

# ***Finite-Range Corrections to Hydrodynamics***

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# Gradient Terms and Nuclear Physics

Attractive non-zero-range potential:

$$V = \frac{1}{2} \int d^3r d^3r' \rho(r) v(r - r') \rho(r')$$

$$= \frac{1}{2} \int d^3r \rho(r) \int d^3\delta r v(\delta r) \left[ \rho(r) + \frac{(\delta r)^2}{6} \nabla^2 \rho|_{r=0} \right],$$

$$\Delta V = -\frac{1}{2} \int d^3r \kappa \rho \nabla^2 \rho,$$

$$\kappa = -\frac{1}{6} \int d^3r r^2 v(r).$$

Free energy:

$$f = \bar{f}(\epsilon, \rho) - \frac{\kappa}{2} \rho \nabla^2 \rho$$

# Gradient Terms and Nuclear Physics

## Applications:

- ▶ Nuclear Structure: Skyrme, DFT
- ▶ Phase interfaces: Liquid-Gas, QGP-hadron

$$\delta F = 0 = \delta \int dx \left[ P_0 - P + (\mu - \mu_0)\rho + \frac{\kappa}{2}(d\rho/dx)^2 \right]$$

$$\frac{d\rho}{dx} = \sqrt{2[P_0 - P(\rho) + (\mu(\rho) - \mu_0)\rho]/\kappa}$$

- ▶ Critical Phenomena: Correlation length

$$\ell^2 = \beta\kappa\chi$$

- ▶ Spinodal Decomposition  
Damps short wavelength modes
- ▶ Higher  $\kappa \rightarrow$  diffuse surfaces,  
higher surface energy, higher nucleation barriers  
longer correlation lengths,  
slower growth of unstable modes

# **GOAL: Consistent Set of Equations**

## **I. Equations of Motion:**

$$\begin{aligned}\partial_\alpha T^{\alpha\beta} &= 0, \\ \partial_\alpha J^\alpha &= 0\end{aligned}$$

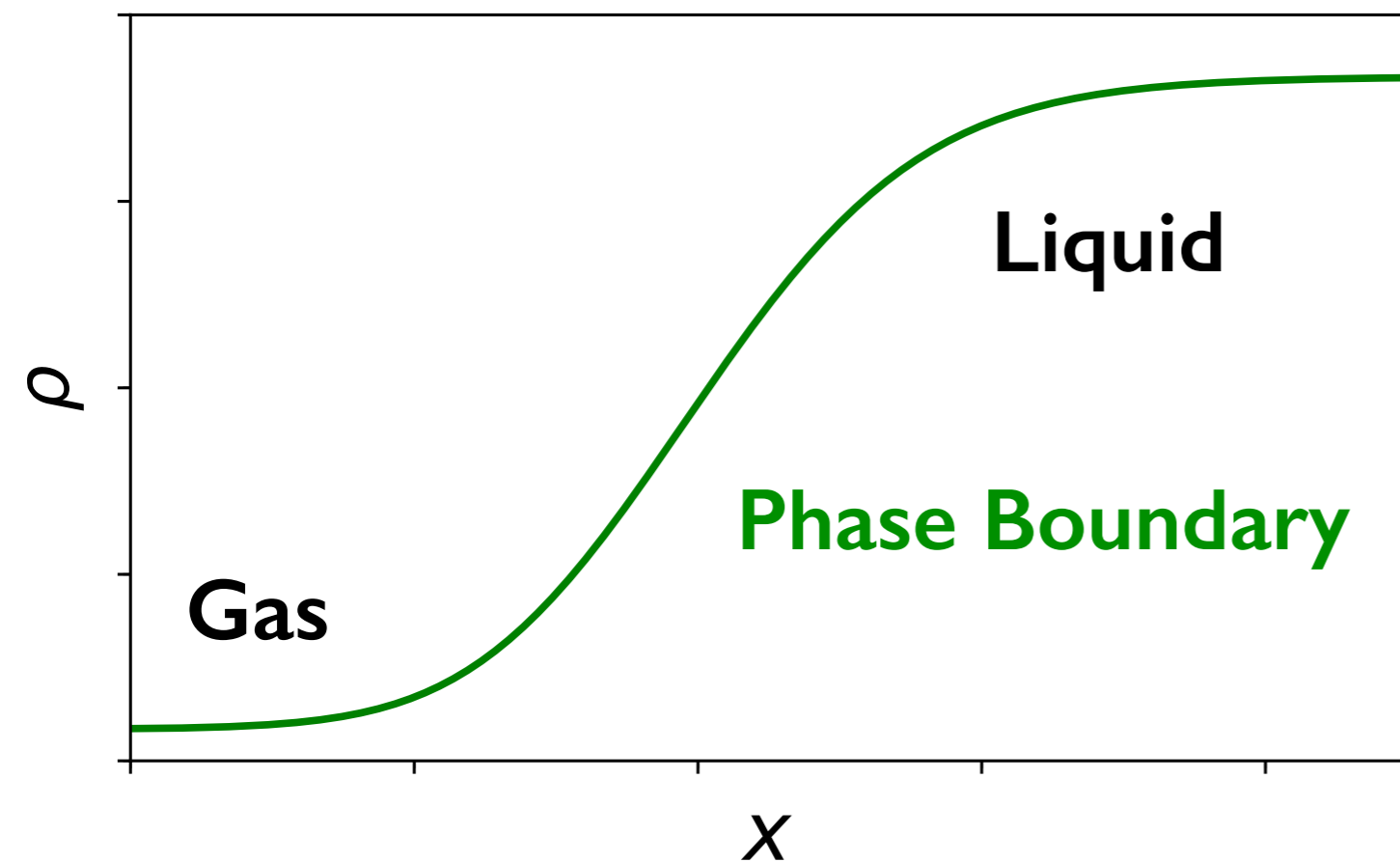
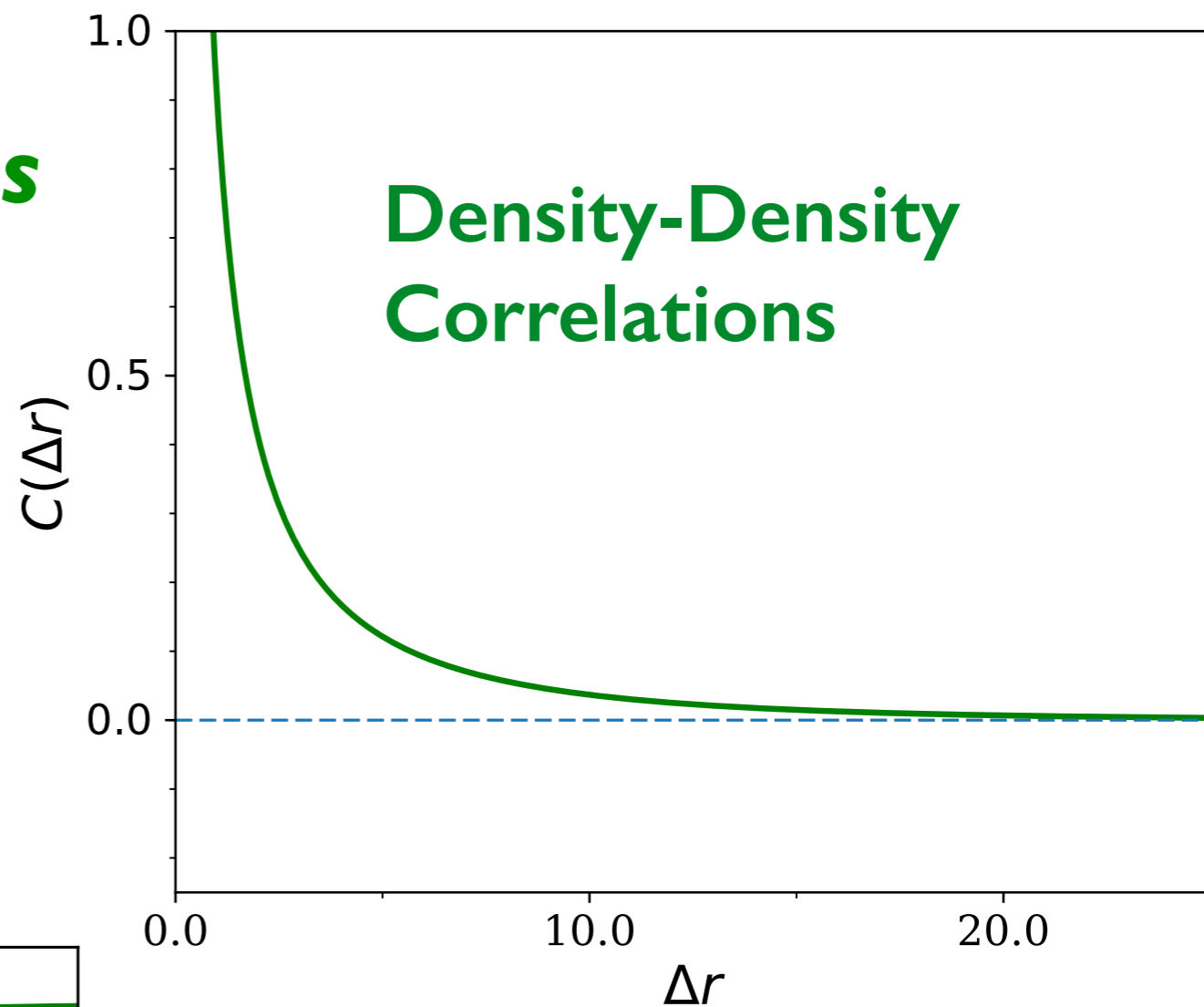
**$T$ ,  $\mu$ ,  $s$ ,  $T^{0i}$  and  $T_{ij}$  in terms of  $\epsilon$ ,  $\rho$ ,  $\nabla\rho$  and  $\nabla v$**

## **2. Conserve Entropy**

## **3. Reproduce Equilibrium**

**GOAL:**  
**Consistent Set of Equations**

Reproduce Equilibrium  
according to  
Landau Field Theory



# Starting Point (Choice #1)

$$s = \bar{s}(\epsilon_\kappa, \rho), \quad \epsilon_\kappa = \epsilon + \frac{\kappa}{2} \rho \nabla^2 \rho$$

More general

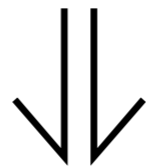
- ▶  $\kappa$  would be  $\kappa(\epsilon, \rho)$
- ▶ Add terms like

$$\epsilon_\kappa = \epsilon + \frac{\kappa_{\rho\rho}}{2} \rho \nabla^2 \rho + \frac{\kappa_{\epsilon\epsilon}}{2} \epsilon \nabla^2 \epsilon + \frac{\kappa_{\rho\epsilon}}{2} \rho \nabla^2 \epsilon + \frac{\kappa_{\epsilon\rho}}{2} \epsilon \nabla^2 \rho$$

# Gradient Corrections to $\mu = -\alpha T$ & $\beta = T^{-1}$

Given

$$S = \bar{S}(\epsilon_\kappa, \rho), \quad \epsilon_\kappa = \epsilon + \frac{\kappa}{2} \rho \nabla^2 \rho$$



$$\delta S = \int d^3 r \left\{ \frac{\partial \bar{S}}{\partial \epsilon} \delta \epsilon + \frac{\partial \bar{\beta}}{\partial \rho} \delta \rho \right\},$$

$$= \int d^3 r \{ \beta \delta \epsilon + \alpha \delta \rho \},$$

$$\beta = \bar{\beta}(\epsilon_\kappa, \rho),$$

$$\alpha = \bar{\alpha}(\epsilon_\kappa, \rho) + \frac{\kappa \bar{\beta}(\epsilon_\kappa, \rho)}{2} \rho \nabla^2 \rho + \frac{\kappa}{2} \rho \nabla^2 (\bar{\beta}(\epsilon_\kappa, \rho) \rho)$$

## Starting Point (Choice #2)

Entropy moves with charge

Energy (momentum density) moves separately



energy in this cell affected by  $v$  in neighboring cells

$$\frac{D}{Dt}\rho = -\rho\nabla \cdot \mathbf{v},$$

$$\frac{D}{Dt}s = -s\nabla \cdot \mathbf{v},$$

$$\frac{D}{Dt}\epsilon = -\epsilon\nabla \cdot \mathbf{v} - T_{ij}\partial_i v_j - \nabla \cdot \mathbf{M}$$

$M_i$  is additional momentum density,  $\Delta T^{0i}$



## From here on: everything determined

$$\begin{aligned} D_t s &= \bar{\beta} \left[ D_t \epsilon + D_t \left( \frac{\kappa}{2} \rho \nabla^2 \rho \right) \right] - \bar{\beta} \bar{\mu} D_t \rho \\ &= -\bar{\beta} T_{ij} \partial_i v_j + [\bar{\beta} \bar{\mu} \rho - \bar{\beta} \epsilon] \nabla \cdot \mathbf{v} + \bar{\beta} D_t \left( \frac{\kappa}{2} \rho \nabla^2 \rho \right) - \bar{\beta} \nabla \cdot \mathbf{M} \end{aligned}$$

**Mess**

# From here on: everything determined

$$\begin{aligned}
 D_t s &= \bar{\beta} \left[ D_t \epsilon + D_t \left( \frac{\kappa}{2} \rho \nabla^2 \rho \right) \right] - \bar{\beta} \bar{\mu} D_t \rho \\
 &= -\bar{\beta} T_{ij} \partial_i v_j + [\bar{\beta} \bar{\mu} \rho - \bar{\beta} \epsilon] \nabla \cdot \mathbf{v} + \bar{\beta} D_t \left( \frac{\kappa}{2} \rho \nabla^2 \rho \right) - \bar{\beta} \nabla \cdot \mathbf{M}
 \end{aligned}$$

lots of algebra ↓

$$\begin{aligned}
 &= -\bar{\beta} T_{ij} \partial_i v_j + \bar{\beta} \bar{\mu} \rho \nabla \cdot \mathbf{v} - \bar{\beta} \epsilon \nabla \cdot \mathbf{v} \\
 &+ \frac{\kappa \bar{\beta}}{2} (\partial_i (\rho \partial_j \rho)) (\partial_i v_j + \partial_j v_i) + \frac{\kappa \bar{\beta}}{2} (\partial_i (\rho \partial_i \rho)) \nabla \cdot \mathbf{v} \\
 &- \frac{\kappa \bar{\beta}}{2} (\nabla^2 (\rho^2)) \nabla \cdot \mathbf{v} - \bar{\beta} \kappa \rho (\nabla^2 \rho) \nabla \cdot \mathbf{v} - \bar{\beta} \kappa \rho (\partial_i \partial_j \rho) \partial_i v_j \\
 &= -s \nabla \cdot \mathbf{v},
 \end{aligned}$$

$$M_i = -\frac{\kappa}{2} \rho (\partial_j \rho) (\partial_i v_j + \partial_j v_i) - \frac{\kappa}{2} \rho^2 \partial_i \nabla \cdot \mathbf{v} + \frac{\kappa}{2} \rho (\partial_i \rho) \nabla \cdot \mathbf{v}$$

Read off  $T_{ij}$

# SUMMARY OF EQUATIONS

$$s = \bar{s}(\epsilon_\kappa, \rho), \quad \epsilon_k = \epsilon + \frac{\kappa}{2} \rho \nabla^2 \rho$$

$$\beta = \bar{\beta}(\epsilon_\kappa, \rho),$$

$$\alpha = \bar{\alpha}(\epsilon_\kappa, \rho) + \frac{\bar{\beta}\kappa}{2} \nabla^2 \rho + \frac{\kappa}{2} \nabla^2 (\bar{\beta} \rho),$$

$$M_i = -\frac{\kappa}{2} \rho (\partial_j \rho) (\partial_i v_j + \partial_j v_i) - \frac{\kappa}{2} \rho^2 \partial_i \nabla \cdot v + \frac{\kappa}{2} \rho (\partial_i \rho) \nabla \cdot v,$$

$$T_{ij} = \bar{P} \delta_{ij} - \kappa \left[ \rho \nabla^2 \rho + \frac{1}{2} (\nabla \rho)^2 \right] \delta_{ij} + \kappa (\partial_i \rho) (\partial_j \rho),$$

$$J^{(D)} = -D \nabla \rho + \frac{\kappa \chi D}{2} \nabla \left[ \bar{\beta} \nabla^2 \rho + \nabla^2 (\bar{\beta} \rho) \right].$$

- ▶ Additional momentum density  $M$  requires velocity gradients
- ▶  $T_{ij}$  has off-diagonal elements
- ▶ Diffusion altered from altering  $\mu$

# Consistency between $T_{ij}$ , $T$ , $\mu$

Entropy maximized if  $\beta$  and  $\mu$  are uniform

$$\Rightarrow \partial_i T_{ij} = 0!$$

Otherwise, you can't reach equilibrium!

$$\partial_i T_{ij}^{(\text{fixed } T)} = \partial_i \bar{P} - \kappa \rho \partial_i \nabla^2 \rho,$$

$$\partial_i \bar{P} = \frac{\partial \bar{P}}{\partial \bar{\mu}} \partial_i \bar{\mu} = \rho \partial_i \bar{\mu},$$

$$\partial_i T_{ij}^{(\text{fixed } T)} = \rho \partial_j \mu \quad \checkmark$$

Not possible if  $T_{ij} \sim \delta_{ij}$

# Example 1: Liquid-Gas Phase Interface (at equilibrium)

$$\bar{P}(\epsilon_\kappa, \rho) = \frac{\rho \bar{T}(\epsilon_\kappa, \rho)}{1 - \rho/\rho_s} - a\rho^2.$$

Maximize entropy,

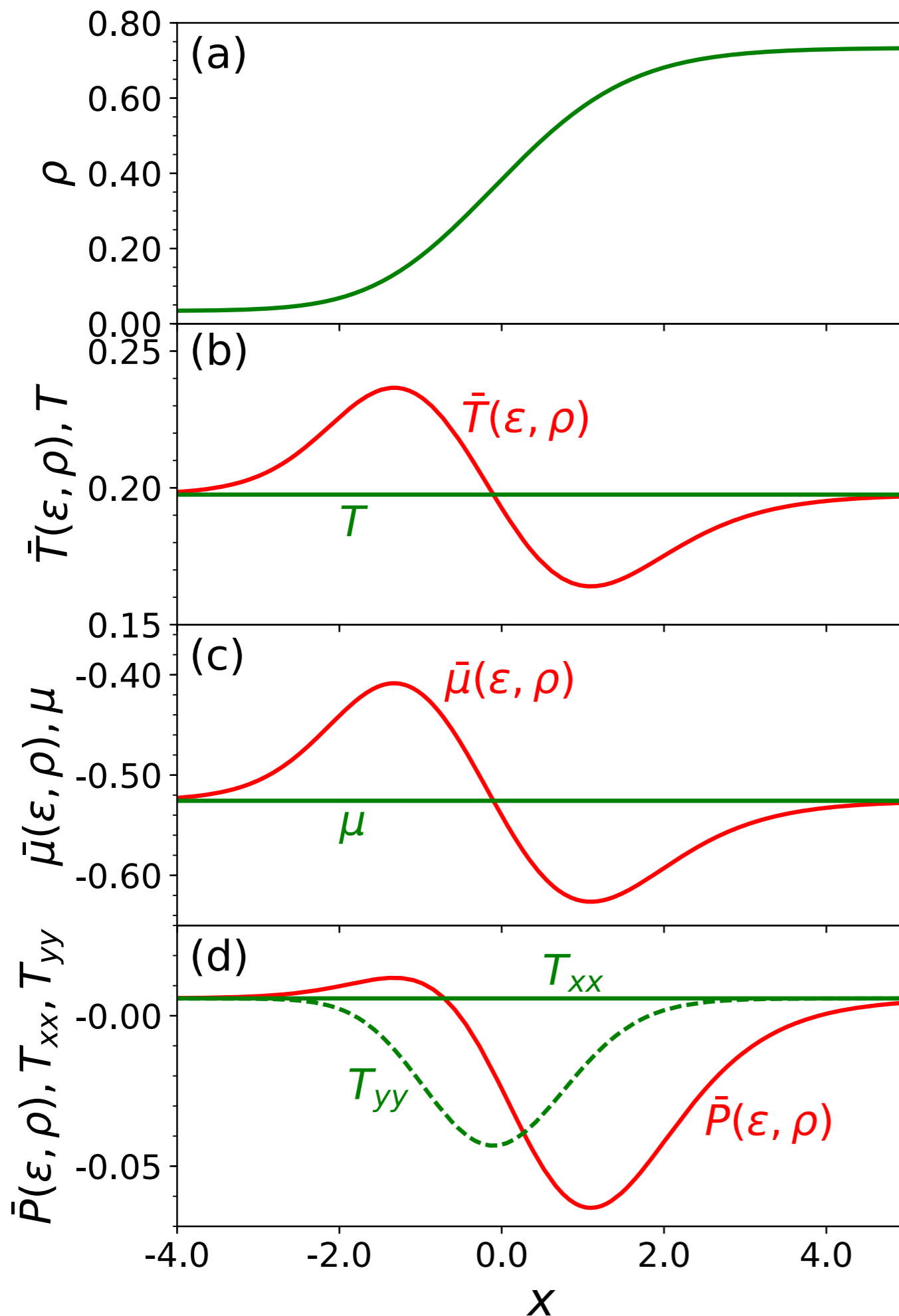
$$T\delta S/A = 0$$

$$= \delta \int dx \left[ P_0 - P + (\mu - \mu_0)\rho + \frac{\kappa}{2} (d\rho/dx)^2 \right]$$

$$\frac{d\rho}{dx} = \sqrt{2[P_0 - P(\rho) + (\mu(\rho) - \mu_0)\rho]}/\kappa$$

$$\Rightarrow x(\rho)$$

# Liquid-Gas Phase Transition



Gradient-modified properties are uniform!

$$T = \bar{T}(\epsilon_{\kappa}, \rho),$$

$$\mu = \bar{\mu}(\epsilon_{\kappa}, \rho) - \kappa \nabla^2 \rho,$$

$$T_{xx} = \bar{P}(\epsilon_{\kappa}, \rho) - \kappa \rho \nabla^2 \rho + \frac{1}{2} (\nabla \rho)^2$$

## Example 2: Density-Density Correlations

$$\langle \delta\rho(0)\delta\rho(r) \rangle = (\chi - \chi_0) \frac{e^{-r/\ell}}{4\pi\ell^2 r} + \chi_0 \delta^3(r),$$

$$\langle \delta\rho(0)\delta T(r) \rangle = 0,$$

$$\langle \delta\rho(0)\delta\mu(r) \rangle = 0,$$

$$\langle \delta\rho(0)\delta T_{ij}(r) \rangle = 0,$$

$$\langle \delta\rho(0)\delta\bar{\mu}[\epsilon(r), \rho(r)] \rangle = \frac{T}{\chi} \langle \delta\rho(0)\delta\rho(r) \rangle,$$

$$\langle \delta\rho(0)\delta\bar{\beta}[\epsilon(r), \rho(r)] \rangle = \frac{1}{\chi_{QE}} \langle \delta\rho(0)\delta\rho(r) \rangle,$$

$$\langle \delta\rho(0)\delta\bar{P}[\epsilon(r), \rho(r)] \rangle = \frac{\rho T}{\chi} \langle \delta\rho(0)\delta\rho(r) \rangle$$

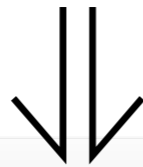
# Altering Hydrodynamic Noise

Noisy current:

$$\langle j_i^{(n)}(0) j_k^{(n)}(r) \rangle = 2\sigma T \delta_{ik} \delta^4(r) + \sigma \kappa \chi_0 \partial_i \partial_k \delta^4(r),$$

$$D = \sigma T / \chi$$

NEW



$$\langle \delta \rho(0) \delta \rho(r) \rangle = \chi_0 \delta^3(r) + \frac{(\chi - \chi_0) T}{\kappa \chi} \frac{e^{-r/\ell}}{4\pi r},$$

$$\ell^2 = \beta \kappa \chi$$

As  $T \rightarrow T_c$ ,

$\chi \rightarrow \infty$ ,  $\ell \rightarrow \infty$ ,  $D \rightarrow 0$

**But,  $\kappa$ ,  $\chi_0$ ,  $\sigma$  and noise are well behaved!**



# Summary

## Alter Hydrodynamic Equations:

- Gradient terms for  $s$ ,  $T_{ij}$ ,  $\mu$ ,  $T$
- Additional momentum density,  $\sim$  to  $\partial v$
- Adjust thermal noise

## Advantages

- Conserve entropy (aside from diff.) / energy / momentum
- Reproduce thermal correlations (not just fluc.s)
- L.G. Interface properties
- Damp high-momentum modes

## Needs Work:

- Highly non-causal diffusion,  $\omega \sim k^4$