Chiral kinetic theory from world-lines in quantum field theory

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Based on Niklas Mueller and RV,
arXiv:1701.03331 and arXiv:1702.01233

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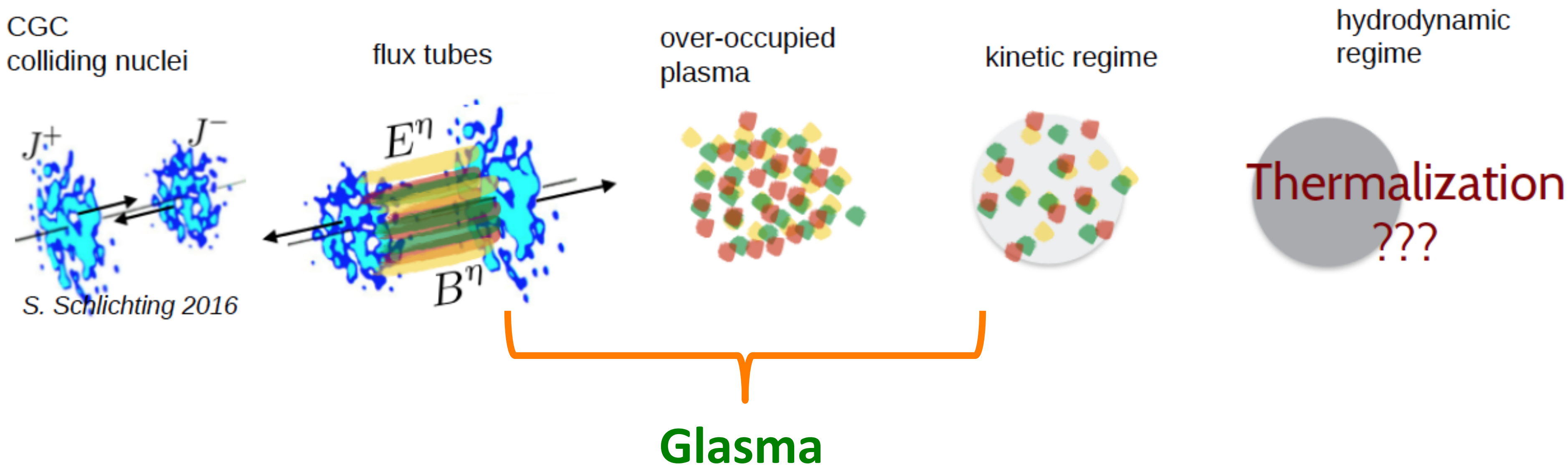
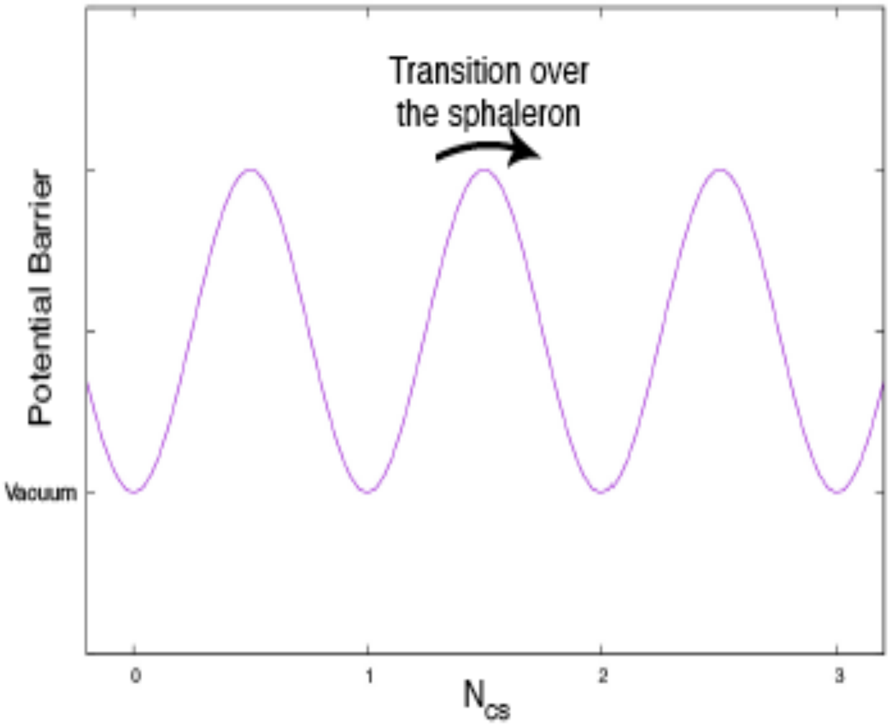
Topological transitions in the Glasma



Sphaleron transitions in QCD

Sphaleron: spatially localized, unstable finite energy classical solutions
(σφαλερος -``ready to fall")

EW theory: Klinkhamer, Manton, PRD30 (1984) 2212
QCD: McLerran,Shaposhnikov,Turok,Voloshin, PLB256 (1991) 451



Chiral Anomaly:

$$\partial_\mu J^\mu_{5,f} = 2m_f \bar{q}\gamma_5 q - \frac{g^2}{16\pi^2} F^a_{\mu\nu}\tilde{F}^{\mu\nu,a}$$

Chern-Simons current:

$$K^\mu = \frac{g^2}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}\left(F^a_{\nu\rho}A^a_\sigma - \frac{g}{3} f_{abc}A^b_\nu A^c_\rho A^c_\sigma\right)$$

Chern-Simons #:

$$N_{CS}(t) = \int d^3x\, K^0(t,\mathbf{x})$$

Rate of change of CS #

$$\frac{dN_{CS}(t)}{dt} = \frac{g^2}{8\pi^2}\int d^3x\, E^a_i(\mathbf{x})B^a_i(\mathbf{x})$$

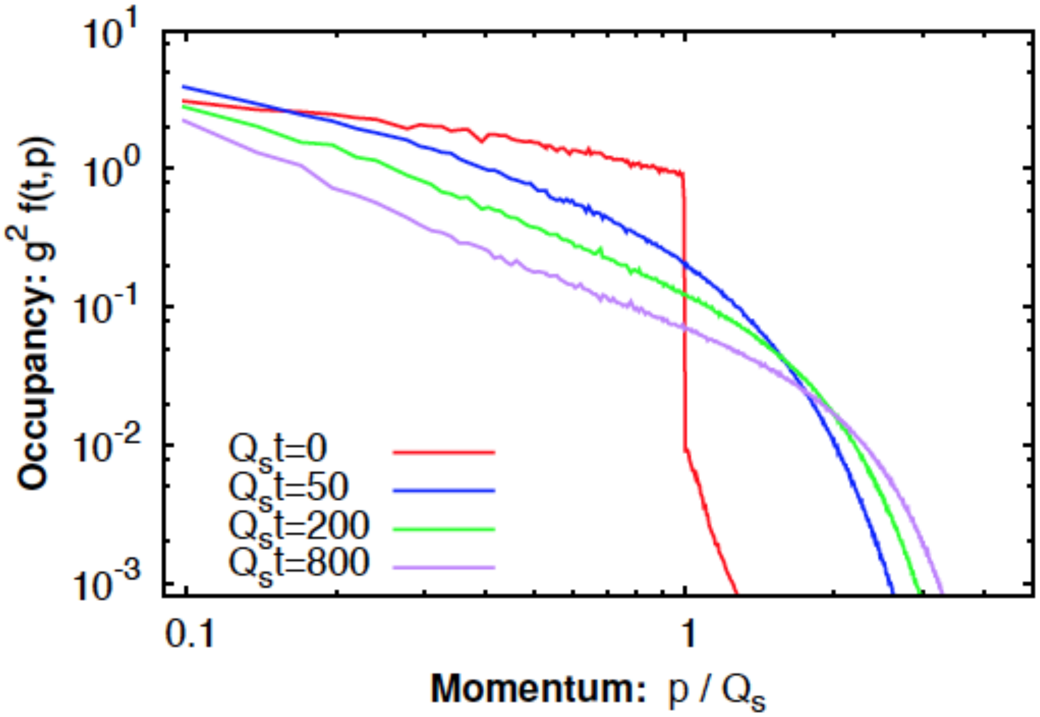
- **Sphaleron transition rate – significant literature on thermal rate**

$$\Gamma^{eq} = \lim_{\delta t -> \infty} \frac{\langle (N_{\rm CS}(t+\delta t) - N_{\rm CS}(t))^2\rangle_{\rm eq}}{V\,\delta t}$$

State of the art at finite T,

Moore, Tassler, arXiv:1011.1167

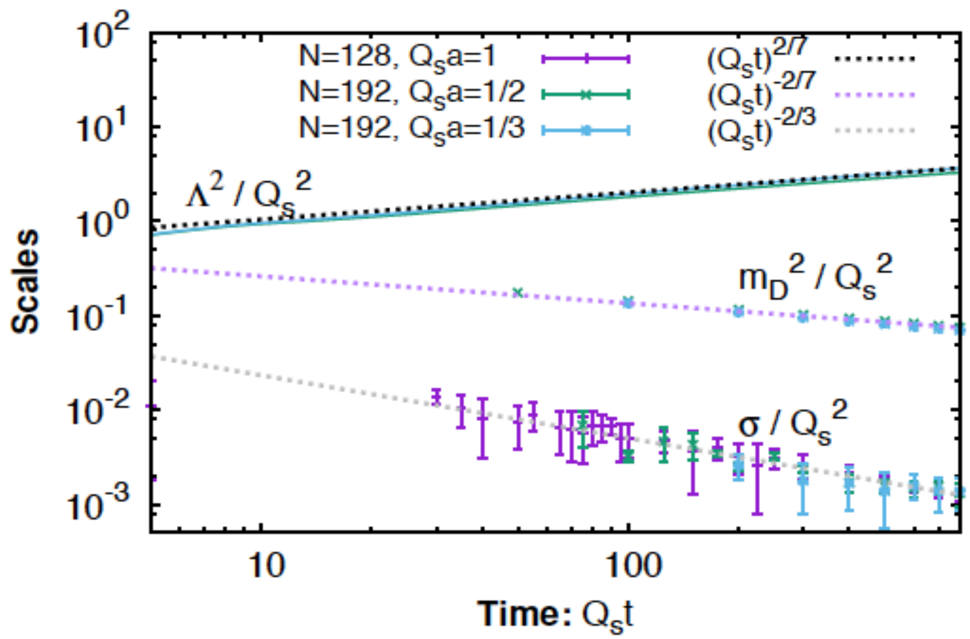
Overoccupied gauge fields in a box



Thermalization extensively studied in this context employing classical-statistical simulations

Berges,Schlichting,Sexty, PRD86 (2012) 074006
Schlichting PRD86 (2012) 065008
York,Kurkela,Lu,Moore, PRD89 (2014) 074036

Overoccupied gauge fields in a box



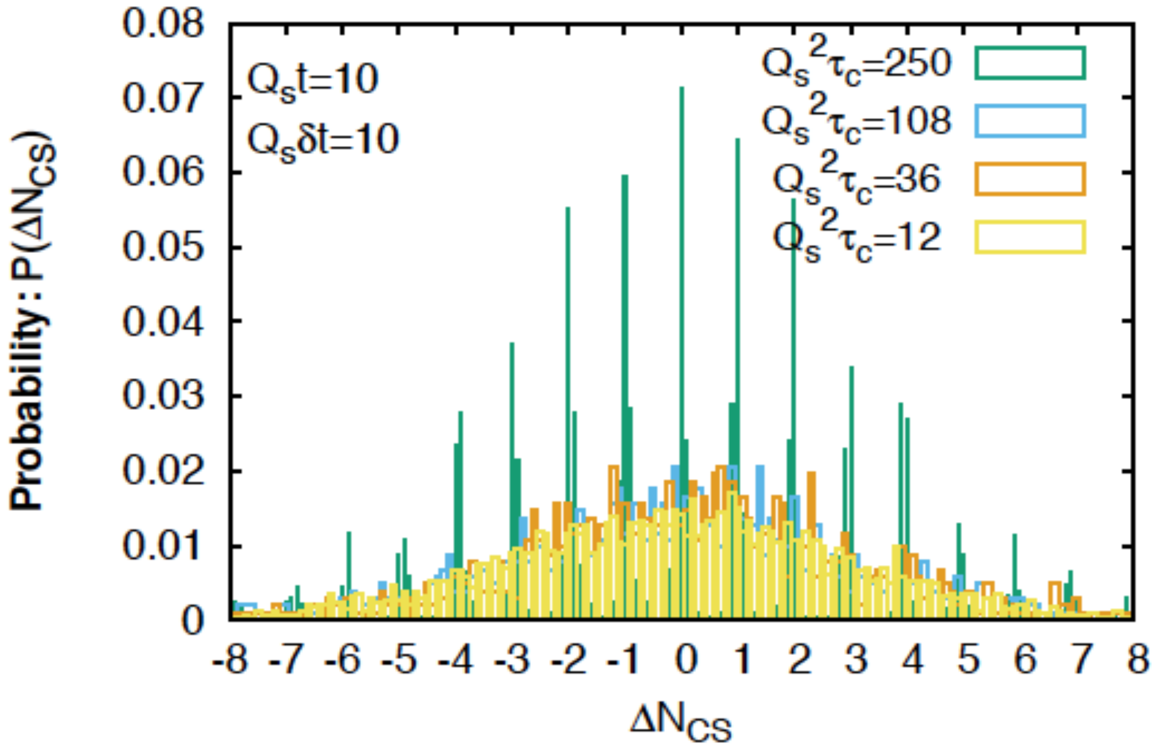
Clean separation of scales develop *a la* thermal field theory:

Temperature (T)
Electric (Debye) screening (gT)
Magnetic screening (g^2 T) scales

Berges,Scheffler,Sexty, PRD77 (2008) 034504
Mace,Schlichting,Venugopalan, PRD93 (2016), 074036
Berges,Mace,Schlichting, PRL118 (2017)

Topological transitions in the Glasma

Mace,Schlichting,Venugopalan, PRD93 (2016), 074036



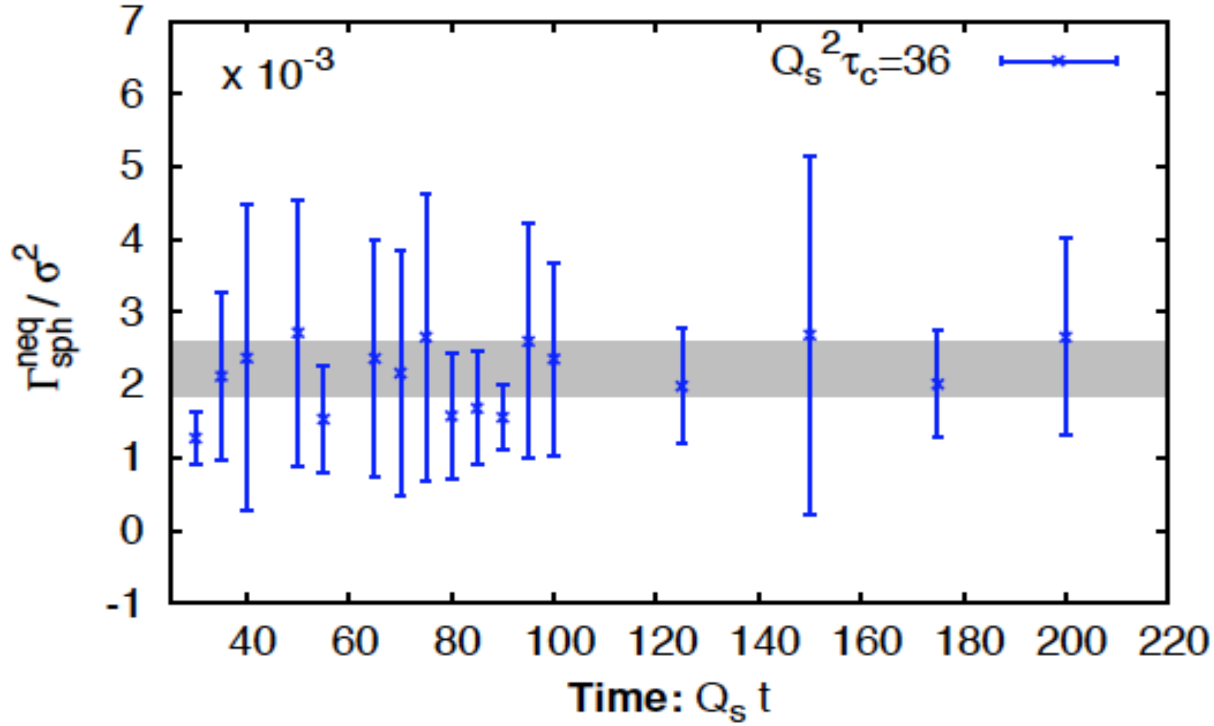
Distribution of Chern-Simons charge localizes around integer values as UV modes are removed

First ever ab initio computation of sphaleron transition off equilibrium…

"Cooled" Glue configurations in the Glasma are topological!

Topological transitions in the Glasma

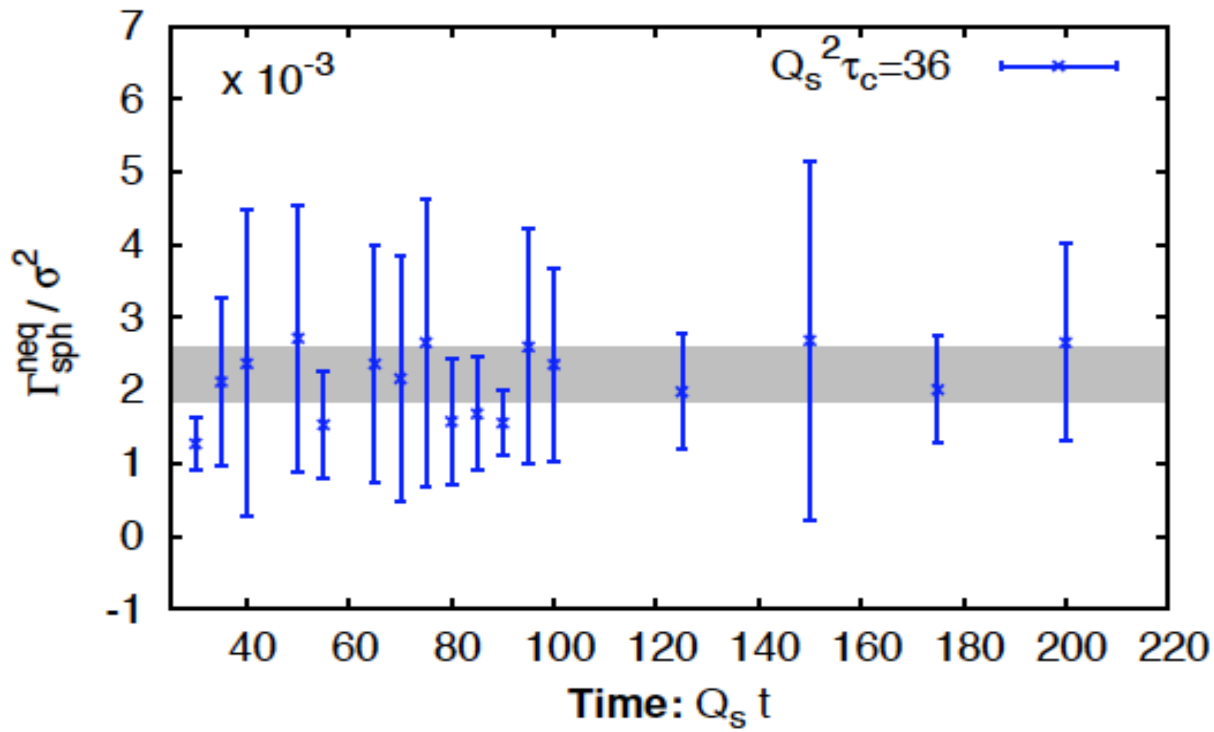
Mace,Schlichting,Venugopalan, PRD93 (2016), 074036



Sphaleron transition rate scales with string tension squared

Topological transitions in the Glasma

Mace,Schlichting,Venugopalan, PRD93 (2016), 074036



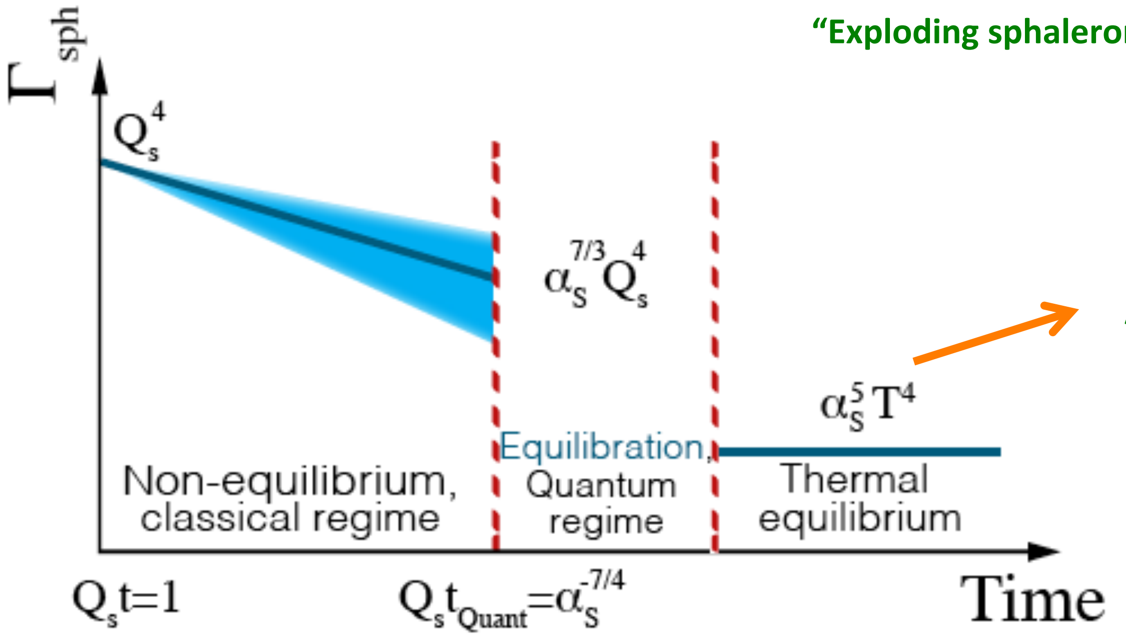
Sphaleron transition rate scales with string tension squared

Improved determination of scaling with σ^2 and SU(3) in progress

M. Mace

Exploding sphalerons

"Exploding sphalerons": Shuryak, Zahed, PRD67 (2003) 014006



Arnold,Son,Yaffe,PRD55(1997)6264

- **Sphaleron transition rate very large in the Glasma**

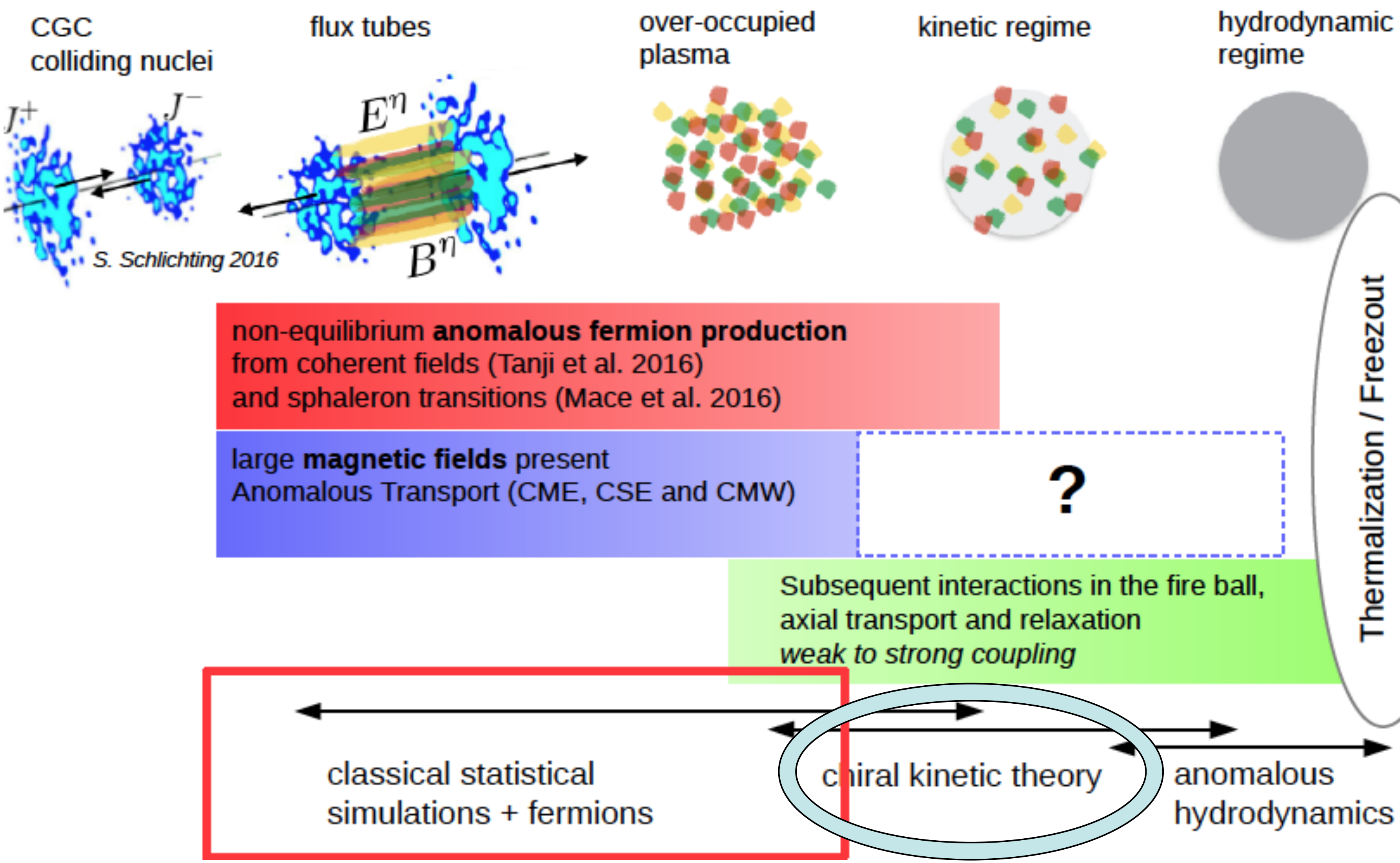
- much larger than equilibrium rate

- **Couple sphaleron background with fermions & external EM fields to simulate *ab initio* the Chiral Magnetic Effect!**

Mueller,Schlichting,Sharma,PRL117 (2016) 142301
Mace,Mueller,Schlichting,Sharma, arXiv: 1612.02477

The limits of classical-statistical simulations

Slide by Niklas Mueller



Towards a chiral kinetic theory:

I. World-line formulation of QFT and the chiral anomaly

Kinetic evolution of the chiral magnetic current

- **Quantum kinetic evolution of the chiral magnetic current depends on typical time scales for *scattering*, for *sphaleron transitions*, and *E&M conductivity* in the system**

- **Significant work on chiral kinetic theory**

Son,Yamamoto, PRL109 (2012), 181602; PRD87 (2013) 085016
Stephanov, Yin, PRL109 (2012) 162001
Chen, Son, Stephanov, Yee, Yin, PRL 113 (2014) 182302
Chen, Son, Stephanov, PRL115 (2015) 021601
Chen,Pu,Wang,Wang, PRL110 (2013) 262301
Gao,Liang,Pu,Wang,Wang, PRL109 (2012) 232301
Stone,Dwivedi,Zhou, PRD91 (2015) 025004
Zahed, PRL109 (2012) 091603;
Basar,Kharzeev,Zahed, PRL111 (2013)161601
Stephanov,Yee,Yin, PRD91 (2015) 125014
Fukushima, PRD92 (2015) 054009
Manuel, Torres-Rincon, PRD90 (2014) 074018
Hidaka,Pu,Yang, arXiv:1612.04630
…. Number of 2017 papers…

- **We will discuss here a novel approach based on the world-line formalism in QCD**

Niklas Mueller and RV, arXiv: 1701.03331 and 1702.01233

World-line formalism: preliminaries

Review: Corradini, Schubert, arXiv:1512.08694
Also, Strassler, NPB385 (1992) 145

Based on Schwinger's proper time trick:

$$\log(\sigma) = \int_1^\sigma \frac{dy}{y} \equiv \int_1^\sigma dy \int_0^\infty dt\, e^{-yt} = -\int_0^\infty \frac{dt}{t}\left(e^{-\sigma t} - e^{-t}\right)$$

One loop effective action of massless scalar field coupled to background Abelian field

$$\mathcal{L} = \Phi^\dagger D^2 \Phi$$

$$D_\mu = \partial_\mu - igA_\mu$$

$$\Gamma[A] = -\log\left[\det(-D^2)\right] \equiv -\mathrm{Tr}\left(\log(-D^2)\right) = \int_0^\infty \frac{dT}{T} \mathrm{Tr}\, \exp(-TD^2)$$

$$= \int_0^\infty \frac{dT}{T} \mathcal{N} \int \mathcal{D}x\, \mathcal{P} \exp\left[-\int_0^T d\tau \left(\frac{1}{2\varepsilon}\dot{x}^2 + igA[x(\tau)]\cdot\dot{x}\right)\right]$$

with $\mathcal{N} = \int \mathcal{D}p\, \exp(-\frac{1}{2}\int_0^T d\tau \epsilon\, p^2)$ ε is the Einbein: square root of 1D metric

World-line formalism: vector and axial vector fields

$$S[A,B] = \int d^4x\, \bar{\psi}\, \left(i\not{\partial} + \not{A} + \gamma_5 \not{B}\right)\, \psi$$

A is a vector gauge field and B is an auxiliary axial vector gauge field

Fermion effective action:

$$-W[A,B] = \log\, \det\,(\theta) \quad \textbf{with} \quad \theta = i\not{\partial} + \not{A} + \gamma_5 \not{B}$$

$$W[A,B] = W_R + i\, W_I$$

Focus first on the real part:

$$W_R = -\frac{1}{8} \log\, \det\left(\tilde{\Sigma}^2\right) \equiv -\frac{1}{8} \mathrm{Tr}\, \log\left(\tilde{\Sigma}^2\right)$$

$$\tilde{\Sigma}^2 = (p - \mathcal{A})^2\, \mathbb{I}_8 + \frac{i}{2} \Gamma_\mu \Gamma_\nu F_{\mu\nu}[\mathcal{A}] \qquad \mathcal{A} = \begin{pmatrix} A+B & 0 \\ 0 & A-B \end{pmatrix}$$

$$F_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu - [\mathcal{A}_\mu, \mathcal{A}_\nu]$$

World-line formalism: coherent states

Doubling dimension of Dirac matrices & extension of Clifford algebra essential for coherent state spinor representation

$$\Gamma_\mu = \begin{pmatrix} 0 & \gamma_\mu \\ \gamma_\mu & 0 \end{pmatrix}, \quad \Gamma_5 = \begin{pmatrix} 0 & \gamma_5 \\ \gamma_5 & 0 \end{pmatrix}, \quad \Gamma_6 = \begin{pmatrix} 0 & i\mathbb{I}_4 \\ -i\mathbb{I}_4 & 0 \end{pmatrix}$$

D'Hoker, Gagne, hep-ph/9508131

$$\Gamma_7 = -i \prod_{A=1}^{6} \Gamma_A = \begin{pmatrix} \mathbb{I}_4 & 0 \\ 0 & -\mathbb{I}_4 \end{pmatrix} \qquad \{\Gamma_7, \Gamma_A\} = 0$$

Coherent states can be used to generate finite dimensional representations of internal symmetry groups

Berezin,Marinov, Annals Phys. 104 (1977) 336

$$a_r^{\pm} = \frac{1}{2}(\Gamma_r \pm i\Gamma_{r+3}), \qquad \{a_r^+, a_s^-\} = \delta_{rs}, \qquad \{a_r^+, a_s^+\} = \{a_r^-, a_s^-\} = 0,$$

$$\langle\theta| a_r^- = \langle\theta|\theta_r \qquad a_r^-|\theta\rangle = \theta_r|\theta\rangle \qquad \langle\bar\theta|a_r^+ = \langle\bar\theta|\bar\theta_r \qquad a_r^+|\bar\theta\rangle = \bar\theta_r|\bar\theta\rangle$$

$$r = 1, 2, 3$$

$$\int |\theta\rangle\langle\theta|\, d^3\theta = \int d^3\bar\theta\, |\bar\theta\rangle\langle\bar\theta| = \mathbb{I}.$$

Grassmanian path integral representation

$$W_R = -\frac{1}{8}\mathrm{Tr}\,\log\left(\tilde{\Sigma}^2\right) = \frac{1}{8}\int_0^\infty \mathrm{Tr}_{16}\,\exp\left(-\frac{\varepsilon}{2}T\tilde{\Sigma}^2\right)$$

In the fermionic coherent state Grassmanian representation,
the trace can be rewritten as the quantum mechanical path integral…

Ohnuki,Kashiwa Prog.Theo.Phys.60 (1978)548
D'Hoker, Gagne, hep-ph/9512080

$$\Longrightarrow \quad \frac{1}{8}\int_0^\infty \frac{dT}{T}\mathcal{N}\int_P \mathcal{D}x\int_{AP}\mathcal{D}\psi\,\mathrm{tr}_c\mathcal{P}\left(e^{-\int_0^T d\tau\,\mathcal{L}_L(\tau)} + e^{-\int_0^T d\tau\,\mathcal{L}_R(\tau)}\right)$$

with the "point particle" Lagrangian

$$\mathcal{L}_{L/R}(\tau) = \frac{\dot{x}^2}{2\mathcal{E}} + \frac{1}{2}\psi_a\dot{\psi}_a - i\dot{x}_\mu(A\pm B)_\mu + \frac{i\mathcal{E}}{2}\psi_\mu\psi_\nu F_{\mu\nu}[A\pm B]$$

Switched here from 3-D complex θ basis to that of 6-D Majorana fermions ψ_a
- simple mnemonic: $\Gamma \Rightarrow \sqrt{2}\psi$

Phase of the determinant and the chiral anomaly

- The phase of the complex determinant is well known to be the origin of the chiral anomaly

K. Fujikawa, PRL42 (1979)1195; PRD21 (1980) 2848

- Its treatment in a world line path integral framework is also well known

L. Alvarez-Gaume, E. Witten, NPB234 (1984) 269
A.M. Polyakov, *Gauge fields and strings* (1987), section 6.3

- We will adopt a different regularization (due to D'Hoker&Gagne) and apply it to derive the quantity of interest

Mueller, Venugopalan, 1701.03331 & 1702.01233

Phase of the determinant and the chiral anomaly

$$W_I = -\frac{1}{2}\arg\det[\Omega]$$

$$\Omega = \begin{pmatrix} 0 & \theta \\ \theta & 0 \end{pmatrix}$$

with

$$\Omega = \Gamma_\mu(p_\mu - A_\mu) - i\Gamma_7\Gamma_\mu\Gamma_5\Gamma_6 B_\mu$$

Using a trick due to D'Hoker & Gagne, W_I can be rewritten in a form ***very much like that for the real part…***

$$: \frac{i\mathcal{E}}{64}\int_{-1}^{1} d\alpha \int_0^\infty dT\,\mathrm{Tr}\left\{\hat{M}e^{-\frac{\mathcal{E}}{2}T\tilde{\Sigma}^2_{(\alpha)}}\right\}$$

where the trace insertion is $\hat{M} = \Gamma_7\Lambda$

$$\Lambda = (2\Gamma_5\Gamma_6[\partial_\mu, B_\mu] + [\Gamma_\mu, \Gamma_\nu]\{\partial_\mu, B_\nu\}\Gamma_5\Gamma_6)$$

The parameter α breaks chiral symmetry explicitly - setting it to ±1 restores it

Phase of the determinant and the chiral anomaly

The axial vector current can be expressed as

$$\langle j^5_\mu(y)\rangle = \frac{i\delta W_I}{\delta B_\mu(y)}\Big|_{B=0} = -\frac{\mathcal{E}}{32}\int_0^\infty dT\,\mathrm{Tr}\left\{\frac{\delta\hat{M}}{\delta B_\mu(y)}e^{-\frac{\mathcal{E}}{2}T\tilde{\Sigma}^2}\right\}_{B=0}$$

An advantage of the D'Hoker-Gagné construction is that the axial current has the ***same world-line structure as its vector counterpart***

After some algebra:

i) expressing above as a path integral,

ii) separating zero & non-zero modes (the Γ_7 insertion makes PBC and Fermion zero modes feasible in the path integral construction)

iii) and fixing Fock-Schwinger gauge about the zero modes,

one can show that

See our paper 1702.01233 for details

$$\partial_\mu\langle J^5_\mu(y)\rangle = \frac{1}{8\pi^2}\mathrm{Tr}\left(\tilde{F}_{\mu\nu}F^{\mu\nu}\right)$$

which is the well known anomaly equation

Towards a chiral kinetic theory:

II. Pseudo-classical equations of motion

Spinning (& colored) pseudo-classical world-lines

Brink, Di Vecchia, Howe, NPB118 (1977) 76
Balachandran, Salomonson, Skagerstam, Winnberg, PRD15 (1978) 2308
Barducci, Casalbuoni, Lusanna, NPB124 (1977) 93

For a consistent treatment of the Hamiltonian dynamics, introduce Lagrange multipliers in action to impose physical constraints

$$S = \int_0^T d\tau \left\{ p_\mu \dot{x}^\mu + \frac{i}{2}\left[\psi_\mu \dot{\psi}^\mu + \psi_5 \dot{\psi}_5\right] - H \right\}$$

with $H = \frac{\varepsilon}{2}\left(\pi^2 + m^2 + i\psi^\mu F_{\mu\nu}\psi^\nu\right) + \frac{i}{2}\left(\pi_\mu \psi^\mu + m\psi_5\right)\chi$

Here ε and χ are the vierbein fields that impose the mass shell and helicity constraints of the theory

$Q=\pi_\mu \psi^\mu + m\psi_5$ is a supersymmetric charge generating an N=1 SUSY algebra

Canonical momenta:

$$p^\mu \equiv \frac{\partial \mathcal{L}}{\partial \dot{x}_\mu} \quad \text{with} \quad \pi^\mu \equiv p^\mu - A^\mu = m_R u^\mu - \frac{i m_R}{2z}\left(1 - \frac{m^2}{2m_R^2}\right)\left[\psi^\mu + u_\nu \psi^\nu u^\mu\right]\chi$$

where u_μ is the "anomalous" velocity $\qquad m_R^2 = m^2 + i\psi^\mu F_{\mu\nu}\psi^\nu.$

Spinning (& colored) pseudo-classical world-lines

Pseudo-classical equations of motion for spinning particles in the (x,p,ψ) phase space:

$$m_R\,\ddot{x}^\mu + \frac{i}{2\,m_R}\psi^\alpha \partial^\mu F_{\alpha\beta}\,\psi^\beta + F^{\mu\nu}\dot{x}_\nu = 0$$

$$\dot{\psi}^\mu - \frac{1}{m_R}F_{\mu\nu}\psi^\nu = 0 \qquad \dot{\psi}_5 = 0$$

One can also obtain EOM for the Pauli-Lubanski vector

$$\Sigma_\mu = -\frac{i}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu\psi^\rho\psi^\sigma \qquad S_\mu = \Sigma_\mu/\hbar$$

$$\dot{\Sigma}_\mu = \frac{g}{m}\left(F_{\mu\nu}\Sigma^\nu + \partial^\nu F_{\mu\nu}\Gamma_5\right) \qquad \Gamma_5 = \psi_0\psi_1\psi_2\psi_3$$

- **For homogeneous fields, this is the covariant form of the Bargmann-Michel-Telegdi equation**

Extension of framework to colored Grassmanians gives the Wong equations for precessing color charges

Barducci,Casalbuoni,Lusanna, NPB124 (1977) 93

Non-relativistic limit & Berry phase

Carefully taking non-relativistic limit v/c << 1 of the world-line action

Note: Also holds for large μ

Thomas precession term **Larmor term**

$$H \equiv mc^2 + \frac{\left(p - \frac{A}{c}\right)^2}{2m} + A^0(x) - \frac{S\cdot\left(\left[v/c - A/(mc^2)\right]\times E\right)}{2mc} - \frac{B\cdot S}{m}$$

In the adiabatic limit $\frac{\mathbf{B}\cdot\mathbf{S}}{m} \approx 0$ **spin flips are suppressed and particle spin is "slaved" to its motion**

Transition amplitude from initial to final states:

$$T(p_f, p_i.+) = \langle p_f, \psi^+(p_f)|e^{-iH(t_f - t_i)}|p_i, \psi^+(p_i)\rangle$$

$$T(\mathbf{p}_f, \mathbf{p}_i, +) = \int \left(\prod_{k=1}^{N-1} d^3p_k\right)\left(\prod_{l=1}^{N} d^3x_l\right)$$

$$\times \prod_{j=1}^{N} \frac{1}{(2\pi)^3} e^{-i\mathbf{x}_j\cdot(\mathbf{p}_j - \mathbf{p}_{j-1}) - iH_j\Delta} \langle\psi^+(\mathbf{p}_j)|\psi^+(\mathbf{p}_{j-1})\rangle$$

$$\exp\left(i\int dt\dot{p}\cdot\mathcal{A}(p)\right)$$

where the Berry connection is

$$\mathcal{A}(p) = -i\langle\psi^+(p)|\nabla_p|\psi^+(p)\rangle$$

Non-relativistic limit & Berry phase

Transition amplitude from initial to final states:

$$T(p_f, p_i.+) = \langle p_f, \psi^+(p_f)|e^{-iH(t_f-t_i)}|p_i, \psi^+(p_i)\rangle$$

$$\Downarrow$$

$$T(\mathbf{p}_f, \mathbf{p}_i, +) = \int \mathcal{D}x\mathcal{D}p\ \exp\left(i\int dt\left[\dot{\mathbf{x}}\cdot\mathbf{p} - \tilde{H}\right]\right)$$

$$\tilde{H} = mc^2 + \frac{(\mathbf{p}-\mathbf{A}/c)^2}{2m} + A^0(x) - \boxed{\dot{\mathbf{p}}\cdot\mathcal{A}(\mathbf{p})}$$

Note: identical derivation if mass is replaced by large chemical potential

- **Note further that to recover this dynamics one has to take non-relativistic, adiabatic limits of the real part of the effective action…**

- **The chiral anomaly in contrast arises from the imaginary phase and is independent of any kinematic limits…**

Relation of Berry phase and anomaly?

Fujikawa's lament… hep-ph/0501166

The notion of Berry's phase is known to be useful in various physical contexts [17]-[18], and the topological considerations are often crucial to obtain a qualitative understanding of what is going on. Our analysis however shows that the topological interpretation of Berry's phase associated with level crossing generally fails in practical physical settings with any finite T. The notion of "approximate topology" has no rigorous meaning, and it is important to keep this approximate topological property of geometric phases associated with level crossing in mind when one applies the notion of geometric phases to concrete physical processes. This approximate topological property is in sharp contrast to the Aharonov-Bohm phase [8] which is induced by the time-independent gauge potential and topologically exact for any finite time interval T. The similarity and difference between the geometric phase and the Aharonov-Bohm phase have been recognized in the early literature [1, 8], but our second quantized formulation, in which the analysis of the geometric phase is reduced to a diagonalization of the effective Hamiltonian, allowed us to analyze the topological properties precisely in the infinitesimal neighborhood of level crossing.

and… hep-ph/0511142

What we have shown in the present paper is that this expectation is not realized, and the similarity between the two is superficial.

Fujikawa's example

Magnetic moment in external B field: $H = -\mu\hbar\sigma \cdot B$

$$\vec{B}(t) = B\,(\sin\theta\cos\phi(t), \sin\theta\sin\phi(t), \cos\theta)$$

Solution of the Schrödinger equation $i\hbar\partial_t\psi(t) = H\psi(t)$

gives $\psi_\pm(T) = w_\pm(T)\exp\left[\frac{-i}{\hbar}\int_0^T dt\, w_\pm^\dagger(t)Hw_\pm(t)\right]\exp\left[\frac{-i}{\hbar}\int_0^T \mathcal{A}_\pm(\vec{B})\frac{d\vec{B}}{dt}dt\right]$

Berry connection $\vec{\mathcal{A}}_\pm(\vec{B}) \equiv w_\pm^\dagger(t)(-i\hbar\frac{\partial}{\partial\vec{B}})w_\pm(t)$

For one full period of the motion, $\exp\left[-\frac{i}{\hbar}\oint \vec{\mathcal{A}}_\pm(\vec{B})\frac{d\vec{B}}{dt}dt\right]$ = $\exp\{-\frac{i}{2}\Omega_\pm\}$

Fujikawa's example

In the adiabatic limit: $\frac{2\pi}{\mu BT} \to 0$ **the phase is nontrivial** $\Omega_{\pm} \to 2\pi(1 \mp \cos\theta)$

Nonadiabatic limit: $\frac{2\pi}{\mu BT} \to \infty$ **the phase is trivial** $\Omega_{\pm} = 2\pi(1 \mp 1)$

Towards a Chiral Kinetic Theory
III. Keldysh formulation

Minding one's P's & Q's : from the one loop effective action in quantum field theory to classical transport theory

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February 1, 2008

Abstract

The one loop effective action in quantum field theory can be expressed as a quantum mechanical path integral over world lines, with internal symmetries represented by Grassmanian variables. In this paper, we develop a real time, many body, world line formalism for the one loop effective action. In particular, we study hot QCD and obtain the classical transport equations which, as Litim and Manuel have shown, reduce in the appropriate limit to the non–Abelian Boltzmann–Langevin equation first obtained by Bödeker. In the Vlasov limit, the classical kinetic equations are those that correspond to the hard thermal loop effective action. We also discuss the imaginary time world line formalism for a hot ϕ^4 theory, and elucidate its relation to classical transport theory.

Real time Schwinger-Kelydysh formulation in the first quantized formalism

S. Mathur, hep-th/9306090,hep-th/9311025

- **The framework can be extended to construct a covariant quantum kinetic description of the transport of chiral fermions in the presence of topological fluctuations**

For alternative framework, see also

Akamatsu,Yamamoto, PRD90 (2014)125031; Akamatsu,Rothkopf,Yamamoto,JHEP1603 (2016)210

Schwinger-Keldysh framework for CKT

Mueller,Venugopalan,Yin, in progress

I) Effective action on the closed-time path:

$$\Gamma = -\log Z[A^+,A^-] = -\log \int [d\chi d\bar{\chi}]_{\mathcal{C}}\, \rho(\chi, A^+, A^-])\, \exp\left(i \int_{\mathcal{C}} d^4x\, \bar{\chi}(i\not{D}[A] - m)\chi \right)$$

with

$$\Gamma[A,B] = \Gamma_{\mathbb{R}}[A,B] + i\Gamma_{\mathbb{I}}[A,B]$$

as we discussed previously, giving

$$\Gamma[A] = -\log Z[A^+,A^-] \propto \int d^4x^+ d^4x^- \int d^5\psi^+ d^5\psi^-\, \hat{\rho}(x^+,x^-,\psi^+,\psi^-) \int_0^\infty \frac{dT}{T} \int_{x^+}^{x^-} Dx \int_{\psi^+}^{\psi^-} D\psi\, \exp\left(i \int_0^T d\tau \mathcal{L} \right)$$

For pseudo-classical world-line trajectories,

$$\Gamma_{cl} = \int d^4x^+ d^4x^- \int d^5\psi^+ d^5\psi^-\, \hat{\rho}(x^+,x^-,\psi^+,\psi^-)\, \exp\left(i \int d\tau \mathcal{L}(x_{cl}, \psi_{cl}, A) \right)$$

For a given initial condition, can write down the corresponding "Liouville eqn" using Dirac brackets

Schwinger-Keldysh framework for CKT

$$\{f, H\}_D = f\left(\frac{\overleftarrow{\partial}}{\partial x_\mu}\dot{x}_\mu + \frac{\overleftarrow{\partial}}{\partial p_\mu}\dot{p}_\mu + \frac{\overleftarrow{\partial}}{\partial \psi_\mu}\dot{\psi}_\mu + \frac{\overleftarrow{\partial}}{\partial \psi_5}\dot{\psi}_5\right)$$

$$= f\left(\frac{\overleftarrow{\partial}}{\partial x_\mu}\frac{P^\mu}{m} + \frac{\overleftarrow{\partial}}{\partial P_\mu}\left[\frac{P^\mu}{m}F^{\nu\mu} - \frac{i}{m}\psi^\alpha\partial^\mu F_{\alpha\beta}\psi^\beta\right] - \frac{2}{m}\frac{\overleftarrow{\partial}}{\partial \psi_\mu}F_{\mu\nu}\psi^\nu\right) = 0$$

with

$$p^\mu = P^\mu + \bar{A}^\mu$$

"Anomalous velocity"

Can be expressed in terms of the Pauli-Lubanski vector as

$$\frac{\partial f}{\partial \Sigma^\mu}\,\frac{\partial \Sigma^\mu}{\partial t}$$

where $\partial \Sigma^\mu / \partial t$ **satisfies the BMT equation**

Schwinger-Keldysh framework for CKT

Include averaged quantities and fluctuations in the world-line path integral

$$f(x,P,\psi) = \bar{f}(x,P,\psi) + \delta f(x,P,\psi)$$

$$F_{\mu\nu} = \langle F_{\mu\nu}\rangle + \delta F_{\mu\nu}$$

Boltzmann-Langevin formulation:

$$\boxed{\begin{aligned} &\frac{P_\mu}{m}\frac{\partial}{\partial x_\mu}\bar{f} + \left[\frac{P_\alpha \bar{F}^{\mu\alpha}}{m} - \frac{i}{m}\psi^\alpha\psi^\beta\partial^\mu \bar{F}_{\alpha\beta}\right]\frac{\partial}{\partial P_\mu}\bar{f} - \frac{2}{m}\bar{F}_{\mu\nu}\psi^\nu\frac{\partial}{\partial \psi_\mu}\bar{f} \\ &\quad = -\frac{P_\alpha}{m}\frac{\partial}{\partial P_\mu}\langle \delta F^{\mu\alpha}\,\delta f\rangle + \frac{i}{m}\psi^\alpha\psi^\beta\langle \partial^\mu \delta F_{\alpha\beta}\,\frac{\partial}{\partial P_\mu}\delta f\rangle + \frac{2}{m}\psi^\nu\langle \delta F_{\mu\nu}\,\frac{\partial}{\partial \psi_\mu}\delta f\rangle \end{aligned}}$$

where LHS are the Vlasov terms and RHS gives the (Balescu-Lenard) collision terms

Schwinger-Keldysh framework for CKT

Include averaged quantities and fluctuations in the world-line path integral

Litim, Manuel, NPB562 (1999)237

$$f(x,P,\psi) = \bar{f}(x,P,\psi) + \delta f(x,P,\psi) \qquad F_{\mu\nu} = \langle F_{\mu\nu}\rangle + \delta F_{\mu\nu}$$

Likewise, for fluctuations in a 2nd moment approximation (see Landau-Lifshitsz):

$$\begin{aligned}\frac{P_\mu}{m}\frac{\partial}{\partial x_\mu}\delta f + \left[\frac{P_\alpha \bar{F}^{\mu\alpha}}{m} - \frac{i}{m}\psi^\alpha\partial^\mu\bar{F}_{\alpha\beta}\psi^\beta\right]\frac{\partial}{\partial P_\mu}\delta f - \frac{2}{m}\bar{F}_{\mu\nu}\psi^\nu\frac{\partial}{\partial\psi_\mu}\delta f \\ = -\left[\frac{P_\alpha\delta F^{\mu\alpha}}{m} - \frac{i}{m}\psi^\alpha\partial^\mu\delta F_{\alpha\beta}\psi^\beta\right]\frac{\partial}{\partial P_\mu}\bar{f} + \frac{2}{m}\delta F_{\mu\nu}\psi^\nu\frac{\partial}{\partial\psi_\mu}\bar{f}\end{aligned}$$

This has to be supplemented by

$$\partial_\mu F^{\mu\nu}[A] = \frac{\delta\Gamma_{cl}}{\delta A_\nu} = eJ^\nu \qquad \Leftrightarrow \qquad \partial_\mu(\bar{F}^{\mu\nu} + \delta F^{\mu\nu}) = e(\bar{J}^\nu + \delta J^\nu)$$

- **Analogous expression for the difference of left and right handed currents given by world-lines derivation for J_μ^5**

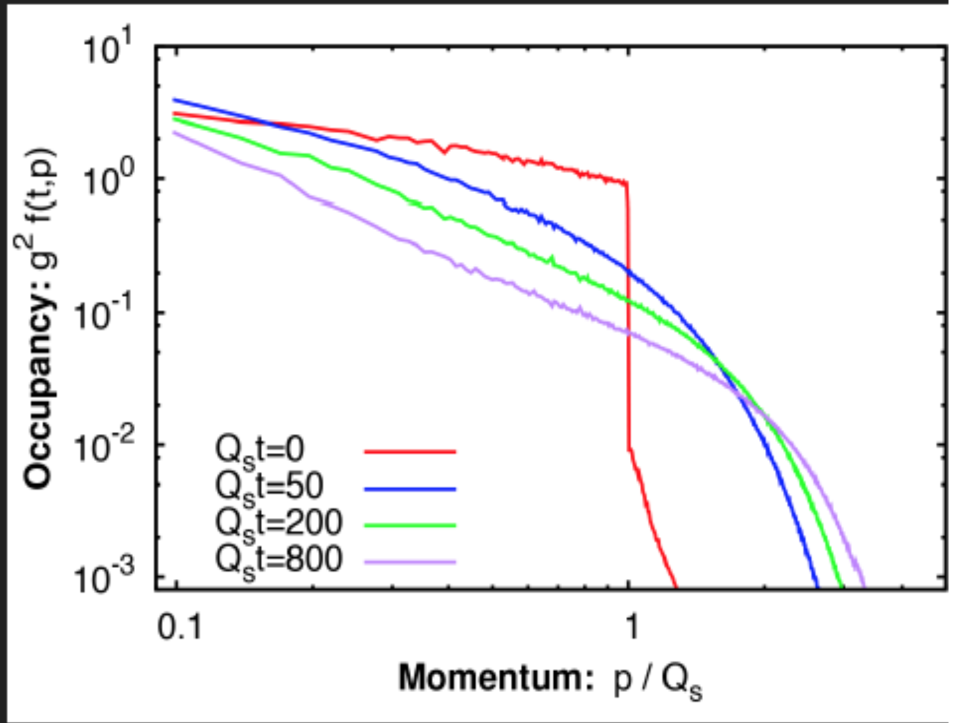
Thank you for your attention!

Summary

- World-line structure is embedded in the structure of gauge theories
- Powerful intuitive framework for construction of kinetic theories with internal degrees of freedom, including thermal and topological fluctuations
- Systematic covariant treatment of the anomaly and dynamics from the e.o.m
- Missing link between first principles classical-statistical simulations and anomalous hydrodynamics

NON-EQUILIBRIUM SPHALERONS

- Initial conditions
  - Over-occupied, far from equilibrium
  - Doesn't interact with cutoff
  - Initially, Q_s is only scale in the problem (CGC), evolves into multiple scales in a finite temperature plasma
  - Quasi-particle picture
    - Superposition of transversely polarized gluons
  - Need many configurations (classical statistical)

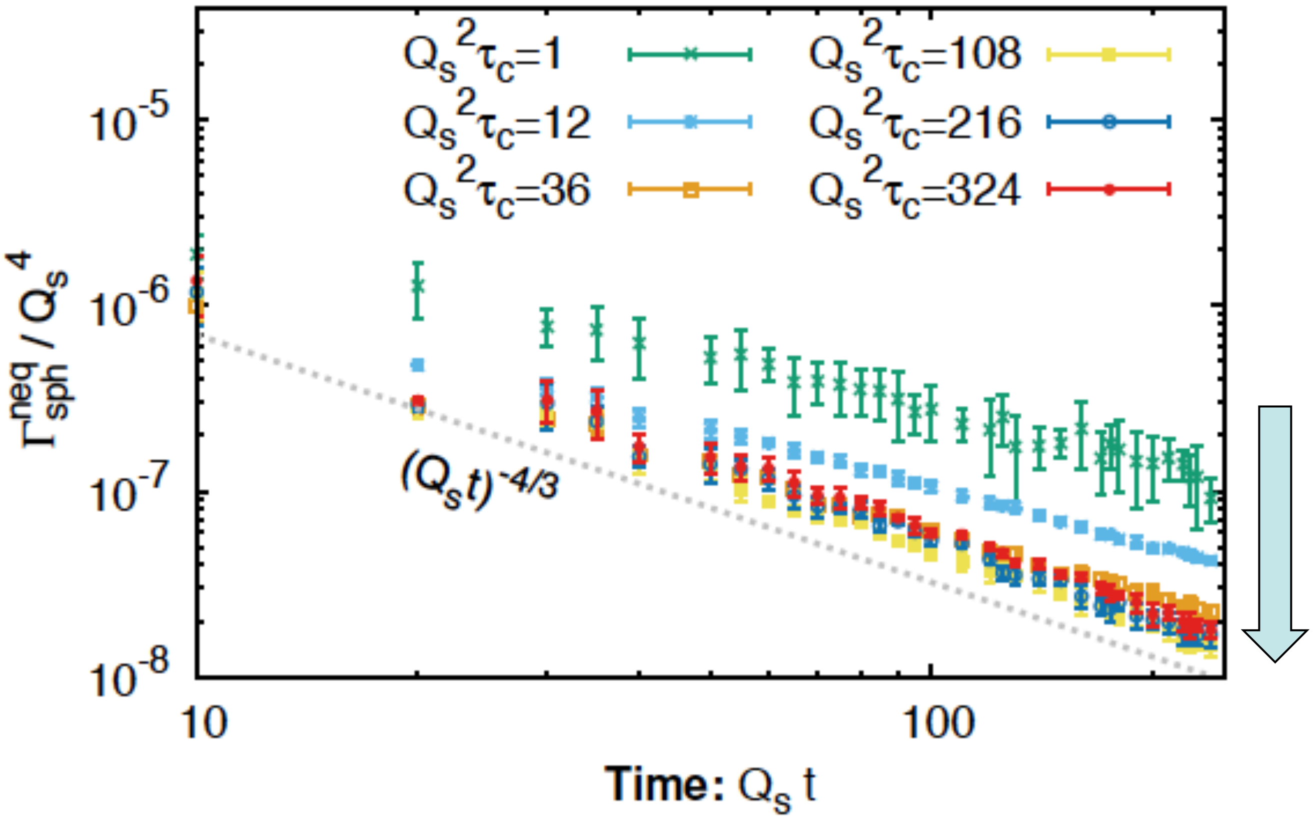


$$A^a_\mu(t_0, x) = \sum \int \frac{d^3k}{(2\pi)^2} \frac{1}{2k} \sqrt{f(t_0, k)} \left[c^a_k \xi^\lambda_\mu(k) e^{ikx} + c.c.\right]$$

$$E^a_\mu(t_0, x) = \sum \int \frac{d^3k}{(2\pi)^2} \frac{1}{2k} \sqrt{f(t_0, k)} \left[c^a_k \dot{\xi}^\lambda_\mu(k) e^{ikx} + c.c.\right]$$

MM, S. SCHLICHTING, R. VENUGOPALAN ARXIV:1601.07342 [HEP-PH]

Overoccupied gauge fields in a box



Increasing cooling time

Mace,Schlichting,Venugopalan, PRD93 (2016), 074036

CLASSICAL YANG-MILLS DYNAMICS

- Continuum Lagrangian $\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a$
- Recast in terms of lattice variables
  - Gauge fields A -> U
  - Electric Fields E -> E

$$\frac{\delta U_\mu(x)}{\delta A^a_\nu(y)} = -iga\ \tau^a\ U_\mu(x)\ \delta_\mu{}^\nu a u^a$$

- Use Kogut-Susskind YM Hamiltonian in terms of products of gauge links (plaquettes)

$$H = \frac{a^3}{2} \sum_{j,x} E^a_j(x) E^a_j(x) + \frac{2}{g^2 a} \sum_{\square} \mathrm{Re}\mathrm{Tr}\left[\mathbf{I} - U_\square\right]$$

$$\boxed{\partial_t E^\mu_a(x) = -\frac{\delta H}{\delta A^a_\mu(x)}}$$

Eqns of motion

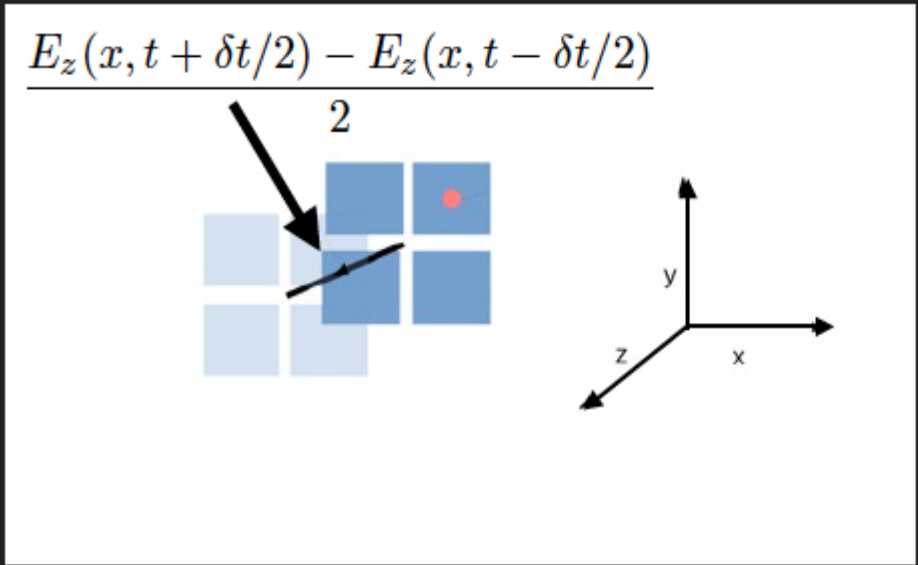
$$\boxed{\partial_t U_\mu(x) = -iga\ \tau^a\ \frac{\delta H}{\delta E^\mu_a(x)} U_\mu(x)}$$

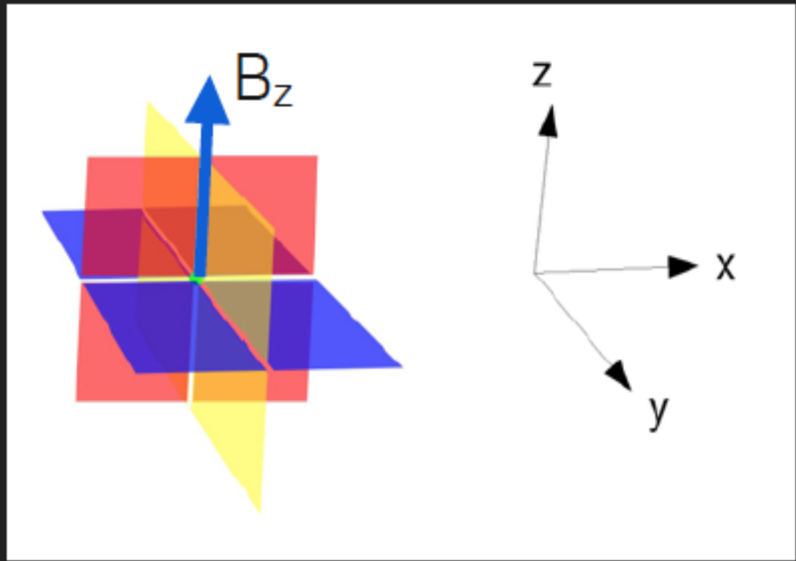
KOGUT AND SUSSKIND PHYS. REV. D 11, 395

TOPOLOGY ON THE LATTICE

- From earlier expressions we can get gauge invariant quantity $\dfrac{dN_{CS}(t)}{dt} = \int d^3x \dfrac{g^2}{8\pi^2} E_i^a B_i^a$
- Then on the lattice, this becomes

$$\frac{dN_{CS}}{dt} \simeq \frac{g^2}{8\pi^2} \sum_{\mathbf{x} \in lattice, i} \frac{(E_i^a(\mathbf{x}, t+\delta t/2) + E_i^a(\mathbf{x}, t-\delta t/2))}{2} 2 \sum_{8\Box} \mathrm{tr}\left(\frac{i\tau^a U_\Box(\mathbf{x})}{4}\right)$$



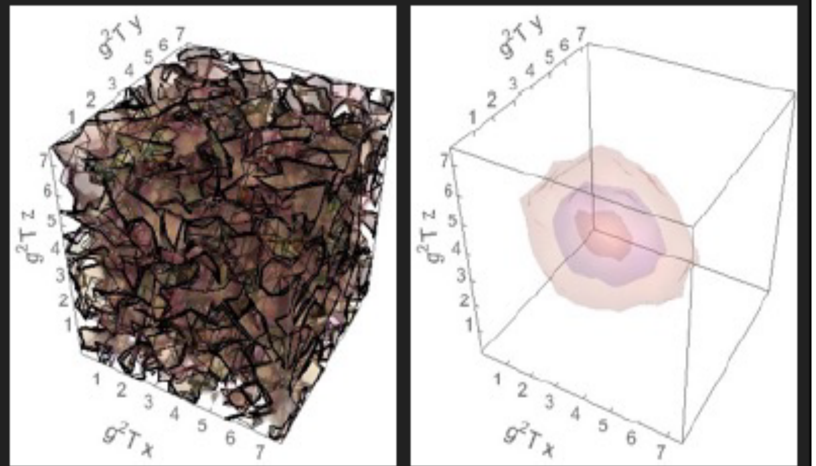


TOPOLOGY ON THE LATTICE

$$\partial_\mu K^\mu = \frac{g^2}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{\mu\nu}_a = \frac{g^2}{8\pi^2} \mathbf{E} \cdot \mathbf{B}$$

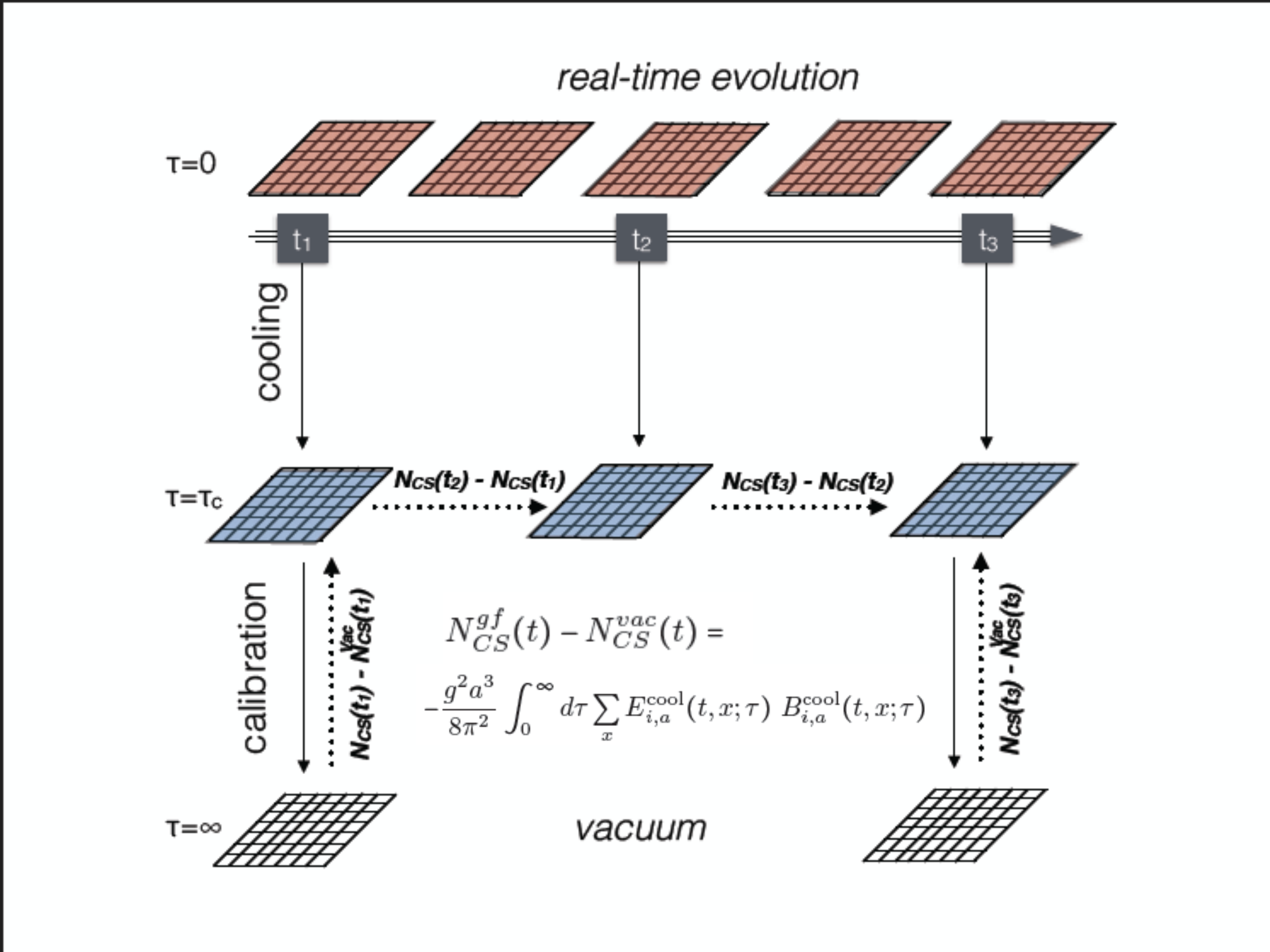
- Can now define ΔN_{CS} on lattice
- One problem
  - Anomaly eqn. is a total derivative, E.B on lattice is not = poor on the lattice
    - Highly susceptible to UV noise
    - Want a topological definition
    - Need smoother configurations
- Need methods like cooling $$\frac{\partial A^a_i(x)}{\partial \tau} = -\frac{\partial H}{\partial A^a_i(x)}$$

Cooling a configuration



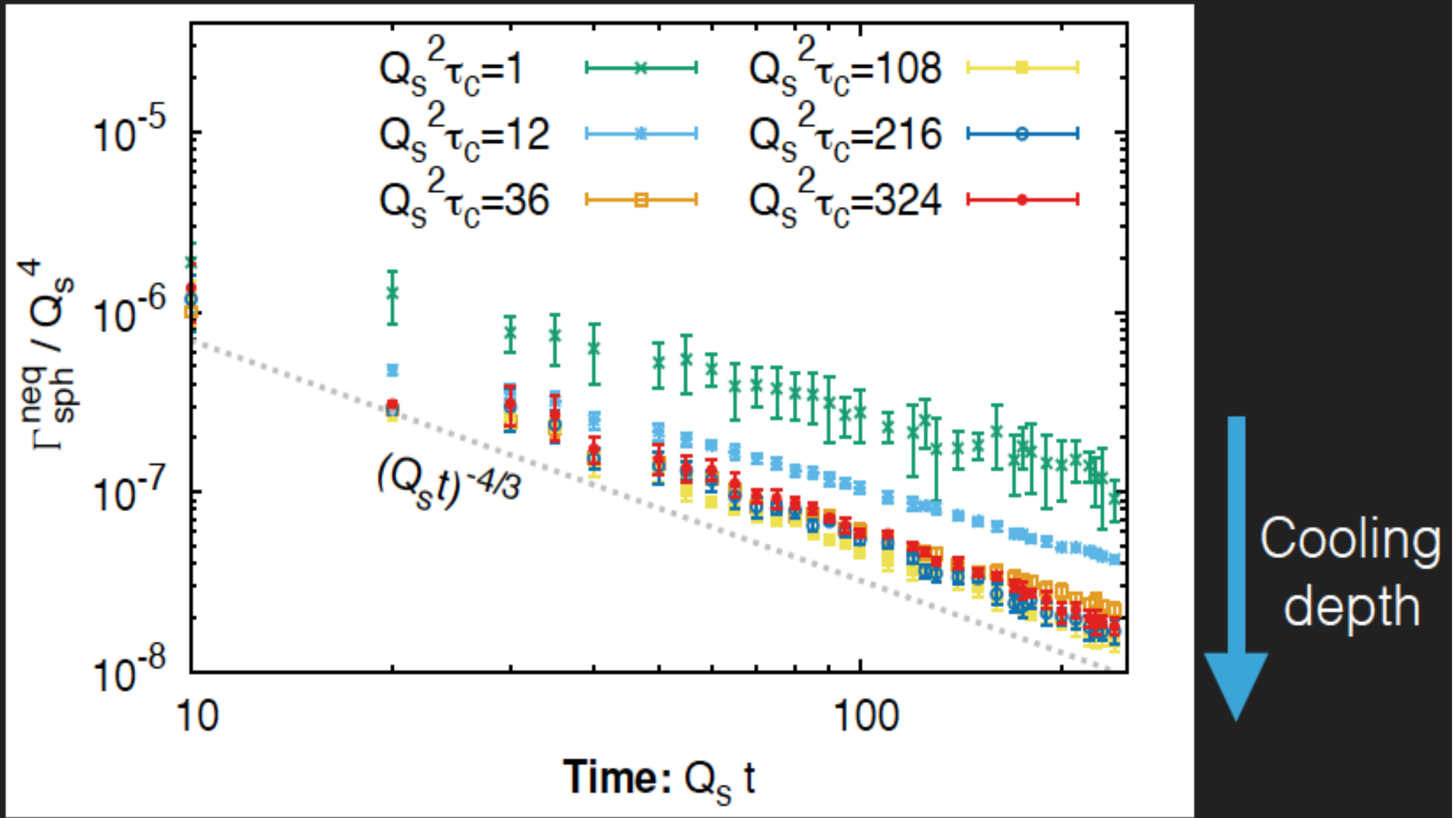
AMBJORN AND KRASNITZ, NUCL. PHYS. B506, 387 (1997). P. WOIT, NUCL.PHYS. B262, 284 (1985). MOORE AND TUROK, PHYS.REV. D56, 6533 (1997).

COOLING: A CARTOON



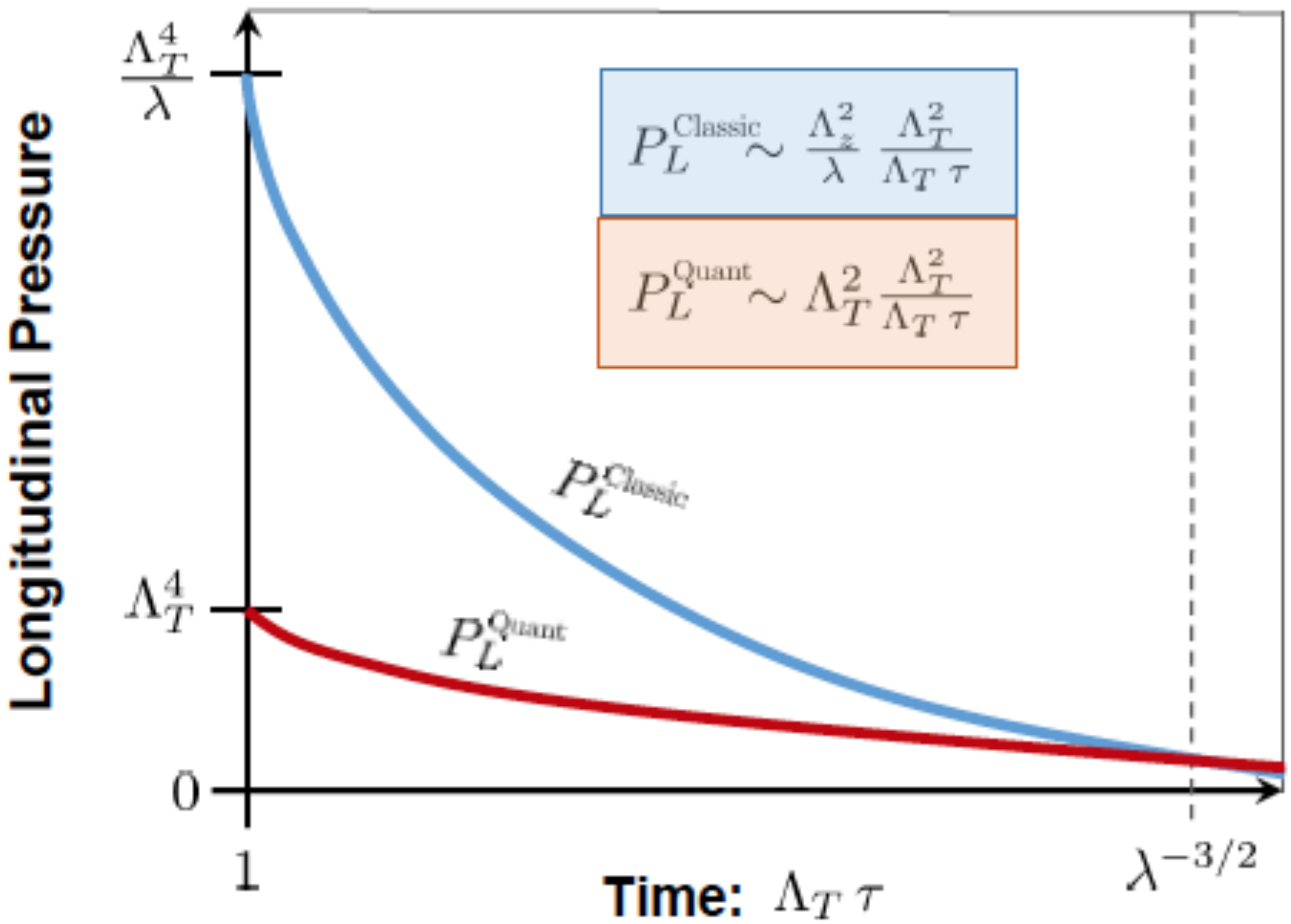
THE NON-EQUILIBRIUM SPHALERON TRANSITION RATE

- From $\Gamma^{neq}_{sph}(t) = \left\langle \frac{(N_{CS}(t+\delta t) - N_{CS}(t))^2}{V\ \delta t} \right\rangle_{Q_s \delta t < 10}$
- We find $\Gamma \sim Q_s^4 (Q_s t)^{-4/3}$



Quantum vs Classical contributions to the Pressure

Berges,Boguslavski,Schlichting,Venugopalan, 1508.03073



Back to the real part: semi-classical world-lines

Eg., G. Dunne, C. Schubert, hep-th/0507174

Consider our simpler case of scalar particles in a background field

$$\Gamma[A] = \int_0^\infty \frac{dT}{T}\, \mathrm{e}^{-m^2 T} \int_{x(T)=x(0)} \mathcal{D}x \exp\left[-\int_0^T d\tau \left(\frac{\dot{x}^2}{4} + ieA\cdot\dot{x}\right)\right]$$

Rewrite exactly as

$$\Gamma[A] \;=\; 2\int_{x(1)=x(0)} \mathcal{D}x\, K_0\left(m\sqrt{\int_0^1 du\,\dot{x}^2}\right) \exp\left[-ie\int_0^1 du\, A\cdot\dot{x}\right]$$

$$\simeq \sqrt{\frac{2\pi}{m}} \int \mathcal{D}x\, \frac{1}{\left(\int_0^1 du\,\dot{x}^2\right)^{1/4}} \exp\left[-\left(m\sqrt{\int_0^1 du\,\dot{x}^2} + ie\int_0^1 du\, A\cdot\dot{x}\right)\right]$$

for $m\sqrt{\int_0^1 du\,\dot{x}^2} \gg 1$

Stationary phase "world-line instanton" of functional integral

$$m\frac{\ddot{x}_\mu}{\sqrt{\int_0^1 du\,\dot{x}^2}} = ieF_{\mu\nu}\dot{x}_\nu$$

Equation of motion of scalar particle in background Abelian field

Relation of Berry phase to anomaly

- **A reading of the work of Stone et al.**

suggests that the content of the chiral kinetic equations can be obtained from the covariant BMT equation

Stone,Dwivedi,Zhou, PRD91 (2015) 025004

- **In our work, this arises entirely from the**

real part of the effective action…

Berry connection and chiral kinetic theory

Son, Yamamoto,…

Canonical example: two component spinor satisfying the Weyl equation $(\boldsymbol{\sigma} \cdot \mathrm{p})u_{\mathbf{p}} = \pm|\mathrm{p}|u_{\mathbf{p}}$

has Berry connection $i\mathcal{A}_{\mathbf{p}} \equiv u_{\mathbf{p}}^{\dagger}\boldsymbol{\nabla}_{\mathbf{p}}u_{\mathbf{p}}$

and curvature $\Omega_{\mathbf{p}} \equiv \boldsymbol{\nabla}_{\mathbf{p}} \times \mathcal{A}_{\mathbf{p}} = \pm\dfrac{\hat{\mathrm{p}}}{2|\mathrm{p}|^2}$

Fictitious magnetic field associated with a "magnetic monopole"

Son and Yamamoto consider the action

D.Xiao et al., PRL95 (2005)137204
Duval et al., PLB20 (2006) 373

$$S = \int dt[p^i\dot{x}^i + A^i(x)\dot{x}^i - \mathcal{A}^i(p)\dot{p}^i - \epsilon_{\mathbf{p}}(x) - A^0(x)]$$

with

$$\mathrm{j} = -\int \frac{d^3p}{(2\pi)^3}\left[\epsilon_{\mathbf{p}}\frac{\partial n_{\mathbf{p}}}{\partial \mathrm{p}} + \left(\Omega_{\mathbf{p}} \cdot \frac{\partial n_{\mathbf{p}}}{\partial \mathrm{p}}\right)\epsilon_{\mathbf{p}}\mathrm{B} + \epsilon_{\mathbf{p}}\Omega_{\mathbf{p}} \times \frac{\partial n_{\mathbf{p}}}{\partial \mathrm{x}}\right] + \mathrm{E} \times \boldsymbol{\sigma},$$

$$\boldsymbol{\sigma} = \int \frac{d^3p}{(2\pi)^3}\Omega_{\mathbf{p}}n_{\mathbf{p}}$$