Out-of-equilibrium hydrodynamic fluctuations in the expanding QGP

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Outline

Motivation and separation of scales

Toy example: Brownian motion

Hydrodynamics with noise: Bjorken expansion

Out-of-equilibrium noise contributions to $\langle T^{\mu\nu} \rangle$

Bulk viscosity induced by fluctuations

Motivation and separation of scales

Hydrodynamics with noise in heavy ion collisions

Kovtun-Yaffe (03), Gavin-AbdelAziz (06), Kovtun-Moore-Romatschke (11), Kapusta-Muller-Stephanov (12), Young-Kapusa-Gale-Jeon-Schenke (15), Nagai-Kurita-Murase-Hirano (16), Yan-Grönqvist (16), Gavin-Moschelli-Zin (16), Akamatsu-AM-Teaney (17), . . .

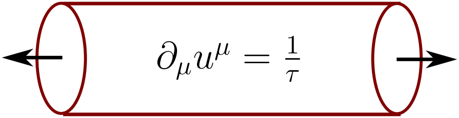
1. Initial state fluctuations not discussed in this talk
2. Thermal fluctuations see Landau-Lifshitz

$$N_{gg}(t,k) \equiv \underbrace{\langle g^i(t,k) g^{j*}(t,k)\rangle}_{\text{momentum, } g^i \equiv T^{0i}} = \underbrace{(e+p)T\delta^{ij}}_{\text{equilibrium}}$$

   - Required by the Fluctuation-Dissipation Theorem
   - Close to equilibrium determined by the background
   - Larger in smaller systems: $N_{\text{particles}} \sim 10000$ in the heavy-ions
   - Fluctuations enhanced near a critical point
3. Rapid expansion drives hydrodynamic fluctuations out of equilibrium.

How do hydrodynamic fluctuations evolve during a Bjorken expansion?
What is the effect of out-of-equilibrium fluctuations to the expansion?

Evolution of background without fluctuations



Equation of motion for Bjorken expansion:

$ds^2 = -d\tau^2 + dx^2 + dy^2 + \tau^2 d\eta^2$

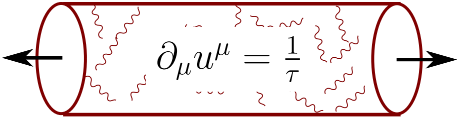
$T^{zz} = \tau^2 T^{\eta\eta}$

$$\partial_\tau T^{\tau\tau} = -\frac{T^{\tau\tau} + T^{zz}}{\tau}$$

- Constitutive equations without hydrodynamic fluctuations:

$$T^{zz} = \underbrace{p}_{\text{ideal}} - \underbrace{\frac{4\eta}{3\tau}}_{\text{1st order}} + \underbrace{(\lambda_1 - \eta\tau_\pi)\frac{8}{9\tau^2}}_{\text{2nd order}}$$

Evolution of background with fluctuations



Statistically averaged equation of motion for Bjorken expansion:

$$\partial_\tau \langle T^{\tau\tau} \rangle = -\frac{\langle T^{\tau\tau} \rangle + \langle T^{zz} \rangle}{\tau}$$

- Hydrodynamic fluctuations contribute to $\langle T^{zz} \rangle$

$$\langle T^{zz} \rangle_{\text{fluct}} = (e_0 + p_0)\langle u^z u^z \rangle = \frac{\langle g^z g^z \rangle}{e_0 + p_0}$$

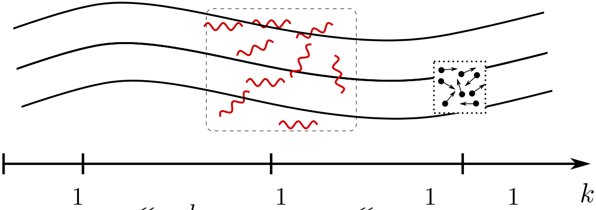
Expansion drives hydrodynamic fluctuations away from equilibrium!

Kinetic regime of hydrodynamic fluctuations – a new scale k_*

1. For hydrodynamic fluctuations with wavenumber k:
   - Equilibration rate $\sim \gamma_\eta k^2$ — $\gamma_\eta \equiv \eta/(e+p)$
   - Expansion rate $\sim 1/\tau$ for a Bjorken expansion
2. Compete at a critical scale k_*: — $\epsilon \equiv \gamma_\eta/\tau \ll 1$

$$k_* \sim \frac{1}{\sqrt{\gamma_\eta \tau}} \sim \frac{1}{c_s \tau \sqrt{\epsilon}}$$

3. Separation of scales in hydrodynamic expansion

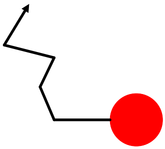


$$\frac{1}{c_s\tau} \ll k_* \sim \frac{1}{c_s\tau\sqrt{\epsilon}} \ll \frac{1}{l_{\mathrm{mfp}}} \sim \frac{1}{c_s\tau\epsilon}$$

We derive an effective description for the kinetic regime k_*

Toy example: Brownian motion

Hydro-kinetic equation: an analogy with Brownian motion



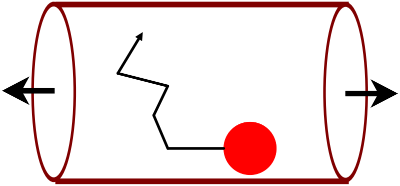
1. Langevin equation

$$\frac{dp}{dt} = \underbrace{-\gamma p}_{\text{drag}} + \underbrace{\xi}_{\text{noise}}, \quad \langle \xi(t)\xi(t')\rangle = 2TM\gamma\delta(t-t')$$

2. Calculate momentum diffusion $N_{pp} = \langle p^2 \rangle$

$$\frac{d}{dt}N_{pp} = \underbrace{-2\gamma\left[N_{pp} - MT\right]}_{\text{equilibration}}$$

Hydro-kinetic equation: an analogy with Brownian motion



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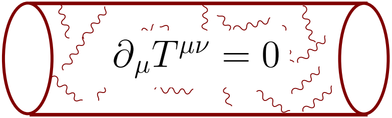
Follow the same steps for hydrodynamics with external forcing

Hydrodynamics with noise: Bjorken expansion



$T^{\mu\nu} = T^{\mu\nu}_{\text{ideal}} + T^{\mu\nu}_{\text{visc.}} + S^{\mu\nu}_{\text{noise}}$

1. Linearized hydro-kinetic equations

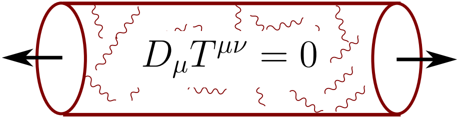
$$\partial_\tau \delta T^{\tau\nu} = -\partial_i \delta T^{i\nu}_{\text{ideal}} - \partial_i \delta T^{i\nu}_{\text{visc.}} - \partial_i S^{i\nu}_{\text{noise}}, \quad \langle S^{\mu\nu} S^{\rho\sigma} \rangle \sim 2T\eta_0 \delta(t - t')$$

2. Calculate transverse momentum diffusion $N_{g_T g_T} = \langle g_T g_T^* \rangle$ $\quad g^i = \delta T^{\tau i}$

$$\frac{d}{dt} N_{g_T g_T} = \underbrace{-2\gamma_\eta k^2 \left[N_{g_T g_T} - T(e_0 + p_0) \right]}_{\text{equilibration}}$$

3. Hydrodynamics additionally has sound modes $c_s^2 \delta e \pm \hat{k}_i g^i$

Hydro-kinetic equations



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3. Hydrodynamics additionally has sound modes $c_s^2 \delta e \pm \hat{k}_i g^i$

Eigenbasis for hydro-kinetic equations

1. Euclidean 4-vector ϕ_a of perturbations: $e = e_0 + \delta e,\ \vec{g} = (e_0 + p_0)\vec{v}$

$$\phi_a(\tau, \vec{k}) \equiv (c_s \delta e, \vec{g})$$

2. Hydro-kinetic equations for $\phi_a(\tau, \vec{k})$

$$D_\mu \delta T^{\mu\nu} = 0 \Leftrightarrow -\dot{\phi}_a(\tau, \vec{k}) = \underbrace{i\mathcal{L}_{ab}\phi_b}_{\text{ideal}} + \underbrace{\mathcal{D}_{ab}\phi_b}_{\text{viscous}} + \underbrace{\xi_a}_{\text{noise}} + \underbrace{\mathcal{P}_{ab}(\tau)\phi_b}_{\text{expansion}}$$

3. Four eigenmodes of $\mathcal{L}_{\text{ideal}}$: ϕ_+, ϕ_-, ϕ_{T_1}, ϕ_{T_2}

$$\underbrace{\text{left moving sound}}_{\lambda_- = -c_s k} \qquad \underbrace{\text{right moving sound}}_{\lambda_+ = c_s k} \qquad \underbrace{\text{transverse modes}}_{\lambda_T = 0}$$

Diagonalize the hydro-kinetic equations in $\mathcal{L}_{\text{ideal}}$ eigenbasis

Hydro-kinetic equations

$$\phi_\pm \sim c_s^2 \delta e \pm \hat{\vec{k}} \cdot \vec{g}, \quad \phi_{T_1} \sim \vec{g}_{T1}, \quad \phi_{T_2} \sim \vec{g}_{T2}$$

1. Analyze the squares of the eigenmodes – for example

$$N_{++}(\tau, \vec{k}) \equiv \langle \phi_+(\tau, \vec{k}) \phi_+^*(\tau, \vec{k}) \rangle$$

2. Hydro-kinetic equations for N_{++}

$$\dot{N}_{++} = \underbrace{-\frac{4}{3}\gamma_\eta k^2 \left[N_{++} - T(e_0 + p_0)\right]}_{\text{equilibration}} - \underbrace{\frac{1}{\tau}\left[2 + c_s^2 + \cos^2\theta_k\right] N_{++}}_{\text{expansion}}$$

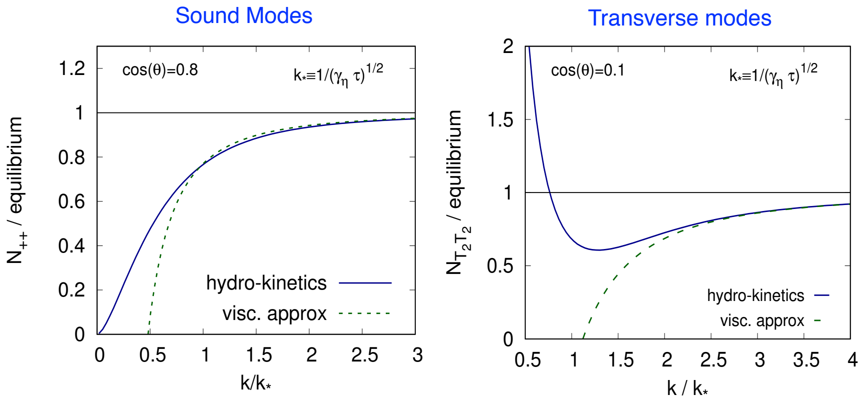
3. Only diagonal components of density matrix contribute

$$N_{+-} \sim \underbrace{e^{-i(\lambda_+ - \lambda_-)t} \sim 0}_{\text{rotating wave approx}}$$

Two point correlators N_{AA} are driven out of equilibrium at k_*

Solution of the hydro-kinetic equations

Bjorken expansion at late times

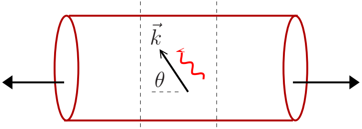


k_* is the critical scale: For larger $k \gg k_*$, closer to equilibrium

$$N_{AA} \sim N_{\text{eq}} \left[1 + \frac{\#}{\gamma_\eta \tau k^2} + \cdots \right]$$

Out-of-equilibrium noise contributions to $\langle T^{\mu\nu} \rangle$

Nonlinear contributions from k_* to the background



1. Compute the contribution from fluctuation

$$\langle T^{zz}\rangle_{\text{fluct.}} = \langle (g^z(t,\vec{x}))^2\rangle = \int^{\Lambda} d^3k \underbrace{\left[N_{++}\cos^2\theta + N_{T_2T_2}\sin^2\theta\right]}_{\sim 1 + \#/(\gamma_\eta \tau k^2) + \cdots \text{ for } k \gg k_*}$$

- Regularize cubic and linear UV divergences by a cutoff Λ

2. Renormalize the divergences c.f. Kovtun-Moore-Romatschke (11)

$$\langle T^{zz}\rangle = \underbrace{p_0(\Lambda) + \frac{\Lambda^3 T}{6\pi^2}}_{\equiv p_{\text{phys}}} - \frac{4}{3\tau}\underbrace{\left[\eta_0(\Lambda) + \frac{17\Lambda T}{120\pi^2}\frac{e_0+p_0}{\eta_0}\right]}_{\equiv \eta_{\text{phys}}} + \text{finite}$$

The cutoff dependence is absorbed by renormalization of p_0 and η_0

Finite contributions: Long-time tails

Evaluate the finite parts after renormalization

$$\langle T^{zz} \rangle = \underbrace{p_{\text{phys}}}_{\text{ideal}} - \underbrace{\frac{4\eta_{\text{phys}}}{3\tau}}_{\text{1st order}} + \underbrace{1.08318\ T \left(\frac{1}{4\pi\gamma_\eta\tau}\right)^{3/2}}_{\text{long-time tail}} + \cdots$$

Simple understanding of the scaling

$$\langle T^{zz} \rangle_{\text{fluct.}} \sim \underbrace{\tfrac{1}{2}k_B T}_{\text{equipart}} \underbrace{\int^{k_*} d^3k}_{\text{\# of modes}} \sim T k_*^3 \sim T \left(\frac{1}{\gamma_\eta \tau}\right)^{3/2}$$

The finite contribution from k_* gives the long-time tails

Implications for heavy-ion collisions

Plugging in typical values

$$\frac{T^3}{s} \simeq \frac{1}{13.5}, \qquad \frac{\lambda_1 - \eta\tau_\pi}{e+p} \simeq -0.8\left(\frac{\eta}{e+p}\right)^2$$

Compute $4\langle T^{zz}\rangle/(e+p)$

$$\frac{\eta}{s} = \frac{1}{4\pi}: \qquad 1 - 0.092\left(\frac{4.5}{\tau T}\right) + 0.034\left(\frac{4.5}{\tau T}\right)^{3/2} - 0.00085\left(\frac{4.5}{\tau T}\right)^2$$

$$\frac{\eta}{s} = \frac{2}{4\pi}: \qquad \underbrace{1}_{\text{ideal}} \underbrace{- 0.185\left(\frac{4.5}{\tau T}\right)}_{\text{1st order}} \underbrace{+ 0.013\left(\frac{4.5}{\tau T}\right)^{3/2}}_{\text{1.5th order}} \underbrace{- 0.0034\left(\frac{4.5}{\tau T}\right)^2}_{\text{2nd order}}$$

Thermal fluctuation is practically larger than 2nd order viscous correction

Bulk viscosity induced by fluctuations

Renormalization of bulk viscosity in non-conformal fluid $c_s^2 \neq 1/3$

Metric perturbation in a static system

$$ds^2 = -dt^2 + (\delta_{ij} + h_{ij})dx^i dx^j \quad h \ll 1$$

1. Apply a bulk perturbation

$$h_{ij}(t) = h\delta_{ij}e^{-i\omega t}$$

2. Compute two point correlator evolution away from equilibrium:

$$\dot{N}_{++} = -\gamma_\eta k^2[N_{++} - N_{\text{eq}}] + \dot{h}(t)N_{++}$$

3. Find non-linear noise contribution to constitutive equations $\langle T^{\mu\nu}\rangle_{\text{fluct}}$
4. Bulk viscosity modification due hydrodynamic fluctuations:

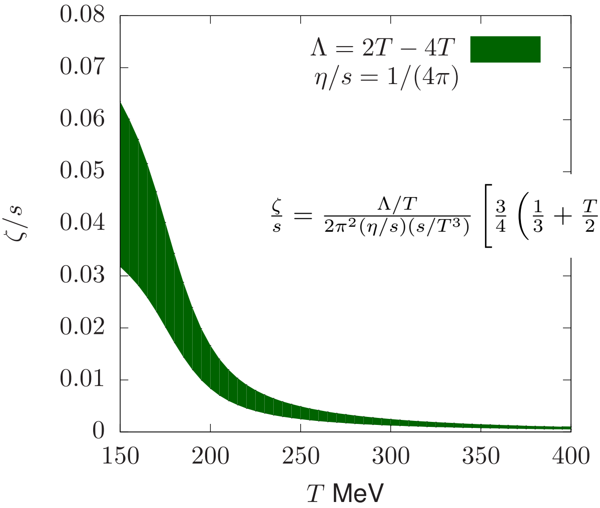
$$\zeta = \zeta_0(\Lambda) + \frac{\Lambda T^2}{2\pi^2}\left[\left(\frac{1}{3} + \frac{T}{2}\frac{dc_s^2}{dT} - c_s^2\right)^2 \frac{s}{\zeta_0 + \frac{4}{3}\eta_0} + \left(\frac{1}{3} - c_s^2\right)^2 \frac{2s}{\eta_0}\right]$$

Akamatsu-AM-Teaney, in preparation

c.f. diagramatic approaches: Kovtun-Yaffe (03), Kovtun-Moore-Romatschke (11), Martinez-Schäfer (17)

Fluctuation induced bulk viscosity

Lattice equation of state $c_s^2(T)$, $\eta/s = 1/(4\pi)$ and $\zeta_0 = 0$



$$\frac{\zeta}{s} = \frac{\Lambda/T}{2\pi^2(\eta/s)(s/T^3)}\left[\frac{3}{4}\left(\frac{1}{3} + \frac{T}{2}\frac{dc_s^2}{dT} - c_s^2\right)^2 + 2\left(\frac{1}{3} - c_s^2\right)^2\right]$$

Small, but conceptionally important contribution!

Summary & Outlook

- Hydro-kinetic equations for k_*, advantageous in expanding systems
- Universal renormalization of pressure $p_0(\Lambda)$ and viscosity $\eta_0(\Lambda)$
- Background-dependent long-time tails $\propto \tau^{-3/2}, \omega^{3/2}$
- Alternative way to solve the hydrodynamics with noise

$$D_\mu T^{\mu\nu} = 0, \quad \dot{N}_{AA} = \cdots, \quad T^{\mu\nu} = T^{\mu\nu}_{\mathsf{bkg}} + \int_k N_{AA},$$

- Bulk viscosity renormalization for non-conformal fluid

Akamatsu-AM-Teaney, in preparation

$$\zeta = \zeta_0(\Lambda) + \frac{\Lambda T^2}{2\pi^2}\left[\left(\frac{1}{3} + \frac{T}{2}\frac{dc_s^2}{dT} - c_s^2\right)^2 \frac{s}{\zeta_0 + \frac{4}{3}\eta_0} + \left(\frac{1}{3} - c_s^2\right)^2 \frac{2s}{\eta_0}\right]$$