THE OHIO STATE UNIVERSITY

Hydrodynamic Approximation Schemes and Gubser Solution: Comparative Studies

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Phys. Rev. C 95, 054907 (2017)

BEST COLLABORATION

Overview

- Compare a wide variety of hydrodynamic models and test their validity
  - Anisotropic hydrodynamics (primary focus)
  - DNMR viscous hydrodynamics
- Testing ground: 0+1d Gubser flow
  - Massless Boltzmann gas subject to Gubser flow
  - Compare to exact solution of Boltzmann kinetic equation in Relaxation Time Approximation (RTA)

Gubser Flow

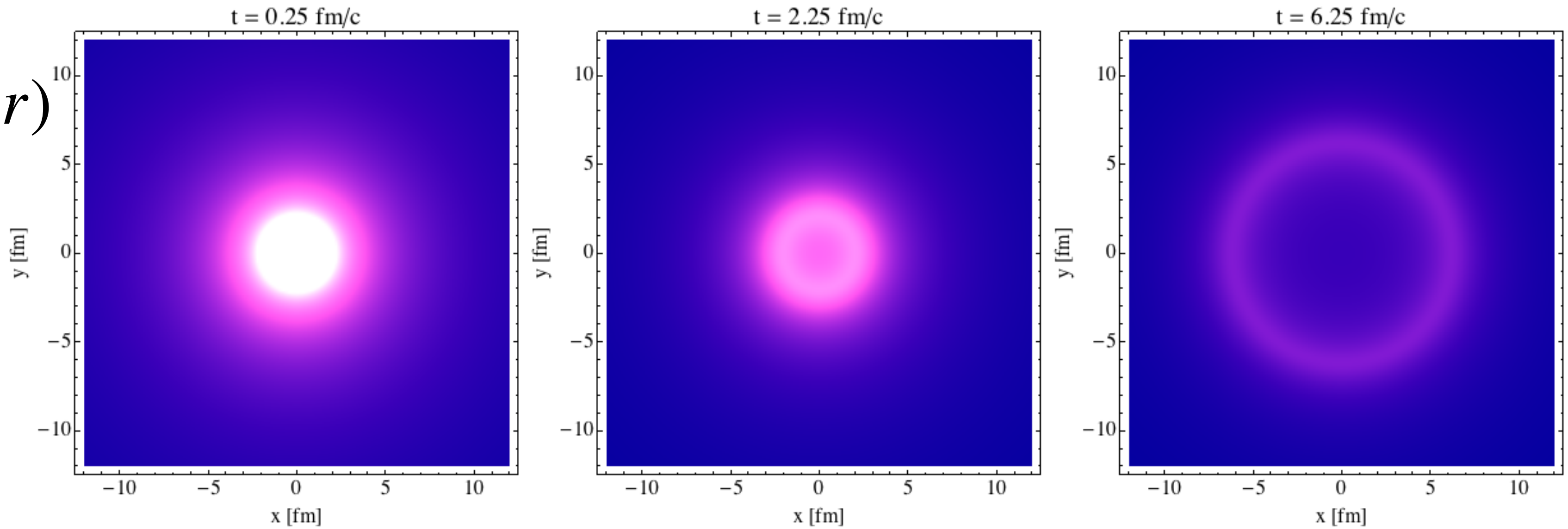
- Flow profile characteristics
  - Like Bjorken flow it has boost - invariant longitudinal expansion
  - Azimuthally symmetric radial flow in transverse plane

$$v_z = \frac{z}{t} \qquad v_r = \frac{2q^2\tau r}{1 + q^2\tau^2 + q^2r^2}$$

q = transverse size scale

"ring of fire"

$T(\tau, r)$



Credit: M. Strickland

Properties

- Boost-invariant, cylindrically symmetric and conformal

$$(\tau, r, \phi, \eta) \ \rightarrow \ (\rho, \theta, \phi, \eta)$$

Milne polar deSitter

$$\sinh\rho = -\frac{1 - q^2\tau^2 + q^2 r^2}{2q\tau}$$

$$\tan\theta = \frac{2qr}{1 + q^2\tau^2 - q^2 r^2}$$

$$\tau = 0^+ \rightarrow \rho = -\infty$$

$$\tau = \infty \rightarrow \rho = \infty$$

$$\hat{u}^\mu = (1, 0, 0, 0)$$

Gubser flow static in deSitter space. Macroscopic quantities only depend on "time" variable ρ. (0+1d)

$$\hat{\theta} = 2\tanh\rho \qquad \hat{\sigma}^\mu_\nu = \tanh\rho \ \mathrm{diag}(0, \frac{1}{3}, \frac{1}{3}, -\frac{2}{3})$$

- Gubser flow is highly anisotropic!

$P_L / P_T << 1$ for early ρ

$P_L / P_T >> 1$ for late ρ

Properties

- Distribution function takes the form: $f(\rho, \hat{p}_\Omega^2, |\hat{p}_\eta|)$
- $T^{\mu\nu}$ is diagonal, two independent components $(\hat{\epsilon}, \hat{\pi}_\eta^\eta)$

$$\hat{T}^\mu_\nu = \text{diag}(-\hat{\epsilon}, \hat{\mathcal{P}}_\perp, \hat{\mathcal{P}}_\perp, \hat{\mathcal{P}}_L)$$

$$\hat{\epsilon} = 2\hat{\mathcal{P}}_\perp + \hat{\mathcal{P}}_L$$

Conformal EoS

$$\hat{\pi}^\mu_\nu = \text{diag}(0, -\hat{\pi}^\eta_\eta/2, -\hat{\pi}^\eta_\eta/2, \hat{\pi}^\eta_\eta)$$

- $\hat{\pi} = \hat{\pi}^\eta_\eta = \frac{2}{3}\left(\hat{\mathcal{P}}_L - \hat{\mathcal{P}}_\perp\right)$ can grow large due to strong anisotropic flow

$$\partial_\rho \ln \hat{\epsilon} = \frac{4}{3}(\hat{\bar{\pi}} - 2)\tanh\rho \quad \text{(energy conservation law)} \qquad \hat{\bar{\pi}} = \frac{3\hat{\pi}}{4\hat{\epsilon}}$$

- Remaining task: determine evolution of $\hat{\bar{\pi}}(\rho)$

Exact Solution

- RTA Boltzmann equation is solvable for Gubser flow

$$\partial_\rho f(\rho, \hat{p}_\Omega^2, |\hat{p}_\eta|) = -\frac{f(\rho, \hat{p}_\Omega^2, |\hat{p}_\eta|) - f_{eq}(\rho, \hat{p}_\Omega^2, |\hat{p}_\eta|)}{\hat{\tau}_r} \qquad \hat{\tau}_r = \frac{5\,\hat{\eta}/\hat{s}}{\hat{T}(\rho)}$$

- Solve ordinary differential equation for $f(\rho, \hat{p}_\Omega^2, |\hat{p}_\eta|)$
- Compute $\hat{\epsilon}(\rho)$ and $\hat{\bar{\pi}}(\rho)$ directly
- We'll compare hydrodynamic models with exact solution.

Denicol et al., Phys. Rev. D 90, 125026 (2014)

Viscous Hydrodynamics

- DNMR viscous hydrodynamics $f = f_{eq} + \delta f$
  - Derived from Boltzmann equation in the 14 moment approximation

$$f_{eq}(\hat{x},\hat{p}) = \exp\left[-\frac{1}{\hat{T}(\hat{x})}\sqrt{\frac{\hat{p}_\Omega^2}{\cosh^2\rho} + \hat{p}_\eta^2}\right]$$

$$\hat{\tau}_r \dot{\hat{\pi}}^{\langle\mu\nu\rangle} + \hat{\pi}^{\mu\nu} = -2\hat{\eta}\hat{\sigma}^{\mu\nu} - \frac{4}{3}\hat{\pi}^{\mu\nu}\hat{\theta} - \frac{10}{7}\hat{\pi}^{\lambda\langle\mu}\hat{\sigma}_\lambda^{\nu\rangle}$$

- deSitter coordinates

$$\partial_\rho \hat{\bar{\pi}} + \frac{\hat{\bar{\pi}}}{\hat{\tau}_r} = \frac{4}{3}\tanh\rho\left(\frac{1}{5} + \frac{5}{14}\hat{\bar{\pi}} - \hat{\bar{\pi}}^2\right)$$

$$\partial_\rho \ln\hat{\epsilon} = \frac{4}{3}(\hat{\bar{\pi}} - 2)\tanh\rho$$

May break down for large shear forces

$$\hat{\bar{\pi}} = \frac{3\hat{\pi}}{4\hat{\epsilon}}$$

Denicol et al., Phys. Rev. D 90, 125026 (2014)

Anisotropic Hydrodynamics

- Capture large shear stress with a leading order anisotropic distribution function f_a (Romatschke-Strickland (RS))

$$f = f_a + \delta\tilde{f} \qquad f_a(\hat{x},\hat{p}) = \exp\left[-\frac{1}{\hat{\Lambda}(\hat{x})}\sqrt{\frac{\hat{p}_\Omega^2}{\cosh^2\rho} + (1+\xi(\hat{x}))\hat{p}_\eta^2}\right]$$

- ξ(ρ): momentum scaling factor characterizes pressure anisotropy
  - A dynamical variable sensitive to shear forces
  - f_a optimized for cylindrical flow
- Λ(ρ): effective temperature
  - Energy conservation law provides 1 of 2 equations

$$\hat{\bar{\pi}}_{RS}(\xi) = \frac{1}{4}\left(\frac{3\hat{\mathcal{R}}_{220}(\xi)}{\hat{\mathcal{R}}_{200}(\xi)} - 1\right)$$

$-1 < \xi < \infty$

$$\hat{\Lambda} = \hat{T}\,\hat{\mathcal{R}}_{200}^{1/4}(\xi) \quad \text{(Landau matching condition)}$$

$\frac{1}{2} > \hat{\bar{\pi}}_{RS} > -\frac{1}{4}$

The Closing Kinetic Equation

- Select a kinetic moment to evolve ξ(ρ) with Boltzmann Eq.
- Look at various kinetic equations
  - P_L moment, 0^{th} and 2^{nd} moments of Boltzmann equation
  - Include residual shear correction to improve accuracy

$$f = f_a + \delta\tilde{f}$$

$$\hat{\bar{\pi}}_{\text{tot}} = \hat{\bar{\pi}}_{RS} + \hat{\tilde{\bar{\pi}}}$$

$$\partial_\rho \hat{\tilde{\bar{\pi}}} = \partial_\rho \left(\int_{\hat{p}} \hat{p}^{\langle\eta}\hat{p}^{\eta\rangle}\, \delta\tilde{f} \right)$$

Equation of motion for $\hat{\tilde{\bar{\pi}}}$

- Important guideline: recover viscous hydrodynamics

For $f = f_a \to f_{eq} + (f_a - f_{eq})$	For $f = f_a + \delta\tilde{f} \to f_{eq} + \delta\tilde{f}$
It's |ξ| << 1	It's ξ = 0

0^{th} Moment

- What you shouldn't be doing…
- You have a conformal system with no chemical potential…but you try to evolve ξ with 0^{th} moment of the Boltzmann equation

$$\hat{D}_\mu \, \hat{n}_a^\mu = -\frac{(\hat{n}_a - \hat{n}_{eq})}{\hat{\tau}_r} \rightarrow 0 \qquad \hat{n}_a = \int_{\hat{P}} (\hat{u} \cdot \hat{p}) f_a$$

- Particle production rate vanishes for ξ = 0 ($\hat{\bar{\pi}}_{\text{tot}} = 0$)
  - Requirement satisfied at leading order ($\hat{\bar{\pi}}_{RS}(0) = 0$)
  - Disaster once you add $\hat{\tilde{\bar{\pi}}}$ as a dynamical degree of freedom

$$\hat{\bar{\pi}}_{\text{tot}} = \hat{\bar{\pi}}_{RS} + \hat{\tilde{\bar{\pi}}} \qquad \partial_\rho \ln \hat{\epsilon} = \frac{4}{3}(\hat{\tilde{\bar{\pi}}} - 2) \tanh \rho \qquad \hat{\tilde{\bar{\pi}}} \rightarrow 0$$

- Unable to map smoothly to vHydro: solution fails (backup slide)
- *0^{th} moment should be used to evolve chemical potentials, not ξ*

P_L Matching

- Let ξ capture all of the pressure anisotropy: $\delta\hat{\tilde{\mathcal{P}}}_L \equiv 0$ or $\hat{\tilde{\pi}} \equiv 0$
  - Evolve P_L (or $\hat{\pi}$) for the pressure anisotropy like in DNMR

$$\partial_\rho \hat{\mathcal{P}}_L = \partial_\rho \left(\int_{\hat{p}} \hat{p}_\eta^2 \, f \right)$$

- In terms of $\hat{\bar{\pi}}$:

$$\partial_\rho \hat{\bar{\pi}} + \frac{\hat{\bar{\pi}}}{\hat{\tau}_r} = \frac{4}{3} \tanh\rho \left(\frac{5}{16} + \hat{\bar{\pi}} - \hat{\bar{\pi}}^2 - \frac{9}{16} \mathcal{F}(\hat{\bar{\pi}}) \right)$$

Linearizes to DNMR in |ξ| << 1 limit

$$\partial_\rho \xi + \frac{1}{\hat{\tau}_r} \frac{\hat{\bar{\pi}}(\xi)}{\partial_\xi \hat{\bar{\pi}}(\xi)} = -2 \tanh\rho \, (1+\xi)$$

(xi version)

L. Tinti, Phys. Rev. C 94, 044902 (2016)

E. Molnar, H. Niemi, and D.H Rischke, Phys. Rev. D 94, 125003 (2016)

NRS*

- Evolve ξ with the 2nd moment of the Boltzmann equation

$$\hat{D}_\mu \hat{\mathcal{I}}_a^{\mu\nu\lambda} = -\frac{\hat{u}_\mu\left(\hat{\mathcal{I}}_a^{\mu\nu\lambda} - \hat{\mathcal{I}}_{eq}^{\mu\nu\lambda}\right)}{\hat{\tau}_r}$$

$$\hat{\mathcal{I}}_a^{\mu\nu\lambda} = \int_{\hat{P}} \hat{p}^\mu \hat{p}^\nu \hat{p}^\lambda f_a$$

- Combine 2 independent projections

$$\hat{\mathcal{I}}_\eta = \int_{\hat{P}} (\hat{u}\cdot\hat{p})\hat{p}_\eta^2 f_a$$

$$(1+\xi)\partial_\rho \hat{\mathcal{I}}_\eta - \frac{1}{2}\partial_\rho \hat{\mathcal{I}}_\Omega$$

$$\hat{\mathcal{I}}_\Omega = \int_{\hat{P}} (\hat{u}\cdot\hat{p})\frac{\hat{p}_\Omega^2}{\cosh^2\rho} f_a$$

$$\boxed{\partial_\rho \xi + \frac{\xi(1+\xi)^{3/2}\,\hat{\mathcal{R}}_{200}^{5/4}(\xi)}{\hat{\tau}_r} = -2\tanh\rho\,(1+\xi)}$$

* M. Nopoush, R. Ryblewski, and M. Strickland, Phys. Rev. D 91, 045007 (2015)

Next-to-Leading Order NRS

- Must include residual shear stress $\hat{\bar{\pi}}$ to NRS (NLO NRS)

$$\partial_\rho \hat{\bar{\pi}} = \partial_\rho \left(\int_{\hat{p}} \hat{p}^{\langle\eta}\hat{p}^{\eta\rangle}\, \delta\tilde{f} \right)$$

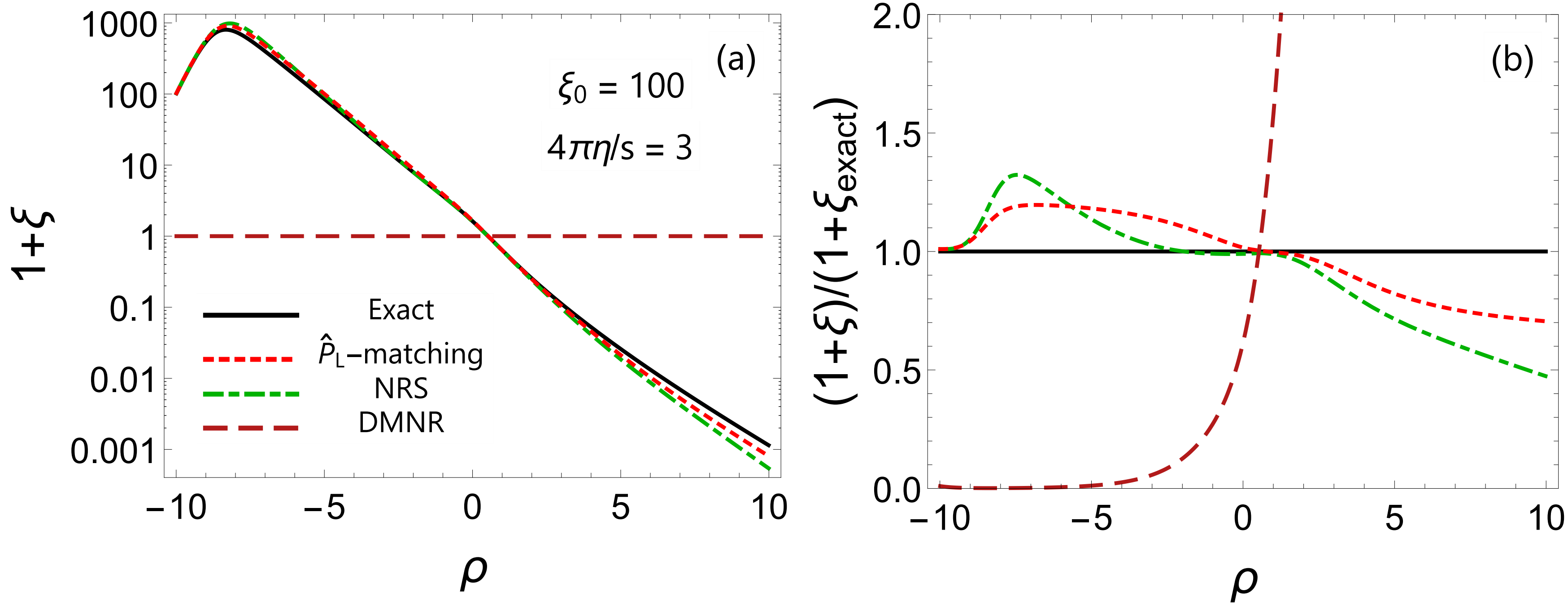
(Evolve using Boltzmann equation)

$$\partial_\rho \hat{\bar{\pi}} = -\frac{\hat{\pi}_{\rm RS}+\hat{\bar{\pi}}}{\hat{\tau}_r} - \tanh\rho \left[\frac{4}{3}\hat{\bar{\pi}} + \hat{\alpha}\,\hat{I}_{240} - \hat{\beta}\,\hat{I}_{340} + \frac{4}{3}\hat{\omega}\,\hat{I}_{440} + \frac{1}{2}\hat{\omega}_{\langle\eta\eta\rangle}\left(3\,\hat{I}_{460} - \hat{I}_{440}\right)\right]$$
$$- \frac{\partial_\rho\hat{\Lambda}}{\hat{\Lambda}^2}\left(\hat{\mathcal{H}}_{221} - \frac{1}{3}\hat{\mathcal{H}}_{201}\right) - \frac{\tanh\rho}{\hat{\Lambda}}\left(\frac{4}{3}\hat{\mathcal{H}}_{42-1} - \hat{\mathcal{H}}_{44-1} - \frac{1}{3}\hat{\mathcal{H}}_{40-1}\right) + \frac{\partial_\rho\xi}{2\hat{\Lambda}}\left(\hat{\mathcal{H}}_{44-1} - \frac{1}{3}\hat{\mathcal{H}}_{42-1}\right)$$

- Lengthy…but only a small correction
- Viscous hydrodynamics recovered in $\xi = 0$ limit

Results

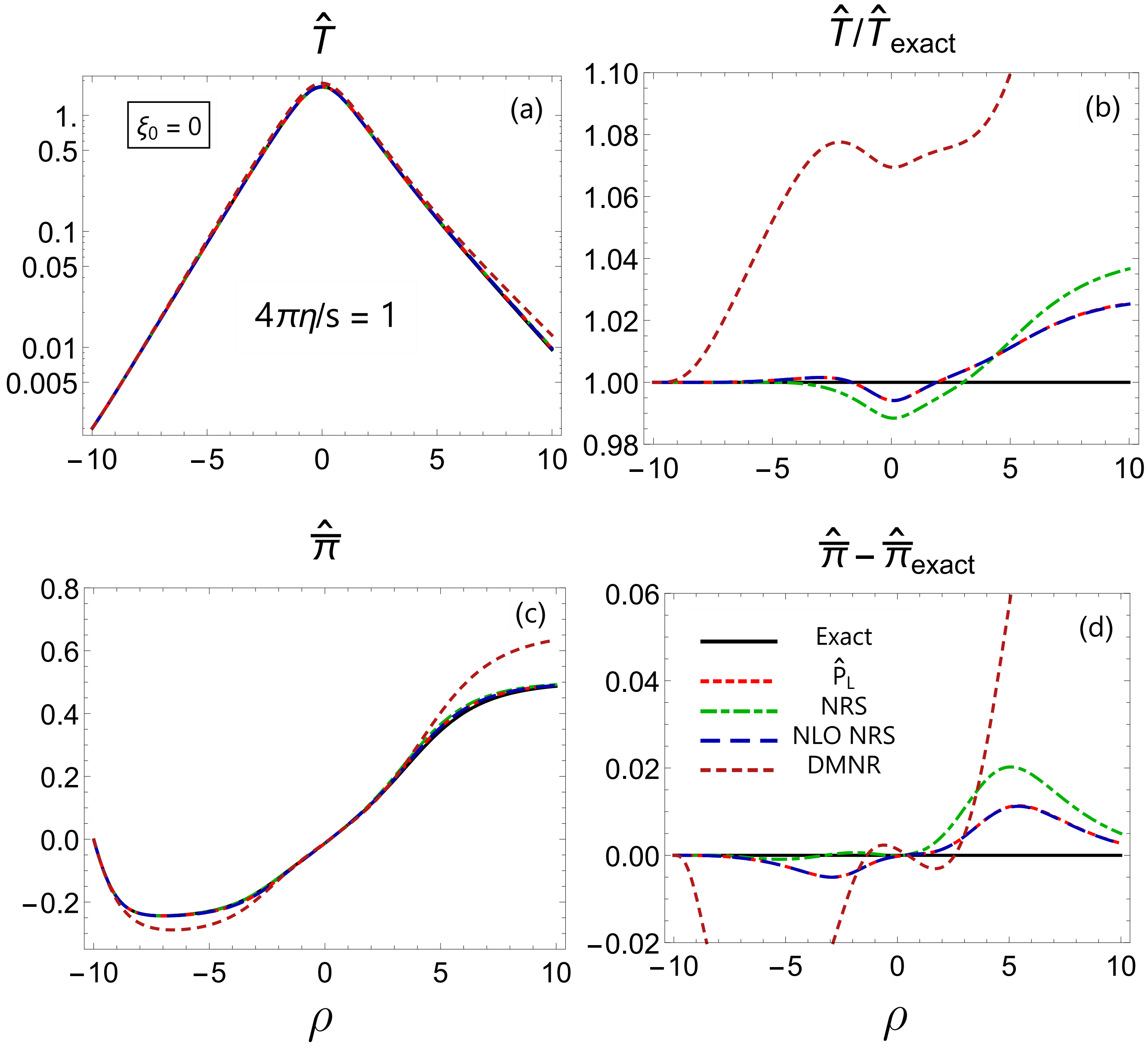
ξ Evolution



$$\hat{\bar{\pi}}(\xi) = \frac{1}{4}\left(\frac{3\hat{\mathcal{R}}_{220}(\xi)}{\hat{\mathcal{R}}_{200}(\xi)} - 1\right)$$

$\hat{\bar{\pi}} \to -1/4 \qquad \xi \to \infty$

$\hat{\bar{\pi}} \to 1/2 \qquad \xi \to -1$



Early ρ behavior

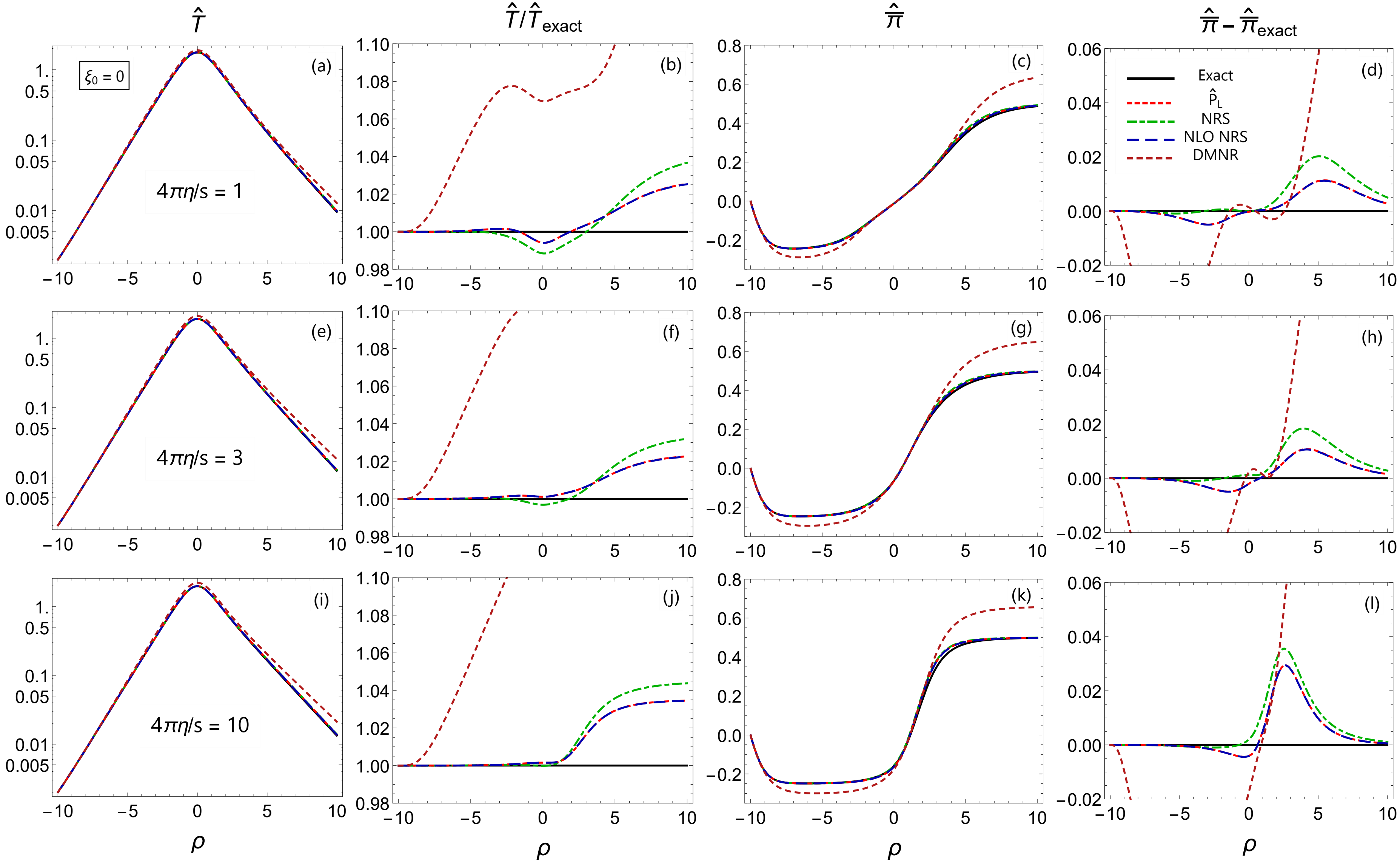
$$\hat{T} \propto (\cosh\rho)^{-3/4}$$

$$\hat{\bar{\pi}} \to -1/4$$

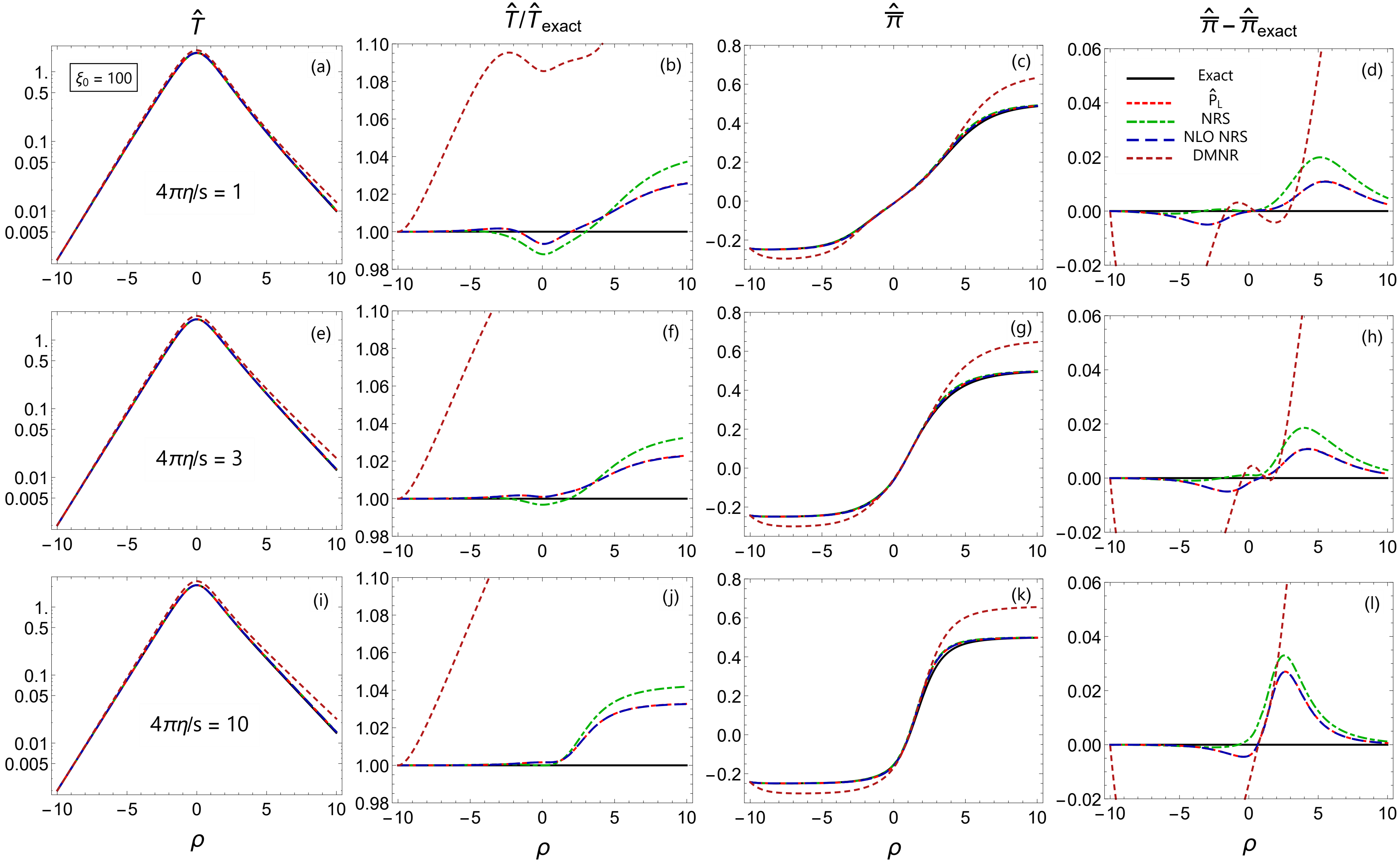
Late ρ behavior

$$\hat{T} \propto (\cosh\rho)^{-1/2}$$

$$\hat{\bar{\pi}} \to 1/2$$



- Rapid information loss of initial state
- Far from equilibrium for most of the evolution



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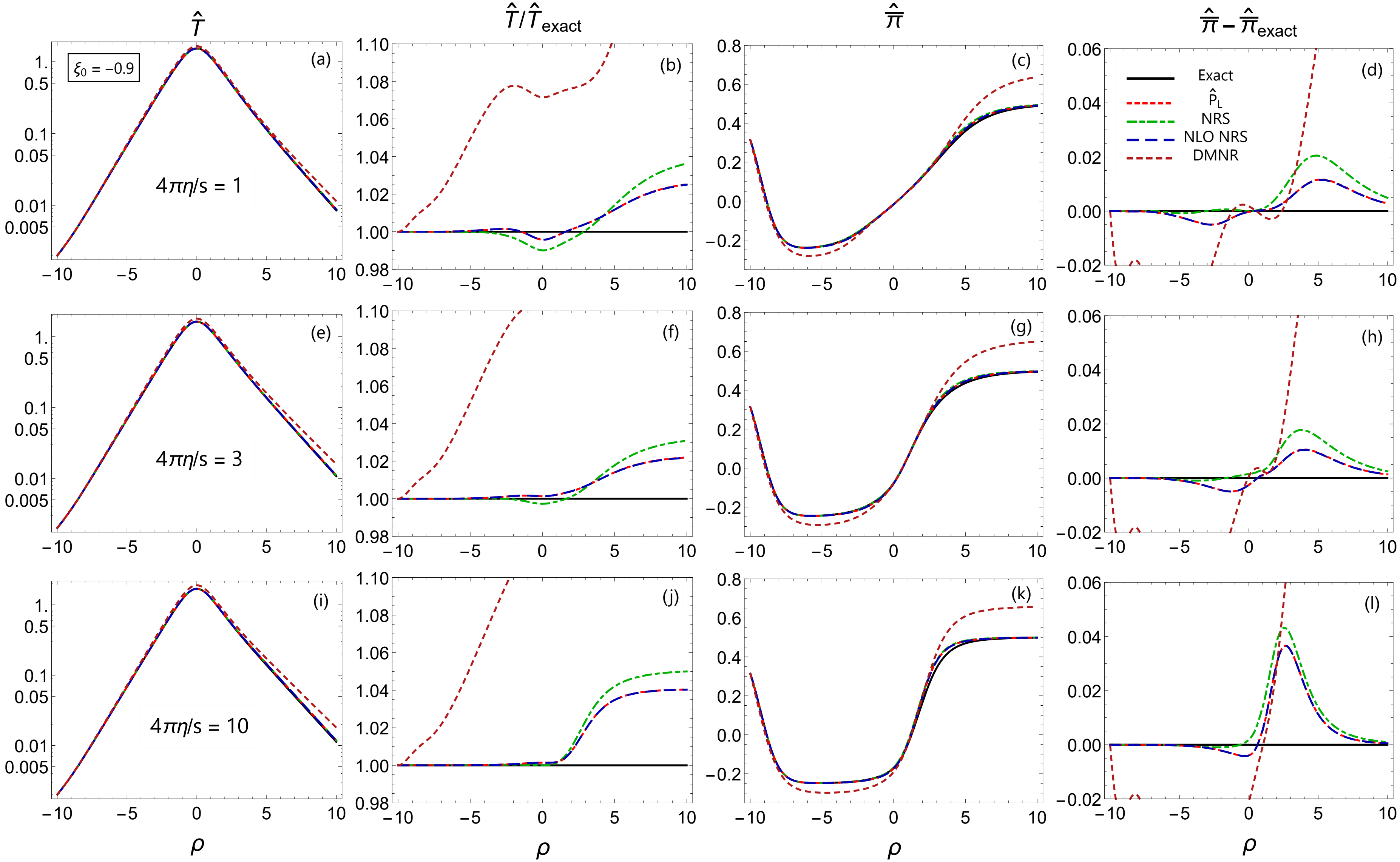

P_L = NLO NRS > NRS > DNMR

Conclusion and Outlook

- The P_L matching is the easiest and most effective anisotropic hydrodynamic model for capturing cylindrical-type flows.
- A first version of 3+1d viscous anisotropic hydro code has been developed (D. Bazow)
- Continue to develop and test it:
  - Anisotropic Cooper Frye interface with residual shear corrections
  - Evolve the transverse pressure P_T dynamically
  - Include net baryon transport and diffusion

Thanks! 😉

Backup Slides

0^{th} Moment Failure at Next-to-Leading Order