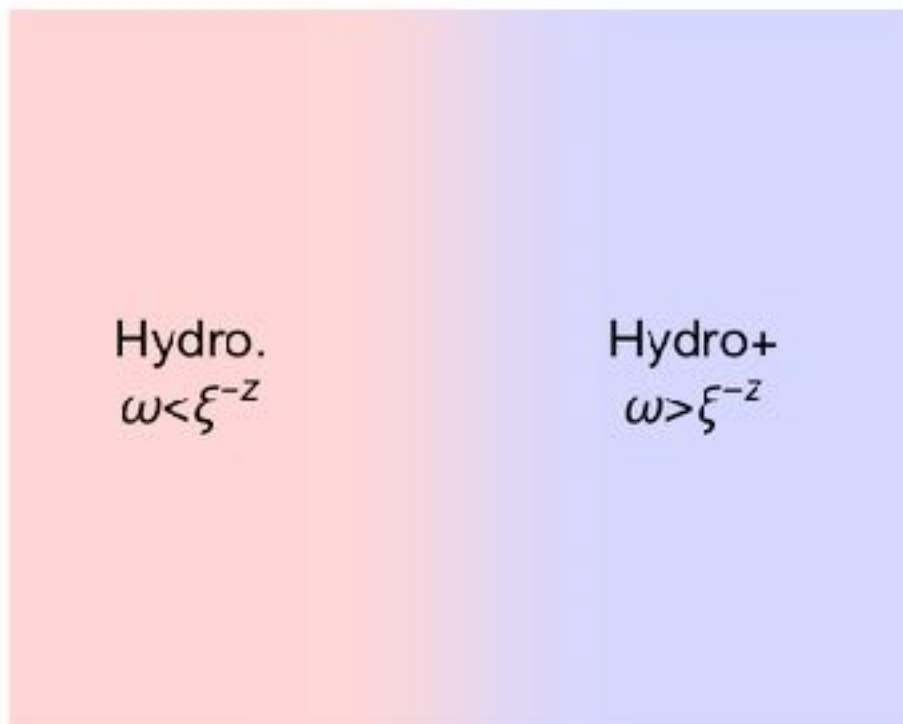
Hydrodynamics with critical slowing down (“Hydro+”)



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(M. Stephanov and YY, in preparation)

StonyBrook, Aug.5, 2017

Motivations

- Critical fluctuations relax slowly (relaxation time ~ $(\xi)^z$, $z=3-\alpha/\nu\approx3$): additional time scale near the critical point (C.P.) .

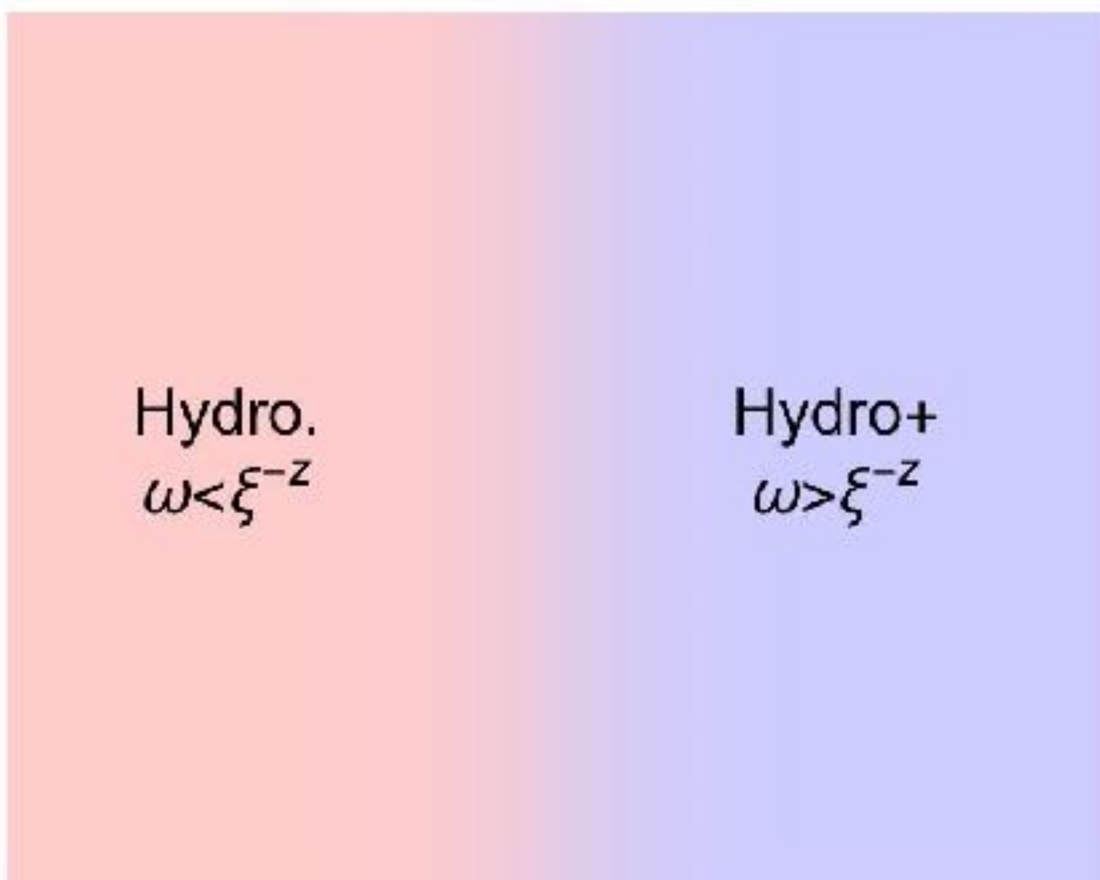
- Hydro. is an effective theory in long time limit. Its applicability near C.P. is limited by the critical slowing down time scale ($\omega< 1/\xi^3$).

  - The critical E.o.S is applicable when critical fluctuations are in equilibrium.

  - Another symptom: the growth of bulk viscosity $\zeta_{Kubo} \sim \xi^3$.

The goal of hydro+critical slowing (or "Hydro+")

- Formulating a hydro-like theory which is applicable at scale $\omega > 1/\xi^z$.
  - Basic idea: adding critical slow modes to hydro.



Outline

- Construction of "Hydro+".
- An example in the expanding background (briefly).
- Discussion: interface "hydro+" with hydro. codes. (talk here, the "User's manual" of "hydro+"; at CPOD, the trailer of "hydro+")

A warm-up exercise: adding one slow mode to hydro.

Overall strategy

- Step I: writing down a general local theory with additional slow mode(s) (constrained by 2nd law) ϕ and relaxation rate Γ_ϕ .

- Step II: Fixing inputs of "hydro+" as much as possible from static/dynamical critical universality.

Quasi-static entropy

- Non-equilibrium (or quasi-static) entropy (E.o.S) : $s_{(+)}(\varepsilon,n,\phi)$.
  - Equilibrium entropy is the maximum of $s_{(+)}(\varepsilon,n,\phi)$:

$$s(\varepsilon,n) = s_{(+)}(\varepsilon,n,\phi)\,|_{\pi=0}$$

where $\pi(\varepsilon,n,\phi) = \partial s_{(+)}(\varepsilon,n,\phi)/\partial\phi$

Relaxation equation for ϕ

- E.o.M for ϕ (slow equilibration of ϕ) :

$$(u^\mu \partial_\mu)\phi = - \gamma_\phi \pi - A_\phi (\partial u) + (\partial)^2 \qquad \gamma_\phi \propto \Gamma_\phi$$

(γ_ϕ, A_ϕ *are functions of* ε, n, ϕ)

- A specific example: if ϕ were axial charge n_A. ($\pi \Rightarrow \mu_A$).

$$(u^\mu \partial_\mu)\phi = - \gamma_\phi \pi - A_\phi (\partial u) \quad \Rightarrow \quad (u^\mu \partial_\mu) n_A = - \Gamma_{\rm sphaleron} \mu_A - n_A (\partial u)$$

Hydro part of "hydro+"

- $T^{\mu\nu} = \varepsilon u^\mu u^\nu + p_{(+)}(\varepsilon,n,\phi) g^{\mu\nu} + \Delta T^{\mu\nu}$.

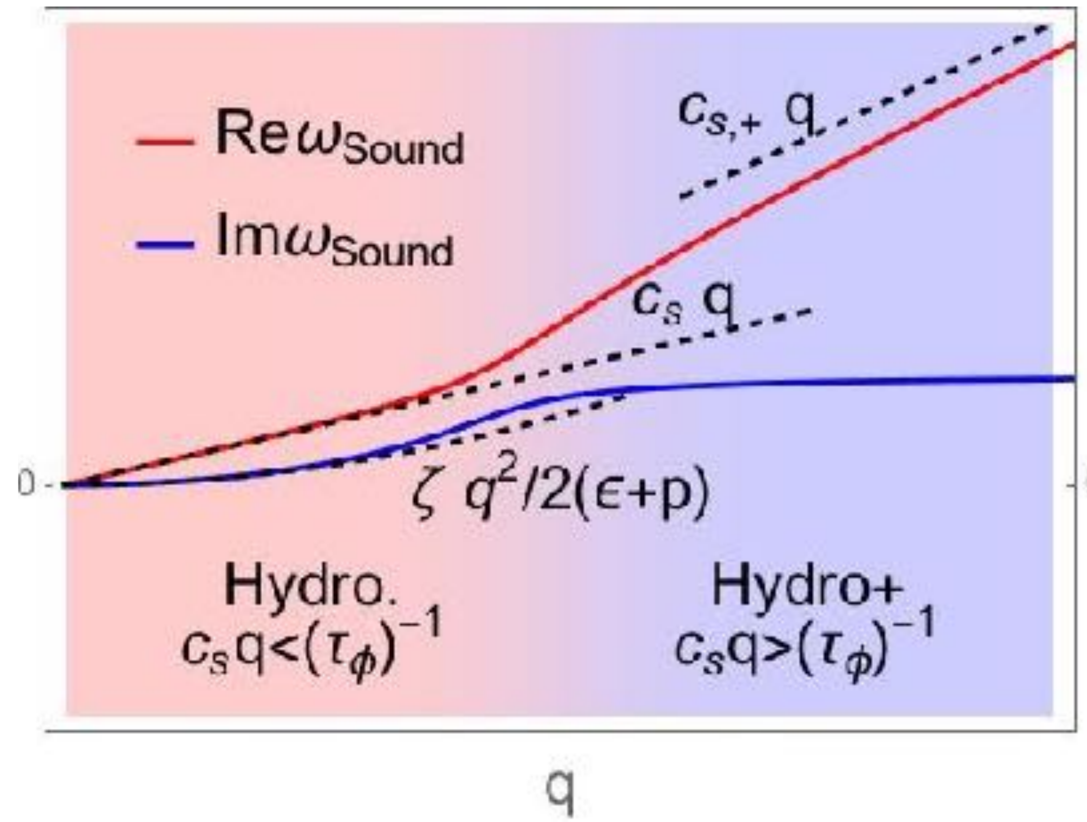
  $s_{(+)}(\varepsilon,n,\phi), A_\phi(\varepsilon,n,\phi) \longrightarrow p_{(+)}(\varepsilon,n,\phi)$

- $\Delta T^{\mu\nu} = - \eta_+ \Delta^{\mu\nu} - \zeta_{(+)} (\partial u)$ (similar expression for ΔJ^μ)

- Constraints:
  - 2nd law of thermodynamics: $\Gamma_\phi, \eta_{(+)}, \zeta_{(+)} > 0$.
  - Reproducing hydro. limit ($\omega < \Gamma_\phi$):

$\zeta_{Kubo} = \zeta_{(+)}$ + contributions from ϕ ($\propto 1/\Gamma_\phi$)

Sound in Linearized Hydro+one mode: propagation



- At hydro. limit, i.e, $\omega(q)<\Gamma_\phi$, sound velocity is given by equilibrium E.o.S:

$$c_s^2 = \left(\frac{\partial p(\epsilon, n)}{\partial \epsilon}\right)_{s/n}$$

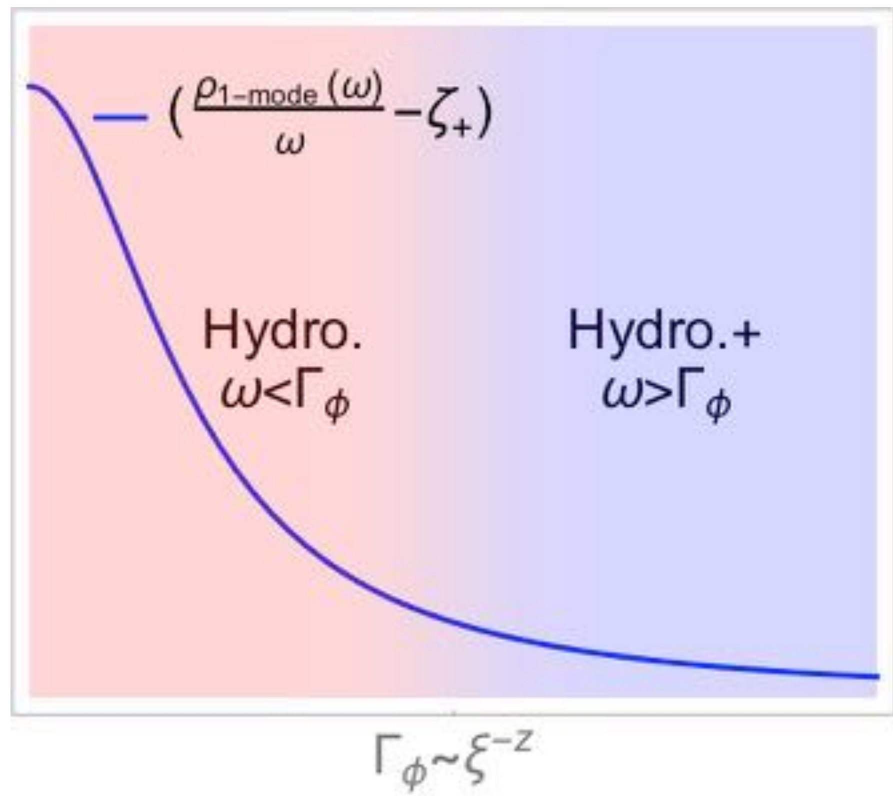
- When $\omega(q)>\Gamma_\phi$, ϕ is off-equilibrium and the "effective" sound velocity is given by the quasi-static E.o.S :

$$c_+^2 = \left(\frac{\partial p_{(+)}(\epsilon, n, \phi)}{\partial \epsilon}\right)_{s/n,\phi}$$

Fixing inputs of hydro+ from matching

- Computing bulk spectrum function from linearized hydro+ : $\zeta_{Kubo}-\zeta_{+}=[(c_{s,+})^2-(c_s)^2]/\Gamma_\phi$

- Matching to critical behavior of bulk viscosity: $\zeta_{Kubo} \sim \xi^3 \quad \rightarrow \quad \Gamma_\phi \sim \xi^{-3}$
    - Advantage: the input of "hydro+" ζ_+ is finite.



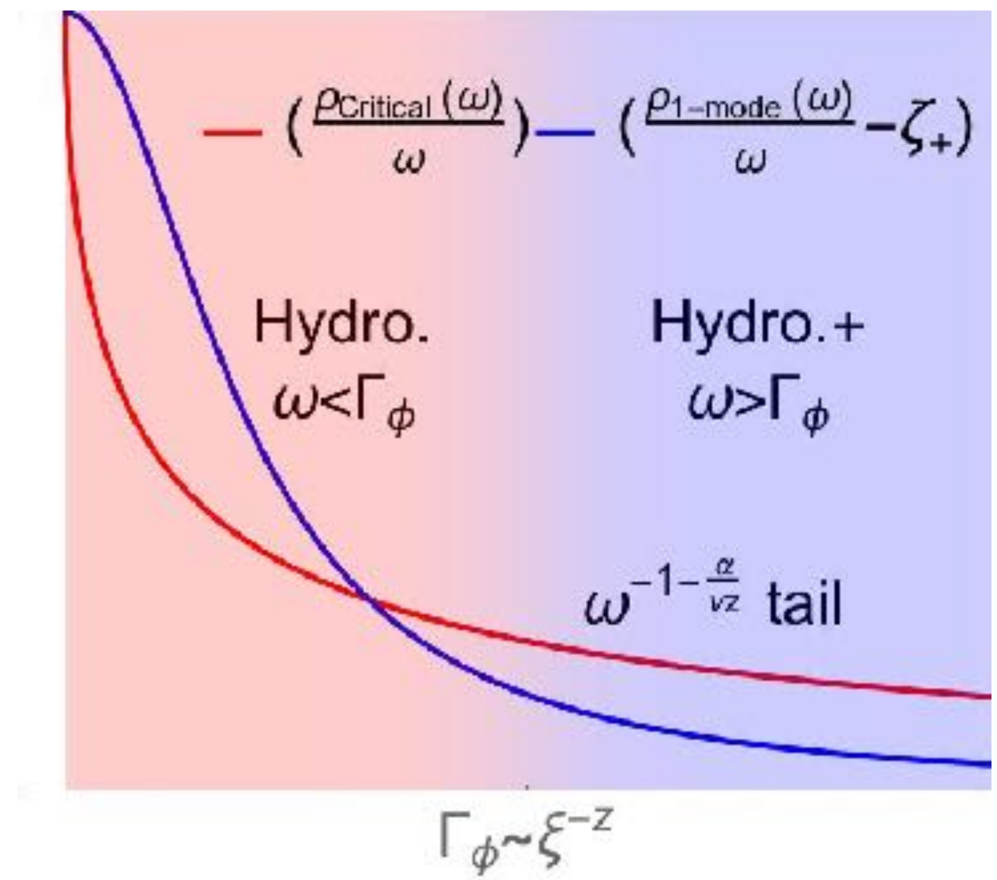
($\rho_{Bulk}(\omega) \sim \mathrm{Im} \langle T^i_i T^i_i \rangle$ vs ω.)

- A sum rule: $[(c_{s,+})^2-(c_s)^2] \propto \int d\omega \, (\rho(\omega)/\omega - \zeta_{(+)})$.

Matching to the critical behavior of $\rho_{Bulk}(\omega)$ (computed by Onuki for model H) using sum-rule fixes $[(c_{s,+})^2-(c_s)^2]$.

What can be achieved and what can not from "Hydro+" one mode

- "Hydro+" one mode qualitatively captures the transition from hydro regime $\omega < 1/\xi^3$ to "hydro+" regime $\omega > 1/\xi^3$.
  - Naively using $\zeta_{Kubo} \sim \xi^3$ in the regime $\omega > \Gamma_\phi$ would overestimate entropy production.



- One mode is not enough to fully capture the critical dynamic behavior.

- Next step: Hydro+ a spectrum of slow modes.

Hydro+a spectrum of slow modes

Construction of hydro+$\phi(t,x;Q)$

$s_+(\varepsilon,n,\phi)$ $\longrightarrow$ $s_+(\varepsilon,n,\phi(Q))$ (thus $p_+(\varepsilon,n,\phi(Q))$)

$\pi = \partial s_+(\varepsilon,n,\phi)/\partial\phi$ $\longrightarrow$ $\pi(Q) = \delta s_+(\varepsilon,n,\phi(Q))/\delta\phi(Q)$

$(u^\mu \partial_\mu)\phi = -\gamma_\phi \pi - A_\phi (\partial u)$ $\longrightarrow$ $(u^\mu \partial_\mu)\phi(Q) = -\gamma_\phi(Q)\,\pi(Q) - \ldots$

To proceed, a more "microscopic" understanding of critical slow modes is needed

Slow modes near a critical point

- Order parameter M and the fluctuations of order parameter (<δMδM>, etc) relax slowly (critical slowing down).
  - A general critical point: those slow modes include order parameter (M), and <δMδM> (and potentially higher cumulants…).
  - QCD critical point: M is a linear combination of ϵ, n and chiral condensate σ. σ equilibrates at microscopic time scale and the evolution of σ simply traces the evolution of ϵ, n ⇒the equation for M is redundant.

    (Son-Stephanov, 04')
  - Therefore: hydro + <δMδM> .

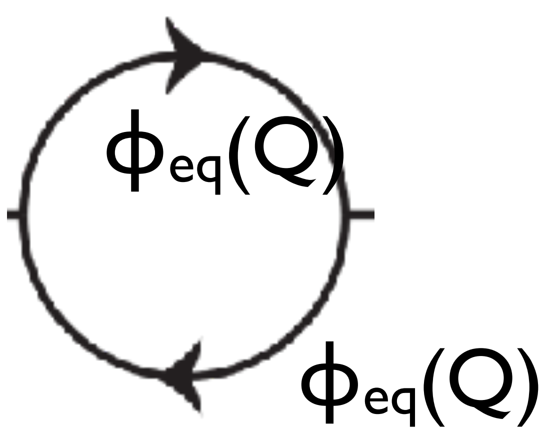
Connection between $\langle \delta M \delta M \rangle$ and $\phi(t, x ; Q)$

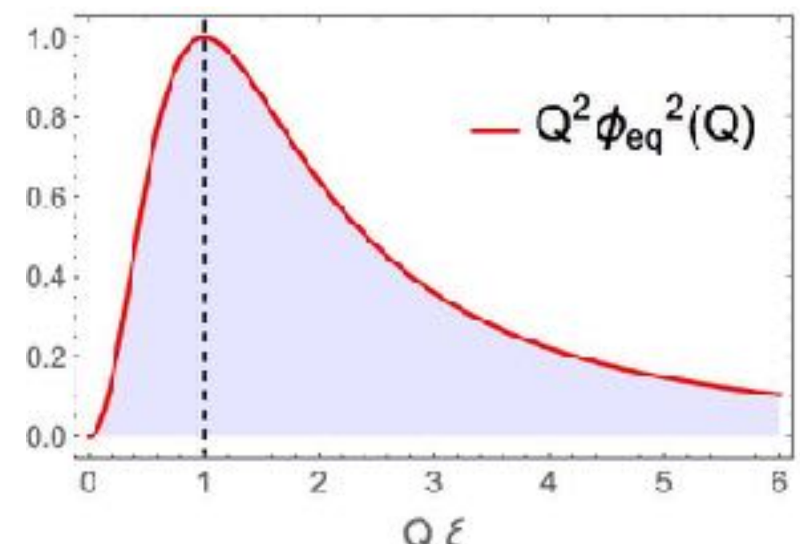
- The Wigner transform of $\langle \delta M \delta M \rangle$

$$\phi(t, x ; Q) = \int d^3\Delta x \, \langle \delta M(t, x+\Delta x) \, \delta M(t, x-\Delta x) \rangle \, e^{-i Q \Delta x}$$

NB: viewing $\phi(t,x ;Q)$ as many local slow modes with label Q at a fluid cell (t,x).

- In equilibrium: $\phi_{eq}(Q) = 1/[(\chi_M)^{-1}+Q^2]$ $(\phi_{eq}(Q=0)=\chi_M \sim \kappa_2)$ and contributes to the critical behavior of E.o.S .





$(c_s)^{-2}$ near C.P.~shaded area

Generalized Entropy s+(ϵ,n,ϕ(Q))

- $s_+(\epsilon, n, \phi(Q))$ is the analogue of 2PI effective action in QFT.

  (J. M. Cornwall, R. Jackiw, E. Tomboulis, 1974')

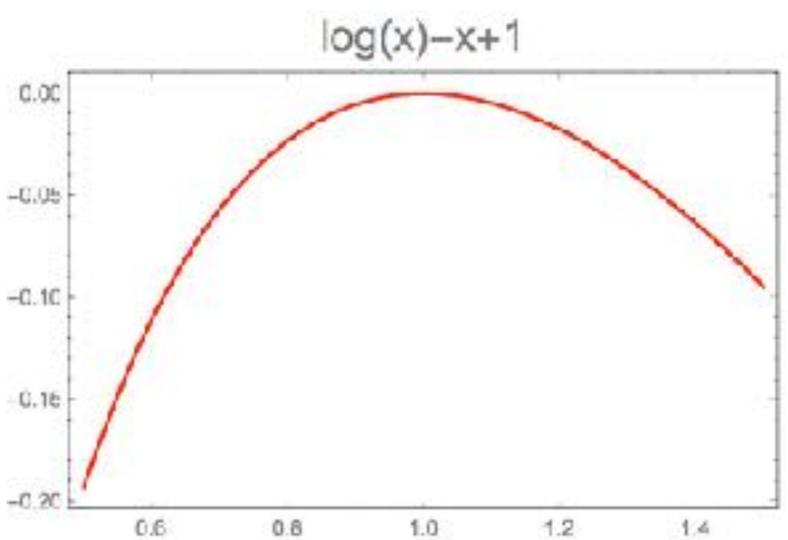
  - NB: 2PI effective action is a useful tool to study non-equilibrium effects.

    (e.g. J. Berges et al, hep-ph/0409123)

- A simple form at the leading order in "loop expansion":

$$s_+(\epsilon, n, \phi(Q)) = s_{\rm eq}(\epsilon, n) + \frac{1}{2}\int_Q \left\{ \log\left(\frac{\phi(Q)}{\phi_{eq}(Q)}\right) - \frac{\phi(Q)}{\phi_{eq}(Q)} + 1 \right\},$$

$$\pi(Q) = \frac{\delta s_+}{\delta \phi(Q)} = \phi_{\rm eq}^{-1}(Q) - \phi^{-1}(Q)$$



“Hydro+” critical slowing down

- A Q-dependent (phenomenological) relaxation equation for ϕ:

$$(u^\mu \partial_\mu)\phi = -\gamma_\phi \pi - A_\phi (\partial u) \longrightarrow (u^\mu \partial_\mu)\,\phi(Q) = -2\,\gamma(Q)\pi(Q)$$

  - $\Gamma(Q)=\gamma(Q)/(\phi_{eq}(Q))^2$ is known from model H.
  - The dilution of critical fluctuations can be included by keeping $A_\phi(Q)$ (∂u) term (dropped for simplicity).

- Successfully reproducing critical behavior of $\rho_{Bulk}(\omega) \sim \mathrm{Im} \langle T^i_i T^i_i \rangle$

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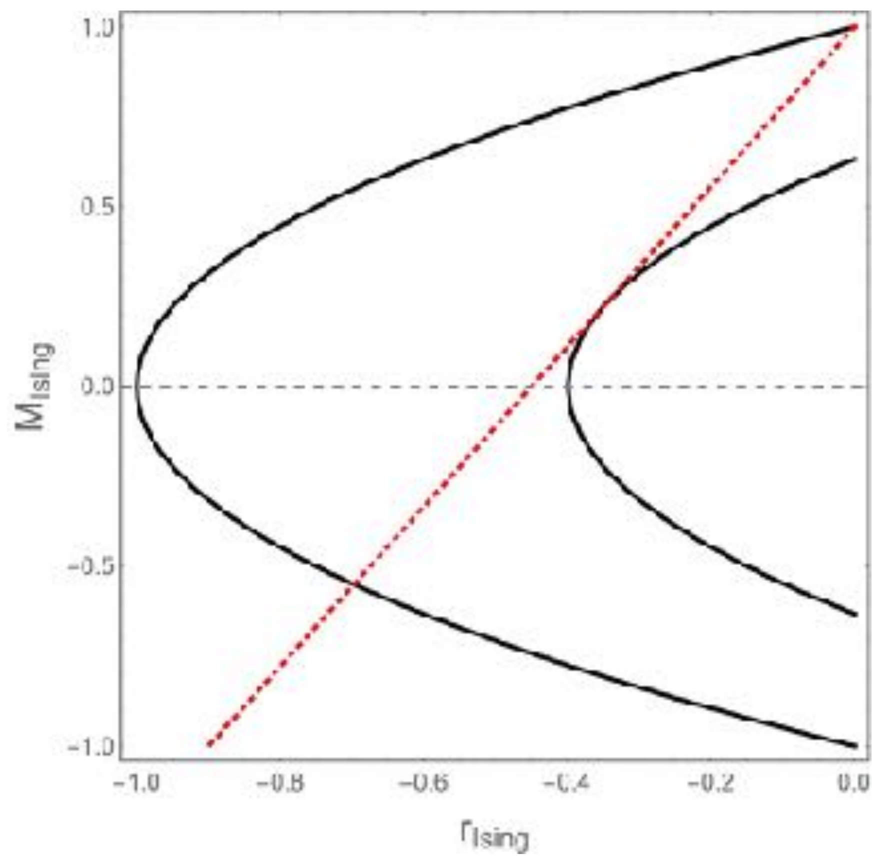
(given in Onuki’s textbook)

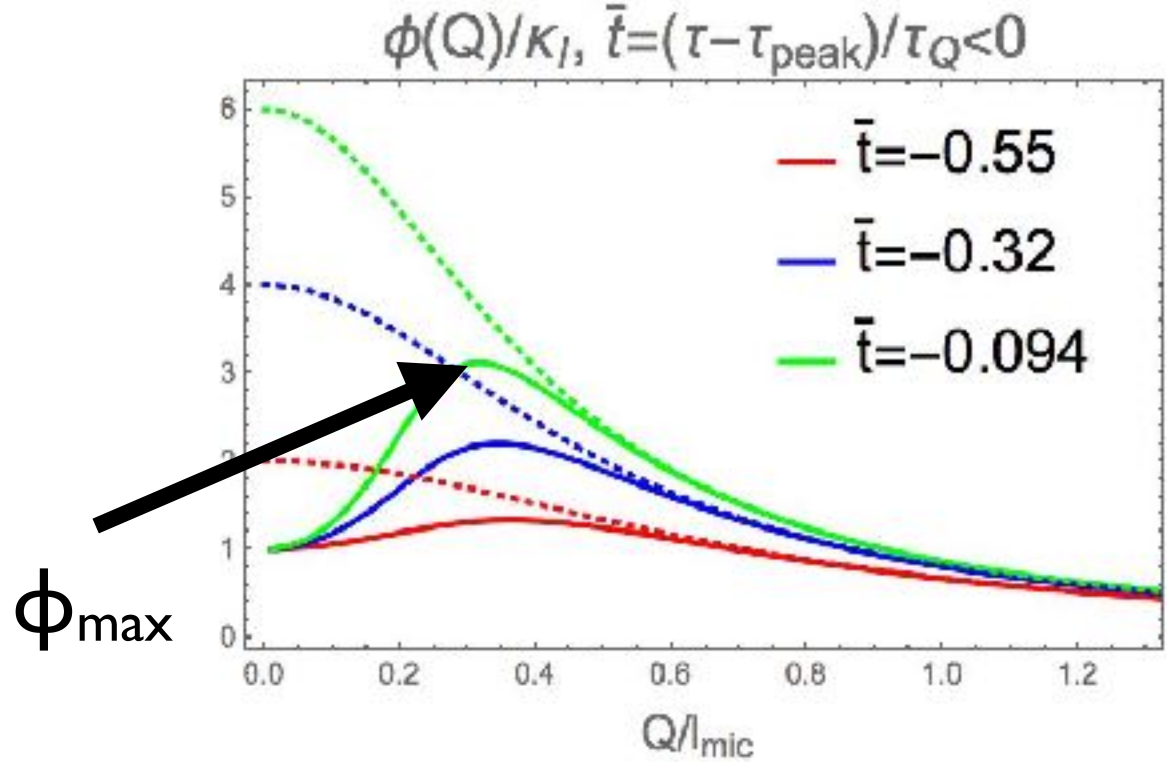
An example of hydro+ in an expanding QGP

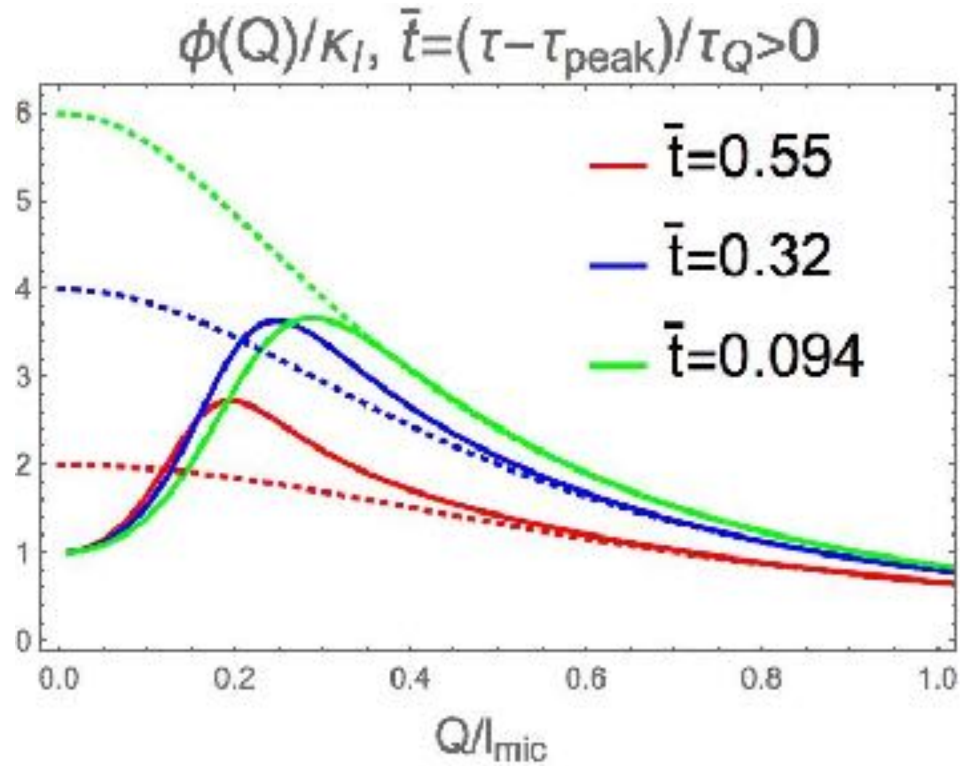
Solving equation for $\phi(Q)$ along a trajectory



$M \sim T - T_c$, $r_{Ising} \sim \mu - \mu_c$

"Hydro+" describes the slow relaxation of critical fluctuations

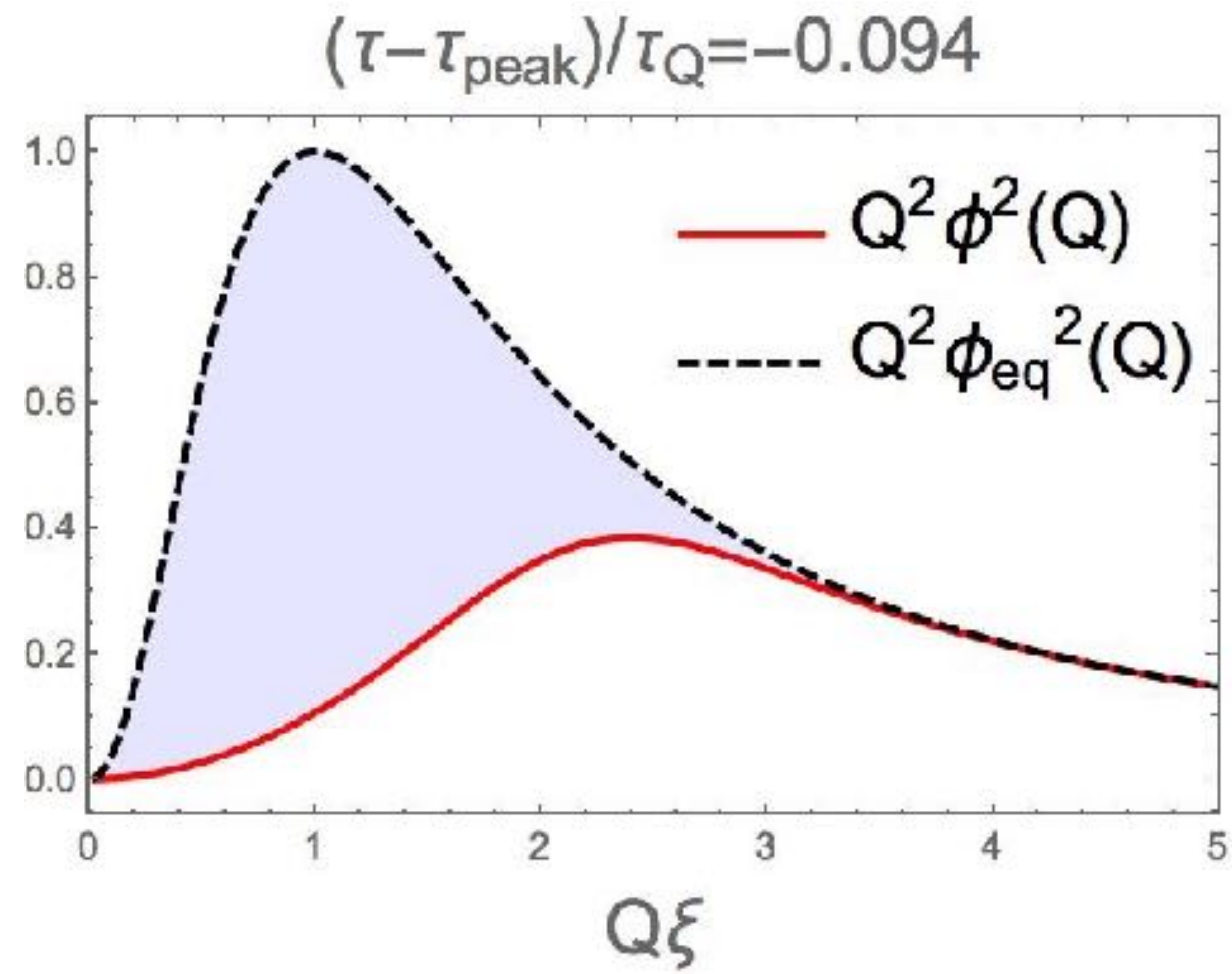




$\boldsymbol{\tau < \tau_{peak}}$, Fall out of equilibrium. $\boldsymbol{\tau > \tau_{peak}}$, memory.

- NB: ϕ(Q) can be related to the baryon number balance function (if supplemented with mapping and freeze-out prescription).

Hydro+ describes off-equilibrium modification of E.o.S



$(c_s)^{-2} - (c_{s,\ \text{off-equilibrium}})^{-2}$ ∝shaded area

- Finite time effect limits the divergence of specific heat .

Discussion: interface “hydro+” with hydro. codes

A summary of key equations and inputs

$$s_+(\epsilon, n, \phi(Q)) = s_{eq}(\epsilon, n) + \frac{1}{2}\int_Q \{\log\left(\frac{\phi(Q)}{\phi_{eq}(Q)}\right) - \frac{\phi(Q)}{\phi_{eq}(Q)} + 1\},$$

$$\pi(Q) = \frac{\delta s_+}{\delta\phi(Q)} = \phi_{eq}^{-1}(Q) - \phi^{-1}(Q)$$

$$(u^\mu\partial_\mu)\,\phi(Q) = -2\,\gamma(Q)\pi(Q)$$

- From mapping+critical E.o.S : fix $\phi_{eq}(Q) = 1/[(\chi_M)^{-1}+Q^2]$ and $s_{eq}(\varepsilon, n)$.
- From model H (critical dynamic scaling): fix $\Gamma(Q)=\gamma(Q)/(\phi_{eq}(Q))^2$.
- "Transport coefficients +" is related to "Transport coefficient Kubo", e.g., $\zeta_{Kubo} = \zeta_+$ + contributions from $\phi(Q)$ (also known form model H)
- Question for hydro. group: is it convenient to implement numerically ? What else shall we input?

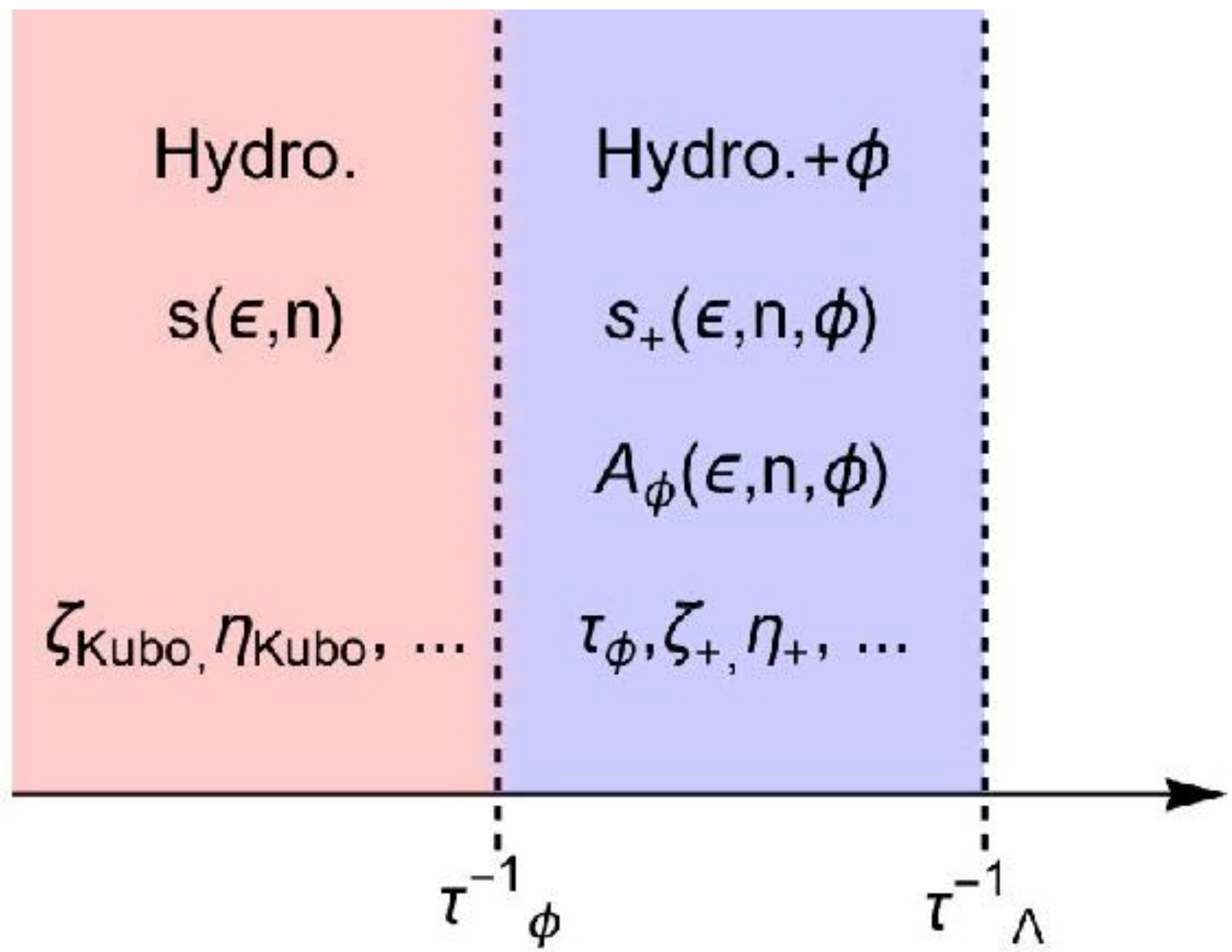
Alternative approach

- Similar to gluing lattice E.o.S with critical E.o.S, shall we glue "Hydro+non-critical fluctuations" with "Hydro+critical slowing down"?

- An alternative proposal: implement "hydro+non-critical fluctuations" into the present hydro codes first, then move to "hydro+critical slowing down".

Back-up

Summary of inputs of hydro+ϕ



The extension of "Hydro+" and BEST

- Hydro+ (broadly speaking): a versatile framework to describe non-equilibrium evolution of non-hydrodynamic (but slow) mode(s).
  - Stochastic hydro. in expanding QGP (thermal fluctuation equilibrates slowly when η is small).
  - Hydro. + axial charge and MHD+axial charge (axial charge relaxes slowly).

Evolution of the peak value of $\phi(Q)$