Parametrized Equation of State for QCD from 3D Ising Model

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A Critical EoS for QCD

The task

- Current knowledge of QCD EoS (from first principles) at finite μ_B is a Taylor expansion from Lattice QCD around $\mu_B = 0$, up to $\mathcal{O}(\mu_B^6)$:

$$\mathrm{P}_{\mathrm{QCD}} = T^4 \sum_n c^n(T) \left(\frac{\mu_B}{T}\right)^n \quad , \quad c^n(T) = \frac{1}{n!} \frac{\partial(P/T^4)}{\partial(\mu_B/T)}\bigg|_{\mu_B=0}$$

- An EoS for QCD including critical behavior would be an important ingredient in hydrodynamical simulations of heavy ion collisions

- The expected critical behavior of QCD is in the same static universality class as 3D Ising model

$\Rightarrow$ **Build an EoS that matches Lattice QCD results and includes the correct critical behavior**

A Critical EoS for QCD

The strategy

- Choose a suitable parametrization for the scaling EoS of 3D Ising model
- Define a mapping of the 3D Ising phase diagram onto the QCD one
- Use 3D Ising EoS to estimate critical contribution to $c^n(T)$:

$$c^n(T) = c^n_{\rm reg}(T) + c^n_{\rm crit}(T)$$

- Expand over the whole phase diagram:

$$P(T, \mu_B \neq 0) = T^4 \sum_n c^n_{\rm reg}(T) \left(\frac{\mu_B}{T}\right)^n + f(T, \mu_B) P_{\rm crit}\left(T, \mu_B\right)$$

where $f(T, \mu_B)$ is a regular function of T and μ_B, with dimension 4.

Scaling EoS for 3D Ising model

Scaling EoS can be given in parametric form for magnetization M, magnetic field h and reduced temperature $r = (T - T_C)/T_C$ in 3D Ising model:

$$(R, \theta) \longmapsto (r, h) : \qquad \begin{aligned} M &= M_0 \, R^{\beta} \theta \\ h &= h_0 \, R^{\beta\delta} \tilde{h}(\theta) \\ r &= R(1 - \theta^2) \end{aligned}$$

where:

- M_0, h_0 are normalization constants;
- $\tilde{h}(\theta) = \theta(1 + a\,\theta^2 + b\,\theta^4)$ with $(a = -0.76201, b = 0.00804)$;
- $R \geq 0$ and $|\theta| \leq 1.154$ (second zero of $\tilde{h}(\theta)$);
- $\beta \simeq 0.326$, $\delta \simeq 4.80$ are critical exponents.

C. Nonaka and M. Asakawa, Phys.Rev. C71 (2005) 044904, R. Guida and J. Zinn-Justin, Nucl.Phys. B489 (1997) 626-652, P. Schofield, Phys. Rev. Lett. 23 (1969) 109

Scaling EoS for 3D Ising

Construct (Helmoltz) and thus Gibbs free energy densities:

$$F(M,r) = h_0 M_0 R^{2-\alpha} g(\theta) \quad \longrightarrow \quad G(r,h) = F(M,r) - Mh$$

Thanks to the map:

$$(R,\theta) \longmapsto (\mathbf{r},\mathbf{h}) \longleftrightarrow (\mathbf{T},\mu_{\mathbf{B}})$$

we can write the pressure in QCD as:

$$\boxed{P^{QCD}_{\mathrm{crit}}(T,\mu_B) = -G(T(R,\theta),\mu_B(R,\theta)) = h_0 M_0 R^{2-\alpha}\left[g(\theta) - \theta\tilde{h}(\theta)\right]}$$

NOTE: Explicit functional form of $G(T(R,\theta),\mu_B(R,\theta))$ ONLY as a function of (R,θ). Evaluation will require numerical inversion of :

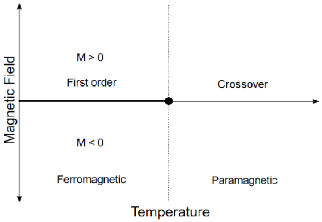
$$T(R,\theta) = T^* \qquad \mu_B(R,\theta) = \mu_B^*$$

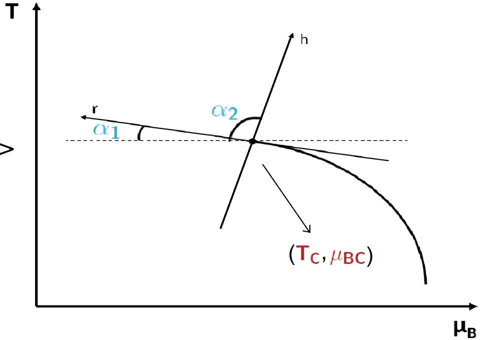
C. Nonaka and M. Asakawa, Phys.Rev. C71 (2005) 044904, R. Guida and J. Zinn-Justin, Nucl.Phys. B489 (1997) 626-652

Map the phase diagram

Relation between scaling variables (h, r) and thermodynamic coordinates (T, μ_B) can be expressed in linear form, and needs 6 parameters:

$$(r, h) \longmapsto (T, \mu_B): \qquad \frac{T - T_C}{T_C} = \mathbf{w}\,(r\rho \sin\alpha_1 + h \sin\alpha_2)$$
$$\frac{\mu_B - \mu_{BC}}{T_C} = \mathbf{w}\,(-r\rho \cos\alpha_1 - h \cos\alpha_2)$$





Map the phase diagram

Comments on the parameters

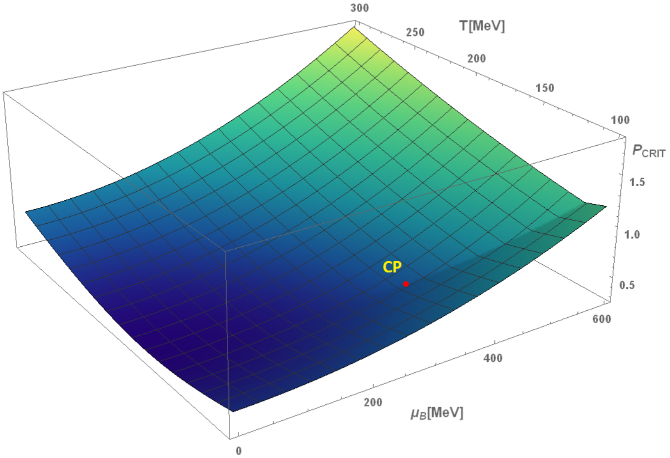
- The purpose of the project is to exploit the parametric nature of the EoS, in order to use theoretical arguments and future BES-II experimental data to **constrain the value of the parameters**.
- While some of the parameters have a straightforward interpretation, others' role is less intuitive
- How is the choice of the parameters driven?
  - From Lattice QCD: $T_C \lesssim 150\,\mathrm{MeV}$, $\mu_{BC} \geq 2T_C$, α_1 somewhat constrained by choice of T_C, μ_{BC}
  - We would like to place the critical point in the region of the phase diagram accessible to BES-II

For illustrative purpose: a choice of parameters

$$\alpha_1 = \pi/30 \qquad T_C = 140\,\mathrm{MeV} \qquad w = 1$$
$$\alpha_2 = \pi/2 + \pi/30 \qquad \mu_{BC} = 350\,\mathrm{MeV} \qquad \rho = 2$$

The critical pressure

The scaling form of the pressure mapped onto the QCD phase diagram.



NOTE: symmetrized around $\mu_B = 0$ in order to ensure $c^n(T) = 0, \forall n$ odd

Matching the Lattice

We make the simple choice $f(T, \mu_B) = T_C^4$ for the normalization of P_{crit}, and then define:

$$c^n_{\text{LAT}}(T) = c^n_{\text{reg}}(T) + \left(\frac{T_C}{T}\right)^4 c^n_{\text{crit}}(T) \ .$$

Remembering the map:

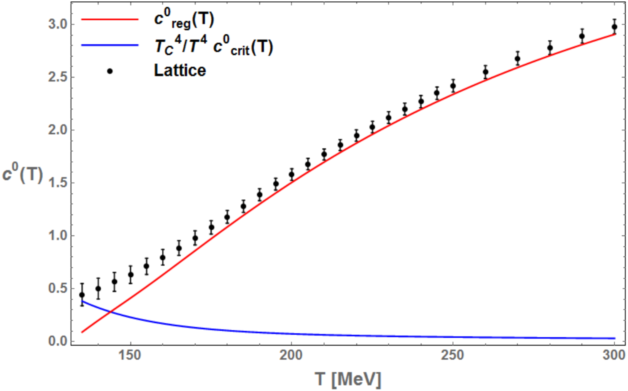
$$(R, \theta) \longmapsto (r, h) \longleftrightarrow (T, \mu_B)$$

one can express any derivative of the critical pressure over the whole phase diagram:

$$P(T, \mu_B = 0) = -G(T(R, \theta), \mu_B(R, \theta) = 0)$$
$$n!\, c^n(T, \mu_B = 0) = -\left(\frac{\partial G}{\partial \mu_B}\right)_T \Big|_{\mu_B = 0}$$

Matching the Lattice: 0^{th} order

- The c^n_{reg} resulting from this procedure might be negative, if the critical contribution exceeds the Lattice results



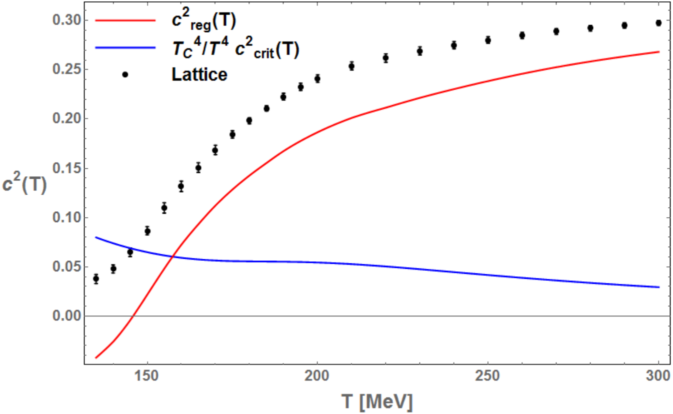
R. Bellwied et al., Phys. Rev. D 92, 114505

- If this happens, for one or more of the Taylor coefficients, it might result in the pressure being negative (or pathologically behaved) for some value of T, μ_B, and will therefore be discarded

Matching the Lattice: 2^{th} order

The full expression for the second coefficient is:

$$2!\, c^2(T, \mu_B = 0) = -\left(\frac{\partial^2 G}{\partial r^2}\right)_h \left(\frac{\partial r}{\partial \mu_B}\right)^2 - \left(\frac{\partial^2 G}{\partial h^2}\right)_r \left(\frac{\partial h}{\partial \mu_B}\right)^2 - 2\frac{\partial^2 G}{\partial h \partial r}\frac{\partial h}{\partial \mu_B}\frac{\partial r}{\partial \mu_B}$$

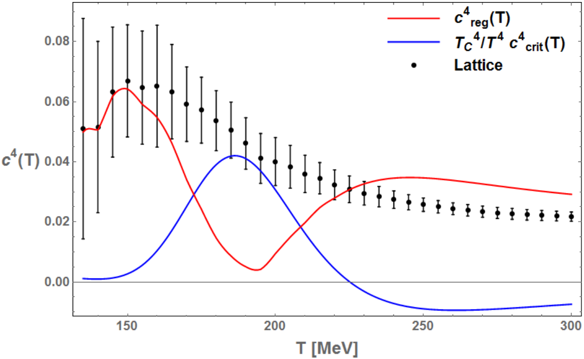


R. Bellwied et al., Phys. Rev. D 92, 114505

Matching the Lattice: 4^{th} order

And for the fourth coefficient is:

$$4!\,c^4(T,\mu_B) = -\left(\frac{\partial^4 G}{\partial r^4}\right)_h \left(\frac{\partial r}{\partial \mu_B}\right)^4 - \left(\frac{\partial^4 G}{\partial h^4}\right)_r \left(\frac{\partial h}{\partial \mu_B}\right)^4 - 4\frac{\partial^4 G}{\partial h^3 \partial r}\left(\frac{\partial h}{\partial \mu_B}\right)^3 \frac{\partial r}{\partial \mu_B} + \\ - 4\frac{\partial^4 G}{\partial h \partial r^3}\frac{\partial h}{\partial \mu_B}\left(\frac{\partial r}{\partial \mu_B}\right)^3 - 6\frac{\partial^4 G}{\partial h^2 \partial r^2}\left(\frac{\partial h}{\partial \mu_B}\right)^2 \left(\frac{\partial r}{\partial \mu_B}\right)^2$$

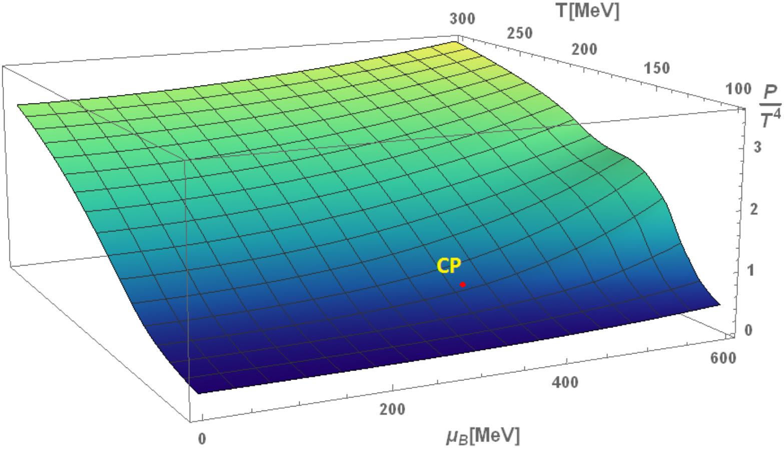


R. Bellwied et al., Phys. Rev. D 92, 114505

Critical EoS: the total pressure

With these ingredients, one can build the total pressure:

$$P(T, \mu_B) = T^4 \sum_n c^n_{\text{reg}}(T) \left(\frac{\mu_B}{T}\right)^n + T_C^4 \, P_{\text{crit}}\,(T, \mu_B)$$



Critical EoS: some comments

By construction, the reconstructed EoS will have the pressure and its derivatives wrt μ_B match Lattice QCD at $\mu_B = 0$. However, it will need to satisfy thermodynamical conditions.

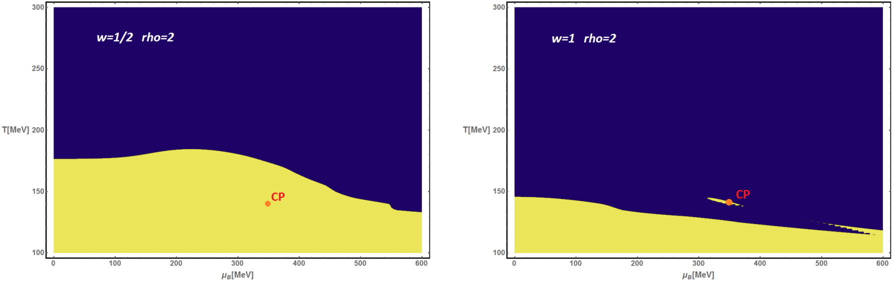
Constraints on the parameters

- Systematically span the space of parameters, requiring that thermodynamic inequalities are satisfied at all (T, μ_B)
  - Positivity of pressure, entropy density, baryon density + further conditions on second order derivatives
  - **NOTE:** a rigorous analysis of thermodynamical inequalities will need the analysis of uncertainties from Lattice QCD data
- The application of the EoS to fluid dynamical simulations will produce results that can further constraint the choice of parameters

Determine the size of the critical region

Comparing the critical contribution to $\chi_2(T, \mu_B)$ with the Taylor reconstruction from Lattice can give us an idea of the size and shape of the critical region for different w, ρ:

$$\chi_2^{\mathrm{LAT}}(T, \mu_B) - \chi_2^{\mathrm{crit}}(T, \mu_B)$$



In yellow the region where the critical contribution exceeds the Lattice one.

$\Rightarrow$ A smaller value of w corresponds to a larger critical contribution and a larger critical region

Summary

Comments

- By means of a parametrized form of the scaling EoS and a non universal mapping of the scaling variables onto QCD coordinates, it is possible to build an expression for the pressure and any derivative over the whole phase diagram
- The choice of some parameters can be somehow driven by what is already known, but for others a systematic analysis will be necessary
- The interplay of many different conditions on the EoS can result in a strong constraint on (among others) the location of the critical point

Outlook

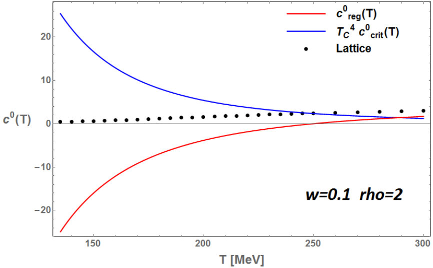
Further improvements

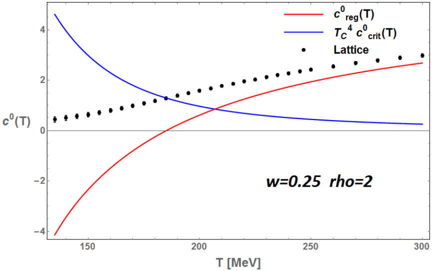
- Run a systematic analysis over the space of parameters of the non-universal mapping
  - Already explored many values of w, ρ ($w = 0.1, 0.2, ..., 3,\ \rho = 2, 4, 8$), and the role of the angles
- Include 6^{th} order coefficient from Lattice in the expansion
- Include temperatures down to $T < 100\,\text{MeV}$ (needed for hydro simulations)
  - For temperatures below the reach of Lattice, one can rely on a smooth merging with models (e.g HRG model)
- Systematically perform the analysis of thermodynamical inequalities carefully including uncertainties from Lattice data

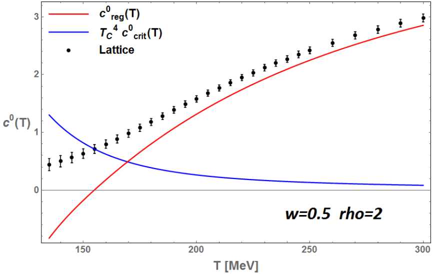
BACKUP

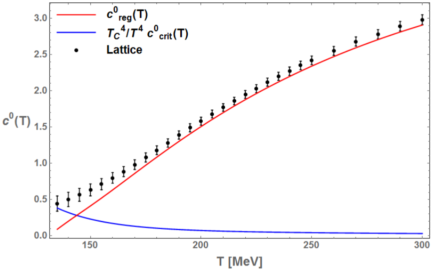
Exploring the parameters space

Matching the Lattice: 0^{th} order



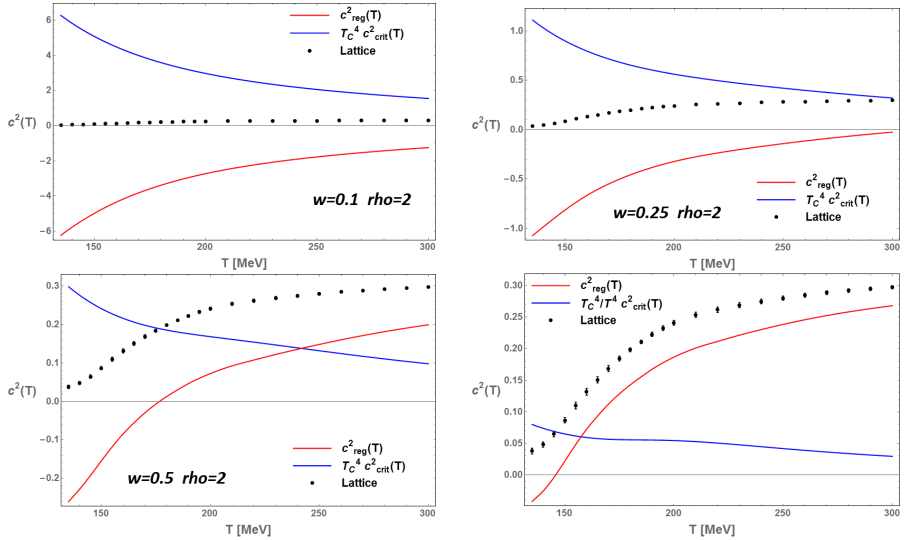






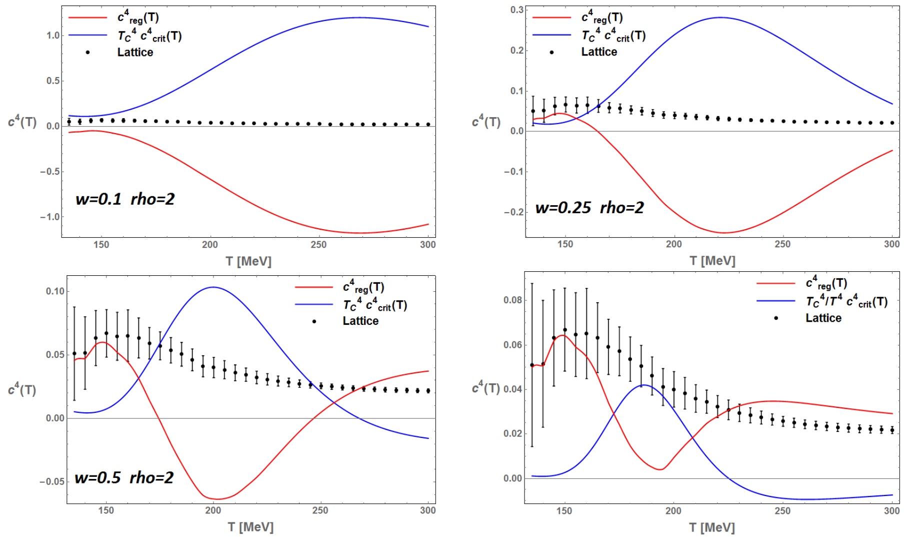
Exploring the parameters space

Matching the Lattice: 2^{th} order



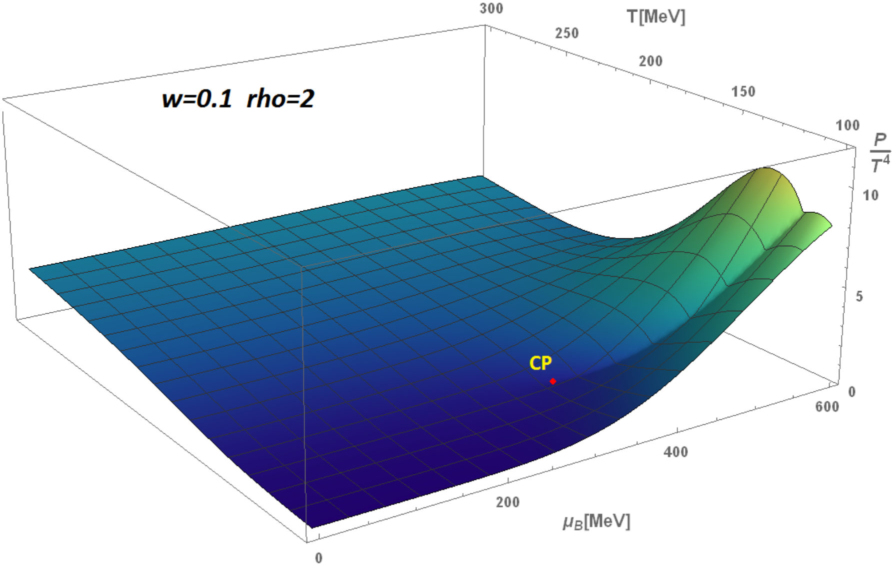
Exploring the parameters space

Matching the Lattice: 4^{th} order



Exploring the parameters space

The reconstructed pressure



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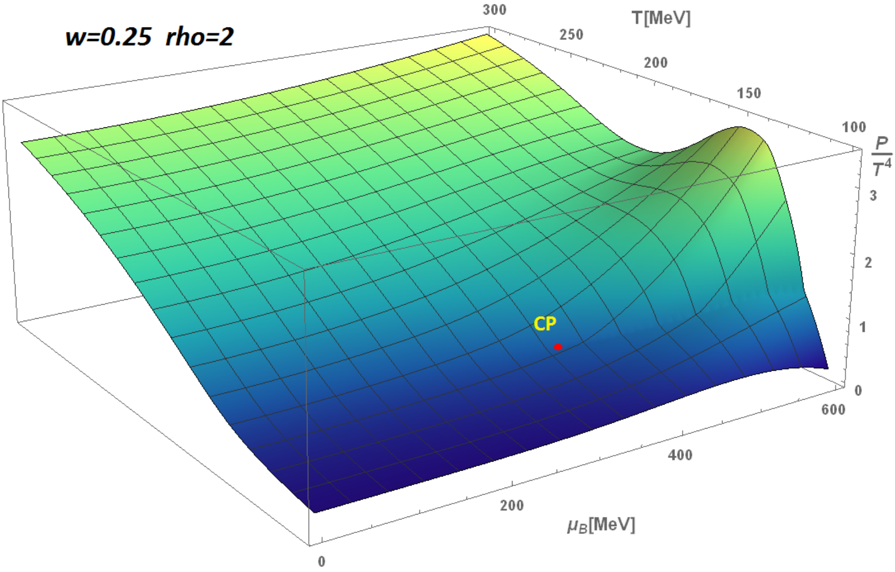

Exploring the parameters space

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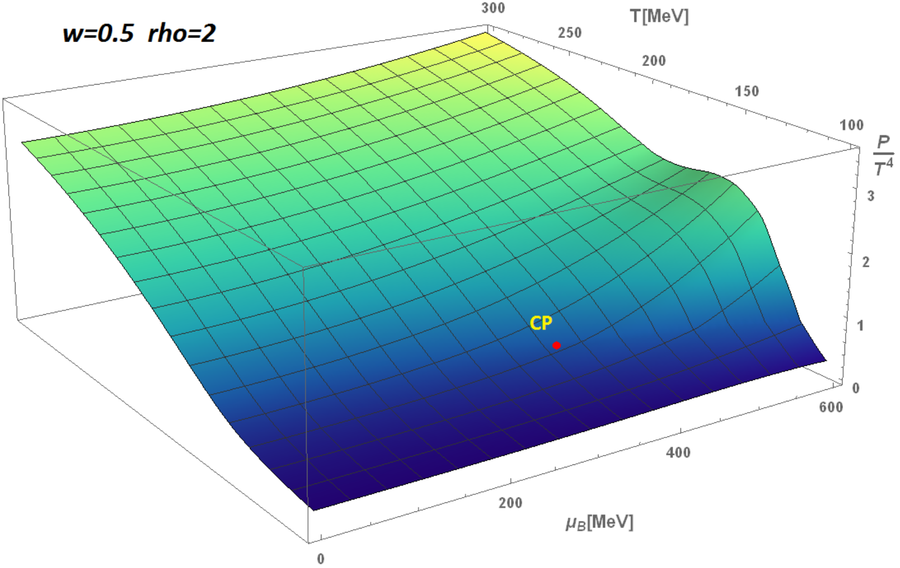
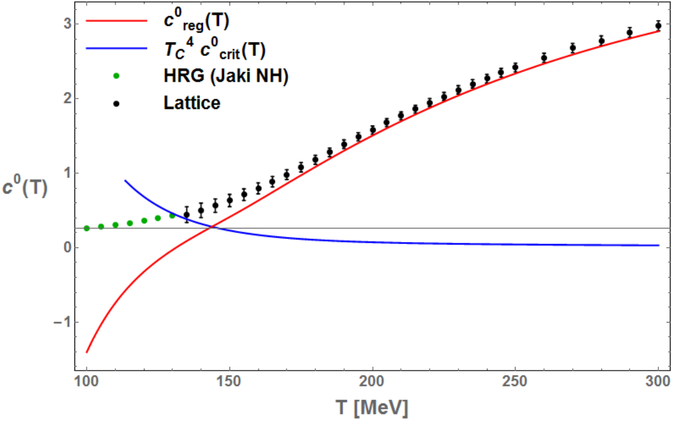


Temperatures below the reach of Lattice QCD

Matching the Lattice: 0^{th} order



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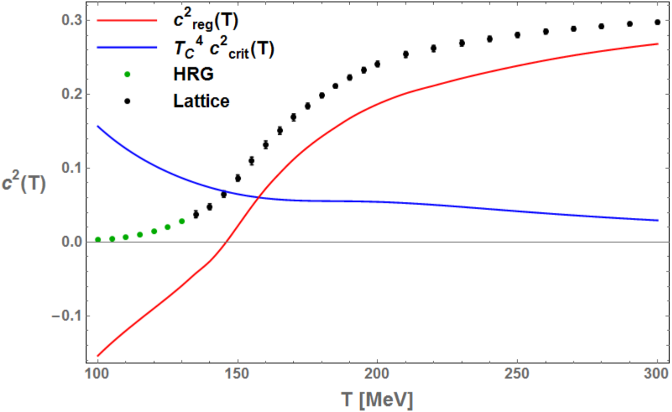
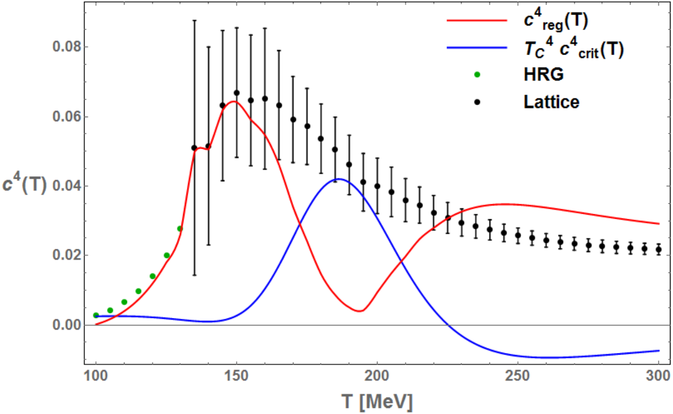
Matching the Lattice: 2^{th} order

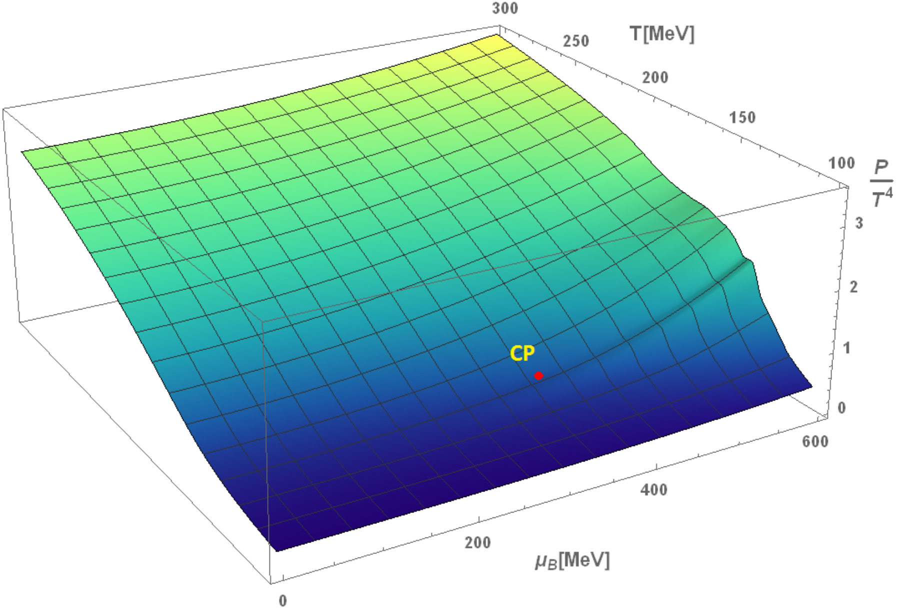


Temperatures below the reach of Lattice QCD

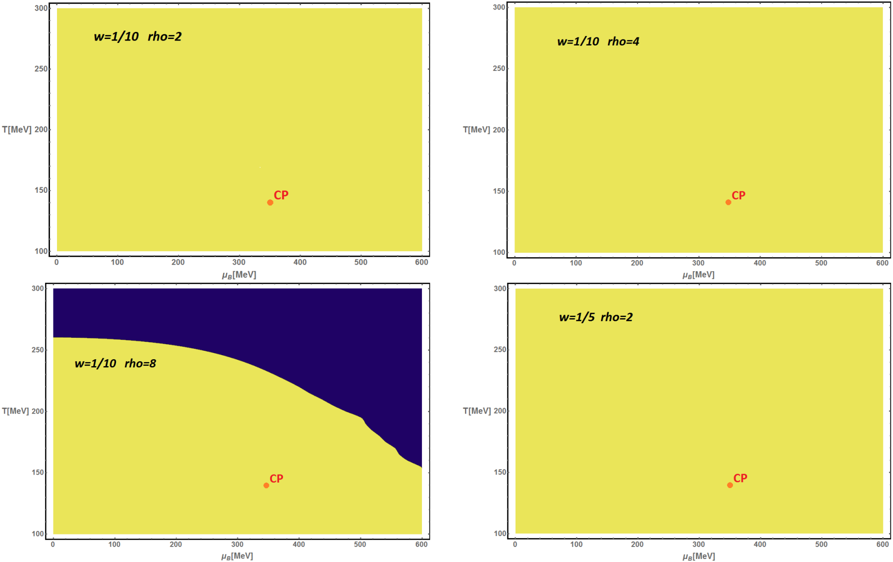
Matching the Lattice: 4^{th} order



Smoothing out Lattice data



Determine the size of the critical region



In yellow the region where the critical contribution exceeds the Lattice one.

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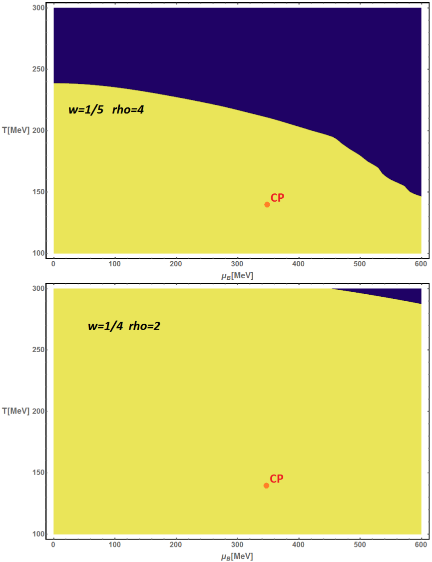

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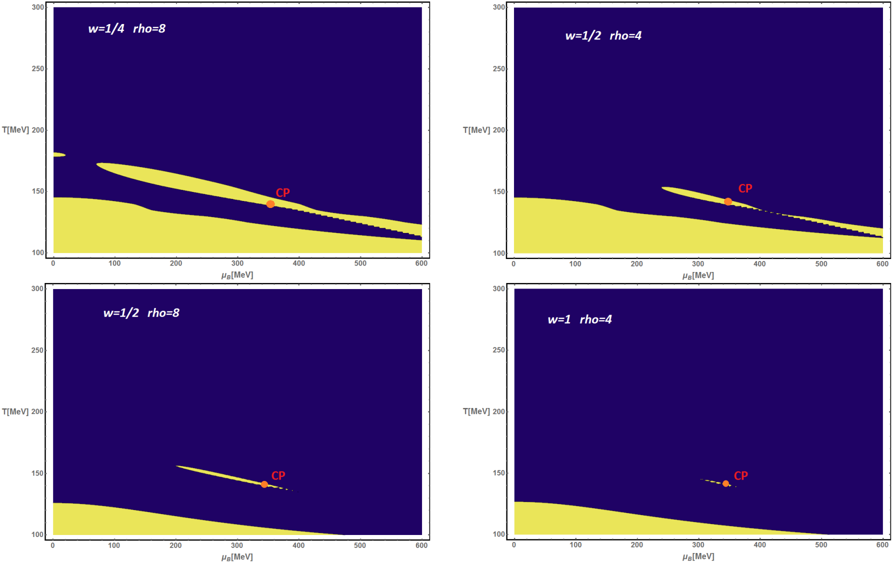

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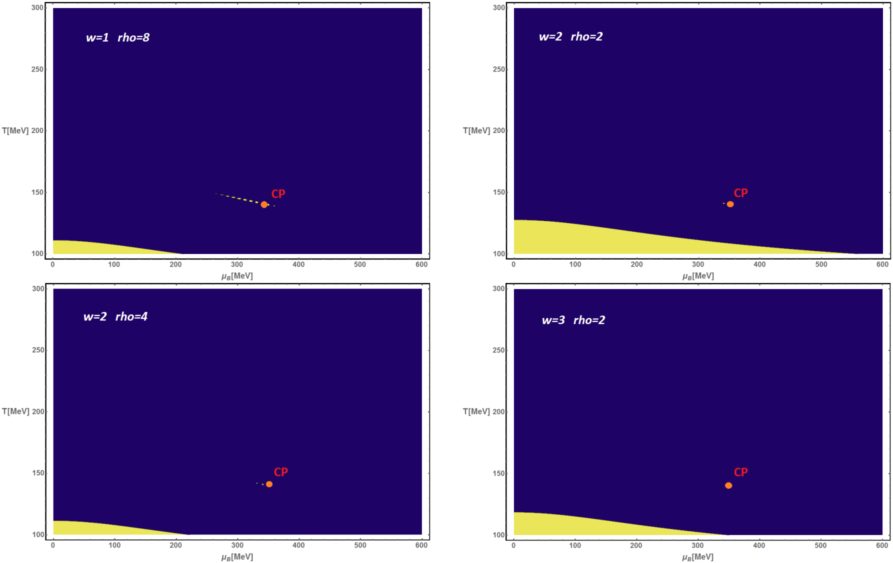

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