Finite-Range Corrections to Hydrodynamics

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Gradient Terms and Nuclear Physics

Attractive non-zero-range potential:

$$\begin{aligned}
V &= \frac{1}{2}\int d^3r d^3r' \; \rho(r)v(r-r')\rho(r') \\
&= \frac{1}{2}\int d^3r \; \rho(r)\int d^3\delta r \; v(\delta r)\left[\rho(r) + \frac{(\delta r)^2}{6}\nabla^2\rho|_{r=0}\right], \\
\Delta V &= -\frac{1}{2}\int d^3r \; \kappa\rho\nabla^2\rho, \\
\kappa &= -\frac{1}{6}\int d^3r \; r^2 v(r).
\end{aligned}$$

Free energy:

$$f = \bar{f}(\epsilon, \rho) - \frac{\kappa}{2}\rho\nabla^2\rho$$

Gradient Terms and Nuclear Physics

Applications:

- Nuclear Structure: Skyrme, DFT
- Phase interfaces: Liquid-Gas, QGP-hadron

$$\delta F = 0 = \delta \int dx \left[P_0 - P + (\mu - \mu_0)\rho + \frac{\kappa}{2}(d\rho/dx)^2 \right]$$

$$\frac{d\rho}{dx} = \sqrt{2[P_0 - P(\rho) + (\mu(\rho) - \mu_0)\rho]/\kappa}$$

- Critical Phenomena: Correlation length

$$\ell^2 = \beta\kappa\chi$$

- Spinodal Decomposition
  Damps short wavelength modes
- Higher κ → diffuse surfaces,
  higher surface energy, higher nucleation barriers
  longer correlation lengths,
  slower growth of unstable modes

GOAL: Consistent Set of Equations

1. Equations of Motion:

$$\partial_\alpha T^{\alpha\beta} = 0,$$
$$\partial_\alpha J^\alpha = 0$$

T, μ, s, T^{0i} and T_{ij} in terms of ε, ρ, $\nabla\rho$ and ∇v

2. Conserve Entropy

3. Reproduce Equilibrium
   Phase boundaries, fluctuations
   (requires diffusion)

Starting Point (Choice #1)

$$s = \bar{s}\left(\epsilon_\kappa, \rho\right), \quad \epsilon_\kappa = \epsilon + \frac{\kappa}{2}\rho\nabla^2\rho$$

More general

- **κ would be κ(ε,ρ)**
- **Add terms like**

$$\epsilon_\kappa = \epsilon + \frac{\kappa_{\rho\rho}}{2}\rho\nabla^2\rho + \frac{\kappa_{\epsilon\epsilon}}{2}\epsilon\nabla^2\epsilon + \frac{\kappa_{\rho\epsilon}}{2}\rho\nabla^2\epsilon + \frac{\kappa_{\epsilon\rho}}{2}\epsilon\nabla^2\rho$$

Gradient Corrections to μ=-αT & β=T⁻¹

Given

$$s = \bar{s}\left(\epsilon_\kappa, \rho\right), \quad \epsilon_\kappa = \epsilon + \frac{\kappa}{2}\rho\nabla^2\rho$$

$$\Downarrow$$

$$\delta S = \int d^3r \left\{ \frac{\partial \bar{s}}{\partial \epsilon}\delta\epsilon + \frac{\partial \bar{\beta}}{\partial \rho}\delta\rho \right\},$$

$$= \int d^3r \left\{\beta\delta\epsilon + \alpha\delta\rho\right\},$$

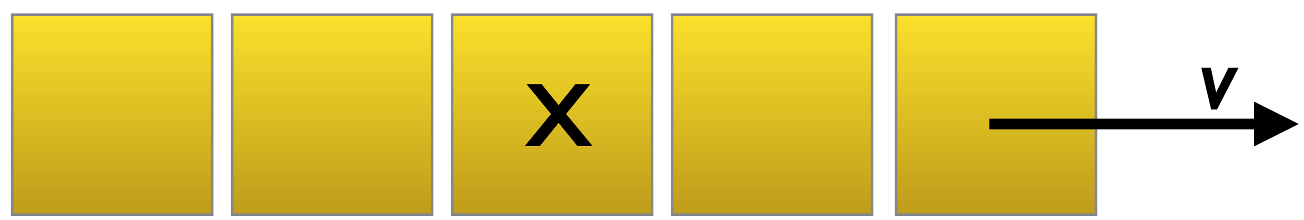
$$\beta = \bar{\beta}(\epsilon_\kappa, \rho),$$

$$\alpha = \bar{\alpha}(\epsilon_\kappa, \rho) + \frac{\kappa\bar{\beta}(\epsilon_\kappa, \rho)}{2}\rho\nabla^2\rho \quad + \frac{\kappa}{2}\rho\nabla^2(\bar{\beta}(\epsilon_\kappa, \rho)\rho)$$

Starting Point (Choice #2)

Entropy moves with charge

Energy (momentum density) moves separately



energy in this cell affected by *v* in neighboring cells

$$\frac{D}{Dt}\rho = -\rho \boldsymbol{\nabla} \cdot \boldsymbol{v},$$

$$\frac{D}{Dt}s = -s \boldsymbol{\nabla} \cdot \boldsymbol{v},$$

$$\frac{D}{Dt}\epsilon = -\epsilon \boldsymbol{\nabla} \cdot \boldsymbol{v} - T_{ij}\partial_i v_j - \boldsymbol{\nabla} \cdot M$$

M_i is additional momentum density, ΔT^{0i}

From here on: everything determined

$$D_t s = \bar{\beta}\left[D_t \epsilon + D_t\left(\frac{\kappa}{2}\rho\nabla^2\rho\right)\right] - \bar{\beta}\bar{\mu}D_t\rho$$

$$= -\bar{\beta}T_{ij}\partial_i v_j + \left[\bar{\beta}\bar{\mu}\rho - \bar{\beta}\epsilon\right]\boldsymbol{\nabla}\cdot\boldsymbol{v} + \bar{\beta}D_t\left(\frac{\kappa}{2}\rho\nabla^2\rho\right) - \bar{\beta}\boldsymbol{\nabla}\cdot\boldsymbol{M}$$

Mess

From here on: everything determined

$$
\begin{aligned}
D_t s &= \bar\beta \left[D_t \epsilon + D_t \left(\frac{\kappa}{2} \rho \nabla^2 \rho \right) \right] - \bar\beta \bar\mu D_t \rho \\
&= -\bar\beta T_{ij} \partial_i v_j + \left[\bar\beta \bar\mu \rho - \bar\beta \epsilon \right] \boldsymbol{\nabla} \cdot \boldsymbol{v} + \bar\beta D_t \left(\frac{\kappa}{2} \rho \nabla^2 \rho \right) - \bar\beta \boldsymbol{\nabla} \cdot \boldsymbol{M} \\
&= -\bar\beta T_{ij} \partial_i v_j + \bar\beta \bar\mu \rho \boldsymbol{\nabla} \cdot \boldsymbol{v} - \bar\beta \epsilon \boldsymbol{\nabla} \cdot \boldsymbol{v} \\
&+ \frac{\kappa \bar\beta}{2} (\partial_i (\rho \partial_j \rho))(\partial_i v_j + \partial_j v_i) + \frac{\kappa \bar\beta}{2} (\partial_i (\rho \partial_i \rho)) \boldsymbol{\nabla} \cdot \boldsymbol{v} \\
&- \frac{\kappa \bar\beta}{2} (\nabla^2 (\rho^2)) \boldsymbol{\nabla} \cdot \boldsymbol{v} - \bar\beta \kappa \rho (\nabla^2 \rho) \boldsymbol{\nabla} \cdot \boldsymbol{v} - \bar\beta \kappa \rho (\partial_i \partial_j \rho) \partial_i v_j \\
&= -s \boldsymbol{\nabla} \cdot \boldsymbol{v}, \\
M_i &= -\frac{\kappa}{2} \rho (\partial_j \rho)(\partial_i v_j + \partial_j v_i) - \frac{\kappa}{2} \rho^2 \partial_i \boldsymbol{\nabla} \cdot \boldsymbol{v} + \frac{\kappa}{2} \rho (\partial_i \rho) \boldsymbol{\nabla} \cdot \boldsymbol{v}
\end{aligned}
$$

lots of algebra ↓

Read off T_{ij}

SUMMARY OF EQUATIONS

$$s = \bar{s}(\epsilon_\kappa, \rho), \quad \epsilon_k = \epsilon + \frac{\kappa}{2}\rho\nabla^2\rho$$

$$\beta = \bar{\beta}(\epsilon_\kappa, \rho),$$

$$\alpha = \bar{\alpha}(\epsilon_\kappa, \rho) + \frac{\bar{\beta}\kappa}{2}\nabla^2\rho + \frac{\kappa}{2}\nabla^2(\bar{\beta}\rho),$$

$$M_i = -\frac{\kappa}{2}\rho(\partial_j\rho)(\partial_i v_j + \partial_j v_i) - \frac{\kappa}{2}\rho^2\partial_i\nabla\cdot v + \frac{\kappa}{2}\rho(\partial_i\rho)\nabla\cdot v.,$$

$$T_{ij} = \bar{P}\delta_{ij} - \kappa\left[\rho\nabla^2\rho + \frac{1}{2}(\nabla\rho)^2\right]\delta_{ij} + \kappa(\partial_i\rho)(\partial_j\rho),$$

$$J^{(\mathrm{D})} = -D\nabla\rho + \frac{\kappa\chi D}{2}\nabla\left[\bar{\beta}\nabla^2\rho + \nabla^2(\bar{\beta}\rho)\right].$$

- Additional momentum density *M* requires velocity gradients
- T_{ij} has off-diagonal elements
- Diffusion altered from altering μ

Consistency between T_{ij}, T, μ

Entropy maximized if β and μ are uniform

⇒ $\partial_i T_{ij}=0$!

Otherwise, you can't reach equilibrium!

$$\partial_i T_{ij}^{(\text{fixed } T)} = \partial_i \bar{P} - \kappa \rho \partial_i \nabla^2 \rho,$$

$$\partial_i \bar{P} = \frac{\partial \bar{P}}{\partial \bar{\mu}} \partial_i \bar{\mu} = \rho \partial_i \bar{\mu},$$

$$\partial_i T_{ij}^{(\text{fixed } T)} = \rho \partial_j \mu \quad ✓$$

Not possible if $T_{ij} \sim \delta_{ij}$

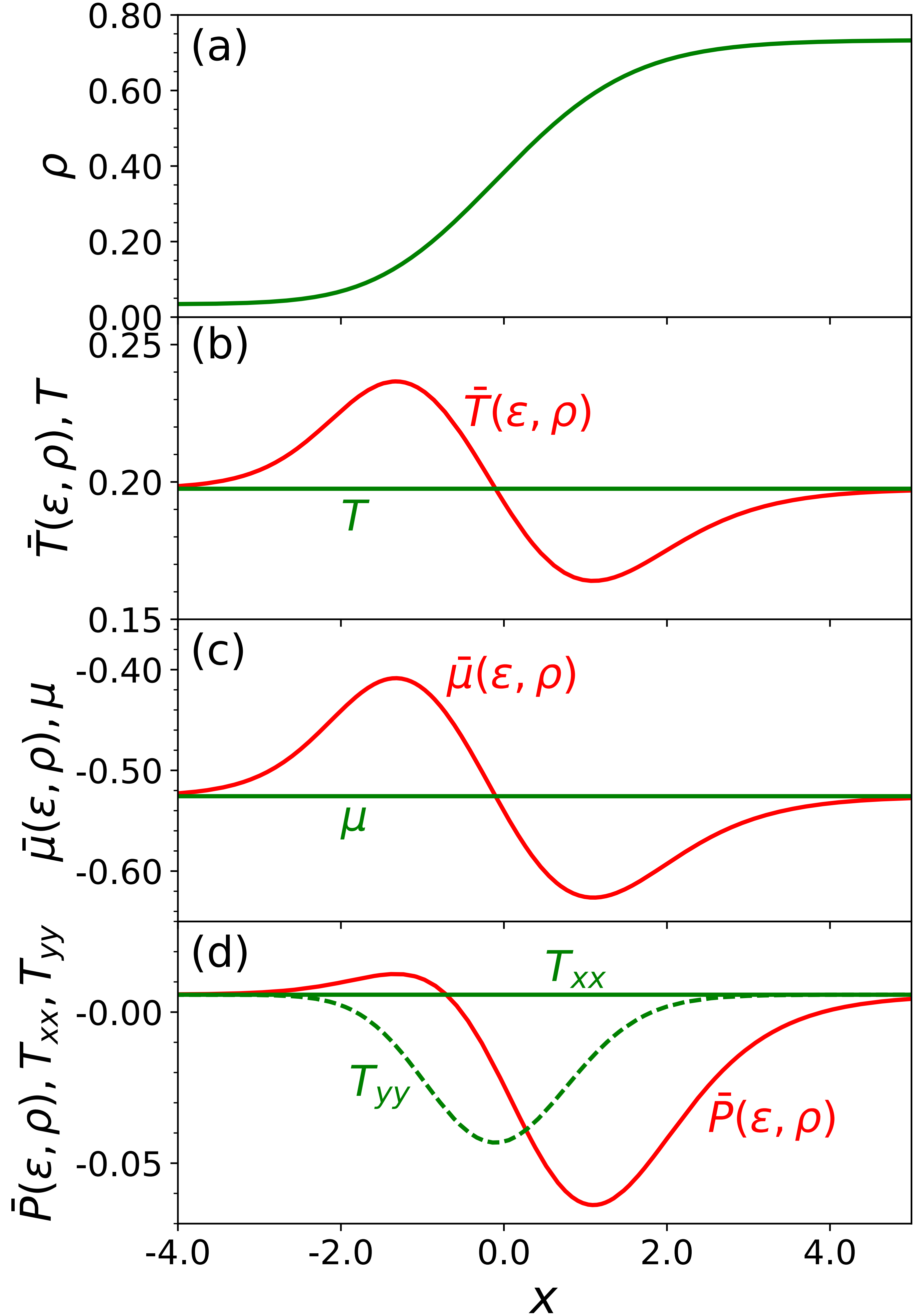
Example 1: Liquid-Gas Phase Interface (at equilibrium)

$$\bar{P}(\epsilon_\kappa, \rho) = \frac{\rho \bar{T}(\epsilon_\kappa, \rho)}{1 - \rho/\rho_s} - a\rho^2.$$

Maximize entropy,

$$\begin{aligned} T\delta S/A &= 0 \\ &= \delta \int dx \left[P_0 - P + (\mu - \mu_0)\rho + \frac{\kappa}{2}(d\rho/dx)^2 \right] \\ \frac{d\rho}{dx} &= \sqrt{2[P_0 - P(\rho) + (\mu(\rho) - \mu_0)\rho]/\kappa} \end{aligned}$$

$\Rightarrow x(\rho)$



Liquid-Gas Phase Transition

Gradient-modified properties are uniform!

$$T = \bar{T}(\epsilon_\kappa, \rho),$$

$$\mu = \bar{\mu}(\epsilon_\kappa, \rho) - \kappa\nabla^2\rho,$$

$$T_{xx} = \bar{P}(\epsilon_\kappa, \rho) - \kappa\rho\nabla^2\rho + \frac{1}{2}(\nabla\rho)^2$$

Example 2: Density-Density Correlations

$$\begin{aligned}
\langle \delta\rho(0)\delta\rho(r)\rangle &= (\chi - \chi_0)\frac{e^{-r/\ell}}{4\pi\ell^2 r} + \chi_0\delta^3(r), \\
\langle \delta\rho(0)\delta T(r)\rangle &= 0, \\
\langle \delta\rho(0)\delta\mu(r)\rangle &= 0, \\
\langle \delta\rho(0)\delta T_{ij}(r)\rangle &= 0, \\
\langle \delta\rho(0)\delta\bar{\mu}[\epsilon(r),\rho(r)]\rangle &= \frac{T}{\chi}\langle \delta\rho(0)\delta\rho(r)\rangle, \\
\langle \delta\rho(0)\delta\bar{\beta}[\epsilon(r),\rho(r)]\rangle &= \frac{1}{\chi_{QE}}\langle \delta\rho(0)\delta\rho(r)\rangle, \\
\langle \delta\rho(0)\delta\bar{P}[\epsilon(r),\rho(r)]\rangle &= \frac{\rho T}{\chi}\langle \delta\rho(0)\delta\rho(r)\rangle
\end{aligned}$$

Altering Hydrodynamic Noise

Noisy current:

$$\langle j_i^{(\mathrm{n})}(0) j_k^{(\mathrm{n})}(r)\rangle = 2\sigma T \delta_{ik}\delta^4(r) + \sigma\kappa\chi_0\partial_i\partial_k\delta^4(r),$$

NEW

$$D = \sigma T/\chi$$

$$\Downarrow$$

$$\langle \delta\rho(0)\delta\rho(r)\rangle = \chi_0\delta^3(r) + \frac{(\chi-\chi_0)T}{\kappa\chi}\frac{e^{-r/\ell}}{4\pi r},$$

$$\ell^2 = \beta\kappa\chi$$

As $T \to T_c$,

$$\chi \to \infty, \quad \ell \to \infty, \quad D \to 0$$

But, κ, χ_0, σ and noise are well behaved!

Summary

Alter Hydrodynamic Equations:

— Gradient terms for s, T_{ij}, μ, T

— Additional momentum density, ~ to ∂v

— Adjust thermal noise

Advantages

— Conserve entropy(aside from diff.) / energy / momentum

— Reproduce thermal correlations (not just fluc.s)

— L.G. Interface properties

— Damp high-momentum modes

Needs Work:

— Highly non-causal diffusion, $\omega \sim k^4$

— Implementing noise beyond linear approx.