On Lee-Yang Edge Singularities and Spinodal Points

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UIC PHYSICS

Outline

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   - Nonperturbative Analysis: Functional Renormalization Group
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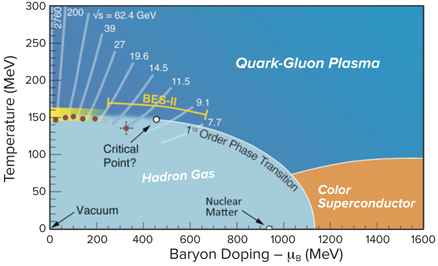
Introduction and Motivation

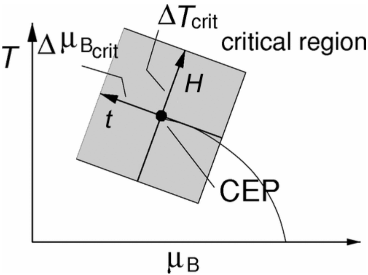
Critical Point: From QCD to Ising Theory

- Coordinate mapping: $(\Delta\mu_{\mathrm{B}}, \Delta T) \longrightarrow (t, H)$.



QCD phase diagram

(The 2015 Long Range Plan for Nuclear Science)



Mapping of critical region

(C. Nonaka and M. Asakawa, 2005)

Mean-field Equation of State

Mean-field EoS

- In terms of conveniently rescaled variables Φ and H, the scalar Φ^4 theory in d dimensions can be defined by the Euclidean action

$$\mathcal{S} = \frac{6}{u_0} \int d^d x \left[\frac{1}{2} (\partial_\mu \Phi)^2 + V(\Phi) \right],$$

with the potential

$$V(\Phi) = \frac{t}{2}\Phi^2 + \frac{1}{4}\Phi^4 - H\Phi.$$

- When $u_0 \to 0$, saddle-point method could be applied, which yields the mean-field equation of state (EoS), $V'(M) = 0$, i.e.,

$$H = tM + M^3.$$

- In terms of the properly renormalized scaling variables

$$w = Ht^{-\beta\delta} \quad \text{and} \quad z = Mt^{-\beta}$$

where $\beta = 1/2$, $\delta = 3$, the EoS can be expressed as

$$w = z(1 + z^2).$$

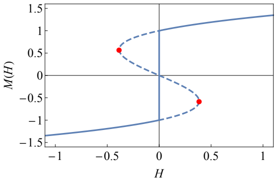
LY edge Singularities and Spinodal Points

- According to the Lee-Yang Theorem (T.D. Lee and C.N. Yang, 1952), in the high-t phase, the EoS features a pair of branch cuts, the Lee-Yang (LY) cuts, which terminate at the Lee-Yang (LY) edge singularities, where the isothermal susceptibility diverges (i.e., $w'(z) = 0$) and one can find

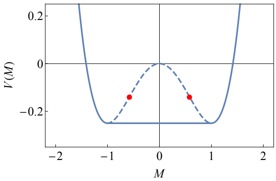
$$w_{\mathrm{LY}} = \pm \frac{2i}{3\sqrt{3}}.$$

- The low-t images of the LY points are called spinodal points, i.e.,

$$H_{\mathrm{sp}} = \pm w_{\mathrm{LY}} t^{3/2}, \quad t < 0.$$



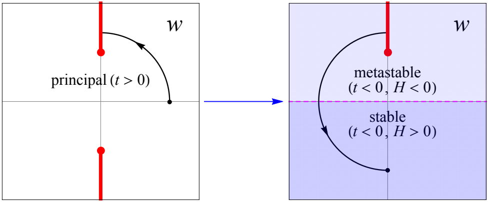
Low-t EoS $M(H)$



Low-t potential $V(M)$

Analytic Continuation to Low-t Phase

- Starting from any point on the Riemann surface in the stable high-t phase, one can analytically continue the high-t EoS to the low-t phase by rotating an angle $\pi\beta\delta$.



Analytic continuation to low-t phase in mean-field approximation

- The metastable low-t phase is reached by an additional rotation by an angle π.

Beyond the Mean-field Equation of State

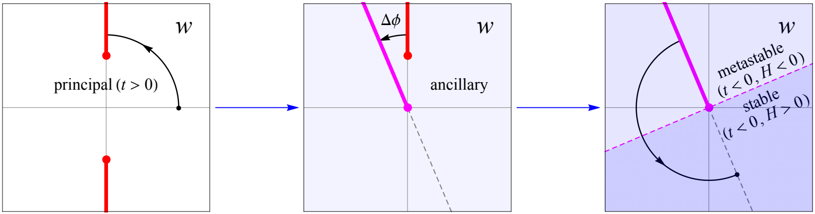
Singularities Beyond the Mean-field EoS

- The mean-field EoS doesn't capture a weak essential singularity at $H = 0$ associated with the Langer cut $(-\infty, 0]$ (J.S. Langer, 1967) in complex H plane. $\mathrm{Im} M \sim \exp\left(-\frac{\mathrm{const}}{u_0 |w|^3}\right)$ for $H \to 0$ and $d \to 4$.
- For $d < 4$, the mean-field approximation no longer applies and the "gap" critical exponent $\beta\delta > 3/2$. Accordingly, the spinodal points,

$$H_{\mathrm{sp}} = w_{\mathrm{LY}} t^{\beta\delta} = \pm |w_{\mathrm{LY}} t^{\beta\delta}| e^{\pm i\Delta\phi}, \quad t < 0,$$

shift from the real H axis (Langer cut) by an angle

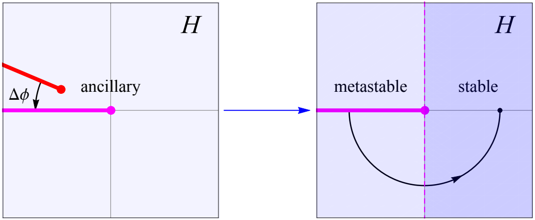
$$\Delta\phi = \pi(\beta\delta - 3/2).$$



Analytic continuation to the low-t phase beyond mean-field approximation

FZ Conjecture

- Standard Analyticity Assumption: $M_{t<0}(H)$ is analytic in full complex H plane with Langer cut.
- Fonseca-Zamolodchikov (FZ) Conjecture (P. Fonseca and A. Zamolodchikov, 2001): $M(t)$, connected with $M(H)$ via scaling relation $H \sim t^{\beta\delta}$, is analytic in full complex t plane with LY cuts.
- According to the FZ conjecture, the spinodal points are the nearest singularities under the Langer cut.



FZ conjecture

Ginzburg Criterion

- The LY edge singularities are described by the Φ^3 theory (M.E. Fisher, 1978), by shifting the field such that the quadratic term of the Φ^4 theory vanishes, which is non-mean-field like for $d < 6$.
- The cubic fluctuation is negligible when

$$|w - w_{\mathrm{LY}}| \gg (\widetilde{u}_0)^{4/(6-d)},$$

  where $\widetilde{u}_0 \equiv u_0 t^{-(4-d)/2}$ is the dimensionless quartic coupling. This condition is similar to the Ginzburg criterion in the theory of superconductors.
- Small ε $(= 4 - d)$ limit:

$$|w - w_{\mathrm{LY}}| \gg \varepsilon^2.$$

Complex Singularities of Φ^4 Theory

EoS for $d = 4 - \varepsilon$

- To order ε^2, the "gap" exponent is given by (E. Brezin *et al*, 1972)

$$\beta\delta = \frac{3}{2} + \frac{1}{12}\varepsilon^2 + \mathcal{O}(\varepsilon^3),$$

  and the scaling function $F(z)$ reads (B.G. Nickel, 1972)

$$F(z) = \sum_{n=0}^{\infty} F_n(z)\varepsilon^n,$$

  with $F_0(z) = z + z^3$, etc.
- $F(z)$ is valid for small z hence cannot be applied to the full scaling regime.

Parametric Representation of scaling EoS

- In terms of the resummed critical exponents, the scaling EoS could be parametrized to match the ε expanded EoS, while the analyticity is manifest in the full scaling regime.
- JS Parametric Representation (B. Josephson and P. Schofield, 1969):

$$\begin{cases} t(R,\theta) = Rk(\theta), \\ M(R,\theta) = R^{\beta} m(\theta), \\ H(R,\theta) = R^{\beta\delta} h(\theta), \end{cases}$$

where $k(\theta) = 1 - \theta^2, m(\theta) = \bar{m}\theta, h(\theta) = \bar{h}(\theta + h_3\theta^3)$.

- The scaling variables w and z can be expressed in terms of θ alone, i.e.,

$$z = \frac{\bar{z}\theta}{(1-\theta^2)^{\beta}} \quad \text{and} \quad w = \frac{\bar{w}(\theta + h_3\theta^3)}{(1-\theta^2)^{\beta\delta}}.$$

$\bar{m}$, $\bar{h}$, $\bar{z}$ and $\bar{w}$ are normalization factors depending on ε.

Singularities of the Parametric EoS

- Now we arrive at the scaling form for the inverse susceptibility

$$F'(\theta) = \frac{w'(\theta)}{z'(\theta)} = \frac{\bar{w}}{\bar{z}}(1-\theta^2)^{-\gamma}\frac{1 + (2\beta\delta + 3h_3 - 1)\,\theta^2 + (2\beta\delta - 3)h_3\theta^4}{1 - (1 - 2\beta)\theta^2}.$$

- The poles/zeros of $F'(\theta)$ must have the following form

$$\theta_n^2 = \frac{c_n}{\varepsilon}\left[1 + \mathcal{O}(\varepsilon)\right].$$

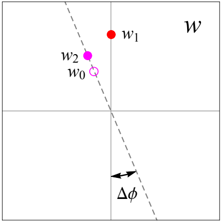
- The corresponding poles/zeros can be expressed as

$$w_n = \pm\frac{2i\,(-\hat{c}_n)^{\frac{3}{2}-\beta\delta}}{3\sqrt{3}}\left\{1+\mathcal{O}(\varepsilon^2)\right\},$$

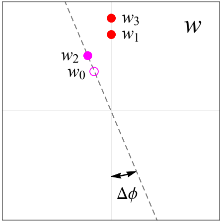
where $\hat{c}_n \equiv c_n/|c_n|$.

Singularities of the Parametric EoS

- In complex w plane, poles ($n = 0$) and zeros ($n = 1, 2, ...$) could either reside on imaginary axis or shift from it by an angle $\Delta\phi$ nonperturbatively, which, in the spirit of padé approximation, indicates the existence of Langer cut.



$\mathcal{O}(\varepsilon^2)$



$\mathcal{O}(\varepsilon^3)$

- $\Delta\phi \sim \mathcal{O}(\varepsilon^2)$ while nonperturbative domain $|w - w_{\mathrm{LY}}| \sim \mathcal{O}(\varepsilon^2) \Longrightarrow$ FZ conjecture cannot be verified since we can not rule out possible singularities in the angle $\Delta\phi$.

Singularities in the O(N) Theory: $N \to \infty$

- In the $N \to \infty$ limit the critical exponents are known (E. Brezin, 1972)

$$\beta = \frac{1}{2}, \quad \delta = \frac{d+2}{d-2}, \quad \text{and} \quad \gamma = \frac{2}{d-2}, \quad \text{for} \quad 2 < d < 4.$$

- Now $\Delta\phi = \pi\beta\delta \sim \mathcal{O}(\varepsilon)$, compared to which the nonperturbative domain $|w - w_{\mathrm{LY}}| \sim \mathcal{O}(\varepsilon^2)$ is negligible.

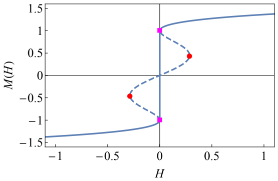
- The scaling EoS is determined as

$$F(z) = z(1+z^2)^{\gamma}.$$

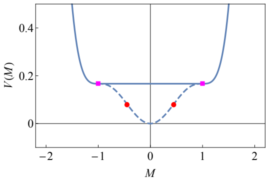
- The solutions of $F'(z) = 0$ turn out to be

$$z_G^2 = -1,\ \ w_G = 0 \quad \text{and} \quad z_{LY}^2 = -\frac{1}{1+2\gamma},\ \ w_{LY} = \pm i\frac{(2\gamma)^{\gamma}}{(1+2\gamma)^{\beta\delta}}.$$

w_G is the Goldstone mode induced singularity associated with the Goldstone cuts. $\mathrm{Im}M \sim H^{(d-2)/2}$ for $H \to 0$ and $t < 0$.



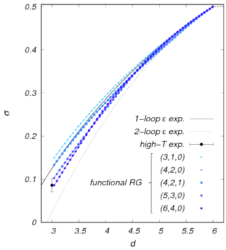
Low-t EoS $M(H)$, $d = 3$



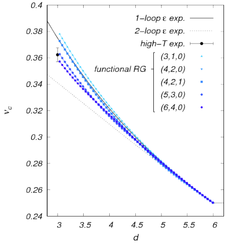
Low-t potential $V(M)$, $d = 3$

Nonperturbative (FRG) Approach to LY Edge Singularity

- From functional renormalization group (FRG) analysis we determined the scaling properties (critical exponents) of LY edge singularity between $3 \leq d \leq 6$ (XA, D. Mesterházy and M. Stephanov, *JHEP* **1607** *(2016) 041)*.



$\sigma(d)$



$\nu_c(d)$

- Compare σ with those obtained from other methods:

Dimension	FRG	4-loop ε exp.	strong coupling	MC	conf. bootstrap
3	0.0742(56)	0.0747	0.076(2)	0.080(7)	0.085(1)
4	0.2667(32)	0.2584	0.258(5)	0.261(12)	0.2685(1)
5	0.4033(12)	0.3981	0.401(9)	0.40(2)	0.4105(5)

Summary and Discussion

Summary

- Beyond mean-field approximation the spinodal points lie off the real H axis.
- The nonperturbative regions around the LY edge singularities, governed by Φ^3 theory, are determined by the Ginzburg criterion.
- FZ conjecture is valid for the $O(N)$ theory in the large-N limit.

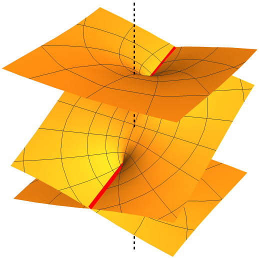
Discussion

- The lack of spinodal singularities at real H may be interpreted as the expression of the fact that the correlation length does not have time to develop due to the decay of the metastable state via nucleation, which differs from the mean-field case where the decay rate is suppressed by vanishing quartic fluctuation.
- The absence of singularities on the real H axis (except $H = 0$) could have implications for the behavior of systems undergoing cooling past the first-order phase transition separating hadron gas and QGP phases of QCD associated with the QCD critical point, which is being searched for using the BES heavy-ion collision experiments.

Thank You

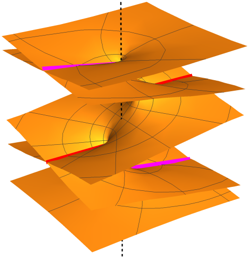
Backup

Riemann Surface of the Scaling Mean-field EoS



Riemann surface of $w = z(1 + z^2)$

($d > 4$, mean-field)

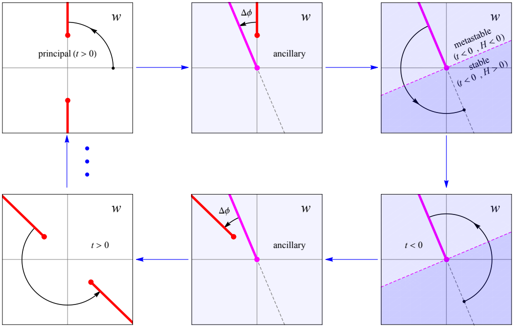


Riemann surface of $w = z(1 + z^2)^2$

($d = 3$, $N \to \infty$)

Riemann Surface of the O(N) Theory, $N \to \infty$

- For $3 \leq d \leq 4$, the structure of the Riemann surface of $z(w)$ relies on the value of $\beta\delta$.



Riemann surface for $3 \leq d \leq 4$

- For $2 < d < 3$, the structure of Riemann surface is much more complicated (i.e., with more Goldstone cuts and ancillary sheets).

$1/N$ Correction

- The $1/N$ corrections can be expressed in terms of momentum integrals coming from a series of bubble diagrams at that order (E. Brezin *et al*, 1972; R. Abe *et al*, 1977):

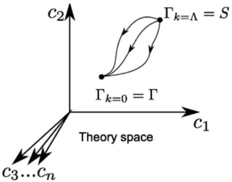
$$F(z) = z\left(1+z^2\right)^{\gamma}\left\{1+\mathcal{O}(N^{-1})\right\}. \tag{1}$$

- When $d = 3$, the aforementioned momentum integrals yield only two branch points at $z^2 = -1$ and $z^2 = -1/5$, which coincide with the same singularities already found in the $N \to \infty$ limit, while the position of the corresponding points in the complex w plane is shifted by an amount of order $1/N$.

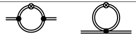
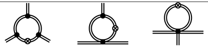
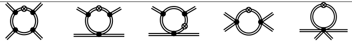
FRG Approach to LY Edge Singularity

- We employ the following *ansatz* for the scale-dependent effective action, where higher order truncations are not negligible:

$$\Gamma_k[\varphi] = \int_x \left\{ U_k(\varphi) + \frac{1}{2}\left[Z_k(\varphi)(\partial_\mu\varphi)^2 + W_k^a(\varphi)(\Box\varphi)^2 \right.\right.$$
$$\left.\left. + W_k^b(\varphi)(\partial_\mu\varphi)^2\,\Box\varphi + W_k^c(\varphi)\left((\partial_\mu\varphi)^2\right)^2 \right]\right\}.$$



FRG flows

Green function type	Contributing diagrams
Two-Point	
Three-Point	
Four-Point	

Contributing diagrams to nonperturbative vertex functions