [ignore contact term]

in practice:

- noisy data ⇒ statistical statements ⇒ hypothesis testing
- # of states infinite ⇒ "data consistent with description of n-states"

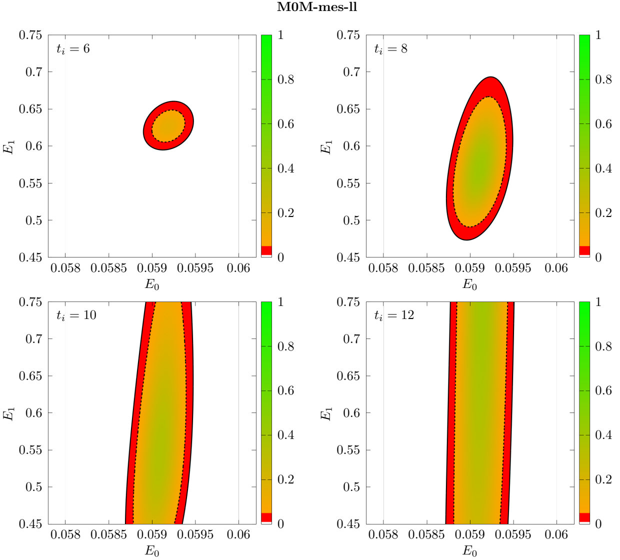
Multi-state hypothesis testing and the spectrum

- For simplicity test 2 states
- Define $\Lambda = \{\lambda_0, \lambda_1\}$ and time range $t \in [t_i, t_f)$ with $t_i > 2$
- Define the hypothesis H_Λ:

$$C_{\Lambda;t_i,t_f} \equiv D_{\lambda_0} D_{\lambda_1} C(t) = 0$$

- Assume sufficient statistics that correlator is normally distributed in interval.
- D_λ is a linear transformation
  $\Rightarrow$ $C_{\Lambda;t_i,t_f}$ is also normally distributed
  $\Rightarrow$ use standard p value and define confidence level $1-\alpha$
- Set $\alpha = 0.05$ and scan the parameter space (λ_0, λ_1).
- Interpret 'solutions' as $(\tilde{E}_0, \tilde{E}_1)$ and map them to (E_0, E_1) for convenience.

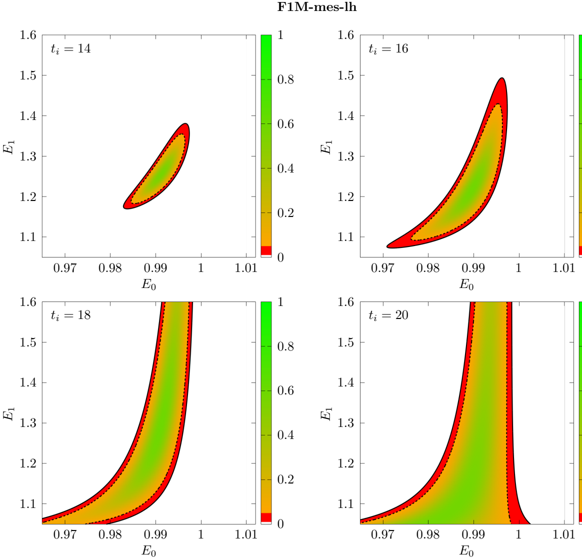
Multi-state hypothesis testing: physical pion



[Plots by A. Portelli]

- $t_f = 48$
- mapped $(\tilde{E}_0, \tilde{E}_1)$ back to (E_0, E_1)
- no allowed region it t_i too small
- decreasing sensitivity to E_1 as t_i increases
- compare to stability analysis: $E_0 = 0.05908(7)$ $E_1 = 0.49(3)$

Multi-state hypothesis testing: heavy light



- $t_f = 36$
- mapped $(\tilde{E}_0, \tilde{E}_1)$ back to (E_0, E_1)
- no allowed region it t_i too small
- decreasing sensitivity to E_1 as t_i increases
- compare to stability analysis: $E_0 = 0.9898(15)$ $E_1 = 1.222(27)$

[Plots by A. Portelli]

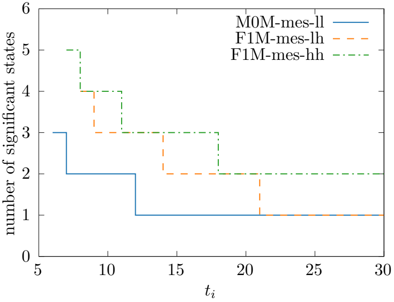
Determination of number of significant states (sketch)

Q: For a given time-range, can we determine the number of significant states from this?

- Define hypothesis H_r:

  $\exists\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ s.t. $C_{\Lambda;t_i,t_f} = 0$

- Compute ΔT^2 between H_{r+1} and H_r and use likelihood ratio test to decide whether H_r should be rejected in favour of H_{r+1}. Repeat until no rejection occurs.
- Check that H_r is not rejected (see previous slides).



[Plot by A. Portelli]

Possible Applications?

1. Use this for existing correlator fit methods:
   - Check whether correct number of states are included in the ansatz, i,e.
     $\Rightarrow$ are they sufficient to describe the data?
     $\Rightarrow$ is the fit actually sensitive to them?

   Bayes Relate determinations of $\tilde{E}_i$ to priors and allowed range to widths

   Frequ Relate determinations of $\tilde{E}_i$ to initial parameter guesses

2. Turn this into a new way to fit correlators:
   - ☺ No non-linear fitting required
   - ☺ No need to fit amplitudes (but can be determined)
   - ? Work in progress: Rigorously assign uncertainties

3. Thinking about excited states...

   Q: Rather than killing <u>all</u> the exponentials, can we only remove the unwanted ones?

Excited states

Recap: Excited states

- Notorious source of systematic uncertainties in precision predictions
- $N\pi$ states in nucleon form factors [Bär; 1812.09191, 1906.03652,1912.05873]
- $B\pi$ excited states in semi-leptonic B decays? [Bär, Broll, Sommer; 2306.02703]
- Fitting excited states
  - need more time slices and more parameters in fit
  - more correlation matrix issues + more involved stability analyses
  - often numerically unstable: need priors and/or sophisticated initial guess determinations
- can be mitigated
  - smearing (additional tuning - hard to get right for many states at once)
  - GEVP [Blossier, Della Morte, von Hippel, Mendes, Sommer; 0902.12675] (additional inversions, more complicated for 3-point functions)

Killing exponentials - removing excited states

Recall: Laplace filtering preserves exponentials. Specifically:

$$D_\lambda \sum_i A_i e^{-m_i t} = \sum_i \left[\tilde{E}_i^2 - \lambda^2\right] A_i e^{-m_i t} \equiv \sum_i A_i'(\lambda) e^{-m_i t}$$

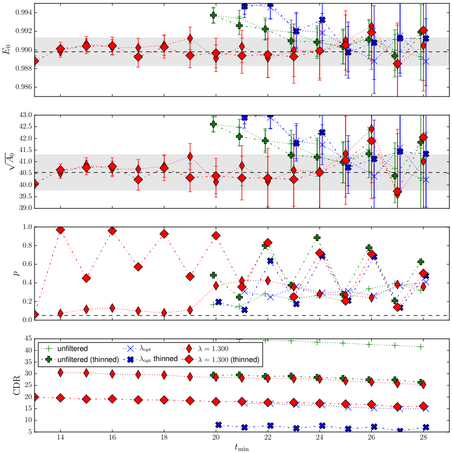
with $\tilde{E}_i^2 = 2(\cosh(m_i) - 1)$

- λ is a free parameter. Can choose it to remove an unwanted exponential, e.g. m_1

$$\lambda = \tilde{E}_1 = \sqrt{2(\cosh(m_1) - 1)}$$

- Known simple relation between A_i and $A_i'(\lambda)$
- Can be iterated – multiple applications of Laplace filtering

Recall from before



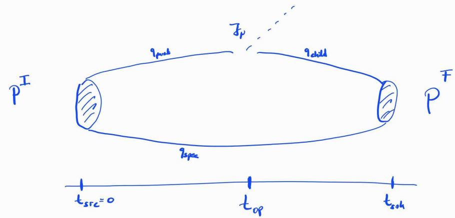
- Choice for the red data points was $\lambda = 1.3$
- No coincidence: recall result from stability analysis:

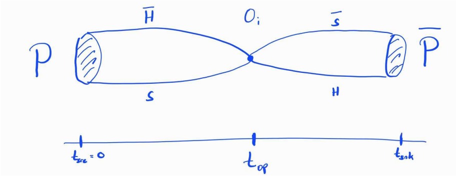
$$E_0 = 0.9898(15), \;\; E_1 = 1.222(27)$$

$$\Rightarrow \tilde{E}_1 = \sqrt{2[\cosh(E_1) - 1]} \sim 1.29946$$

Q: What about more complicated correlation functions?

Applications beyond 2-point functions





- Three-point functions have two (relative) time scales.
  (w.l.o.g. set $t_{\rm src} = 0$)
- Need $0 \equiv t_{\rm src} \ll t_{\rm op} \ll t_{\rm snk} \ll T$
  $\Rightarrow$ hard to fit on a typical temporal extent?
- trade-off: excited state contamination vs signal-to-noise properties

Can Laplace filtering help?

Semi-leptonic decays

- neglecting around the world effects (heavy-light)
- two-point functions for initial (P^I) and final (P^F) states:

$$C_2^X(t) = \sum_i \left(Z_i^X\right)^2 e^{-E_i^X t}$$

- Three-point function for a PS to PS transition:

$$C_{3,\mu}^{I\to F}(t_{\rm op}, t_{\rm snk}) = \sum_{i,j} Z_i^I Z_j^F \left\langle P_j^F \middle| V_\mu \middle| P_i^I \right\rangle e^{-E_j^F t_{\rm op} - E_i^I (t_{\rm snk} - t_{\rm op})}$$

- Define

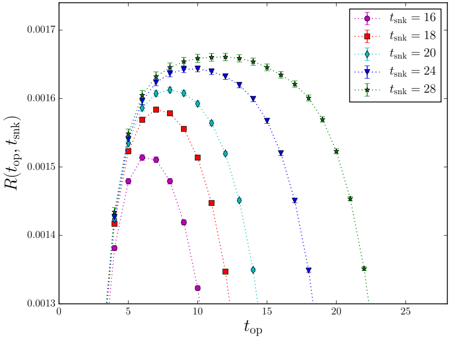
$$R_\mu(t_{\rm op}, t_{\rm snk}) = \frac{C_{3,\mu}^{I\to F}(t_{\rm op}, t_{\rm snk})}{C_2^F(t_{\rm op}) C_2^I(t_{\rm snk} - t_{\rm op})} \xrightarrow{0 \ll t_{\rm op}, t_{\rm snk}} \frac{\left\langle P_0^F \middle| V_\mu \middle| P_0^I \right\rangle}{Z_0^I Z_0^F}$$

What choice of $t_{\rm snk}$ is large enough?

Example: $D_s \to \eta_s$ (raw data)

$$R_\mu(t_{\rm op}, t_{\rm snk}) = \frac{C_{3,\mu}^{I\to F}(t_{\rm op}, t_{\rm snk})}{C_2^F(t_{\rm op})C_2^I(t_{\rm snk}-t_{\rm op})}$$

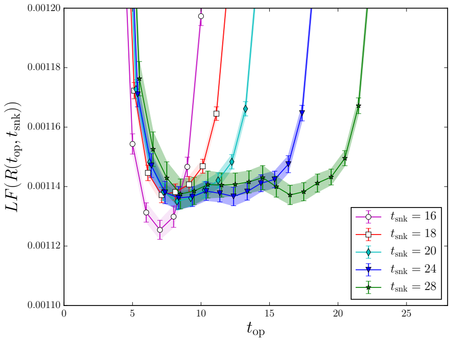
- $a^{-1} \sim 1.7\,\mathrm{GeV}$, $m_\pi \sim 280\,\mathrm{MeV}$, $L^3 \times T = 24^3 \times 64$
- everything at rest (i.e. $q^2_{\rm max}$)
- temporal component of vector current
- 5 source-sink separations
- $N_{\rm conf} = 180$
- 8 $\mathbb{Z}_2$ source-planes/config



All $t_{\rm snk}$ too short to have isolated ground state - are we doomed?

Example: $D_s \to \eta_s$ (Laplace filtered)

$$LF(R_\mu(t_{\rm op}, t_{\rm snk})) \equiv \frac{C_3^{I\to F}(t_{\rm op}, t_{\rm snk})}{D_{\lambda_F}[C_2^F](t_{\rm op})\, D_{\lambda_I}[C_2^I](t_{\rm snk} - t_{\rm op})}$$

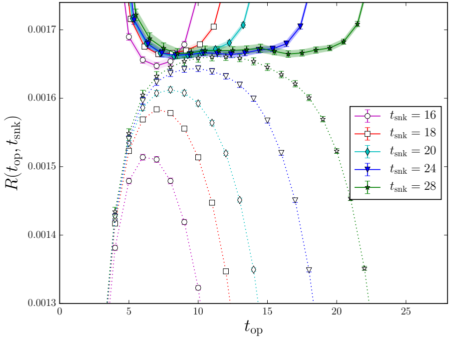


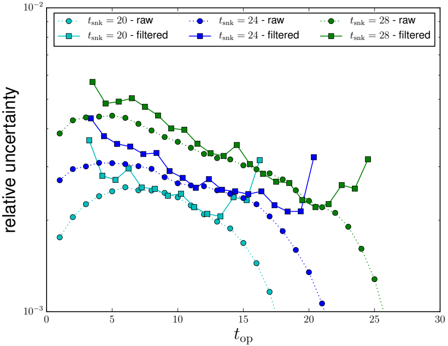
- Same plateau for different $t_{\rm snk}$ $\Leftrightarrow$ removed excited states
- $t_{\rm snk} = 20, 24, 28$ compatible
- Several options: Could instead/also apply D_{λ_3} to C_3
- **different asymptotic value**

$$\lim_{0 \ll t_{\rm op} \ll t_{\rm snk}} LF(R_\mu(t_{\rm op}, t_{\rm snk})) = \frac{\lim_{0 \ll t_{\rm op} \ll t_{\rm snk}} R_\mu(t_{\rm op}, t_{\rm snk})}{(\lambda_I^2 - \tilde{E}_I^2)(\lambda_F^2 - \tilde{E}_F^2)}$$

Example: $D_s \to \eta_s$ (Laplace filtered) cont.

Correct for $(\lambda_I^2 - \tilde{E}_I^2)(\lambda_F^2 - \tilde{E}_F^2)$ by 'reading off' E_0^I and E_0^F from effective mass





- $t_{\rm snk} = 28$ raw within less than 2σ of filtered results
- multiple $t_{\rm snk}$ at same plateau value
- extended plateaus!
- similar statistical precision for filtered and unfiltered!

Neutral meson mixing

- Same initial and final state: "Same spectrum on both sides"
- For $0 < t_{\rm op} < t_{\rm snk}$ we have

$$C_3^{O_k}(t_{\rm op}, t_{\rm snk}) = \sum_{i,j} Z_i Z_j \langle P_j | O_k | P_i \rangle e^{-E_j t_{\rm op} - E_i (t_{\rm snk} - t_{\rm op})}$$

(safely neglecting around the world effects since $E_0 \sim 0.94$)

- at fixed $t_{\rm snk}$: $t_{\rm op}$ dependence cancels if single-state dominated
- historical normalisation: "bag parameters", e.g. for Standard Model bag parameter (VV+AA)

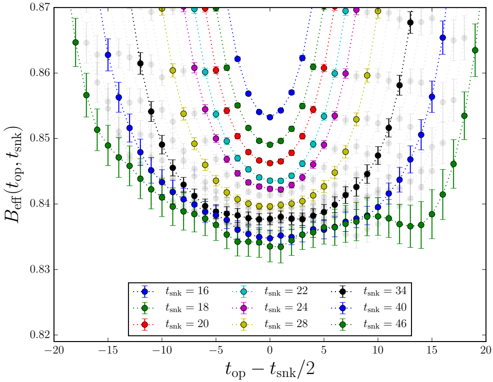
$$B_{\rm eff}(t_{\rm op}, t_{\rm snk}) = \frac{C_3^{O_1}(t_{\rm op}, t_{\rm snk})}{8/3 C_2^{PA}(0) C_2^{AP}(t_{\rm snk})}$$

normalisation does not influence $t_{\rm op}$-dependence.

Example: $\overline{B}_s - B_s$ mixing (raw data)

$$B_{\rm eff}(t_{\rm op}, t_{\rm snk}) = \frac{C_3^{O_1}(t_{\rm op}, t_{\rm snk})}{8/3\, C_2^{PA}(-t_{\rm snk}/2)\, C_2^{AP}(t_{\rm snk}/2)}$$

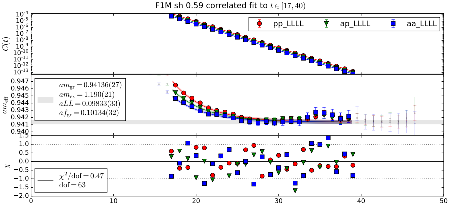
- $a^{-1} \sim 2.7\,\mathrm{GeV}$, $m_\pi \sim 230\,\mathrm{MeV}$, $L^3 \times T = 48^3 \times 96$
- $am_s \sim am_s^{\rm phys}$, $am_h = 0.59$ $\Rightarrow M_{D_s} < M_{H_s} < M_{B_s}$
- $N_{\rm conf} = 72$
- 48 $\mathbb{Z}_2$ source-planes/config
- 16 different $t_{\rm snk}$ values (only plotting some of them for readability!)

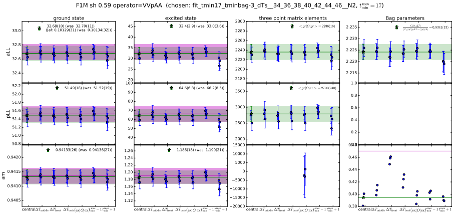


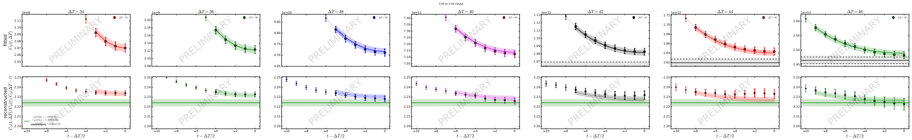
Example: $\overline{B}_s - B_s$ mixing ("traditional" analysis)

Results "known" from extensive stability analysis:

- 2-exponential frequentist fit
- simultaneously fit in both time coordinates $C_2(t_{\rm op})$ and $C_3(t_{\rm op}, t_{\rm snk})$ for many $t_{\rm snk} \in [34, 46]$
- source-place treatment [JTT, RBC/UKQCD, 2404.02297]



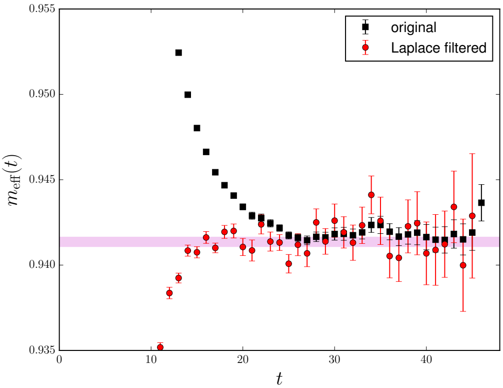




Example: $\overline{B}_s - B_s$ mixing (Laplace filtered - mesons)

$$C_2(t) \sim \sum_i A_i e^{-E_i t}$$

"Known" E_i: Use $\lambda = \tilde{E}_1$



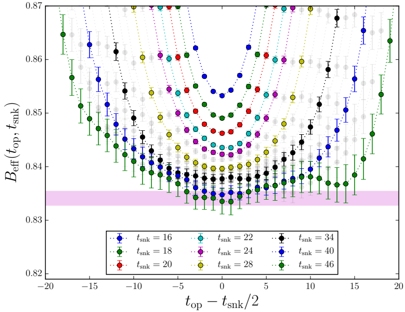
- Individual errors grow, but more independent information
- Reducing correlations between neighbouring time-slices
  $\Rightarrow$ noisier effective mass data.
- plateau starts ~ 10 time-slices earlier

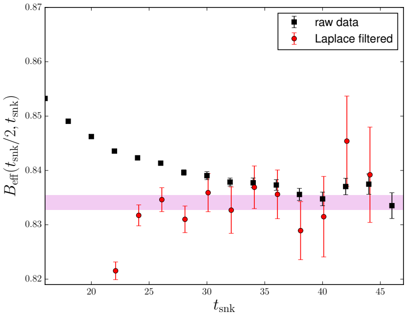
Example: $\overline{B}_s - B_s$ mixing (Laplace filtered - bags)

Recall

$$C_3(0, t_{\rm snk}) \sim \sum_{i,j} Z_{ij} e^{-(E_i+E_j)\frac{t_{\rm snk}}{2}}$$

Use "known" energies in LF: Use $\lambda = \tilde{E}'$ where $E' = (E_1 + E_2)/2$ restrict to $t_{\rm op} = t_{\rm snk}/2$ (i.e. midpoint) for C_3





- Sometimes (other ensembles/operators) even maximal $t_{\rm snk}$ too small.
- Individual errors grow, but more independent information

Conclusions and outlook

Summary

Numerical estimation of correlation matrices

- significant improvement without any ad-hoc assumptions
- simple analysis: transformed data can be fitted to transformed model
- tunable parameter to reduce correlations...

Contamination by excited states

- ... and use tunable parameter to remove excited states
  $\Rightarrow$ gain control over ground state parameters
- no additional data generation + can be applied post-data production
- demonstrated applications for two-point and three-point functions

Alternatives and improvements to non-linear fitting

- get spectrum without needing multi-exponential fits
- determination of number of statistically significant states
- data driven priors & widths (Bayes)/initial guesses (Freq.)

$\Rightarrow$ **Draft in preparation (arXiv:2508.xxxxx)** ☺